Asymptotic separation for independent trajectories of Markov processes

Alexander Grigor'yan *
Imperial College
180 Queen's Gate
London SW7 2BZ
United Kingdom
email: a.grigoryan@ic.ac.uk

Mark Kelbert †
University of Wales, Swansea
Singleton Park,
Swansea SA2 8PP
United Kingdom
email: m.kelbert@swansea.ac.uk

April 2000

Contents

*Supported by the EPSRC Research Fellowship B/94/AF/1782
†Partially supported by the EPSRC Visiting Fellowship GR/M61573

1 Introduction

The question of whether two independent random walks or Brownian motions intersect or not has a long history and has attracted much interest both in Probability Theory and in Mathematical Physics. This problem is related to the question of Triviality of continuous limits in Quantum Field Theory, see [1] and [19]. However, this problem also has a glorious history in the framework of Probability Theory starting from the classical works of Dvoretzky, Erdös, Kakutani and Taylor [15], [16], [17], [33].

The following picture was established in the works cited above. In $\mathbb{R}^2$, any finite number of independent Brownian trajectories intersect with probability 1 (moreover, points of intersection of cardinality of continuum exist almost surely). In $\mathbb{R}^3$, any two independent trajectories still intersect with probability 1, whereas the probability of intersection of three trajectories started apart is equal to 0.

If $d \geq 4$, then two independent Brownian trajectories in $\mathbb{R}^d$ started apart, intersect with probability 0. Nevertheless, in the borderline case $d = 4$, the trajectories do approach arbitrarily close each to other with probability 1, which is not the case when $d > 4$ (see also [21] and [39] for intersections of trajectories quasi everywhere).

A similar but somewhat different picture is established for simple random walks in $\mathbb{Z}^d$. If $d \leq 4$ then two independent walks intersect with probability 1, whereas in the case $d \geq 5$, this probability is smaller than 1 and tends to 0 when the starting points are moved apart. The difference between the continuous and discrete cases is due to the fact that on the lattice there is no difference between the notions of intersection and proximity.

The purpose of this paper is to study the properties of *asymptotic proximity* and *asymptotic separation* of two or more trajectories in a rather general setting of Markov processes, including certain diffusion processes, the α-stable processes and random walks.

Let M be a metric space with a distance function ρ, and let $\xi(t)$ be a stochastic process on M with infinite lifetime. The time t may have the range $\mathbb{R}_+$ or $\mathbb{Z}_+$. Denote by $\mathbb{P}_x$ the distribution law of ξ associated with the starting point $x \in M$. Given a sequence $\overline{x} = (x_1, x_2, ..., x_n)$ of n points of M, we consider independent processes ξ_{x_1}, ξ_{x_2}, ..., ξ_{x_n} with the joint distribution $\mathbb{P}_{\overline{x}} := \mathbb{P}_{x_1} \times \mathbb{P}_{x_2} \times ... \times \mathbb{P}_{x_n}$.

Definition 1.1 We say that two processes ξ_x, ξ_y are *asymptotically separated* if, for some $a > 0$,

$$\mathbb{P}_{x,y}(\exists T \ \ \forall t, s > T : \quad \rho(\xi_x(t), \, \xi_y(s)) \geq a) = 1. \tag{1.1}$$

Otherwise, we say that ξ_x and ξ_y are *asymptotically close.*

Similarly, n processes $\xi_{x_1}, \xi_{x_2}, ..., \xi_{x_n}$ are *asymptotically separated* if, for some $a > 0$,

$$\mathbb{P}_{\overline{x}} \left(\exists T \quad \forall t_1, ..., t_n > T : \quad \max_{1 \leq J, k \leq n} \rho(\xi_{x_j}(t_j), \, \xi_{x_k}(t_k)) \geq a \right) = 1. \tag{1.2}$$

Otherwise, we say that $\xi_{x_1}, \xi_{x_2}, ..., \xi_{x_n}$ are *asymptotically close.*

We have required that the process $\xi(t)$ has an infinite lifetime a.s. that is, ξ is stochastically complete. This is formally necessary in order to write down the conditions (1.1) and (1.2). The definition may be modified to include also stochastically incomplete processes, but we do not consider such processes for the sake of simplicity.

