Heat kernels on metric measure spaces

A.Grigor'yan

Lectures at Cornell Probability Summer School, July 2010

Contents

Chapter 1

The notion of the heat kernel

1.1 Examples of heat kernels

1.1.1 Heat kernel in $\mathbb{R}^n$

The classical Gauss-Weierstrass heat kernel is the following function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right), \qquad ((1))$$

where $x, y \in \mathbb{R}^n$ and $t > 0$. This function is a fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. Moreover, if f is a continuous bounded function on $\mathbb{R}^n$ then the function

$$u(t,x) = \int_{\mathbb{R}^n} p_t(x,y) f(y)\, dy$$

solves the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ u(0,x) = f(x) \end{cases}.$$

This also can be written in the form

$$u(t,\cdot) = \exp(-t\mathcal{L}) f\ ,$$

where $\mathcal{L}$ here is a self-adjoint extension of $-\Delta$ in $L^2(\mathbb{R}^n)$ and $\exp(-t\mathcal{L})$ is understood in the sense of the functional calculus of self-adjoint operators. This means that $p_t(x,y)$ is the integral kernel of the operator $\exp(-t\mathcal{L})$.

The function $p_t(x,y)$ has also a probabilistic meaning: it is the transition density of Brownian motion $\{X_t\}_{t \geq 0}$ in $\mathbb{R}^n$. The graph of $p_t(x,0)$ as a function of x is shown here:

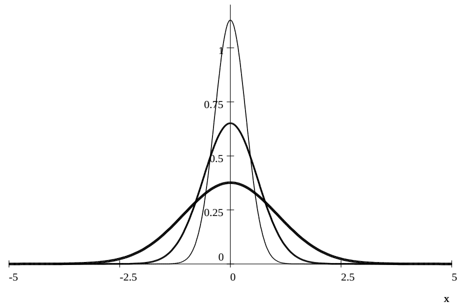

Figure 1.1: The Gauss-Weierstrass heat kernel at different values of t

The term $\frac{|x-y|^2}{t}$ determines the *space/time scaling*: if $|x-y|^2 \leq Ct$ then $p_t(x,y)$ is comparable with $p_t(x,x)$, that is, the probability density in the $C\sqrt{t}$-neighborhood of x is nearly constant.

1.1.2 Heat kernels on Riemannian manifolds

Let (M,g) be a connected Riemannian manifold, and Δ be the Laplace-Beltrami operator on M. Then the heat kernel $p_t(x,y)$ is can be defined as the integral kernel of the heat semigroup $\{\exp(-t\mathcal{L})\}_{t\geq 0}$ where $\mathcal{L}$ is the Dirichlet Laplace operator, that is, the minimal self-adjoint extension of $-\Delta$ in $L^2(M,\mu)$, and μ is the Riemannian volume. Alternatively, $p_t(x,y)$ is the minimal positive fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

The function $p_t(x,y)$ can be used to define Brownian motion $\{X_t\}_{t\geq 0}$ on M. Namely, $\{X_t\}_{t\geq 0}$ is a *diffusion process* (that is, a Markov process with continuous trajectories), such that

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x,y)\,d\mu(y)$$

for any Borel set $A \subset M$.

Let $d(x,y)$ be the geodesic distance on M. It turns out that the *Gaussian* type space/time scaling $\frac{d^2(x,y)}{t}$ appears in heat kernel estimates on general Riemannian manifolds:

1. (*Varadhan*) For an arbitrary Riemannian manifold,

$$\log p_t(x,y) \sim -\frac{d^2(x,y)}{4t} \quad \text{as } t \to 0,$$

2. (*Davies*) For an arbitrary manifold M, for any two measurable sets $A, B \subset M$

$$\int_A \int_B p_t(x,y)\,d\mu(x)\,d\mu(y) \leq \sqrt{\mu(A)\,\mu(B)}\exp\left(-\frac{d^2(A,B)}{4t}\right)$$

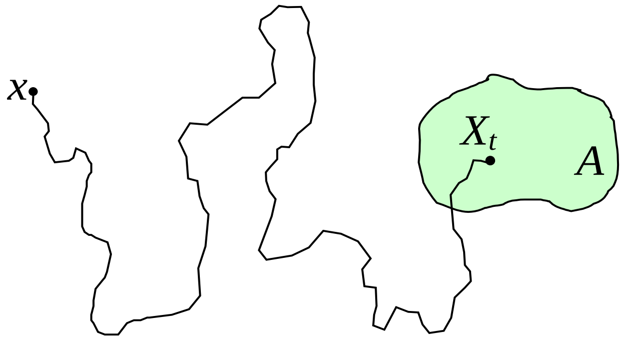

Figure 1.2: The Brownian motion X_t hits a set A

Technically, all these results depend upon the following property of the geodesic distance: $|\nabla d| \leq 1$.

It is natural to ask the following question:

Are there settings where the space/time scaling is different from Gaussian?

1.1.3 Heat kernels of fractional powers of Laplacian

Easy examples can be constructed using another operator instead of the Laplacian. As above, let $\mathcal{L}$ be the Dirichlet Laplace operator on a Riemannian manifold M, and consider the evolution equation

$$\frac{\partial u}{\partial t} + \mathcal{L}^{\beta/2} u = 0,$$

where $\beta \in (0,2)$. The operator $\mathcal{L}^{\beta/2}$ is understood in the sense of the functional calculus in $L^2(M,\mu)$. Let $p_t(x,y)$ be now the heat kernel of $\mathcal{L}^{\beta/2}$, that is, the integral kernel of $\exp\left(-t\mathcal{L}^{\beta/2}\right)$.

The condition $\beta < 2$ leads to the fact that the semigroup $\exp\left(-t\mathcal{L}^{\beta/2}\right)$ is *Markovian*, which, in particular, means that $p_t(x,y) > 0$ (if $\beta > 2$ then $p_t(x,y)$ may be signed). Using the techniques of subordinators, one obtains the following estimate for the heat kernel of $\mathcal{L}^{\beta/2}$ in $\mathbb{R}^n$:

$$p_t(x,y) \asymp \frac{C}{t^{n/\beta}}\left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)} \asymp \frac{C}{t^{n/\beta}}\left(1 + \frac{|x-y|^\beta}{t}\right)^{-\frac{n+\beta}{\beta}}. \qquad ((2))$$

(the symbol $\asymp$ means that both $\leq$ and $\geq$ are valid but with different values of the constant C).

The heat kernel of $\sqrt{\mathcal{L}}$ in $\mathbb{R}^n$ (that is, the case $\beta = 1$) is known explicitly:

$$p_t(x,y) = \frac{c_n}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}} = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}},$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$. This function coincides with the Poisson kernel in the half-space $\mathbb{R}^{n+1}_+$ and with the density of the Cauchy distribution in $\mathbb{R}^n$ with the parameter t.

As we see, the space/time scaling is given by the term $\frac{d^\beta(x,y)}{t}$ where $\beta < 2$. The heat kernel of the operator $\mathcal{L}^{\beta/2}$ is the transition density of a *symmetric stable process of index* β that belongs to the family of Levy processes. The trajectories of this process are discontinuous, thus allowing jumps. The heat kernel $p_t(x,y)$ of such process is nearly constant in a $Ct^{1/\beta}$-neighborhood of y. If t is large then

$$t^{1/\beta} \gg t^{1/2},$$

that is, this neighborhood is much larger than that for the diffusion process, which is not surprising because of the presence of jumps. The space/time scaling with $\beta < 2$ is called *super-Gaussian.*

1.1.4 Heat kernels on fractal spaces

A rich family of heat kernels for diffusion processes has come from Analysis on *fractals.* Loosely speaking fractals are subsets of $\mathbb{R}^n$ with certain self-similarity properties. One of the best understood fractals is *the Sierpinski gasket* (SG). The construction of the Sierpinski gasket is similar to the Cantor set: one starts with a triangle as a closed subset of $\mathbb{R}^2$, then eliminates the open middle triangle (shaded on the diagram), then repeats this procedure for the remaining triangles, etc.

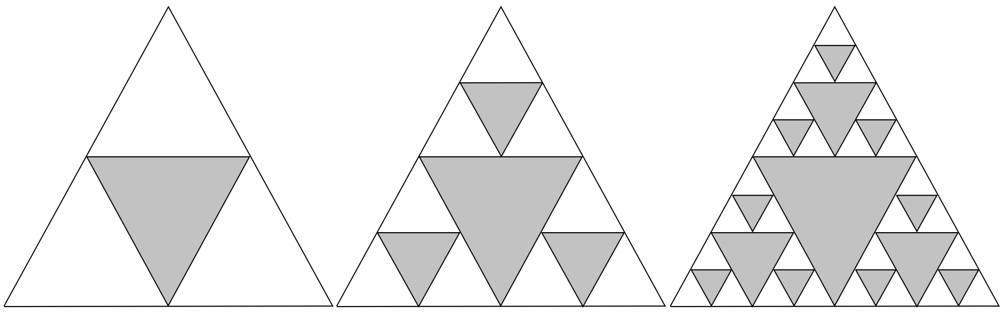

Figure 1.3: Construction of the Sierpinski gasket

Hence, SG is a compact connected subset of $\mathbb{R}^2$. The *unbounded* SG is obtained from SG by merging the latter (at the left lower corner of the next diagram) with two shifted copies and then by repeating this procedure at larger scales.

Barlow and Perkins '88, Goldstein '87 and Kusuoka '87 have independently constructed by different methods a natural diffusion process on SG that has the same self-similarity as SG. Barlow and Perkins considered random walks on the graph approximations of SG and showed that, with an appropriate scaling, the random walks converge to a diffusion process. Moreover, they proved that this process has

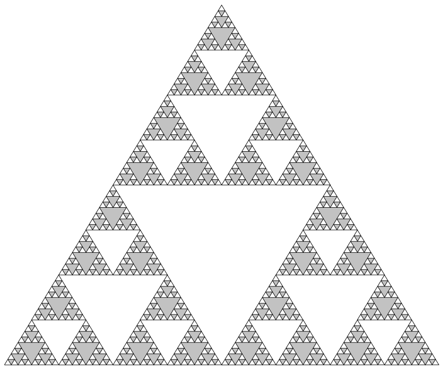

Figure 1.4: The unbounded SG is obtained from SG by merging the latter (at the left lower corner of the diagram) with two shifted copies and then by repeating this procedure at larger scales.

a transition density $p_t(x,y)$ with respect to a proper Hausdorff measure μ of SG, and that p_t satisfies the following estimate:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right), \qquad ((3))$$

where $d(x,y) = |x-y|$ and

$$\alpha = \dim_H SG = \frac{\log 3}{\log 2}, \qquad \beta = \frac{\log 5}{\log 2} > 2.$$

Similar estimates were proved by Barlow and Bass for other families of fractals, including Sierpinski carpets, and the parameters α and β in (3) are determined by the intrinsic properties of the fractal. In all cases, α is the Hausdorff dimension (which is also called the *fractal dimension*). The parameter β, that is called the *walk dimension*, is larger than 2 in all interesting examples.

The heat kernel $p_t(x,y)$, satisfying (3) is nearly constant in a $Ct^{1/\beta}$-neighborhood of y. If t is large then

$$t^{1/\beta} \ll t^{1/2},$$

that is, this neighborhood is much smaller than that for the diffusion process, which is due to the presence of numerous holes-obstacles that the Brownian particle must bypass. The space/time scaling with $\beta > 2$ is called *sub-Gaussian*.

1.1.5 Summary of examples

Observe now that in all the above examples, the heat kernel estimates can be unified as follows:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c\frac{d(x,y)}{t^{1/\beta}}\right), \qquad ((4))$$

where α, β are positive parameters and $\Phi(s)$ is a positive decreasing function on $[0,+\infty)$. For example, the Gauss-Weierstrass function (1) satisfies (4) with $\alpha = n$, $\beta = 2$ and

$$\Phi(s) = \exp\left(-s^2\right)$$

(Gaussian estimate).

The heat kernel (2) of the symmetric stable process in $\mathbb{R}^n$ satisfies (4) with $\alpha = n$, $0 < \beta < 2$, and

$$\Phi(s) = (1+s)^{-(\alpha+\beta)}$$

(super-Gaussian estimate).

The heat kernel (3) of diffusions on fractals satisfies (4) with $\beta > 2$ and

$$\Phi(s) = \exp\left(-s^{\frac{\beta}{\beta-1}}\right)$$

(sub-Gaussian estimate).

There are at least two questions related to the estimates of the type (4):

1. What values of the parameters α, β and what functions Φ can actually occur in the estimate (4)?

2. How to obtain estimates of the type (4)?

To give these question a precise meaning, we must define what is a heat kernel.

1.2 Abstract heat kernels

Let (M,d) be a locally compact separable metric space and μ be a Radon measure on M with full support. The triple (M,d,μ) will be called a *metric measure space*.

Definition. A family $\{p_t\}_{t>0}$ of measurable functions $p_t(x,y)$ on $M \times M$ is called a *heat kernel* if the following conditions are satisfied, for μ-almost all $x, y \in M$ and all $s, t > 0$:

(i) Positivity: $p_t(x,y) \geq 0$.

(ii) The total mass inequality:

$$\int_M p_t(x,y)\,d\mu(y) \leq 1.$$

(iii) Symmetry: $p_t(x,y) = p_t(y,x)$.

(*iv*) The semigroup property:

$$p_{s+t}(x,y) = \int_M p_s(x,z)p_t(z,y)d\mu(z).$$

(*v*) Approximation of identity: for any $f \in L^2 := L^2(M,\mu)$,

$$\int_M p_t(x,y)f(y)d\mu(y) \overset{L^2}{\longrightarrow} f(x) \quad \text{as } t \to 0+.$$

If in addition we have, for all $t > 0$ and almost all $x \in M$,

$$\int_M p_t(x,y)d\mu(y) = 1$$

then the heat kernel p_t is called *stochastically complete* (or *conservative*).

1.3 A heat semigroup

Any heat kernel gives rise to the family of operators$\{P_t\}_{t\geq 0}$ where $P_0 = \mathrm{id}$ and P_t for $t > 0$ is defined by

$$P_t f(x) = \int_M p_t(x,y)f(y)d\mu(y),$$

where f is a measurable function on M. It follows from $(i) - (ii)$ that the operator P_t is *Markovian*, that is, $f \geq 0$ implies $P_t f \geq 0$ and $f \leq 1$ implies $P_t f \leq 1$. It follows that P_t is a bounded operator in L^2 and, moreover, is a contraction, that is, $\|P_t\|_{L^2 \to L^2} \leq 1$.

