Correction to "Sub-Gaussian estimates of heat kernels on infinite graphs" by A.Grigor'yan and A.Telcs

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This note contains a corrected version of Section 10 of the paper [4]. The purpose of that section in [4] was to prove the implication $(G) \Rightarrow (H)$ using $(G) \Rightarrow (HG) \Rightarrow (H)$. However, the proof of the first implication $(G) \Rightarrow (HG)$ contained an error. Despite that, the result $(G) \Rightarrow (H)$ remains true, which is proved below using a modified definition of (HG).

10 The Harnack inequality and the Green kernel

Recall that the weighted graph (Γ, μ) satisfies *the elliptic Harnack inequality* (H) if there exist constants $H, K > 1$ such that, for all $z \in \Gamma$, $R \geq 1$, and for any nonnegative function u in $\overline{B(z, KR)}$ which is harmonic in $B(z, KR)$, the following inequality is satisfied[1]

$$\max_{B(z,R)} u \leq H \min_{B(z,R)} u \,. \tag{H}$$

Note that this inequality always holds for $R < 1$ because in this case $B(z, R) = \{z\}$.

In this section we establish that (H) is implied by the condition (G), where the latter means that

$$C^{-1} d(x, y)^{-\gamma} \leq g(x, y) \leq C d(x, y)^{-\gamma}, \quad \forall x \neq y. \tag{G}$$

Consider the following *Harnack inequality for the Green function*[2] (HG): for some constants $H' > 1$, $M > 2$, for all $z \in \Gamma$, $R \geq 1$, and for any finite set $U \supset B(z, MR)$,

$$\max_{x \in B(z,R)^c} g_U(x, z) \leq H' \min_{y \in B(z,2R)} g_U(y, z). \tag{HG}$$

It is easy to see that (HG) can be equivalently stated as follows:

$$\max_{B(z,2R) \setminus B(z,R)} g_U(\cdot, z) \leq H' \min_{B(z,2R) \setminus B(z,R)} g_U(\cdot, z) \,.$$

Proposition 10.1 *Assume that (p_0) hold and the graph (Γ, μ) is transient. Then*

$$(G) \Longrightarrow (HG) \Longrightarrow (H).$$

The essential part of the proof is contained in the following lemma.

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[1] It seems to be unknown whether in general condition (H) with some value of K implies that for a smaller value of K (but possibly with a larger value of H). However, this is true in the presence of the doubling volume property.

[2] A slightly different version of (HG) – denote it by (HG') – was considered in [5] and [1], where in the right hand side of (HG) one takes the minimum over $y \in B(z, R)$ rather than over $y \in B(z, 2R)$. It was shown in [1] that $(H) \Rightarrow (HG')$. It is easy to see that $(H) + (HG') \Rightarrow (HG)$ so that in fact $(H) \Rightarrow (HG)$. Proposition 10.1 contains the converse to that.

Lemma 10.2 *Let $U_0 \subset U_1 \subset U_2 \subset U_3$ be a sequence of finite sets in Γ such that $\overline{U_i} \subset U_{i+1}$, $i = 0, 1, 2$. Denote $A = U_2 \setminus U_1$, $B = U_0$ and $U = U_3$. Then, for any function u which is nonnegative in $\overline{U}$ and harmonic in U, we have*

$$\max_B u \leq H \min_B u\,, \tag{10.1}$$

where

$$H := \max_{x,y \in B} \max_{z \in A} \frac{g_U(x,z)}{g_U(y,z)} \tag{10.2}$$

(see Fig. 1).

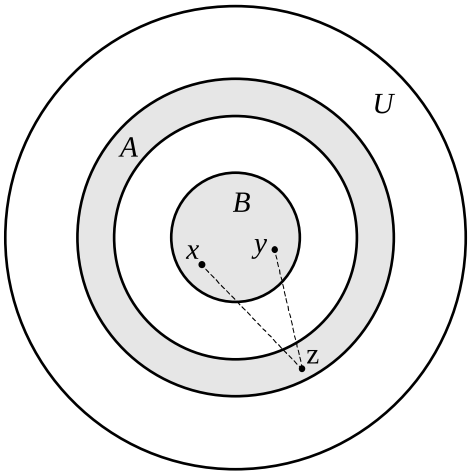

Figure 1: The sets $B = U_0$, $A = U_2 \setminus U_1$ and $U = U_3$

Remark 10.1 Note that no a priori assumption has been made about the graph (Γ, μ) except for connectedness and unboundedness.

Proof. The following potential-theoretic argument is borrowed from [2]. Given a non-negative function u in $\overline{U}$, which is harmonic in U, denote by S_u the following class of super-harmonic functions in U:

$$S_u = \left\{ v : v \geq 0 \text{ in } \overline{U}, \quad v \geq u \text{ in } \overline{U_1}, \quad \text{and} \quad \Delta v \leq 0 \text{ in } U \right\},$$

and define the function w on $\overline{U}$ by

$$w(x) = \min \left\{ v(x) : v \in S_u \right\}. \tag{10.3}$$

Clearly, $w \in S_u$. Since the function u itself is also in S_u, we have $w \leq u$ in $\overline{U}$. On the other hand, by definition of S_u, $w \geq u$ in $\overline{U_1}$, whence we see that $u = w$ in $\overline{U_1}$ (see Fig. 2). In particular, it suffices to prove (10.1) for w instead of u.

Let us show that $w \in c_0(U)$, that is, w vanish on $\overline{U} \setminus U$. Indeed, let $v(x)$ solve the Dirichlet problem

$$\begin{cases} \Delta v = -1 & \text{in } U, \\ v = 0 & \text{on } \overline{U} \setminus U. \end{cases}$$

Since v is superharmonic, by the strong minimum principle v is strictly positive in U. Hence, for a large enough constant C, we have $Cv \geq u$ in $\overline{U_1}$ whence $Cv \in S_u$ and $w \leq Cv$. Since $v = 0$ on $\overline{U} \setminus U$, this implies $w = 0$ on $\overline{U} \setminus U$.