It is easy to see that two processes ξ_x and ξ_y are asymptotically close if, for any $a > 0$,

$$\mathbb{P}_{x,y}(\exists \, \{t_i\} \, , \{s_i\} \to \infty : \quad \rho(\xi_x(t_i), \, \xi_y(s_i)) < a) > 0. \tag{1.3}$$

Similarly, n processes ξ_{x_k} are asymptotically close if, for any $a > 0$,

$$\mathbb{P}_{\overline{x}}\left(\exists\{t_i^{(1)}\}, \{t_i^{(2)}\}, ..., \{t_i^{(n)}\} \to \infty : \quad \max_{1s<j,k\leq n} \rho(\xi_{x_j}(t_i^{(j)}),\, \xi_{x_k}(t_i^{(k)})) < a\right) > 0 \tag{1.4}$$

(see Fig. 1).

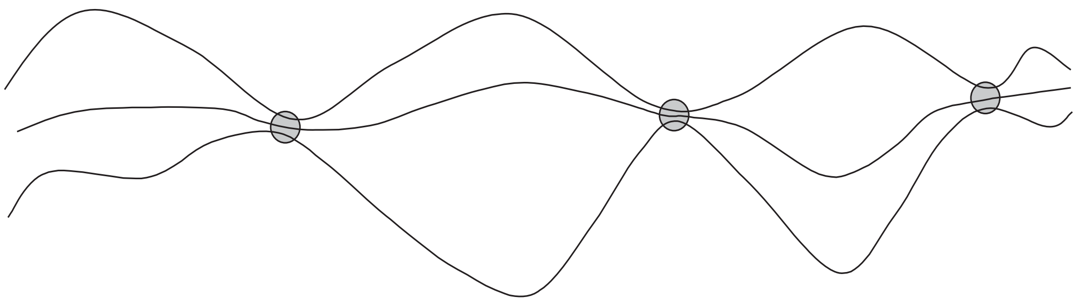

Figure 1: Three trajectories are asymptotically close if, for any $a > 0$, they approach each other within a distance a at arbitrarily large times, with positive probability.

The word "asymptotic" emphasizes that fact that we disregard segments of the trajectories of finite time duration. On the contrary, we concentrate on the global behavior of the trajectories and relations to the geometry "in the large" of the state space. It turns out that the property of the trajectories being asymptotically separated is connected to certain estimates of the Green kernel and of the heat kernel.

Let now M be a Riemannian manifold and ξ denote the Brownian motion on M governed by the Laplace-Beltrami operator Δ. Denote by $p(t, x, y)$ the transition density (=the heat kernel) for the process ξ.

Theorem 1.1 (=Corollary 2.4) *Let M be a manifold with bounded geometry* (see Definition 2.1)*. Assume that, for some $\nu > 0$ and all t large enough,*

$$\sup_{x\in M} p(t, x, x) \leq \frac{C}{t^{\nu/2}}, \tag{1.5}$$

and, for some integer $n \geq 2$,

$$\frac{2}{\nu} + \frac{1}{n} < 1. \tag{1.6}$$

Then n independent processes ξ_{x_1}, ξ_{x_2}, ..., ξ_{x_n} on M are asymptotically separated.

If $M = \mathbb{R}^d$ then $\nu = d$ in (1.5). Therefore, (1.6) holds provided either $n = 2$ and $d > 4$, or $n = 3$ and $d > 3$. In other words, any two trajectories of the Brownian motion in $\mathbb{R}^d$ are asymptotically separated if $d > 4$, and any three trajectories are asymptotically separated if $d > 3$. Of course, these statements are not new and can be deduced from much more detailed information about the properties of the Brownian motion in $\mathbb{R}^d$. However, Theorem 1.1 can be applied on manifolds where the usual Euclidean methods of investigation of trajectories do not work. On the other hand, there are many classes of manifolds where the heat kernel bounds like (1.5) are available (see [30]). Note that the number ν in (1.5) may not be an integer. If $\nu = 4 + \varepsilon$, where $\varepsilon > 0$, then theorem 1.1 implies that two trajectories are asymptotically separated, whereas if $\nu = 3+\varepsilon$ then three trajectories are asymptotically separated. See Section 2 for further discussion about the heat kernel's upper bounds.

Let $\xi^{(\alpha)}$ be the α-process on the manifold M, that is the process generated by the operator $-(-\Delta)^{\alpha/2}$ where $\alpha \in (0, 2]$.