The symmetry property (*iii*) implies that the operator P_t is *symmetric* and, hence, self-adjoint. The semigroup property (*iv*) implies that $P_t P_s = P_{t+s}$, that is, the family $\{P_t\}_{t\geq 0}$ is a *semigroup* of operators. It follows from (*v*) that

$$s\text{-}\lim_{t\to 0} P_t = \mathrm{id} = P_0$$

where s-lim stands for the *strong* limit. Hence, $\{P_t\}_{t\geq 0}$ is a strongly continuous, symmetric, Markovian semigroup in L^2. We say shortly that $\{P_t\}$ is a *heat semigroup*.

Conversely, if $\{P_t\}$ is a heat semigroup and if it has the integral kernel $p_t(x,y)$ then the latter is a heat kernel in the sense of the above Definition.

Given a heat semigroup P_t in L^2, define the *infinitesimal generator* $\mathcal{L}$ of the semigroup by

$$\mathcal{L}f := \lim_{t\to 0} \frac{f - P_t f}{t},$$

where the limit is understood in the L^2-norm. The *domain* $\mathrm{dom}(\mathcal{L})$ of the generator $\mathcal{L}$ is the space of functions $f \in L^2$ for which the above limit exists. By the Hille–Yosida theorem, $\mathrm{dom}(\mathcal{L})$ is dense in L^2. Furthermore, $\mathcal{L}$ is a self-adjoint, positive definite operator, which immediately follows from the fact that the semigroup $\{P_t\}$ is self-adjoint and contractive. Moreover, P_t can be recovered from $\mathcal{L}$ as follows

$$P_t = \exp\left(-t\mathcal{L}\right),$$

where the right hand side is understood in the sense of spectral theory.

Heat kernels and heat semigroups arise naturally from Markov processes. Let $\left(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M}\right)$ be a Markov process on M, that is reversible with respect to measure μ. Assume that it has the transition density $p_t(x,y)$, that is, a function such that, for all $x \in M$, $t > 0$, and all Borel sets $A \subset M$,

$$\mathbb{P}_x\left(X_t \in A\right) = \int_M p_t(x,y)\, d\mu(y).$$

Then $p_t(x,y)$ is a heat kernel in the sense of the above Definition.

1.4 The Dirichlet form

Given a heat semigroup $\{P_t\}$ on a metric measure space (M,d,μ), define for any $t > 0$ a bilinear form $\mathcal{E}_t$ on L^2 by

$$\mathcal{E}_t(u,v) := \left(\frac{u - P_t u}{t}, v\right) = \frac{1}{t}\left((u,v) - (P_t u, v)\right),$$

where $(\cdot,\cdot)$ is the inner product in L^2. Since P_t is symmetric, the form $\mathcal{E}_t$ is also symmetric. Since P_t is a contraction, it follows that

$$\mathcal{E}_t(u) := \mathcal{E}_t(u,u) = \frac{1}{t}\left((u,u) - (P_t u, u)\right) \geq 0,$$

that is, $\mathcal{E}_t$ is a positive definite form.

In terms of the spectral resolution $\{E_\lambda\}$ of the generator $\mathcal{L}$, $\mathcal{E}_t$ can be expressed as follows

$$\mathcal{E}_t(u) = \frac{1}{t}\left((u,u) - (P_t u,u)\right) = \frac{1}{t}\left(\int_0^\infty d\|E_\lambda u\|_2^2 - \int_0^\infty e^{-t\lambda} d\|E_\lambda u\|_2^2\right) = \int_0^\infty \frac{1-e^{-t\lambda}}{t} d\|E_\lambda u\|_2^2,$$

which implies that $\mathcal{E}_t(u)$ is decreasing in t, since the function $t \mapsto \frac{1-e^{-t\lambda}}{t}$ is decreasing. Define for any $u \in L^2$

$$\mathcal{E}(u) = \lim_{t\downarrow 0} \mathcal{E}_t(u)$$

where the limit (finite or infinite) exists by the monotonicity, so that $\mathcal{E}(u) \geq \mathcal{E}_t(u)$. Since $\frac{1-e^{-t\lambda}}{t} \to \lambda$ as $t \to 0$, we have

$$\mathcal{E}(u) = \int_0^\infty \lambda d\|E_\lambda u\|_2^2.$$

Set

$$\mathcal{F} := \left\{u \in L^2 : \mathcal{E}(u) < \infty\right\} = \operatorname{dom}\left(\mathcal{L}^{1/2}\right) \supset \operatorname{dom}(\mathcal{L})$$

and define a bilinear form $\mathcal{E}(u,v)$ on $\mathcal{F}$ by the polarization identity

$$\mathcal{E}(u,v) := \frac{1}{4}\left(\mathcal{E}(u+v) - \mathcal{E}(u-v)\right),$$

which is equivalent to

$$\mathcal{E}(u,v) = \lim_{t\to 0} \mathcal{E}_t(u,v).$$

Note that $\mathcal{F}$ contains dom($\mathcal{L}$). Indeed, if $u \in \operatorname{dom}(\mathcal{L})$ then we have for all $v \in L^2$

$$\lim_{t\to 0} \mathcal{E}_t(u,v) = \left(\lim_{t\to 0} \frac{u - P_t u}{t}, v\right) = (\mathcal{L}u, v).$$

Setting $v = u$ we obtain $u \in \mathcal{F}$. Then choosing any $v \in \mathcal{F}$ we obtain the identity

$$\mathcal{E}(u,v) = (\mathcal{L}u, v) \text{ for all } u \in \operatorname{dom}(\mathcal{L}) \text{ and } v \in \mathcal{F}.$$

The space $\mathcal{F}$ is naturally endowed with the inner product

$$[u,v] := (u,v) + \mathcal{E}(u,v).$$

It is possible to show that the form $\mathcal{E}$ is *closed*, that is, the space $\mathcal{F}$ is *Hilbert*. Furthermore, $\operatorname{dom}(\mathcal{L})$ is dense in $\mathcal{F}$.

The fact that P_t is Markovian implies that the form $\mathcal{E}$ is also *Markovian*, that is

$$u \in \mathcal{F} \Rightarrow \widetilde{u} := \min(u_+, 1) \in \mathcal{F} \text{ and } \mathcal{E}(\widetilde{u}) \le \mathcal{E}(u).$$

Indeed, let us first show that for any $u \in L^2$

$$\mathcal{E}_t(u_+) \le \mathcal{E}_t(u).$$

We have

$$\mathcal{E}_t(u) = \mathcal{E}_t(u_+ - u_-) = \mathcal{E}_t(u_+) + \mathcal{E}_t(u_-) - 2\mathcal{E}_t(u_+, u_-) \ge \mathcal{E}_t(u_+)$$

because $\mathcal{E}_t(u_-) \ge 0$ and

$$\mathcal{E}_t(u_+, u_-) = \frac{1}{t}(u_+, u_-) - \frac{1}{t}(P_t u_+, u_-) \le 0.$$

Assuming $u \in \mathcal{F}$ and letting $t \to 0$ we obtain

$$\mathcal{E}(u_+) = \lim_{t\to 0} \mathcal{E}_t(u_+) \le \lim_{t\to 0} \mathcal{E}_t(u) = \mathcal{E}(u) < \infty$$

whence $\mathcal{E}(u_+) \le \mathcal{E}(u)$ and, hence, $u_+ \in \mathcal{F}$.

Similarly one proves that $\widetilde{u} = \min(u_+, 1)$ belongs to $\mathcal{F}$ and $\mathcal{E}(\widetilde{u}) \le \mathcal{E}(u_+)$.

Conclusion. *Hence,* $(\mathcal{E},\mathcal{F})$ *is a* Dirichlet form*, that is, a bilinear, symmetric, positive definite, closed, densely defined form in* L^2 *with Markovian property.*

If the heat semigroup is defined by means of a heat kernel p_t, then $\mathcal{E}_t$ can be equivalently defined by

$$\mathcal{E}_t(u)=\frac{1}{2t}\int_M\int_M (u(x)-u(y))^2\,p_t(x,y)d\mu(y)d\mu(x)+\frac{1}{t}\int_M (1-P_t1(x))\,u^2(x)d\mu(x). \tag{(5)}$$

Indeed, we have

$$\begin{aligned}u(x)-P_tu(x) &= u(x)\,P_t1(x)-P_tu(x)+(1-P_t1(x))\,u(x)\\ &= \int_M (u(x)-u(y))\,p_t(x,y)d\mu(y)+(1-P_t1(x))\,u(x)\end{aligned}$$

whence

$$\begin{aligned}\mathcal{E}_t(u) &= \frac{1}{t}\int_M\int_M (u(x)-u(y))\,u(x)p_t(x,y)d\mu(y)d\mu(x)\\ &\quad+\frac{1}{t}\int_M (1-P_t1(x))\,u^2(x)d\mu(x).\end{aligned}$$

Interchanging the variables x and y in the first integral and using the symmetry of the heat kernel, we obtain also

$$\begin{aligned}\mathcal{E}_t(u) &= \frac{1}{t}\int_M\int_M (u(y)-u(x))\,u(y)p_t(x,y)d\mu(y)d\mu(x)\\ &\quad+\frac{1}{t}\int_M (1-P_t1(x))\,u^2(x)d\mu(x),\end{aligned}$$

and (5) follows by adding up the two previous lines.

Note that $P_t1\le 1$ so that the second term in the right hand side of (5) is non-negative. If the heat kernel is stochastically complete, that is, $P_t1=1$, then that term vanishes and we obtain

$$\mathcal{E}_t(u)=\frac{1}{2t}\int_M\int_M (u(x)-u(y))^2\,p_t(x,y)d\mu(y)d\mu(x). \tag{(6)}$$

Definition. The form $(\mathcal{E},\mathcal{F})$ is called *local* if $\mathcal{E}(u,v)=0$ whenever the functions $u,v\in\mathcal{F}$ have compact disjoint supports. The form $(\mathcal{E},\mathcal{F})$ is called *strongly local* if $\mathcal{E}(u,v)=0$ whenever the functions $u,v\in\mathcal{F}$ have compact supports and $u\equiv\mathrm{const}$ in an open neighborhood of $\operatorname{supp} v$.

For example, if $p_t(x,y)$ is the heat kernel of the Laplace-Beltrami operator on a complete Riemannian manifold then the associated Dirichlet form is given by

$$\mathcal{E}(u,v)=\int_M\langle\nabla u,\nabla v\rangle d\mu, \tag{(7)}$$

and $\mathcal{F}$ is the Sobolev space $W_2^1(M)$. Note that this Dirichlet form is strongly local because $u = \text{const}$ on $\operatorname{supp} v$ implies $\nabla u = 0$ on $\operatorname{supp} v$ and, hence, $\mathcal{E}(u, v) = 0$.

If $p_t(x, y)$ is the heat kernel of the symmetric stable process of index β in $\mathbb{R}^n$, that is, $\mathcal{L} = (-\Delta)^{\beta/2}$, then

$$\mathcal{E}(u, v) = c_{n,\beta} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\beta}} dxdy,$$

and $\mathcal{F}$ is the Besov space $B_{2,2}^{\beta/2}(\mathbb{R}^n) = \{u \in L^2 : \mathcal{E}(u, u) < \infty\}$. This form is clearly non-local.

Denote by $C_0(M)$ the space of continuous functions on M with compact supports, endowed with sup-norm.

Definition. The form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ (with $[\cdot, \cdot]$-norm) and in $C_0(M)$ (with sup-norm).

All the Dirichlet forms in the above examples are regular.

Assume that we are given a Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$. Then one can define the generator $\mathcal{L}$ of $(\mathcal{E}, \mathcal{F})$ by the identity

$$(\mathcal{L}u, v) = \mathcal{E}(u, v) \quad \text{for all } u \in \operatorname{dom}(\mathcal{L}), \ v \in \mathcal{F} \tag{(8)}$$

where $\operatorname{dom}(\mathcal{L}) \subset \mathcal{F}$ must satisfy one of the following two equivalent requirements:

1. $\operatorname{dom}(\mathcal{L})$ is a maximal possible subspace of $\mathcal{F}$ such that (8) holds

2. $\mathcal{L}$ is a densely defined self-adjoint operator.

Clearly, $\mathcal{L}$ is positive definite so that $\operatorname{spec} \mathcal{L} \subset [0, +\infty)$. Hence, the family of operators $P_t = e^{-t\mathcal{L}}$, $t \geq 0$, forms a strongly continuous, symmetric, contraction semigroup in L^2. Moreover, using the Markovian property of the Dirichlet form $(\mathcal{E}, \mathcal{F})$, it is possible to prove that $\{P_t\}$ is Markovian, that is, $\{P_t\}$ is a heat semigroup. The question whether and when P_t has the heat kernel requires additional investigation.

1.5 More examples of heat kernels

Let us give some examples of stochastically complete heat kernels that do not satisfy (4).

Example. *(A frozen heat kernel)* Let M be a countable set and let $\{x_k\}_{k=1}^{\infty}$ be the sequence of all distinct points from M. Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence of positive reals and define measure μ on M by $\mu(\{x_k\}) = \mu_k$. Define a function $p_t(x, y)$ on $M \times M$ by

$$p_t(x, y) = \begin{cases} \frac{1}{\mu_k}, & x = y = x_k \\ 0, & \text{otherwise.} \end{cases}$$

Easy to check that $p_t(x,y)$ is a heat kernel. For example, let us check the approximation of identity: for any function $f \in L^2(M,\mu)$, we have

$$P_t f(x) = \int_M p_t(x,y) f(y)\, d\mu(y) = p_t(x,x) f(x) \mu(\{x\}) = f(x).$$

This identity implies also the stochastic completeness. The Dirichlet form is

$$\mathcal{E}(f) = \lim_{t\to 0}\left(\frac{f - P_t f}{t}, f\right) = 0.$$

The Markov process associated with the frozen heat kernel is very simple: $X_t = X_0$ for all $t \geq 0$ so that it is a frozen diffusion.

Example. *(The heat kernel in $\mathbb{H}^3$)* The heat kernel of the Laplace-Beltrami operator on the 3-dimensional hyperbolic space $\mathbb{H}^3$ is given by the formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{3/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right),$$

where $r = d(x,y)$ is the geodesic distance between x, y. The Dirichlet form is (7).