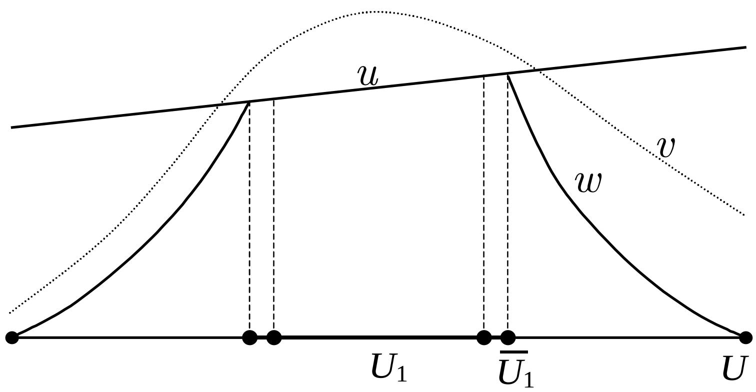

Figure 2: The function u, a function $v \in S_u$ and the function $w = \min_{S_u} v$. The latter is harmonic in U_1 and in $U \setminus \overline{U_1}$.

Set $f := -\Delta w$ and observe that by construction $f \geq 0$ in U. Since $w \in c_0(U)$, we have, for any $x \in U$,

$$w(x) = \sum_{z \in U} g_U(x, z) f(z) \mu(z). \tag{10.4}$$

Next we will prove that $f = 0$ outside A so that the summation in (10.4) can be restricted to $z \in A$. Given that much, we obtain, for all $x, y \in B$,

$$\frac{w(x)}{w(y)} = \frac{\sum_{z \in A} g_U(x, z) f(z) \mu(z)}{\sum_{z \in A} g_U(y, z) f(z) \mu(z)} \leq H,$$

whence (10.1) follows.

We are left to verify that w is harmonic in U_1 and outside $\overline{U_1}$. Indeed, if $x \in U_1$ then

$$\Delta w(x) = \Delta u(x) = 0,$$

because $w = u$ in $\overline{U_1}$. Let $\Delta w(x) \neq 0$ for some $x \in U \setminus \overline{U_1}$. Since w is superharmonic, we have $\Delta w(x) < 0$ and

$$w(x) > Pw(x) = \sum_{y \sim x} P(x, y) w(y).$$

Consider the function w' which is equal to w everywhere in $\overline{U}$ except for the point x, and w' at x is defined to satisfy

$$w'(x) = \sum_{y \sim x} P(x, y) w'(y).$$

Clearly, $w'(x) < w(x)$, and w' is superharmonic in U. Since $w' = w = u$ in $\overline{U_1}$, we have $w' \in S_u$. Hence, by the definition (10.3) of w, $w \leq w'$ in $\overline{U}$ which contradicts $w(x) > w'(x)$. ■

Proof of Proposition 10.1. Let us prove $(G) \Rightarrow (HG)$. It will be sufficient to prove that if $U \supset B(z, MR)$ (where $M > 2$ is to be specified below) then

$$g_U(y, z) \geq \frac{1}{2} g(y, z) \quad \text{for all } y \in B(z, 2R). \tag{10.5}$$

Since also $g_U \leq g$, hypothesis (G) and (10.5) will imply

$$\max_{x \in B(z,R)^c} g_U(x, z) \leq \max_{x \in B(z,R)^c} g(x, z) \leq C \min_{y \in B(z,2R)} g(y, z) \leq 2C \min_{y \in B(z,2R)} g_U(x, z).$$

The proof of (10.5) follows the approach of [3]. Consider the function $u = g(\cdot, z) - g_U(\cdot, z)$ which is nonnegative and harmonic in U. Since outside U the function u coincides with $g(\cdot, z)$, we obtain by the maximum principle and (G) that

$$\max_U u = \max_{U^c} u = \max_{U^c} g(\cdot, z) \le C(MR)^{-\gamma}.$$

Therefore, for $y \in B(x, 2R)$,

$$g(y, z) \ge C^{-1}(2R)^{-\gamma} \ge 2C(MR)^{-\gamma} \ge 2\max u$$

provided M is large enough, whence it follows that

$$g_U(y, z) \ge g(y, z) - \max u \ge \frac{1}{2} g(y, z).$$

Let us prove $(HG) \Rightarrow (H)$. Fix a point $x_0 \in \Gamma$ and write for shortness $B_r := B(x_0, r)$. Let u be a nonnegative harmonic function in $U := B_{6MR}$, where $R > 1$. By Lemma 10.2, we have

$$\max_{B_R} u \le H \min_{B_R} u, \tag{10.6}$$

where

$$H := \max_{x,y\in B_R} \max_{z\in A} \frac{g_U(x, z)}{g_U(y, z)}, \tag{10.7}$$

and $A = B_{5R} \setminus B_{4R}$ (see Fig. 1). Let us show that $H \le H'$ where H' is the constant from (HG). Indeed, if $x, y \in B_R$ and $z \in A$ then it is easy to see that $x \in B(z, 3R)^c$ and $y \in B(z, 6R)$. Since $5R + 3MR < 6MR$, we see that $B(z, 3MR) \subset U$. By (HG) we obtain, for all $x, y \in B_R$,

$$g_U(x, z) \le H' g_U(y, z).$$

Substituting into (10.7), we obtain that (H) holds with $K = 6M$ and $H = H'$. ■

References

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