Theorem 1.2 (=Corollary 4.3) *Let M be a manifold with bounded geometry. Assume that, for some $\nu > 0$ and all t large enough,*

$$\sup_{x \in M} p(t, x, x) \le \frac{C}{t^{\nu/2}}, \tag{1.7}$$

and, for some integer $n \ge 2$,

$$\frac{\alpha}{\nu} + \frac{1}{n} < 1. \tag{1.8}$$

Then n independent α-processes $\xi_{x_1}^{(\alpha)}$, $\xi_{x_2}^{(\alpha)}$, ..., $\xi_{x_n}^{(\alpha)}$ on M are asymptotically separated.

Let us emphasize that the condition (1.7) is given in terms of the transition density $p(t, x, y)$ of the Brownian motion ξ, rather than the α-process $\xi^{(\alpha)}$. In contrast to obtaining estimates of the transition density for the process $\xi^{(\alpha)}$, the heat kernel $p(t, x, y)$ can be effectively estimated in many interesting cases - see Section 2.

If $M = \mathbb{R}^d$ then $\nu = d$ and $\xi^{(\alpha)}$ is the α-stable process in $\mathbb{R}^d$. Let us compare the condition (1.8) with the results of S.J.Taylor [45] on self-intersections of the α-stable process in $\mathbb{R}^d$. The theorem of Taylor guaranties that if

$$\frac{\alpha}{d} + \frac{1}{n} > 1$$

then the set of n-multiple points of ξ is rather rich, which implies that n trajectories are asymptotically close. In the borderline case

$$\frac{\alpha}{d} + \frac{1}{n} = 1,$$

n trajectories already do not intersect, but they are still asymptotically close. Finally, under the condition (1.8), n trajectories are asymptotically separated.

The structure of this paper is the following. We first present in Sections 2, 3 and 4 the results for processes on Riemannian manifolds. Another particular case is random walks on graphs, which is treated in Section 5. In Section 6 we consider Markov processes on abstract metric measure spaces and state our results in the most general setting (including diffusions on fractals). In Section 7 we show how the particular processes mentioned here fit into the abstract scheme. Finally, we prove all the theorems in Section 8.

The dependences of the results are presented in the diagram below.

$$\begin{tikzcd}
\boxed{\text{Theorem 5.1}} & \boxed{\text{Theorem 6.2}} \arrow[l] \arrow[r] \arrow[dl] \arrow[d] & \boxed{\text{Theorem 2.1}} \arrow[d] \\
\boxed{\text{Theorem 4.2}} \arrow[d] & \boxed{\text{Theorem 6.3}} \arrow[d] & \boxed{\text{Theorem 2.3}} \arrow[d] \\
\boxed{\text{Corollary 4.3}} & \boxed{\text{Theorems 2.2, 4.1, 5.2}} & \boxed{\text{Corollary 2.4}}
\end{tikzcd}$$

2 Diffusion on Riemannian manifolds

Let M be a Riemannian manifold and ξ be the diffusion on M generated by the operator

$$L = \sigma^{-1} \operatorname{div}(\sigma \nabla), \tag{2.1}$$

where div and ∇ are the Riemannian divergence and the gradient respectively, and σ is a smooth function on M. For example, if $\sigma \equiv 1$ then $L = \Delta$ - the Laplace-Beltrami operator on M.

It is known that the operator L is formally self-adjoint with respect to the measure μ defined by

$$d\mu = \sigma d\mu_0,$$

where μ_0 is the Riemannian measure on M. The operator L with the domain $C_0(M)$ can be shown to be essentially self-adjoint in $L^2(M,\mu)$. Then, by the spectral theory, it is possible to define the operator semigroup e^{tL}. It possesses a smooth symmetric kernel $p(t,x,y)$ with respect to the measure μ, which simultaneously is the transition density of the diffusion ξ generated by L. We will refer to ξ as the *L-diffusion*. In particular, if $\sigma \equiv 1$ then $L = \Delta$ is the Laplace-Beltrami operator on M and ξ is the *Brownian motion* on M.

Denote by $\rho(x,y)$ the geodesic distance on M, by $B(x,r)$ the open geodesic ball of radius r centered at $x \in M$, and $V(x,r) := \mu(B(x,r))$.

For a subset $\Omega \subset M$, we will frequently use the notation $|\Omega| := \mu(\Omega)$. On any hypersurface S, we introduce *the surface area μ'* which is the measure on S having the density σ with respect to the Riemannian measure of co-dimension 1.