Example. *(The Mehler heat kernel)* Let $M = \mathbb{R}$, measure μ be defined by

$$d\mu = e^{x^2} dx,$$

and the operator $\mathcal{L}$ be given by

$$\mathcal{L} = -e^{-x^2}\frac{d}{dx}\left(e^{x^2}\frac{d}{dx}\right) = -\frac{d^2}{dx^2} - 2x\frac{d}{dx}.$$

Then the heat kernel of $\mathcal{L}$ is given by the formula

$$p_t(x,y) = \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(\frac{2xye^{-2t} - x^2 - y^2}{1 - e^{-4t}} - t\right).$$

The associated Dirichlet form is given by (7).

Similarly, for the measure

$$d\mu = e^{-x^2} dx$$

and for the operator

$$\mathcal{L} = e^{x^2}\frac{d}{dx}\left(e^{-x^2}\frac{d}{dx}\right) = -\frac{d^2}{dx^2} + 2x\frac{d}{dx},$$

we have

$$p_t(x,y) = \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(\frac{2xye^{-2t} - (x^2+y^2) e^{-4t}}{1 - e^{-4t}} + t\right).$$

1.6 Summary of Chapter 1

Given a metric measure space (M, d, μ), we have defined the notion of a heat kernel $p_t(x, y)$ as a family of functions on $M \times M$ that satisfies certain properties. It gives rise to the family of operators $P_t : L^2 \to L^2$

$$P_t f = \int_M p_t(x, y) f(y) \, d\mu(y)$$

that forms a heat semigroup $\{P_t\}_{t \geq 0}$. The latter determines the generator $\mathcal{L}$, defined by

$$\mathcal{L}u = L^2\text{-}\lim_{t \to 0} \frac{u - P_t u}{t}$$

with $\operatorname{dom}(\mathcal{L}) = \{u \in L^2\text{: the above limit exists}\}$, which is a positive definite self-adjoint operator in L^2.

The heat semigroup determines also a Dirichlet form $(\mathcal{E}, \mathcal{F})$ where $\mathcal{E}$ as a quadratic form is defined by

$$\mathcal{E}(u) := \lim_{t \to 0} \mathcal{E}_t(u) = \lim_{t \to 0} \left(\frac{u - P_t u}{t}, u \right)$$

and $\mathcal{F} = \{u \in L^2\text{:the above limit is finite}\} \supset \operatorname{dom}(\mathcal{L})$.

Note that $\mathcal{E}_t(u)$ monotone increases as $t \downarrow$. In terms of the heat kernel we have

$$\mathcal{E}_t(u) = \frac{1}{2t} \int_M \int_M (u(x) - u(y))^2 p_t(x, y) d\mu(y) d\mu(x) + \frac{1}{t} \int_M (1 - P_t 1(x)) \, u^2(x) d\mu(x).$$

Conversely, given a Dirichlet form $(\mathcal{E}, \mathcal{F})$ in L^2, one defines the generator $\mathcal{L}$ by the identity

$$(\mathcal{L}u, v) = \mathcal{E}(u, v) \quad \text{for all } u \in \operatorname{dom}(\mathcal{L}), \ v \in \mathcal{F}$$

and the requirement that $\operatorname{dom}(\mathcal{L})$ must be a maximal possible subspace of $\mathcal{F}$. Then the heat semigroup is defined by $P_t = e^{-t\mathcal{L}}$. The existence of the heat kernel (=the integral kernel of P_t) requires additional investigation.

The Dirichlet form (or the heat semigroup) determines a Markov process $(\{X_t\}, \{\mathbb{P}_x\})$ on M such that

$$\mathbb{E}_x(f(X_t)) = P_t f(x)$$

for all $f \in \mathcal{B}_b(M)$, $t > 0$ and almost all $x \in M$. The process X_t is a diffusion if and only if $(\mathcal{E}, \mathcal{F})$ is local.

We have seen examples of heat kernels satisfying the estimates of the type

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{d(x, y)}{t^{1/\beta}} \right)$$

with the following functions Φ:

1. $\Phi(s) = \exp(-s^2)$ (Gaussian estimates, Brownian motion in $\mathbb{R}^n$)
2. $\Phi(s) = (1+s)^{-(\alpha+\beta)}$, $0 < \beta < 2$ (super-Gaussian estimates, symmetric stable processes in $\mathbb{R}^n$)
3. $\Phi(s) = \exp(-s^{\frac{\beta}{\beta-1}})$, $\beta > 2$ (sub-Gaussian estimates, diffusions on fractals)

Chapter 2

Consequences of heat kernel bounds

In this Chapter we assume that $p_t(x,y)$ is a heat kernel on a metric measure space (M,d,μ) that satisfies certain upper and/or lower estimates, and investigate the consequences of these estimates.

2.1 Identifying Φ in the non-local case

Fix two positive parameters α and β and a monotone decreasing function $\Phi : [0,+\infty) \to [0,+\infty)$.

Theorem 2.1. *(AG, T.Kumagai '09) Let p_t be a heat kernel on (M,d,μ).*

(a) *If the heat kernel satisfies the estimate*

$$p_t(x,y) \le \frac{1}{t^{\alpha/\beta}} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right),$$

for all $t>0$ and almost all $x,y \in M$, then either the associated Dirichlet form $(\mathcal{E},\mathcal{F})$ is local or

$$\Phi(s) \ge c(1+s)^{-(\alpha+\beta)}$$

for all $s>0$ and some $c>0$.

(b) *If the heat kernel satisfies the estimate*

$$p_t(x,y) \ge \frac{1}{t^{\alpha/\beta}} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right),$$

then

$$\Phi(s) \le C(1+s)^{-(\alpha+\beta)}$$

for all $s>0$ and some $C>0$.

(c) *Consequently, if the heat kernel satisfies the estimate*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c\frac{d(x,y)}{t^{1/\beta}}\right),$$

then either the Dirichlet form $\mathcal{E}$ is local or

$$\Phi(s) \simeq (1+s)^{-(\alpha+\beta)}.$$

(The symbol $\simeq$ means that the ratio of the left hand side and right hand side is bounded between two positive constants).

Proof of (b)**.** Let u be a non-constant function from $L^2(M,\mu)$. Choose a ball $Q \subset M$ where u is non-constant and let $a > b$ be two real values such that the sets

$$A = \{x \in Q : u(x) > a\} \text{ and } B = \{x \in Q : u(x) < b\}$$

have positive measures.

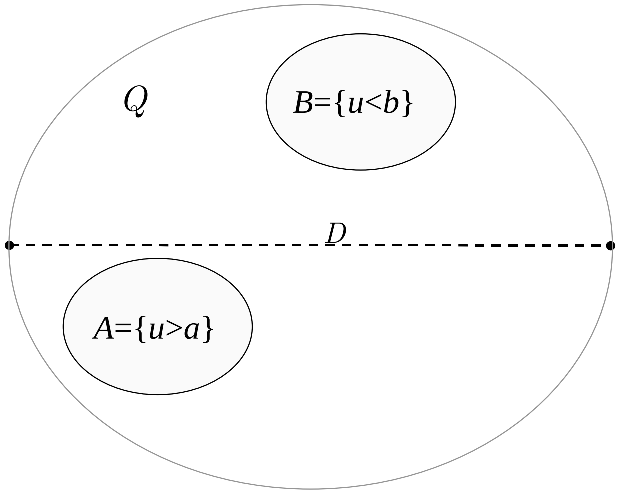

Figure 2.1: Sets A and B

$D = \operatorname{diam} Q$ then we have, for almost all $x, y \in Q$,

$$p_t(x,y) \geq \frac{1}{t^{\alpha/\beta}} \Phi\left(\frac{D}{t^{1/\beta}}\right),$$

whence for any $t > 0$

$$\begin{aligned} \mathcal{E}(u) &\geq \mathcal{E}_t(u) \geq \frac{1}{2t} \int_A \int_B (u(x) - u(y))^2 p_t(x,y) d\mu(y) d\mu(x) \\ &\geq (a-b)^2 \mu(A) \mu(B) \frac{1}{2t^{1+\alpha/\beta}} \Phi\left(\frac{D}{t^{1/\beta}}\right) \\ &= \frac{c'}{t^{1+\alpha/\beta}} \Phi\left(\frac{D}{t^{1/\beta}}\right), \end{aligned}$$

where $c' > 0$. If the inequality $\Phi(s) \leq C(1+s)^{-(\alpha+\beta)}$ fails then there exists a sequence $\{s_k\} \to \infty$ such that

$$s_k^{\alpha+\beta}\Phi(s_k) \to \infty \quad \text{as } k \to \infty.$$

Define a sequence $\{t_k\}$ from the condition

$$s_k = \frac{D}{t_k^{1/\beta}}$$

so that $t_k \to 0$ as $k \to \infty$. We have

$$\frac{1}{t_k^{1+\alpha/\beta}}\Phi\left(\frac{D}{t_k^{1/\beta}}\right) = D^{-(\alpha+\beta)}s_k^{\alpha+\beta}\Phi(s_k) \to \infty \quad \text{as } k \to \infty,$$

whence $\mathcal{E}(u) = \infty$.

Hence, we have arrived at the conclusion that the domain $\mathcal{F}$ of the form $\mathcal{E}$ may contain only constants. Since $\mathcal{F}$ is dense in $L^2(M,\mu)$, this is not possible, which finishes the proof. ■

2.2 Volume of balls

Denote by $B(x,r)$ a metric ball in (M,d), that is

$$B(x,r) := \{y \in M : d(x,y) < r\}.$$

Theorem 2.2. *(AG, J.Hu, K.-S. Lau '03) Let $p_t(x,y)$ be a heat kernel on (M,d,μ). Assume that it is stochastically complete and that it satisfies the two-sided estimate*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\Phi\left(c\frac{d(x,y)}{t^{1/\beta}}\right). \tag{(1)}$$

Then, for all $x \in M$ and $r > 0$,

$$\mu(B(x,r)) \simeq r^\alpha,$$

that is, μ is α-regular.

Consequently, $\dim_H(M,d) = \alpha$ *and $\mu \simeq H^\alpha$ on all Borel subsets of M, where H^α is the Hausdorff measure of the dimension α in M.*

In particular, the parameter α is the invariant of the metric space (M,d), and measure μ is determined (up to a factor $\simeq 1$) by the metric space (M,d).

Proof. For all $r,t > 0$ and for almost all $x \in M$ we have

$$\int_{B(x,r)} p_t(x,y)d\mu(y) \leq 1. \tag{(2)}$$

It follows from (2) that

$$\mu(B(x,r)) \leq \left(\operatorname*{einf}_{y\in B(x,r)} p_t(x,y) \right)^{-1}.$$

Choose $\varepsilon > 0$ so that $\Phi(\varepsilon) > 0$. Choosing t from the identity $r = c^{-1}\varepsilon t^{1/\beta}$ we obtain

$$\operatorname*{einf}_{y\in B(x,r)} p_t(x,y) \geq \frac{C}{t^{\alpha/\beta}} \Phi\left(c\frac{r}{t^{1/\beta}} \right) = c' r^{-\alpha} \varepsilon^{\alpha} \Phi(\varepsilon),$$

whence

$$\mu(B(x,r)) \leq C r^{\alpha}. \tag{(3)}$$

To prove the lower bound for $\mu(B(x,r))$, we first show that for all $0 < t \leq \varepsilon r^{\beta}$ and almost all $x \in M$,

$$\int_{M\setminus B(x,r)} p_t(x,y) d\mu(y) \leq \frac{1}{2}, \tag{(4)}$$

provided $\varepsilon > 0$ is sufficiently small. Setting $r_k = 2^k r$ and using the monotonicity of Φ and (3) we obtain

$$\begin{aligned}
\int_{M\setminus B(x,r)} p_t(x,y) d\mu(y) &= \sum_{k=0}^{\infty} \int_{B(x,r_{k+1})\setminus B(x,r_k)} p_t(x,y)\, d\mu(y) \\
&\leq \sum_{k=0}^{\infty} \int_{B(x,r_{k+1})\setminus B(x,r_k)} C t^{-\alpha/\beta} \Phi\left(\frac{r_k}{t^{1/\beta}} \right) d\mu(y) \\
&\leq \sum_{k=0}^{\infty} C r_{k+1}^{\alpha} t^{-\alpha/\beta} \Phi\left(\frac{r_k}{t^{1/\beta}} \right) \\
&= C' \sum_{k=0}^{\infty} \left(\frac{2^k r}{t^{1/\beta}} \right)^{\alpha} \Phi\left(\frac{2^k r}{t^{1/\beta}} \right) \\
&\leq C' \int_{\frac{1}{2} r/t^{1/\beta}}^{\infty} s^{\alpha} \Phi(s) \frac{ds}{s}.
\end{aligned} \tag{(5)}$$

By Theorem 2.1(b), the integral (5) converges. Hence, its value can be made arbitrarily small provided r^{β}/t is large enough, whence (4) follows.

By the stochastic completeness of the heat kernel and (4) we conclude that, under the condition $0 < t \leq \varepsilon r^{\beta}$,

$$\int_{B(x,r)} p_t(x,y) d\mu(y) \geq \frac{1}{2},$$

whence

$$\mu(B(x,r)) \geq \frac{1}{2} \left(\operatorname*{esup}_{y\in B(x,r)} p_t(x,y) \right)^{-1}.$$

Finally, choosing $t = \varepsilon r^\beta$ and using the upper bound

$$p_t(x,y) \leq C t^{-\alpha/\beta}\Phi(0) = C r^{-\alpha}\varepsilon^{-\alpha/\beta}\Phi(0),$$

we obtain

$$\mu(B(x,r)) \geq c r^\alpha,$$

which finishes the proof. ■

2.3 Besov spaces

Fix $\alpha > 0$, $\sigma > 0$ and introduce the following seminorms on $L^2 = L^2(M,\mu)$:

$$N_{2,\infty}^{\alpha,\sigma}(u) = \sup_{0<r\leq 1} \frac{1}{r^{\alpha+2\sigma}} \iint\limits_{\{x,y\in M: d(x,y)<r\}} |u(x)-u(y)|^2 \, d\mu(x)d\mu(y)$$

and

$$N_{2,2}^{\alpha,\sigma}(u) = \int_0^\infty \frac{dr}{r}\frac{1}{r^{\alpha+2\sigma}} \iint\limits_{\{x,y\in M: d(x,y)<r\}} |u(x)-u(y)|^2 \, d\mu(x)d\mu(y)$$

Define the space

$$\Lambda_{2,\infty}^{\alpha,\sigma} = \left\{u \in L^2 : N_{2,\infty}^{\alpha,\sigma}(u) < \infty\right\}$$

and the norm in this space by

$$\|u\|^2_{\Lambda_{2,\infty}^{\alpha,\sigma}} = \|u\|_2^2 + N_{2,\infty}^{\alpha,\sigma}(u).$$

Similarly, one defines the space $\Lambda_{2,2}^{\alpha,\sigma}$. More generally one can define $\Lambda_{p,q}^{\alpha,\sigma}$ for $p \in [1,+\infty)$ and $q \in [1,+\infty]$.