Denote by $g(x,y)$ the Green kernel of ξ, which is defined by

$$g(x,y) = \int_0^\infty p(t,x,y)dt.$$

Unlike the heat kernel, the Green kernel may be identically equal to infinity, which is equivalent to the recurrence of the process ξ. If $g \not\equiv \infty$ then $g(x,y) < \infty$ for distinct x,y, and g is the smallest positive fundamental solution to the operator L.

Throughout the paper, we assume that the operator L is *uniformly elliptic*, that is, for some $C > 1$,

$$C^{-1} \le \sigma(x) \le C, \quad \forall x \in M. \tag{2.2}$$

Definition 2.1 We say that the manifold M has *bounded geometry* if the Ricci curvature of M is uniformly bounded from below, and if its injectivity radius is positive.

Assuming that M has bounded geometry, denote by r_0 its injectivity radius. Then all balls $B(x, r_0/2)$ are uniformly quasi-isometric to the Euclidean ball of radius $r_0/2$ of the same dimension. This allows us to use the technique of uniformly elliptic and parabolic equations in $\mathbb{R}^d$ in order to locally estimate $p(t,x,y)$ and $g(x,y)$. Note also that manifolds of bounded geometry are geodesically complete.

The following three theorems are our main results for diffusions on manifolds.

Theorem 2.1 *Let M be a manifold with bounded geometry and L be uniformly elliptic. Assume that, for some integer $n > 1$, a point $x \in M$ and $\varepsilon > 0$,*

$$\int_{M\setminus\Omega_\varepsilon} g^n(x,y)\, d\mu(y) < \infty \tag{2.3}$$

where Ω_ε is the ε-neighborhood of x. Then the independent processes ξ_{x_1}, ξ_{x_2}, ..., ξ_{x_n} are asymptotically separated, for all $x_1, x_2, ..., x_n \in M$.

Let us observe that if the process ξ is recurrent, that is $g \equiv \infty$, then the trajectories $\xi_{x_1}(t)$, $\xi_{x_2}(t)$, ..., $\xi_{x_n}(t)$ are automatically asymptotically close, for any n. This can be regarded as a limiting case for the divergence of the integral in (2.3).

Note that the condition (2.3) is generally *not* necessary for the asymptotic separation of ξ_{x_1}, ξ_{x_2}, ..., ξ_{x_n}- see Section 7.1.

The purpose of the following statements is to provide simpler sufficient conditions for (2.3), in terms of the heat kernel decay.

Theorem 2.2 *Let M be a manifold with bounded geometry and L be uniformly elliptic. Assume that, for some $x \in M$,*

$$\int_1^\infty t\, p(t,x,x)\, dt < \infty. \tag{2.4}$$

Then the independent processes ξ_{x_1} and ξ_{x_2} are asymptotically separated, for all $x_1, x_2 \in M$.

Note that, for arbitrary points $x, y \in M$, the ratio

$$\frac{p(t,x,x)}{p(t,y,y)}$$

remains bounded as $t \to \infty$ (see [12]). Therefore, the convergence of the integral (2.4) for *some* x and for *all* x is the same.

Theorem 2.3 *Let M be a manifold with bounded geometry, and L be uniformly elliptic. Assume that, for an integer $n > 1$, and for a point $x \in M$,*

$$\sup_{y \in M} \int_1^\infty t^{\frac{1}{n-1}}\, p(t,x,y)\, \theta(t)\, dt < \infty, \tag{2.5}$$

where $\theta(t)$ is a continuous positive increasing function on $(0,\infty)$ such that

$$\int_1^\infty \frac{dt}{t\,\theta^{n-1}(t)} < \infty. \tag{2.6}$$

Then the condition (2.3) holds, and hence, the independent processes ξ_{x_1}, ξ_{x_2},..., ξ_{x_n} are asymptotically separated, for all n-tuples $x_1, x_2, ..., x_n \in M$.

It is plausible that the hypothesis (2.6) may only be needed for technical reasons. We conjecture that the statement of theorem 2.3 is true if, for some $x \in M$,

$$\int_1^\infty t^{\frac{1}{n-1}}\, p(t,x,x)\ dt < \infty,$$

similarly to theorem 2.2.

Corollary 2.4 (=Theorem 1.1) *Let M be a manifold with bounded geometry, and L be uniformly elliptic. Let us assume that, for all t large enough,*

$$\sup_{x \in M} p(t,x,x) \le \frac{C}{t^{\nu/2}}, \tag{2.7}$$

for some ν such that

$$\nu > \frac{2n}{n-1}. \tag{2.8}$$

Then the hypothesis (2.3) holds, and hence, the independent processes ξ_{x_1}, ξ_{x_2}, ..., ξ_{x_n} are asymptotically separated, for all n-tuples $x_1, x_2, ..., x_n \in M$.

Proof. The hypotheses (2.8) implies $\frac{\nu}{2}-\frac{1}{n-1}>1$. Let us set $\theta(t)=t^{\varepsilon}$, for a small enough positive ε so that
$$\frac{\nu}{2}-\frac{1}{n-1}-\varepsilon>1. \tag{2.9}$$
The semigroup property of the heat kernel yields
$$p(t,x,y)\leq[p(t,x,x)p(t,y,y)]^{1/2}$$
(see (8.25) below). Thus, by the Cauchy-Schwarz inequality and (2.7),
$$\begin{aligned}\int_1^\infty t^{\frac{1}{n-1}}\,p(t,x,y)\,\theta(t)dt &\leq \left[\int_1^\infty t^{\frac{1}{n-1}}p(t,x,x)\theta(t)dt\right]^{1/2}\left[\int_1^\infty t^{\frac{1}{n-1}}p(t,y,y)\theta(t)dt\right]^{1/2}\\ &\leq \int_1^\infty t^{\frac{1}{n-1}}\frac{C}{t^{\nu/2}}t^{\varepsilon}dt.\end{aligned}$$
By (2.9), the above integral is finite, whence we obtain (2.5). Hence, Corollary 2.4 follows from Theorem 2.3. ■

The supremum of numbers ν satisfying (2.7) is called *the asymptotic dimension* of the state space, associated with the process ξ. The geometric background of the hypothesis (2.7) is well understood – see [7], [27], [47] and the discussion below.

Examples. 1. If $n=2$ then (2.8) yields $\nu>4$. Thus, if the asymptotic dimension is $4+\varepsilon$ where $\varepsilon>0$, then any two trajectories are asymptotically separated. As was mentioned above, in the 4-dimensional Euclidean space two trajectories are asymptotically close (see, for example, [2]).

2. If $n=3$ then (2.8) yields $\nu>3$. Hence, if the asymptotic dimension is $3+\varepsilon$ where $\varepsilon>0$ then any three independent trajectories are asymptotically separated. Let us observe that the asymptotic dimension may be fractional, unlike the topological dimension. It is well-known that three trajectories in $\mathbb{R}^3$ are asymptotically close[1].

3. If
$$\sup_{x\in M}p(t,x,x)\leq\frac{C}{t^2\log^{\gamma}t}, \tag{2.10}$$
for some $\gamma>1$ and all t large enough, then the condition (2.4) holds, and any two trajectories are asymptotically separated.

The question of obtaining heat kernel upper bounds like (2.7) or (2.10) has been extensively studied (see [27], [30, Section 7.4] and references therein). Let Ω be an open precompact subset of M. Denote by $\lambda_1(\Omega)$ the first Dirichlet eigenvalue for the operator L in Ω. Then the on-diagonal heat kernel upper bound of the form $p(t,x,x)\leq f(t)$ for all $x\in M$ and $t>0$ is equivalent to a certain lower bound for $\lambda_1(\Omega)$ via the volume $|\Omega|$, for all Ω (see [27, Theorems 2.1 and 2.2], [8]). If M has bounded geometry (which is the case now) then one can localize this statement for large t and, respectively, for large volumes $|\Omega|$ (see [28, Theorem 4.2]). For example, the heat kernel estimate (2.7) can be derived from *the Faber-Krahn inequality*
$$\lambda_1(\Omega)\geq c\,|\Omega|^{-2/\nu}, \tag{2.11}$$
for all Ω with a large enough volume and for some $c>0$. Similarly, (2.10) follows from the estimate
$$\lambda_1(\Omega)\geq c\,|\Omega|^{-1/2}\log^{\gamma/2}|\Omega| . \tag{2.12}$$

[1] However, we could not find a good reference for this.