In the case of $\mathbb{R}^n$, we have the following relations

$$\begin{aligned}
\Lambda_{p,q}^{n,\sigma}(\mathbb{R}^n) &= B_{p,q}^{\sigma}(\mathbb{R}^n), \ 0<\sigma<1, \\
\Lambda_{2,\infty}^{n,1}(\mathbb{R}^n) &= W_p^1(\mathbb{R}^n), \\
\Lambda_{2,2}^{n,1}(\mathbb{R}^n) &= \{0\}, \\
\Lambda_{p,q}^{n,\sigma}(\mathbb{R}^n) &= \{0\}, \ \sigma > 1.
\end{aligned}$$

where $B_{p,q}^\sigma$ is the Besov space and W_p^1 is the Sobolev space. The spaces $\Lambda_{p,q}^{\alpha,\sigma}$ will also be called Besov spaces.

Theorem 2.3. *(Jonsson '96, Pietruska-Paluba '00, AG, J.Hu, K.-S. Lau '03) Let p_t be a heat kernel on (M,d,μ). Assume that it is stochastically complete and that it satisfies the following estimate for all $t>0$ and almost all $x,y \in M$:*

$$\frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right) \leq p_t(x,y) \leq \frac{1}{t^{\alpha/\beta}}\Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right), \qquad ((6))$$

where α, β be positive constants, Φ_1 and Φ_2 monotone decreasing functions from $[0, +\infty)$ to $[0, +\infty)$ such that $\Phi_1(s) > 0$ for some $s > 0$ and

$$\int^{\infty} s^{\alpha+\beta}\Phi_2(s)\frac{ds}{s} < \infty. \tag{(7)}$$

Then, for any $u \in L^2$,

$$\mathcal{E}(u) \simeq N_{2,\infty}^{\alpha,\beta/2}(u),$$

and, consequently, $\mathcal{F} = \Lambda_{2,\infty}^{\alpha,\beta/2}$.

By Theorem 2.1, the upper bound in (6) implies that either $(\mathcal{E}, \mathcal{F})$ is local or

$$\Phi_2(s) \geq c(1+s)^{-(\alpha+\beta)}.$$

Since the latter contradicts (7), the form $(\mathcal{E}, \mathcal{F})$ must be local. For non-local forms the statement is not true. For example, for the operator $(-\Delta)^{\beta/2}$ in $\mathbb{R}^n$ we have $\mathcal{F} = B_{2,2}^{\beta/2} = \Lambda_{2,2}^{n,\beta/2}$ that is strictly smaller than $B_{2,\infty}^{\beta/2} = \Lambda_{2,\infty}^{n,\beta/2}$. This case will be covered by another theorem.

Theorem 2.4. *(Stós '00) Let p_t be a stochastically complete heat kernel on (M, d, μ) satisfying estimate (6) with functions*

$$\Phi_1(s) \simeq \Phi_2(s) \simeq (1+s)^{-(\alpha+\beta)}.$$

Then, for any $u \in L^2$,

$$\mathcal{E}(u) \simeq N_{2,2}^{\alpha,\beta/2}(u)$$

and, consequently, $\mathcal{F} = \Lambda_{2,2}^{\alpha,\beta/2}$.

2.4 Subordinated semigroup

Let $\mathcal{L}$ be the generator of a heat semigroup $\{P_t\}$. Then for any $\delta \in (0,1)$ the operator $\mathcal{L}^\delta$ is also a generator of a heat semigroup, that is, the semigroup $e^{-t\mathcal{L}^\sigma}$ is a heat semigroup. Furthermore, there is the following relation between the two semigroups

$$e^{-t\mathcal{L}^\delta} = \int_0^\infty e^{-s\mathcal{L}}\eta_t(s)\,ds$$

where $\eta_t(s)$ is a *subordinator*. Using the known estimates for $\eta_t(s)$, one obtain the following result.

Theorem 2.5. *Let a heat kernel p_t satisfy the estimate (6) where $\Phi_1(s) > 0$ for some $s > 0$ and*

$$\int_0^\infty s^{\alpha+\beta'}\Phi_2(s)\frac{ds}{s} < \infty,$$

where $\beta' = \delta\beta$, $0 < \delta < 1$. *Then the heat kernel* $q_t(x, y)$ *of operator* $\mathcal{L}^\delta$ *satisfies the estimate*

$$q_t(x,y) \simeq \frac{1}{t^{\alpha/\beta'}}\left(1 + \frac{d(x,y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')} \simeq \min\left(t^{-\alpha/\beta'}, \frac{t}{d(x,y)^{\alpha+\beta'}}\right),$$

for all $t > 0$ *and almost all* $x, y \in M$.

2.5 The walk dimension

It follows from definition that the Besov seminorm

$$N_{2,\infty}^{\alpha,\sigma}(u) := \sup_{0<r\le 1} \frac{1}{r^{\alpha+2\sigma}} \iint_{\{x,y\in M: d(x,y)<r\}} |u(x) - u(y)|^2 \, d\mu(x) d\mu(y)$$

increases when σ increases, which implies that the space

$$\Lambda_{2,\infty}^{\alpha,\sigma} := \left\{u \in L^2 : N_{2,\infty}^{\alpha,\sigma}(u) < \infty\right\}$$

shrinks. For a certain value of σ this space can become trivial. For example, as was already mentioned, $\Lambda_{2,\infty}^{n,\sigma}(\mathbb{R}^n) = \{0\}$ for $\sigma > 1$, while $\Lambda_{2,\infty}^{n,\sigma}(\mathbb{R}^n)$ is non-trivial for $\sigma \le 1$.

Definition. Fix $\alpha > 0$ and set

$$\beta^* := \sup\left\{\beta > 0 : \Lambda_{2,\infty}^{\alpha,\beta/2} \text{ is dense in } L^2(M,\mu)\right\}. \tag{(8)}$$

The number $\beta^* \in [0, +\infty]$ is called the *critical exponent* of the family $\left\{\Lambda_{2,\infty}^{\alpha,\beta/2}\right\}_{\beta>0}$ of Besov spaces.

Note that the value of β^* is an intrinsic property of the space (M, d, μ), which is defined independently of any heat kernel. For example, for $\mathbb{R}^n$ with $\alpha = n$ we have $\beta^* = 2$.

Theorem 2.6. *(A.Jonsson '96, K.Pietruska-Paluba '00, AG, J.Hu, K.-S. Lau '03) Let* p_t *be a heat kernel on a metric measure space* (M, d, μ). *If the heat kernel is stochastically complete and satisfies* (6), *that is,*

$$\frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right) \le p_t(x,y) \le \frac{1}{t^{\alpha/\beta}}\Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right), \tag{copy of (6)}$$

where $\Phi_1(s) > 0$ *for some* $s > 0$ *and*

$$\int^\infty s^{\alpha+\beta+\varepsilon}\Phi_2(s)\frac{ds}{s} < \infty \tag{(9)}$$

for some $\varepsilon > 0$, *then* $\beta = \beta^*$.

By Theorem 2.1, (9) implies that the Dirichlet form $(\mathcal{E},\mathcal{F})$ is local. For non-local forms the statement is not true: for example, in $\mathbb{R}^n$ for symmetric stable processes we have $\beta < 2 = \beta^*$.

Corollary 2.7. *Under the hypotheses of Theorem 2.6, the values of the parameters α and β are the invariants of the metric space (M,d) alone. Moreover, we have*

$$\mu \simeq H^{\alpha} \quad \text{and} \quad \mathcal{E} \simeq N_{2,\infty}^{\alpha,\beta/2}.$$

Consequently, both measure μ and the energy form $\mathcal{E}$ are determined (up to a factor $\simeq 1$) by the metric space (M,d) alone.

Example. Consider in $\mathbb{R}^n$ the Gauss-Weierstrass heat kernel

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

and its generator $\mathcal{L} = -\Delta$ in $L^2(\mathbb{R}^n)$ with the Lebesgue measure. Then $\alpha = n$, $\beta = 2$, and

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx.$$

Consider now another elliptic operator in $\mathbb{R}^n$:

$$\mathcal{L} = -\frac{1}{m(x)} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right),$$

where $m(x)$ and $a_{ij}(x)$ are continuous functions, $m(x) > 0$ and the matrix $(a_{ij}(x))$ is positive definite. The operator $\mathcal{L}$ is symmetric with respect to measure

$$d\mu = m(x)\, dx,$$

and its Dirichlet form is

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx.$$

Let $d(x,y) = |x-y|$ and assume that the heat kernel $p_t(x,y)$ of $\mathcal{L}$ satisfies the conditions of Theorem 2.6. Then we conclude by Corollary 2.7 that α and β must be the same as for the Gauss-Weierstrass heat kernel, that is, $\alpha = n$ and $\beta = 2$; moreover, measure μ must be comparable to the Lebesgue measure, which implies that $m \simeq 1$, and the energy form must admit the estimate

$$\mathcal{E}(u) \simeq \int_{\mathbb{R}^n} |\nabla u|^2 \, dx,$$

which implies that the matrix $(a_{ij}(x))$ is uniformly elliptic. Hence, the operator $\mathcal{L}$ is uniformly elliptic.

By Aronson's theorem the heat kernel for uniformly elliptic operators satisfies the estimate

$$p_t(x,y) \simeq \frac{C}{t^{n/2}} \exp\left(-c\frac{|x-y|^2}{t}\right).$$

What we have proved implies the converse to Aronson's theorem: if the Aronson estimate holds for the operator $\mathcal{L}$ then $\mathcal{L}$ is uniformly elliptic.

The next theorem handles the non-local case.

Theorem 2.8. *Let p_t be a heat kernel on a metric measure space (M,d,μ). If the heat kernel satisfies the lower bound*

$$p_t(x,y) \geq \frac{1}{t^{\alpha/\beta}} \Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right),$$

where $\Phi_1(s) > 0$ for some $s > 0$, then $\beta \leq \beta^*$.

Proof. In the proof of Theorem 2.3 one shows that the lower bound of the heat kernel implies $\mathcal{F} \subset \Lambda_{2,\infty}^{\alpha,\beta/2}$ (and the opposite inclusion follows from the upper bound and stochastic completeness). Since $\mathcal{F}$ is dense in L^2, it follows that $\beta \leq \beta^*$. ∎

2.6 Inequalities for the walk dimension

Definition. We say that a metric space (M,d) satisfies the *chain condition* if there exists a (large) constant C such that for any two points $x,y \in M$ and for any positive integer n there exists a sequence $\{x_i\}_{i=0}^n$ of points in M such that $x_0 = x$, $x_n = y$, and

$$d(x_i, x_{i+1}) \leq C\frac{d(x,y)}{n}, \quad \text{for all } i = 0,1,...,n-1. \tag{(10)}$$

The sequence $\{x_i\}_{i=0}^n$ is referred to as a chain connecting x and y.

Theorem 2.9. *(AG, J.Hu, K.-S. Lau '03) Let (M,d,μ) be a metric measure space and assume that*

$$\mu(B(x,r)) \simeq r^\alpha \tag{(11)}$$

for all $x \in M$ and $0 < r \leq 1$. Then

$$\beta^* \geq 2.$$

If in addition (M,d) satisfies the chain condition then

$$\beta^* \leq \alpha + 1.$$

Observe that the chain condition is essential for the inequality $\beta^* \leq \alpha + 1$ to be true. Indeed, assume for a moment that the claim of Theorem 2.9 holds without the chain condition, and consider a new metric d' on M given by $d' = d^{1/\gamma}$ where $\gamma > 1$.

Let us mark by a dash all notions related to the space (M, d', μ) as opposed to those of (M, d, μ). It is easy to see that $\alpha' = \alpha\gamma$ and $\beta^{*\prime} = \beta^*\gamma$. Hence, if Theorem 2.9 could apply to the space (M, d', μ) it would yield $\beta^*\gamma \le \alpha\gamma + 1$ which implies $\beta^* \le \alpha$ because γ may be taken arbitrarily large. However, there are spaces with $\beta^* > \alpha$, for example SG.

Clearly, the metric d' does not satisfy the chain condition; indeed the inequality (10) implies

$$d'(x_i, x_{i+1}) \le C \frac{d'(x, y)}{n^{1/\gamma}},$$

which is not good enough. Note that if in the inequality (10) we replace n by $n^{1/\gamma}$ then the proof of Theorem 2.9 will give $\beta^* \le \alpha + \gamma$ instead of $\beta^* \le \alpha + 1$.

Proof. To prove that $\beta^* \ge 2$, it suffices to show that $\Lambda^{\alpha,1}_{2,\infty}$ is dense in L^2. Let u be a Lipschitz function with a bounded support A and let L be the Lipschitz constant of u. Then, for any $r \in (0, 1]$,

$$\begin{aligned}
& \frac{1}{r^{\alpha+2}} \int_M \int_{B(x,r)} (u(x) - u(y))^2 \, d\mu(y) d\mu(x) \\
\le \; & \frac{1}{r^{\alpha+2}} \int_{A_1} \int_{B(x,r)} L r^2 d\mu(y) d\mu(x) \\
\le \; & C\mu(A_1),
\end{aligned}$$

where A_r denotes the closed r-neighborhood of A. It follows that

$$N^{\alpha,1}_{2,\infty}(u) \le C\mu(A_1) < \infty,$$

whence we conclude that $u \in \Lambda^{\alpha,1}_{2,\infty}$.