On the other hand, (2.11) can be derived from the following isoperimetric inequality:

$$\mu'(\partial\Omega) \geq c\,|\Omega|^{\frac{\nu-1}{\nu}},$$

and (2.12) follows from

$$\mu'(\partial\Omega) \geq c\,|\Omega|^{3/4}\log^{\gamma/4}|\Omega| \tag{2.13}$$

(see [40, Theorem 2.3.2/1]). Recall that in $\mathbb{R}^4$ the following isoperimetric inequality holds

$$\mu'(\partial\Omega) \geq c\;|\Omega|^{3/4}.$$

As is well-known, any two independent trajectories in $\mathbb{R}^4$ are asymptotically close, whereas a slightly better isoperimetric inequality (2.13) implies that any two independent trajectories are asymptotically separated.

In Theorems 2.1 and 2.3, n takes values $2, 3, 4, \ldots$. It would be interesting to find a probabilistic meaning of the hypotheses (2.3) and (2.5) for other values of n. For example, if $n = 1$ then (2.3) implies that the process ξ is stochastically incomplete (see [26]). However, this cannot take place on manifolds of bounded geometry (see [48]). If $n = \infty$ then (2.3) does not make any sense. However, (2.5) can be interpreted for the infinite n as

$$\int^{\infty} p(t, x, x)dt < \infty, \tag{2.14}$$

neglecting θ and $\sup_x$. The condition (2.14) means exactly the transience of ξ. Thus, the hypothesis (2.3) can be thought of as a kind of interpolation between the transience and the stochastic incompleteness of ξ.

3 Asymptotic proximity and volume growth

We consider here some examples of applications of Theorem 2.3 related to the volume growth of the manifold M. Assume for simplicity $L = \Delta$ so that ξ is the Brownian motion on M.

In the first example, let us assume that M has non-negative Ricci curvature and a positive injectivity radius (which, of course, implies that M has bounded geometry). As follows from a theorem of Li-Yau [37], the heat kernel on a complete manifold of non-negative Ricci curvature satisfies the following inequality

$$\sup_y p(t, x, y) \leq \frac{C}{V(x, \sqrt{t})}$$

for all $x \in M$ and $t > 0$. Therefore, the hypothesis (2.5) is implied by

$$\int_1^{\infty} \frac{t^{\frac{1}{n-1}}\;\theta(t)\,dt}{V(x, \sqrt{t})} < \infty.$$

By changing to $s = t^{\frac{n}{n-1}}$, this amounts to

$$\int_1^{\infty} \frac{\tilde{\theta}(s)ds}{V(x, s^{\frac{n-1}{2n}})} < \infty \tag{3.1}$$

where $\tilde{\theta}(s) = \theta(t^{\frac{n-1}{n}})$. For example, let us take $\theta(t) = \log^\gamma t$, where $\gamma > \frac{1}{n-1}$, so that (2.6) holds. Then (3.1) is true provided

$$\int_1^\infty \frac{\log^\gamma s}{V(x, s^{\frac{n-1}{2n}})} ds < \infty. \tag{3.2}$$

We conclude that, under the condition (3.2), any n independent Brownian trajectories are asymptotically separated.

Based on this observation as well as on the arguments below, we conjecture the following.

1. *If M is a manifold of non-negative Ricci curvature and positive injectivity radius and if*

$$\int_1^\infty \frac{ds}{V(x, s^{\frac{n-1}{2n}})} < \infty,$$

then any n independent Brownian trajectories are asymptotically separated.

2. *If M is any manifold of bounded geometry and if*

$$\int_1^\infty \frac{ds}{V(x, s^{\frac{n-1}{2n}})} = \infty, \tag{3.3}$$

then any n independent Brownian trajectories are asymptotically close.

It is known that the recurrence and the stochastic completeness of the Brownian motion on geodesically complete manifolds can be obtained assuming only a volume growth condition. For example, if for some $x \in M$

$$\int_1^\infty \frac{ds}{V(x, \sqrt{s})} = \infty, \tag{3.4}$$

then ξ is recurrent. Moreover, if the Ricci curvature of M is non-negative, then (3.4) is also necessary for the recurrence of ξ (see [11], [23], [37], [46]). On the other hand, if

$$\int_1^\infty \frac{ds}{\log V(x, \sqrt{s})} = \infty, \tag{3.5}$$

then ξ is stochastically complete (see [35] or [24]). The condition (3.3) can be considered as a kind of interpolation between (3.4) and (3.5).