Let now A be any bounded closed subset of M. For any positive integer n, consider the function on M

$$f_n(x) = (1 - nd(x, A))_+,$$

which is Lipschitz and is supported in $A_{1/n}$. Hence, $f_n \in \Lambda^{\alpha,1}_{2,\infty}$. Clearly, $f_n \to 1_A$ in L^2 as $n \to \infty$, whence it follows that $1_A \in \overline{\Lambda^{\alpha,1}_{2,\infty}}$, where the bar means closure in L^2. Since the linear combinations of the indicator functions of bounded closed sets form a dense subset in L^2, it follows that $\overline{\Lambda^{\alpha,1}_{2,\infty}} = L^2$, which was to be proved.

Now let us prove that $\beta^* \le \alpha + 1$ assuming the chain condition. The hypothesis (11) implies that the space $L^2(M, \mu)$ is ∞-dimensional. The inequality $\beta^* \le \alpha + 1$ will be proved if we show that, for any $\beta > \alpha + 1$, the space $\Lambda^{\alpha,\beta/2}_{2,\infty}$ contains only constants, that is, $N^{\alpha,\beta/2}_{2,\infty}(u) < \infty$ implies $u \equiv \text{const}$.

By definition of $N^{\alpha,\beta/2}_{2,\infty}$ we have, for any $0 < r \le 1$,

$$N^{\alpha,\beta/2}_{2,\infty}(u) \ge r^{-\alpha-\beta} \iint_{\{d(x,y)<r\}} |u(x) - u(y)|^2 d\mu(y) d\mu(x).$$

Fix some $0 < r \leq 1$ and assume that we have a sequence of disjoint balls $\{B_k\}_{k=0}^{l}$ of the same radius $0 < \rho < 1$, such that for all $k = 0, 1, ..., l-1$

$$d_{\max}(B_k, B_{k+1}) := \sup\{d(x,y) : x \in B_k, y \in B_{k+1}\} < r. \tag{(12)}$$

Then we have

$$N_{2,\infty}^{\alpha,\beta/2}(u) \geq r^{-\alpha-\beta} \sum_{k=0}^{l-1} \int_{B_k} \int_{B_{k+1}} |u(x) - u(y)|^2 \, d\mu(y) d\mu(x).$$

Now use the following notation

$$u_A := \frac{1}{\mu(A)} \int_A u \, d\mu$$

and observe that, for any two measurable sets $A, B \subset M$ of finite measure, the following inequality takes place

$$\int_A \int_B |u(x) - u(y)|^2 \, d\mu(x) d\mu(y) \geq \mu(A)\mu(B) \, |u_A - u_B|^2 \, ,$$

which is proved by a straightforward computation.

It follows that

$$\begin{aligned} N_{2,\infty}^{\alpha,\beta/2}(u) &\geq r^{-\alpha-\beta} \sum_{k=0}^{l-1} \mu(B_k) \, \mu(B_{k+1}) \left| u_{B_k} - u_{B_{k+1}} \right|^2 \\ &\geq c r^{-\alpha-\beta} \rho^{2\alpha} \sum_{k=0}^{l-1} \left| u_{B_k} - u_{B_{k+1}} \right|^2 \\ &\geq c r^{-\alpha-\beta} \rho^{2\alpha} \frac{1}{l} \left(\sum_{k=0}^{l-1} \left(u_{B_k} - u_{B_{k+1}} \right) \right)^2 \\ &= c r^{-\alpha-\beta} \rho^{2\alpha} \frac{1}{l} \left(u_{B_0} - u_{B_l} \right)^2 . \end{aligned}$$

Now we construct such a sequence $\{B_k\}$. Fix two distinct points $x, y \in M$ and recall that, by the chain condition, for any positive integer n there exists a sequence of points $\{x_i\}_{i=0}^{n}$ such that $x_0 = x$, $x_n = y$, and

$$d(x_i, x_{i+1}) < C \frac{d(x,y)}{n} := \rho, \quad \text{for all } 0 \leq i < n.$$

It is possible to show that, for large enough n, there exists a subsequence $\{x_{i_k}\}_{k=0}^{l}$ such that $x_{i_0} = x$, $x_{i_l} = y$, the balls $\{B(x_{i_k}, \rho)\}$ are disjoint, and

$$d(x_{i_k}, x_{i_{k+1}}) < 5\rho,$$

for all $k = 0, 1, ..., l-1$. Setting $B_k := B(x_{i_k}, \rho)$ and noticing that

$$d_{\max}(B_k, B_{k+1}) < 5\rho + 2\rho = 7\rho =: r$$

and that $r < 1$ provided n is large enough, we obtain

$$\begin{aligned}\left(u_{B(x,\rho)} - u_{B(y,\rho)}\right)^2 &= \left(u_{B_0} - u_{B_l}\right)^2 \\ &\leq CN_{2,\infty}^{\alpha,\beta/2}(u)\, r^{\beta+\alpha}\rho^{-2\alpha} l \\ &\leq CN_{2,\infty}^{\alpha,\beta/2}(u)\, \rho^{\beta-\alpha} n \\ &\leq CN_{2,\infty}^{\alpha,\beta/2}(u)\, \rho^{\beta-\alpha-1} d(x,y).\end{aligned}$$

By the Lebesgue theorem, we have, for almost all $x \in M$,

$$\lim_{\rho \to 0} u_{B(x,\rho)} = u(x),$$

Letting $n \to \infty$ (that is $\rho \to 0$) we obtain, for almost all $x, y \in M$,

$$(u(x) - u(y))^2 \leq C\, N_{2,\infty}^{\alpha,\beta/2}(u) \left(\lim_{\rho \to 0} \rho^{\beta-\alpha-1}\right) d(x,y) = 0.$$

Since $N_{2,\infty}^{\alpha,\beta/2}(u) < \infty$ and $\beta > \alpha + 1$, the above limit is equal to 0 whence $u \equiv \text{const}$.
■

Corollary 2.10. *AG, J.Hu, K.-S. Lau '03) Let p_t be a stochastically complete heat kernel on (M, d, μ) such that*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{d(x,y)}{t^{1/\beta}}\right).$$

(a) *If for some $\varepsilon > 0$*

$$\int^{\infty} s^{\alpha+\beta+\varepsilon} \Phi(s) \frac{ds}{s} < \infty \tag{(11)}$$

then $\beta \geq 2$.

(b) *If (M, d) satisfies the chain condition then $\beta \leq \alpha + 1$.*

Proof. By Theorem 2.2 μ is α-regular so that Theorem 2.9 applies.
(a) By Theorem 2.9 $\beta^* \geq 2$ and by Theorem 2.6, $\beta = \beta^*$, whence $\beta \geq 2$.
(b) By Theorem 2.9 $\beta^* \leq \alpha + 1$ and by Theorem 2.8 $\beta \leq \beta^*$, whence $\beta \leq \alpha + 1$.
■

Note that the condition (11) can occur only for a local Dirichlet form $\mathcal{E}$. If both (11) and the chain condition are satisfied then we obtain

$$2 \leq \beta \leq \alpha + 1. \tag{(12)}$$

Figure 2.2: The set $2 \leq \beta \leq \alpha + 1$

This inequality was stated by M.Barlow '98 without proof.

The set of couples (α, β) satisfying (12) is shown on the diagram:

Barlow '04 proved that any couple of α, β satisfying (12) can be realized for the heat kernel estimate

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{(13)}$$

For a non-local form, we can only claim that

$$0 < \beta \leq \alpha + 1$$

(under the chain condition). In fact, any couple α, β in the range

$$0 < \beta < \alpha + 1$$

can be realized for the estimate

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta'}}\left(1 + \frac{d(x, y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')}.$$

Indeed, if $\mathcal{L}$ is the generator of a diffusion with parameters α and β satisfying (13) then the operator $\mathcal{L}^\delta$, $\delta \in (0, 1)$, generates a jump process with the walk dimension $\beta' = \delta\beta$ and the same α (cf. Theorem 2.5). Clearly, β' can take any value from $(0, \alpha + 1)$.

It is not known whether the walk dimension for a non-local form can be equal to $\alpha + 1$.

2.7 Identifying Φ in the local case

Theorem 2.11. *(AG, T.Kumagai '09) Assume that the metric space (M,d) satisfies the chain condition and all metric balls are precompact. Let p_t be a stochastically complete heat kernel in (M,d,μ). Assume that the associated Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular, and the following estimate holds with some $\alpha,\beta>0$ and $\Phi:[0,+\infty)\to[0,+\infty)$:*

$$p_t(x,y)\asymp\frac{C}{t^{\alpha/\beta}}\Phi\left(c\frac{d(x,y)}{t^{1/\beta}}\right).$$

Then the following dichotomy holds:

- *either the Dirichlet form $\mathcal{E}$ is local, $2\leq\beta\leq\alpha+1$, and $\Phi(s)\asymp C\exp(-cs^{\frac{\beta}{\beta-1}})$.*
- *or the Dirichlet form $\mathcal{E}$ is non-local, $\beta\leq\alpha+1$, and $\Phi(s)\simeq(1+s)^{-(\alpha+\beta)}$.*

Proof. The non-local case as well as the inequality $\beta\leq\alpha+1$ follow from Corollary 2.10 and Theorem 2.1. We only need to treat the local case: to show that $\beta\geq 2$ and $\Phi(s)\asymp C\exp(-cs^{\frac{\beta}{\beta-1}})$.

Integrating the heat kernel over $B(x,r)^c$ we obtain as in the proof of Theorem 2.2

$$\int_{B(x,r)^c}p_t(x,y)\,d\mu(y)\leq C\int_{\frac{1}{2}r/t^{1/\beta}}^{\infty}s^{\alpha}\Phi(s)\frac{ds}{s},$$

where the integral converges by Theorem 2.1(b): $\Phi(s)\leq(1+s)^{-\alpha-\beta}$. Therefore, for any $\varepsilon>0$ there is $K>0$ such that the following estimate holds

$$\int_{B(x,r)^c}p_t(x,y)\,d\mu(y)\leq\varepsilon \qquad ((14))$$

for almost all $x\in M$, whenever $r\geq Kt^{1/\beta}$.

The estimate (14) allows self-improvement similarly to bootstrapping argument of Barlow for

$$\mathbb{P}_x\left(\tau_{B(x,r)}\leq t\right)\leq\varepsilon.$$

Technically we use a modification of the method of Hebisch and Saloff-Coste '01 (here the regularity of the Dirichlet form is used) and obtain the following: for all $t,r,\lambda>0$

$$\int_{B(x,r)^c}p_t(x,y)\,d\mu(y)\leq C\exp\left(\lambda t-cr\lambda^{1/\beta}\right) \qquad ((15))$$

(AG, J.Hu '08 and '10).

If $\beta<1$ then letting in (15) $\lambda\to\infty$, we obtain that the right hand side in (15) goes to 0. Letting then $r\to 0$, we obtain that, for almost all $x\in M$,

$$\int_{M\setminus\{x\}}p_t(x,y)\,d\mu(y)=0,$$

which is not possible under the present hypotheses. This contradiction proves that $\beta \geq 1$.

Setting in (15)

$$\lambda = \begin{cases} \left(\frac{cr}{2t}\right)^{\frac{\beta}{\beta-1}}, & \text{if } \beta > 1, \\ t^{-1}, & \text{if } \beta = 1 \end{cases}$$

we obtain that, for all positive r, t and almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \leq \begin{cases} C \exp\left(-c\left(\frac{r^\beta}{t}\right)^{\frac{1}{\beta-1}}\right), & \text{if } \beta > 1 \\ C \exp\left(-c\frac{r}{t}\right), & \text{if } \beta = 1 \end{cases}$$

Using the semigroup identity, we have, for all $t > 0$, almost all $x, y \in M$, and $r := \frac{1}{2} d(x,y)$,

$$\begin{aligned} p_t(x,y) &= \int_M p_{\frac{t}{2}}(x,z)\, p_{\frac{t}{2}}(z,y)\, d\mu(z) \\ &\leq \left(\int_{B(x,r)^c} + \int_{B(y,r)^c}\right) p_{\frac{t}{2}}(x,z)\, p_{\frac{t}{2}}(z,y)\, d\mu(z) \\ &\leq \operatorname*{esup}_{z\in M} p_{\frac{t}{2}}(z,y) \int_{B(x,r)^c} p_{\frac{t}{2}}(x,z)\, d\mu(z) \\ &\quad + \operatorname*{esup}_{z\in M} p_{\frac{t}{2}}(x,z) \int_{B(y,r)^c} p_{\frac{t}{2}}(y,z)\, d\mu(z). \end{aligned}$$

Using $\operatorname{esup} p_t \leq C t^{-\alpha/\beta}$, we obtain, for almost all $x, y \in M$,

$$p_t(x,y) \leq \begin{cases} \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right), & \text{if } \beta > 1 \\ \frac{C}{t^\alpha} \exp\left(-c\frac{r}{t}\right), & \text{if } \beta = 1 \end{cases}$$

It follows that p_t satisfies the two-sided estimate (6) with functions

$$\Phi_1(s) := C\Phi(cs)$$

and

$$\Phi_2(s) := \begin{cases} C \exp\left(-cs^{\frac{\beta}{\beta-1}}\right), & \text{if } \beta > 1, \\ C \exp(-cs), & \text{if } \beta = 1. \end{cases}$$

Then Theorem 2.6 applies and yields $\beta = \beta^*$. By Theorems 2.2 and 2.9 we have $\beta^* \geq 2$ whence $\beta \geq 2$.

Consequently, we obtain the upper bound

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right).$$

By hypothesis we have the lower bound

$$p_t(x,y) \geq ct^{-\alpha/\beta} \text{ provided } d(x,y) \leq st^{1/\beta}$$

where $s > 0$ is such that $\Phi(s) > 0$. The standard chaining argument using the chain condition yields then

$$p_t(x,y) \geq \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right)$$

for almost all x, y. Combining with the upper bound, we obtain

$$\Phi(s) \asymp C\exp(-cs^{\frac{\beta}{\beta-1}}),$$

which finishes the proof. ■

Chapter 3

Upper bounds of the heat kernel

3.1 Ultracontractive semigroups

Let (M, d, μ) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in $L^2(M, \mu)$ and $\{P_t\}$ be the associated heat semigroup, $P_t = e^{-t\mathcal{L}}$ where $\mathcal{L}$ is the generator of $(\mathcal{E}, \mathcal{F})$. The question to be discussed here is whether P_t possesses the heat kernel, that is, a function $p_t(x, y)$ that is non-negative, jointly measurable in (x, y), and satisfies the identity

$$P_t f(x) = \int_M p_t(x, y) f(y) \, d\mu(y)$$

for all $f \in L^2$, $t > 0$, and almost all $x \in M$. Usually the conditions that ensure the existence of the heat kernel give at the same token some upper bounds.

Given two parameters $p, q \in [0, +\infty]$, define the $L^p \to L^q$ norm of P_t by

$$\|P_t\|_{L^p \to L^q} = \sup_{f \in L^p \cap L^2 \setminus \{0\}} \frac{\|P_t f\|_q}{\|f\|_p}.$$

In fact, the Markovian property allows to extend P_t to an operator in L^p so that the range $L^p \cap L^2$ of f can be replaced by L^p. Also, it follows from the Markovian property that $\|P_t\|_{L^p \to L^p} \leq 1$ for any p.

Definition. The semigroup $\{P_t\}$ is said to be $L^p \to L^q$ *ultracontractive* if there exists a positive decreasing function γ on $(0, +\infty)$, called the *rate* function, such that, for each $t > 0$

$$\|P_t\|_{L^p \to L^q} \leq \gamma(t).$$

By the symmetry of P_t, if P_t is $L^p \to L^q$ ultracontractive, then P_t is also $L^{q^*} \to L^{p^*}$ ultracontractive with the same rate function, where p^* and q^* are the Hölder conjugates to p and q, respectively. In particular, P_t is $L^1 \to L^2$ ultracontractive if and only if it is $L^2 \to L^\infty$ ultracontractive.

Theorem 3.1.

(a) *The heat semigroup* $\{P_t\}$ *is* $L^1 \to L^2$ *ultracontractive with a rate function* γ, *if and only if* $\{P_t\}$ *has the heat kernel* p_t *satisfying the estimate*

$$\operatorname*{esup}_{x,y\in M} p_t(x,y) \le \gamma(t/2)^2 \quad \text{for all } t>0$$

(b) *The heat semigroup* $\{P_t\}$ *is* $L^1 \to L^\infty$ *ultracontractive with a rate function* γ, *if and only if* $\{P_t\}$ *has the heat kernel* p_t *satisfying the estimate*

$$\operatorname*{esup}_{x,y\in M} p_t(x,y) \le \gamma(t) \quad \text{for all } t>0.$$

This result is "well-known" and can be found in many sources. However, there are hardly complete proofs of the measurability of the function $p_t(x,y)$ in (x,y), which is necessary for many applications, for example, to use Fubini. Normally the existence of the heat kernel is proved in some specific setting where $p_t(x,y)$ is continuous in (x,y), or one proves just the existence of a family of functions $p_{t,x} \in L^2$ so that

$$P_t f(x) = (p_{t,x}, f) = \int_M p_{t,x}(y) f(y)\, d\mu(y)$$

for all $t>0$ and almost all x. However, if one defines $p_t(x,y) = p_{t,x}(y)$, then this function does not have to be jointly measurable. The proof of the existence of a jointly measurable version can be found in a preprint AG, J.Hu "Upper bounds of heat kernels on doubling spaces", 2010. Most of the material of this chapter can also be found there.

3.2 Restriction of the Dirichlet form

Let Ω be an open subset of M. Define the function space $\mathcal{F}_\Omega$ by

$$\mathcal{F}_\Omega = \overline{\{f \in \mathcal{F} : \operatorname{supp} f \subset \Omega\}}^{\mathcal{F}}.$$

Clearly, $\mathcal{F}_\Omega$ is a closed subspace of $\mathcal{F}$ and a subspace of $L^2(\Omega)$.

Lemma 3.2. *If* $(\mathcal{E},\mathcal{F})$ *is a regular Dirichlet form in* $L^2(M)$ *then* $(\mathcal{E},\mathcal{F}_\Omega)$ *is a regular Dirichlet form in* $L^2(\Omega)$. *If* $(\mathcal{E},\mathcal{F})$ *is (strongly) local then so is* $(\mathcal{E},\mathcal{F}_\Omega)$.

The regularity is used, in particular, to ensure that $\mathcal{F}_\Omega$ is dense in $L^2(\Omega)$. From now on let us assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. Other consequences of this assumptions are as follows:

1. The existence of cutoff functions: for any compact set K and any open set $U \supset K$, there is a function $\varphi \in \mathcal{F} \cap C_0(U)$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in an open neighborhood of K.

2. The existence of a Hunt process $\left(\{X_t\}_{t\ge 0}, \{\mathbb{P}_x\}_{x\in M}\right)$ associated with $(\mathcal{E},\mathcal{F})$.

(see Fukushima, Oshima, Takeda "Dirichlet forms and symmetric Markov processes").

Hence, for any open subset $\Omega \subset M$, we have the Dirichlet form $(\mathcal{E}, \mathcal{F}_\Omega)$ that is called a restriction of $(\mathcal{E}, \mathcal{F})$ to Ω.

Example. Consider in $\mathbb{R}^n$ the canonical Dirichlet form

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx$$

in $\mathcal{F} = W_2^1(\mathbb{R}^n)$. Then $\mathcal{F}_\Omega = \overline{C_0^1(\Omega)}^{W_2^1} =: H_0^1(\Omega)$.

Using the restricted form $(\mathcal{E}, \mathcal{F}_\Omega)$ corresponds to imposing the Dirichlet boundary conditions on $\partial\Omega$ (or on Ω^c), so that the form $(\mathcal{E}, \mathcal{F}_\Omega)$ could be called the Dirichlet form with the Dirichlet boundary condition.

Denote by $\mathcal{L}_\Omega$ the generator of $(\mathcal{E}, \mathcal{F}_\Omega)$ and set

$$\lambda_{\min}(\Omega) := \inf \operatorname{spec} \mathcal{L}_\Omega = \inf_{u \in \mathcal{F}_\Omega \setminus \{0\}} \frac{\mathcal{E}(u)}{\|u\|_2^2}. \qquad ((1))$$

Clearly, $\lambda_{\min}(\Omega) \geq 0$ and $\lambda_{\min}(\Omega)$ is decreasing when Ω expands.

Example. If $(\mathcal{E}, \mathcal{F})$ is the canonical Dirichlet form in $\mathbb{R}^n$ and Ω is the bounded domain in $\mathbb{R}^n$ then the operator $\mathcal{L}_\Omega$ has the discrete spectrum $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots$ that coincides with the eigenvalues of the Dirichlet problem

$$\begin{cases} \Delta u + \lambda u = 0 \\ u|_{\partial\Omega} = 0, \end{cases}$$

so that $\lambda_1(\Omega) = \lambda_{\min}(\Omega)$.

3.3 Faber-Krahn and Nash inequalities

Continuing the above example, we have by a theorem of Faber-Krahn

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

where Ω^* is the ball of the same volume as Ω. If r is the radius of Ω^* then we have

$$\lambda_1(\Omega^*) = \frac{c'}{r^2} = \frac{c}{|\Omega^*|^{2/n}} = \frac{c}{|\Omega|^{2/n}}$$

whence

$$\lambda_1(\Omega) \geq c_n |\Omega|^{-2/n}.$$

It turns out that this inequality, that we call *the Faber-Krahn inequality*, is intimately related to the existence of the heat kernel and its upper bound.

Theorem 3.3. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. Fix some constant $\nu > 0$. Then the following conditions are equivalent:*

(i) *(*The Faber-Krahn inequality*) There is a constant $a>0$ such that, for all non-empty open sets $\Omega\subset M$,*

$$\lambda_{\min}(\Omega)\geq a\mu(\Omega)^{-\nu}. \tag{(2)}$$

(ii) *(*The Nash inequality*) There exists a constant $b>0$ such that*

$$\mathcal{E}(u)\geq b\|u\|_2^{2+2\nu}\|u\|_1^{-2\nu}, \tag{(3)}$$

for any function $u\in\mathcal{F}\setminus\{0\}$.

(iii) *(*On-diagonal estimate of the heat kernel*) The heat kernel exists and satisfies the upper bound*

$$\operatorname*{esup}_{x,y\in M} p_t(x,y)\leq ct^{-1/\nu} \tag{(4)}$$

for some constant c and for all $t>0$.

The relation between the parameters a,b,c is as follows:

$$a\simeq b\simeq c^{-\nu}$$

where the ratio of any two of these parameters is bounded by constants depending only on ν.

In $\mathbb{R}^n$ $\nu=2/n$.
$(ii)\Rightarrow(iii)$ Nash '58
$(iii)\Rightarrow(ii)$ Carlen-Kusuoka-Stroock '87
$(i)\Leftrightarrow(iii)$ AG '94, Carron '94.

Proof of $(i)\Rightarrow(ii)\Rightarrow(iii)$**.** Observe first that $(ii)\Rightarrow(i)$ is trivial: choosing in (3) a function $u\in\mathcal{F}_\Omega\setminus\{0\}$ and applying the Cauchy-Schwarz inequality

$$\|u\|_1\leq\mu(\Omega)^{1/2}\|u\|_2,$$

we obtain

$$\mathcal{E}(u)\geq b\mu(\Omega)^{-\nu}\|u\|_2^2,$$

whence (2) follow by the variational principle (1).

The opposite inequality $(i)\Rightarrow(ii)$ is a bit more involved, and we prove it for functions $0\leq u\in\mathcal{F}\cap C_0(M)$ (a general $u\in\mathcal{F}$ requires some approximation argument). By the Markovian property, we have $(u-t)_+\in\mathcal{F}\cap C_0(M)$ for any $t>0$ and

$$\mathcal{E}(u)\geq\mathcal{E}\left((u-t)_+\right).$$

Consider for any $s>0$ the set

$$U_s:=\{x\in M:u(x)>s\},$$

which is clearly open and precompact. If $t > s$ then $(u-t)_+$ is supported in U_s, whence $(u-t)_+ \in \mathcal{F}_{U_s}$. It follows from (1)

$$\mathcal{E}\left((u-t)_+\right) \geq \lambda_{\min}(U_s) \int_{U_s} (u-t)_+^2 \, d\mu.$$

Set for simplicity $A = \|u\|_1$ and $B = \|u\|_2^2$. Since $u \geq 0$, we have

$$(u-t)_+^2 \geq u^2 - 2tu,$$

which implies that

$$\int_{U_s} (u-t)_+^2 d\mu = \int_M (u-t)_+^2 d\mu \geq B - 2tA.$$

On the other hand, we have

$$\mu(U_s) \leq \frac{1}{s} \int_{U_s} u \, d\mu \leq \frac{A}{s},$$

which together with the Faber-Krahn inequality implies

$$\lambda_{\min}(U_s) \geq a\mu(U_s)^{-\nu} \geq a\left(\frac{s}{A}\right)^{\nu}.$$

Combining the above lines, we obtain

$$\mathcal{E}(u) \geq \lambda_{\min}(U_s) \int_{U_s} (u-t)_+^2 \, d\mu \geq a\left(\frac{s}{A}\right)^{\nu}(B - 2tA).$$

Letting $t \to s+$ and then choosing $s = \frac{B}{4A}$, we obtain

$$\mathcal{E}(u) \geq a\left(\frac{s}{A}\right)^{\nu}(B - 2sA) = a\left(\frac{B}{4A^2}\right)^{\nu}\frac{B}{2} = \frac{a}{4^{\nu}2}B^{\nu+1}A^{-2\nu},$$

which is exactly (3).

To prove $(ii) \Rightarrow (iii)$, choose $f \in L^2 \cap L^1$, and consider $u = P_t f$. Since $u = e^{-t\mathcal{L}}f$ and $\frac{d}{dt}u = -\mathcal{L}u$, we have

$$\frac{d}{dt}\|u\|_2^2 = \frac{d}{dt}(u,u) = -2(\mathcal{L}u,u) = -2\mathcal{E}(u,u) \leq -2b\|u\|_2^{2+2\nu}\|u\|_1^{-2\nu} \leq -2b\|u\|_2^{2+2\nu}\|f\|_1^{-2\nu},$$

since $\|u\|_1 \leq \|f\|_1$. Solving this differential inequality, we obtain

$$\|P_t f\|_2^2 \leq ct^{-1/v}\|f\|_1^2,$$

that is, the semigroup P_t is $L^1 \to L^2$ ultracontractive with the rate function $\gamma(t) = \sqrt{ct^{-1/v}}$. By Theorem 3.1 we conclude that the heat kernel exists and satisfies (4). ∎

Let M be a Riemannian manifold with the geodesic distance d and the Riemannian volume μ. Let $(\mathcal{E}, \mathcal{F})$ be the canonical Dirichlet form on M. The heat kernel on manifolds always exists and is a smooth function. In this case the estimate (4) is equivalent to the on-diagonal upper bound

$$\sup_{x \in M} p_t(x, x) \leq c t^{-1/\nu}.$$

It is known (but non-trivial) that the on-diagonal estimate implies the Gaussian upper bound

$$p_t(x, y) \leq C t^{-1/\nu} \exp\left(-\frac{d^2(x, y)}{(4+\varepsilon)\, t}\right),$$

for all $t > 0$ and $x, y \in M$, which is due to the specific property of the geodesic distance function that $|\nabla d| \leq 1$.

More about heat kernels on manifolds can be found in

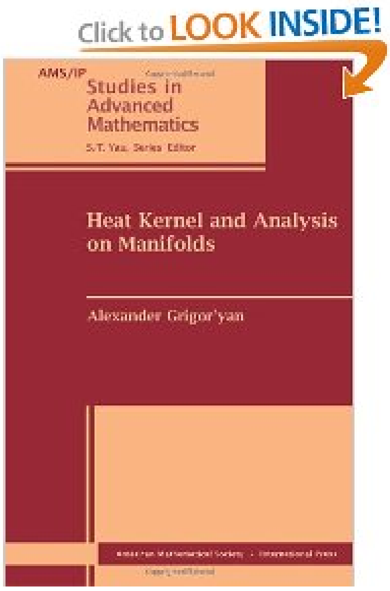

Figure 3.1:

In the context of abstract metric measure space the distance function does not have to satisfy this property and typically it does not (say, on fractals). Consequently, one needs some additional conditions that would relate the distance function to the Dirichlet form and imply the off-diagonal bounds.

3.4 Off-diagonal upper bounds

From now on let $(\mathcal{E}, \mathcal{F})$ be a regular *local* Dirichlet form, so that the associated Hunt process $\left(\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M}\right)$ is a diffusion. Recall that it is related to the heat semigroup $\{P_t\}$ of $(\mathcal{E},\mathcal{F})$ by means of the identity

$$\mathbb{E}_x\left(f\left(X_t\right)\right)=P_tf\left(x\right)$$

for all $f\in\mathcal{B}_b\left(M\right)$, $t>0$ and almost all $x\in M$.