Consider now the second example, with M being a spherically symmetric manifold. As a topological space, $M = \mathbb{R}^d$. Fix a point $x \in \mathbb{R}^d$, consider in $\mathbb{R}^d$ the polar coordinates (r, φ) centered at x, and define the Riemannian metric of M by

$$ds^2 = dr^2 + h^2(r) d\varphi^2, \tag{3.6}$$

where $d\varphi$ is the standard metric on $\mathbb{S}^{d-1}$. At the moment, the function $h(r)$ is any smooth positive function on $(0, \infty)$, such that $h(r) = r$ for $r \leq 1$. The surface area of any sphere $\partial B(x, r)$ can be determined by

$$S(r) = \omega_d h^{d-1}(r),$$

where ω_d is the area of the unit sphere in $\mathbb{R}^d$. The volume $V(x, r)$ is obviously given by

$$V(r) = V(x, r) = \int_0^r S(t) dt.$$

To satisfy the bounded geometry condition, it suffices to assume that $S(r) \to \infty$ as $r \to \infty$ and, for r large enough,

$$\frac{S''(r)}{S(r)} \leq C \quad \text{and} \quad \left|\frac{S'}{S}\right| \leq C \tag{3.7}$$

(see [6]). Assume in addition that, for all $r > 0$,

$$V(2r) \leq CV(r) \quad \text{and} \quad \frac{S(r)}{V(r)} \geq \frac{c}{r}, \tag{3.8}$$

for some $c > 0$. For example, (3.7) and (3.8) are satisfied if $S(r)$ is a power function. Given (3.8), the central value of the heat kernel admits the following upper bound

$$p(t,x,x) \leq \frac{C}{V(\sqrt{t})} \tag{3.9}$$

(see [29, Section 8]).

Let us show that either of the conditions (3.1) or (3.2) implies that any n independent Brownian trajectories on M are asymptotically separated. It will be sufficient to verify the hypothesis (2.5) of Theorem 2.3. Let us introduce the function

$$f(t) = t^{\frac{1}{n-1}}\theta(t), \quad t > 1,$$

and extend f to the interval $(0,1)$ so that $f(t) \equiv 0$ on $(0,1/2)$. Without loss of generality, we may assume that $f \in C^1$ and $f' \geq 0$. Denote also

$$F(y) = \int_0^\infty p(t,x,y)f(t)dt.$$

Then (2.5) will be implied by the boundedness of the function F. The finiteness of $F(x)$ follows from the hypothesis (3.1)/(3.2) and the estimate (3.9). By the local parabolic Harnack inequality (see [12]), $F(y)$ is also finite for all y. We need, however, to show that

$$\sup_y F(y) < \infty.$$

Let us prove a stronger statement that

$$F(y) \leq F(x), \quad \forall y \in M. \tag{3.10}$$

First we check that $\Delta F \leq 0$. Indeed, we have

$$\begin{aligned} \Delta F &= \int_0^\infty \Delta p(t,x,y)f(t)dt = \int_0^\infty \frac{\partial}{\partial t}p(t,x,y)f(t)dt \\ &= -\int_0^\infty p(t,x,y)f'(t)dt \leq 0. \end{aligned}$$

Therefore the function $F(y)$ is superharmonic, and hence satisfies the minimum principle in bounded regions. For a given $y \neq x$, denote $r = \rho(x,y)$. By the minimum principle,

$$\min_{\partial B(x,r)} F = \min_{B(x,r)} F \leq F(x).$$

Since $p(t,x,y)$ is radial, the function F is also radial, that is $F|_{\partial B(x,r)} \equiv F(y)$, whence (3.10) follows.

4 Asymptotic proximity for α-stable processes

Let ξ be the Brownian motion on M and $\xi^{(\alpha)}$ be the α-process on a Riemannian manifold M, that is the process generated by the operator $-(-\Delta)^{\alpha/2}$, where $0<\alpha\leq 2$ (see [22] or [41]). Theorem 2.1 holds again provided $g(x,y)$ is replaced by $g_\alpha(x,y)$, the latter being the Green kernel of $\xi^{(\alpha)}$. However, it is more convenient to have the conditions for the asymptotic separation of the α-process in terms of the heat kernel $p(t,x,y)$ of the Brownian motion ξ, which is much easier to estimate. Let us note that $g_\alpha(x,y)$ and $p(t,x,y)$ are related by