Fix two parameters $\alpha>0$ and $\beta>1$ and introduce some conditions.

(V_α) (Volume regularity) For all $x\in M$ and $r>0$,

$$\mu\left(B\left(x,r\right)\right)\simeq r^{\alpha}.$$

(FK) (The Faber-Krahn inequality) For any open set $\Omega\subset M$,

$$\lambda_{\min}\left(\Omega\right)\geq c\mu\left(\Omega\right)^{-\beta/\alpha}.$$

For any open set $\Omega\subset M$ define the *first exist time* from Ω by

$$\tau_\Omega=\inf\left\{t>0:X_t\notin\Omega\right\}.$$

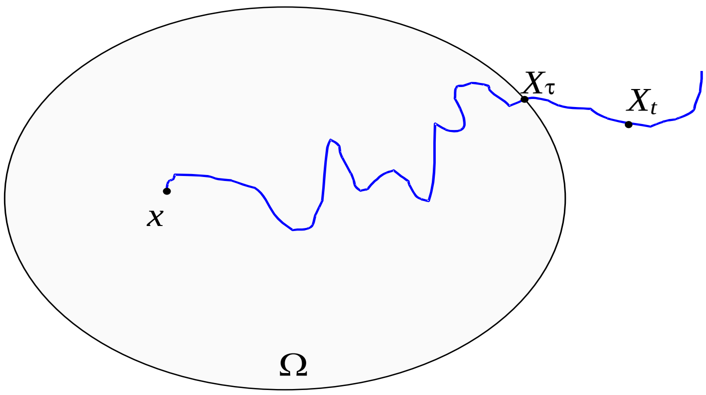

Figure 3.2: First exit time τ

A set $N\subset M$ is called *properly exceptional*, if it is a Borel set of measure 0 that is almost never hit by the process X_t starting outside N. In the next conditions N denotes some properly exceptional set.

(E_β) (*An estimate for the mean exit time from balls*) For all $x\in M\setminus N$ and $r>0$

$$\mathbb{E}_x\tau_{B(x,r)}\simeq r^{\beta}$$

(the parameter β is called the walk dimension of the process).

($E\Omega$) (*An isoperimetric estimate for the mean exit time*) For any open subset $\Omega \subset M$,

$$\sup_{x \in \Omega \setminus N} \mathbb{E}_x (\tau_\Omega) \le C \mu (\Omega)^{\beta/\alpha}.$$

If both (V_α) and (E_β) are satisfied then we obtain for any ball $B \subset M$

$$\sup_{x \in B \setminus N} \mathbb{E}_x (\tau_B) \simeq r^\beta \simeq \mu (B)^{\beta/\alpha}.$$

It follows that the balls are in some sense optimal sets for the condition ($E\Omega$).

Example. If X_t is Brownian motion in $\mathbb{R}^n$ then it is known that

$$\mathbb{E}_x \tau_{B(x,r)} = c_n r^2$$

so that (E_β) holds which satisfies with $\beta = 2$. This can also be rewritten in the form

$$\mathbb{E}_x \tau_B = c_n |B|^{2/n}$$

where $B = B(x,r)$.

It is also known that for any open set $\Omega \subset \mathbb{R}^n$ with finite volume and for any $x \in \Omega$

$$\mathbb{E}_x (\tau_\Omega) \le \mathbb{E}_x \left(\tau_{B(x,r)} \right),$$

provided ball $B(x,r)$ has the same volume as Ω; that is, for a fixed value of $|\Omega|$, the mean exist time is maximal when Ω is a ball and x is its center. It follows that

$$\mathbb{E}_x (\tau_\Omega) \le c_n |\Omega|^{2/n}$$

so that ($E\Omega$) is satisfied with $\beta = 2$ and $\alpha = n$.

Finally, introduce notation for the following estimates of the heat kernel:

(UE) (*Sub-Gaussian upper estimate*) The heat kernel exists and satisfies the estimate

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \exp \left(-c \left(\frac{d^\beta(x,y)}{t} \right)^{\frac{1}{\beta-1}} \right)$$

for all $t > 0$ and almost all $x, y \in M$.

(ΦUE) (Φ-*upper estimate*) The heat kernel exists and satisfies the estimate

$$p_t(x,y) \le \frac{1}{t^{\alpha/\beta}} \Phi \left(\frac{d(x,y)}{t^{1/\beta}} \right)$$

for all $t > 0$ and almost all $x, y \in M$, where Φ is a decreasing positive function on $[0, +\infty)$ such that

$$\int_0^\infty s^\alpha \Phi(s) \frac{ds}{s} < \infty.$$

(DUE) (*On-diagonal upper estimate*) The heat kernel exists and satisfies the estimate

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}$$

for all $t>0$ and almost all $x,y\in M$.

Clearly,

$$(UE) \Rightarrow (\Phi UE) \Rightarrow (DUE).$$

Theorem 3.4. *Let (M,d,μ) be a metric measure space and let (V_α) hold. Let $(\mathcal{E},\mathcal{F})$ be a regular, local, conservative Dirichlet form in $L^2(M,\mu)$. Then, the following equivalences are true:*

$$\begin{aligned}(UE) &\Leftrightarrow (\Phi UE)\\ &\Leftrightarrow (FK)+(E_\beta)\\ &\Leftrightarrow (E\Omega)+(E_\beta)\end{aligned}$$

Let us emphasize the equivalence

$$(UE) \Leftrightarrow (E\Omega)+(E_\beta)$$

where the right hand side means the following: the mean exit time from all sets Ω satisfies the isoperimetric inequality, and this inequality is optimal for balls (up to a constant multiple). Note that the latter condition relates the properties of the diffusion (and, hence, of the Dirichlet form) to the distance function.

Conjecture. *Under the hypotheses of Theorem 3.4,*

$$(UE) \Leftrightarrow (FK)+\left\{\lambda_{\min}(B_r)\simeq r^{-\beta}\right\}$$

Indeed, the Faber-Krahn inequality (FK) can be regarded as an isoperimetric inequality for $\lambda_{\min}(\Omega)$, and the condition

$$\lambda_{\min}(B_r)\simeq r^{-\beta}$$

means that (FK) is optimal for balls (up to a constant multiple).

Theorem 3.4 is an oversimplified version of a result of AG, J.Hu '10 where instead of (V_α) one uses the volume doubling condition, and other hypotheses must be appropriately changed.

The following lemma is used in the proof of Theorem 3.4.

Lemma 3.5. *For any open set $\Omega\subset M$*

$$\lambda_{\min}(\Omega) \geq \frac{1}{\operatorname{esup}_{x\in\Omega}\mathbb{E}_x(\tau_\Omega)}.$$

Proof. Let G_Ω be the Green operator in Ω, that is,

$$G_\Omega = \mathcal{L}_\Omega^{-1} = \int_0^\infty e^{-t\mathcal{L}_\Omega} dt.$$

We claim that

$$\mathbb{E}_x(\tau_\Omega) = G_\Omega 1(x)$$

for almost all $x \in \Omega$. We have

$$\begin{aligned} G_\Omega 1(x) &= \int_0^\infty e^{-t\mathcal{L}_\Omega} 1_\Omega(x)\, dt \\ &= \int_0^\infty \mathbb{E}_x\left(1_\Omega\left(X_t^\Omega\right)\right) \\ &= \int_0^\infty \mathbb{E}_x\left(\mathbf{1}_{\{t<\tau_\Omega\}}\right) dt \\ &= \mathbb{E}_x \int_0^\infty \left(\mathbf{1}_{\{t<\tau_\Omega\}}\right) dt \\ &= \mathbb{E}_x(\tau_\Omega). \end{aligned}$$

Setting

$$m = \operatorname*{esup}_{x\in\Omega} \mathbb{E}_x(\tau_\Omega)$$

we obtain that $G_\Omega 1 \le m$, so that $m^{-1}G_\Omega$ is a Markovian operator. Therefore, $\|m^{-1}G_\Omega\|_{L^2\to L^2} \le 1$ whence $\operatorname{spec} G_\Omega \in [0,m]$. It follows that $\operatorname{spec}\mathcal{L}_\Omega \subset [m^{-1},\infty)$ and $\lambda_{\min}(\Omega) \ge m^{-1}$. ■

Proof of Theorem 3.4 '⇐'. We have the implications

$$(E\Omega) \overset{\text{L.3.5}}{\Rightarrow} (FK) \overset{\text{T.3.3}}{\Rightarrow} (DUE).$$

In particular, we see that the heat kernel exists under any of the hypotheses of Theorem 3.4.

The next observation is that

$$(E_\beta) \Rightarrow \mathbb{P}_x\left(\tau_{B(x,r)} \le t\right) \le \varepsilon \tag{(5)}$$

for some $\varepsilon \in (0,1)$ provided $r \ge Kt^{1/\beta}$ (like in Barlow's lectures), which in turn yields

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \le \varepsilon. \tag{(6)}$$

It is easy to see that also $(\Phi UE) \Rightarrow (6)$ just by direct integration as in the proof of Theorem 2.2.

The condition (6) implies by bootstrapping

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \le C \exp\left(-c\left(\frac{r^\beta}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{(7)}$$

for all $t, r > 0$ and almost all $x \in M$, as it was mentioned in the proof of Theorem 2.11.

Hence, any set of the hypothesis of Theorem 3.4 imply both (DUE) and (7). By an argument in the proof of Theorem 2.11 we conclude

$$(DUE) + (7) \Rightarrow (UE).$$

■

Chapter 4

Two-sided bounds of the heat kernel

4.1 Using elliptic Harnack inequality

Now we would like to extend the results of Ch.3 to obtain also the lower estimates of the heat kernel. As before, (M, d, μ) is a metric measure space, and assume in addition that all metric balls are precompact. Let $(\mathcal{E}, \mathcal{F})$ is a local regular conservative Dirichlet form in $L^2(M, \mu)$.

Definition. We say that a function $u \in \mathcal{F}$ is *harmonic* in an open set $\Omega \subset M$ if

$$\mathcal{E}(u, v) = 0 \text{ for all } v \in \mathcal{F}(\Omega).$$

For example, if $M = \mathbb{R}^n$ and $(\mathcal{E}, \mathcal{F})$ is the canonical Dirichlet form in $\mathbb{R}^n$ then we obtain the following definition: a function $u \in W_2^1(\mathbb{R}^n)$ is harmonic in an open set $\Omega \subset \mathbb{R}^n$ if

$$\int_{\mathbb{R}^n} \langle \nabla u, \nabla v \rangle dx = 0$$

for all $v \in H_0^1(\Omega) \Leftrightarrow v \in C_0^\infty(\Omega)$. This of course implies that $\Delta u = 0$ in a weak sense in Ω and, hence, u is harmonic in Ω in the classical sense. However, unlike the classical definition, we a priori require $u \in W_2^1(\mathbb{R}^n)$.

Definition. We say that M satisfies the *elliptic Harnack inequality* (H) if there exist constants $C > 1$ and $\delta \in (0, 1)$ such that for any ball $B(x, r)$ and for any function $u \in \mathcal{F}$ that is non-negative and harmonic in $B(x, r)$,

$$\operatorname*{esup}_{B(x,\delta r)} u \leq C \operatorname*{einf}_{B(x,\delta r)} u.$$

Theorem 3.6. *(AG, A.Telcs '10) If the hypotheses $(V_\alpha) + (E_\beta) + (H)$ are satisfied, then the heat kernel $p_t(x, y)$ exists, is Hölder continuous in $x, y \in M$, and satisfies for all $t > 0$ and all $x, y \in M$ the following estimates:*

(UE) *a sub-Gaussian upper estimate*

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right),$$

(NLE) *and the* near-diagonal *lower estimate*

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \quad \text{provided } d(x,y) \leq \eta t^{1/\beta},$$

where $\eta > 0$ is a small enough constant.

Furthermore, we have the equivalence

$$(V_\alpha) + (UE) + (NLE) \Leftrightarrow (V_\alpha) + (E_\beta) + (H).$$

This theorem is proved in AG, A.Telcs '10 in a more general setting of volume doubling instead of (V_α).

Approach to the proof. First one shows that $(V_\alpha) + (E_\beta) + (H) \Rightarrow (FK)$, which is quite involved and uses, in particular, Lemma 3.5. Having $(V_\alpha) + (E_\beta) + (FK)$ we obtain (UE) by Theorem 3.4.

Using the elliptic Harnack inequality, one obtain in a standard way the oscillating inequality for harmonic functions and then for functions of the form $u = G_\Omega f$ (that solves the equation $\mathcal{L}_\Omega u = f$) in terms of $\|f\|_\infty$.

If now $u = P_t^\Omega f$ then u satisfies the equation

$$\frac{d}{dt}u = -\mathcal{L}_\Omega u$$

whence

$$u = -G_\Omega\left(\frac{d}{dt}u\right).$$

Knowing an upper bound for u, that follows from the upper bound for the heat kernel, one obtains also an upper bound for $\frac{d}{dt}u$ in terms of u. Applying the oscillation inequality one obtains the Hölder continuity of u and, hence, of the heat kernel.

Let us prove the on-diagonal lower bound

$$p_t(x,x) \geq ct^{-\alpha/\beta}.$$

As in the proof of Theorem 2.2, (UE) and (V_α) imply that

$$\int_{B(x,r)} p_t(x,y)\, d\mu(y) \geq \frac{1}{2}$$

provided $r \geq Kt^{1/\beta}$. Choosing $r = Kt^{1/\beta}$, we obtain

$$\begin{aligned} p_{2t}(x,x) &= \int_M p_t^2(x,y)\,d\mu(y) \\ &\geq \frac{1}{\mu(B(x,r))}\left(\int_{B(x,r)} p_t(x,y)\,d\mu(y)\right)^2 \\ &\geq \frac{c}{r^\alpha} = \frac{c'}{t^{\alpha/\beta}}. \end{aligned}$$

Then (NLE) follows from the upper estimate for

$$|p_t(x,x) - p_t(x,y)|$$

when y close to x, which follows from the oscillation inequality. ■

Assuming that the heat kernel exists, define the Green kernel $g(x,y)$ by

$$g(x,y) = \int_0^\infty p_t(x,y)\,dt.$$

If the Green kernel is finite then it is the integral kernel of the Green operator $G = \mathcal{L}^{-1}$. If the heat kernel satisfies (UE) and (NLE) and $\alpha > \beta$ (a strongly transient case), then it follows that

$$g(x,y) \simeq d(x,y)^{\beta-\alpha}. \qquad ((G))$$

For example, in $\mathbb{R}^n$ we have $g(x,y) = c_n|x-y|^{2-n}$, $n > 2$.

Corollary 3.7. *(The transient case) Assume $\alpha > \beta > 1$. If (V_α) is satisfied then*

$$(G) \Leftrightarrow (UE) + (NLE)$$

In the proof one verifies that $(G) \Rightarrow (H) + (E_\beta)$.

4.2 Matching upper and lower bounds

The purpose of this section is to improve both (UE) and (NLE) in order to obtain matching upper and lower bounds for the heat kernel. The reason why (UE) and (NLE) do not match, in particular, why (NLE) contains no information about lower bound of $p_t(x,y)$ for distant x,y is the lack of *chaining properties* of the distance function, that is an ability to connect any two points $x,y \in M$ by a chain of balls of controllable radii so that the number of balls in this chain is also under control.

For example, the chain condition considered above is one of such properties. If (M,d) satisfies the chain condition then as we have already mentioned, (NLE) implies the full sun-Gaussian lower estimate by the chain argument and the semigroup property.

Here we consider a setting with weaker chaining properties. For any $\varepsilon > 0$ introduce a modified distance $d_\varepsilon(x, y)$ by

$$d_\varepsilon(x, y) = \inf_{\{x_i\} \text{ is } \varepsilon\text{-chain}} \sum_{i=1}^{N} d(x_i, x_{i-1}) \qquad ((8))$$

where *ε-chain* is a sequence $\{x_i\}_{i=0}^N$ of points in M such that

$$x_0 = x, \quad x_N = y, \quad \text{and } d(x_i, x_{i-1}) < \varepsilon \text{ for all } i = 1, 2, ..., N.$$

Clearly, $d_\varepsilon(x, y)$ is decreases as ε increases and $d_\varepsilon(x, y) = d(x, y)$ if $\varepsilon > d(x, y)$. As $\varepsilon \downarrow 0$, $d_\varepsilon(x, y)$ increases and can go to ∞ or even become equal to ∞. It is easy to see that $d_\varepsilon(x, y)$ satisfies all properties of a distance function except for finiteness, so that it is a distance function with possible value $+\infty$.

It is easy to show that

$$d_\varepsilon(x, y) \simeq \varepsilon N_\varepsilon(x, y),$$

where $N_\varepsilon(x, y)$ is the smallest number of balls in a chain of balls of radii ε connecting x and y:

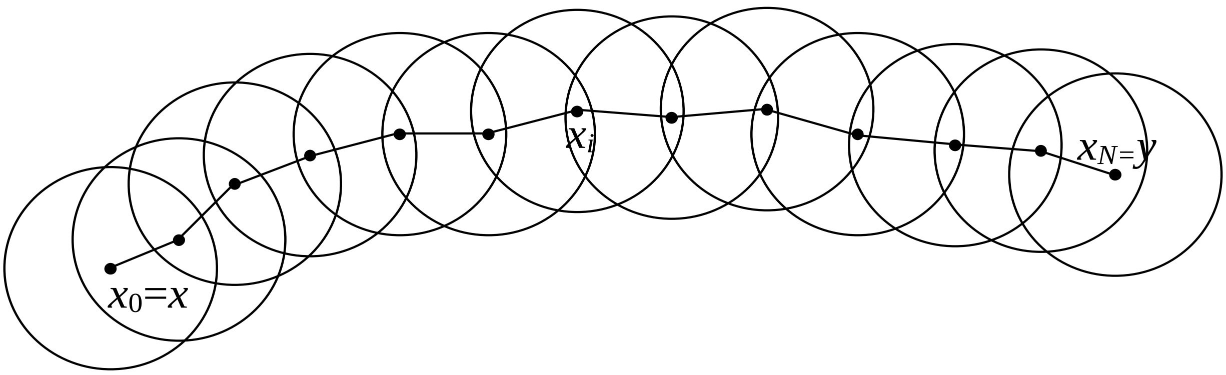

Figure 4.1: Chain of balls connecting x and y

N_ε can be regarded as the graph distance on a graph approximation of M by an ε-net.

If d is geodesic then the points $\{x_i\}$ of an ε-chain can be chosen on the shortest geodesic, whence $d_\varepsilon(x, y) = d(x, y)$. If the distance function d satisfies the chain condition then one can choose in (8) an ε-chain so that $d(x_i, x_{i+1}) \leq C\frac{d(x,y)}{N}$, whence $d_\varepsilon(x, y) \leq Cd(x, y)$. In general, $d_\varepsilon(x, y)$ may go to ∞ as $\varepsilon \to 0$, and the rate of growth of $d_\varepsilon(x, y)$ as $\varepsilon \to 0$ can be regarded as a quantitative description of the chaining properties of d.

We need the following hypothesis

C_β Chaining property: for all $x, y \in M$,

$$\varepsilon^{\beta-1} d_\varepsilon(x,y) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

or equivalently,

$$\varepsilon^{\beta} N_\varepsilon(x,y) \to 0 \text{ as } \varepsilon \to 0.$$

For $x \neq y$ we have $\varepsilon^{\beta-1} d_\varepsilon(x,y) \to \infty$ as $\varepsilon \to \infty$ which implies under (C_β) that there is $\varepsilon = \varepsilon(t,x,y)$ that satisfies the identity

$$\varepsilon^{\beta-1} d_\varepsilon(x,y) = t \tag{(9)}$$

(always take the maximal possible value of ε). If $x = y$ then set $\varepsilon(t,x,x) = \infty$.

Theorem 3.8. *If $(V_\alpha) + (E_\beta) + (H)$ and (C_β) are satisfied then*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d_\varepsilon^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{(10)}$$

$$\asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-cN_\varepsilon(x,y)\right), \tag{(11)}$$

where $\varepsilon = \varepsilon(t,x,y)$.

Since $d_\varepsilon(x,y) \geq d(x,y)$, the upper bound in (10) is an improvement of (UE); similarly the lower bound in (10) is an improvement of (NLE). The proof of the upper bound in (10) follows the same line as the proof of (UE) with careful tracing all places where the distance $d(x,y)$ is used and making sure that it can be replaced by $d_\varepsilon(x,y)$. The proof of the lower bound in (11) uses (NLE) and the semigroup identity along the chain with N_ε balls connecting x and y. Finally, observe that (10) and (11) are equivalent, that is

$$N_\varepsilon \simeq \left(\frac{d_\varepsilon^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}},$$

which follows by substituting here $N_\varepsilon \simeq d_\varepsilon/\varepsilon$ and $t = \varepsilon^{\beta-1} d_\varepsilon(x,y)$.

Example. A good example to illustrate Theorem 3.8 is the class of post critically finite (p.c.f.) fractals. For connected p.c.f. fractals with regular harmonic structure the heat kernel estimate (11) was proved by Hambly and Kumagai '99. In this setting $d(x,y)$ is the resistance metric of the fractal M and μ is the Hausdorff measure of M of dimension $\alpha := \dim_H M$. Hambly and Kumagai proved that (V_α) and (E_β) are satisfied with $\beta = \alpha + 1$. The condition (C_β) follows from their estimate

$$N_\varepsilon(x,y) \leq C\left(\frac{d(x,y)}{\varepsilon}\right)^{\beta/2},$$

because

$$\varepsilon^\beta N_\varepsilon(x,y) \leq Cd(x,y)^{\beta/2} \varepsilon^{\beta/2} \to 0 \text{ as } \varepsilon \to 0.$$

The Harnack inequality (H) on p.c.f. fractals was proved by Kigami '01. Hence, Theorem 2.8 applies and gives the estimates (10)-(11).

The estimate (11) means that the diffusion process goes from x to y in time t in the following way. The process firstly "computes" the value $\varepsilon(t,x,y)$, secondly "detects" a shortest chain of ε-balls connecting x and y, and then goes along that chain.

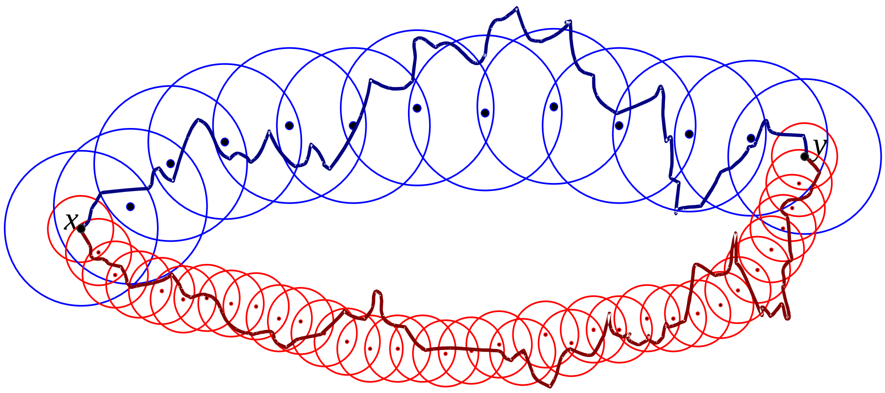

Figure 4.2: Two shortest chains of ε-ball for two distinct values of ε provide different routes for the diffusion from x to y for two distinct values of t.

This phenomenon was first observed by Hambly and Kumagai on p.c.f. fractals, but it seems to be generic. Hence, to obtain matching upper and lower bounds, one needs in addition to the usual hypotheses also the following information, encoded in the function $N_\varepsilon(x,y)$: the graph distance between x and y on any ε-net approximation of M.

Example of computation of ε. Assume that the following bound is known for all $x,y\in M$ and $\varepsilon>0$

$$N_\varepsilon(x,y)\le C\left(\frac{d(x,y)}{\varepsilon}\right)^\gamma,$$

where $0<\gamma<\beta$, so that (C_β) is satisfied (since $N_\varepsilon\ge d(x,y)/\varepsilon$, one must have $\gamma\ge 1$). Since by (9) we have $\varepsilon^\beta N_\varepsilon\simeq t$, it follows that

$$\varepsilon^\beta\left(\frac{d(x,y)}{\varepsilon}\right)^\gamma\ge ct,$$

whence

$$\varepsilon\ge c\left(\frac{t}{d(x,y)^\gamma}\right)^{\frac{1}{\beta-\gamma}}.$$

Consequently, we obtain

$$N_\varepsilon(x,y)\le Cd(x,y)^\gamma\varepsilon^{-\gamma}\le Cd(x,y)^\gamma\left(\frac{d(x,y)^\gamma}{t}\right)^{\frac{\gamma}{\beta-\gamma}}=\left(\frac{d(x,y)^\beta}{t}\right)^{\frac{\gamma}{\beta-\gamma}}$$

and

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d(x,y)^\beta}{ct}\right)^{\frac{\gamma}{\beta-\gamma}}\right).$$

Similarly, the lower estimate of N_ε

$$N_\varepsilon(x,y) \geq c\left(\frac{d(x,y)}{\varepsilon}\right)^\gamma$$

implies an upper bound for the heat kernel

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d(x,y)^\beta}{Ct}\right)^{\frac{\gamma}{\beta-\gamma}}\right).$$

Remark. Assume that (V_α) holds and all balls in M of radius $\geq r_0$ are connected, for some $r_0 > 0$. We claim that (C_β) holds with any $\beta > \alpha$. The α-regularity of measure μ implies by the classical ball covering argument, that any ball B_r of radius r can be covered by at most $C\left(\frac{r}{\varepsilon}\right)^\alpha$ balls of radii $\varepsilon \in (0,r)$. Consequently, if B_r is connected then any two points $x,y \in B_r$ can be connected by a chain of ε-balls containing at most $C\left(\frac{r}{\varepsilon}\right)^\alpha$ balls, so that

$$N_\varepsilon(x,y) \leq C\left(\frac{r}{\varepsilon}\right)^\alpha.$$

Since any two points $x,y \in M$ are contained in a connected ball B_r (say, with $r = r_0 + d(x,y)$), we obtain

$$\varepsilon^\beta N_\varepsilon(x,y) \leq C\varepsilon^{\beta-\alpha} r^\alpha \to 0$$

as $\varepsilon \to 0$, which was claimed.

4.3 Further results

We discuss here some consequences and extensions of the above results.

Corollary 3.9. *If (M,d) satisfies the chain condition then $(V_\alpha) + (E_\beta) + (H)$ is equivalent to the two-sided estimate*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right). \quad ((12))$$

Proof. The implication

$$(V_\alpha) + (E_\beta) + (H) \Rightarrow (12)$$

holds by Theorem 3.8 because $d_\varepsilon \simeq d$. For the opposite implication observe that

$$(12) \Rightarrow (V_\alpha)$$

by Theorem 2.2 and

$$(12) \Rightarrow (UE) + (NLE) \Rightarrow (E_\beta) + (H)$$

by Theorem 3.6. ■

Conjecture. *The condition (E_β) above can be replaced by*

$$\lambda_{\min}\left(B\left(x,r\right)\right) \simeq r^{-\beta}. \qquad ((\lambda_\beta))$$

In fact, (E_β) in all statements can be replaced by the resistance condition:

$$\operatorname{res}(B_r, B_{2r}) \simeq r^{\beta-\alpha} \qquad ((res_\beta))$$

where $B_r = B\left(x,r\right)$. In the strongly recurrent case $\alpha < \beta$ it alone implies the elliptic Harnack inequality (H) so that heat kernel two sided estimates are equivalent to $(V_\alpha)+(res_\beta)$ as was proved by Barlow, Coulhon, Kumagai '05 (in a setting of graphs) and was discussed in M. Barlow's lectures.

An interesting (and obviously hard) question is characterization of the elliptic Harnack inequality (H) in more geometric terms - so far nothing is known, not even a conjecture.

One can consider also a *parabolic* Harnack inequality (PHI), which uses caloric functions instead of harmonic functions. Then in a general setting and assuming the volume doubling condition (VD) (instead of (V_α)), the following holds:

$$(PHI) \Leftrightarrow (UE) + (NLE)$$

(AG, Barlow, Kumagai in preparation). On the other hand, (PHI) is equivalent to

Poincaré inequality + cutoff Sobolev inequality

(Barlow, Bass, Kumagai '05).

Conjecture. *The cutoff Sobolev inequality here can be replaced by (λ_β) and/or (res_β).*