$$g_\alpha(x,y)=\int_0^\infty t^{\alpha/2-1}p(t,x,y)dt. \tag{4.1}$$

Theorem 4.1 *Let M be a manifold with bounded geometry. Assume that, for a number $\alpha\in(0,2]$ and for all $x\in M$,*

$$\int_1^\infty t^{\alpha-1}\,p(t,x,x)\,dt<\infty. \tag{4.2}$$

Then the independent α-processes $\xi_{x_1}^{(\alpha)}$ and $\xi_{x_2}^{(\alpha)}$ are asymptotically separated, for all $x_1,x_2\in M$.

Theorem 4.2 *Let M be a manifold with bounded geometry. Assume that, for an integer $n\geq 2$, for a number $\alpha\in(0,2]$ and for a point $x\in M$,*

$$\sup_{y\in M}\int_1^\infty t^{\beta/2-1}\,p(t,x,y)\,\theta(t)\,dt<\infty, \tag{4.3}$$

where

$$\beta=\frac{\alpha n}{(n-1)} \tag{4.4}$$

and $\theta(t)$ is a continuous positive increasing function on $(0,\infty)$ such that

$$\int_1^\infty \frac{dt}{t\,\theta^{n-1}(t)}<\infty. \tag{4.5}$$

Then the independent α-processes $\xi_{x_1}^{(\alpha)}$, $\xi_{x_2}^{(\alpha)}$,..., $\xi_{x_n}^{(\alpha)}$ are asymptotically separated, for all n-tuples $x_1,x_2,...,x_n\in M$.

Note that for $n=2$, we obtain $\beta=2\alpha$ matching (4.2). Also, (4.4) can be rewritten as

$$\frac{\alpha}{\beta}+\frac{1}{n}=1. \tag{4.6}$$

Hence, we have the following corollary (cf. the proof of Corollary (2.4)).

Corollary 4.3 (=Theorem 1.2) *Let M be a manifold with bounded geometry. Assume that, for some $\nu>0$, the heat kernel of ξ satisfies the following upper bound, for all t large enough:*

$$\sup_{x\in M}p(t,x,x)\leq\frac{C}{t^{\nu/2}}. \tag{4.7}$$

If $\alpha\in(0,2]$ and an integer $n\geq 2$ are such that

$$\frac{\alpha}{\nu}+\frac{1}{n}<1, \tag{4.8}$$

then any n independent α-processes $\xi_{x_1}^{(\alpha)}$, $\xi_{x_2}^{(\alpha)}$, ..., $\xi_{x_n}^{(\alpha)}$ are asymptotically separated.

Since in $\mathbb{R}^d$ the condition (4.7) holds with $\nu = d$, the condition (4.8) becomes

$$\frac{\alpha}{d} + \frac{1}{n} < 1. \tag{4.9}$$

This condition is sharp, as can be seen from the results of S.J.Taylor [45] on n-intersections of stable processes. The following table shows the range of α, n and d for which this condition is satisfied, and hence any n trajectories of the α-stable process in $\mathbb{R}^d$ are asymptotically separated:

$d \mid n$	$n = 2$	$n = 3$	$n \geq 4$
$d \geq 4$	$\alpha < 2$	$\alpha \leq 2$	$\alpha \leq 2$
$d = 3$	$\alpha < \frac{3}{2}$	$\alpha < 2$	$\alpha \leq 2$
$d = 2$	$\alpha < 1$	$\alpha < \frac{4}{3}$	$\alpha < 2 - \frac{2}{n}$

(4.10)

Let M be an *arbitrary* manifold of bounded geometry. The following heat kernel estimate holds *without* any further hypothesis:

$$\sup_{x \in M} p(t, x, x) \leq \frac{C}{t^{1/2}}, \quad \forall t > 1$$

(see [9] and [28]). Hence, (4.7) holds automatically with $\nu = 1$, and we see that any n independent α-processes on M are asymptotically separated provided

$$\alpha + \frac{1}{n} < 1. \tag{4.11}$$

Of course, the condition (4.11) is more restrictive than (4.9). However, it does not involve any further geometric assumption (for instance, dimension).

As follows from (4.1), the process $\xi^{(\beta)}$ is transient if and only if, for some $x, y \in M$,

$$\int^{\infty} t^{\beta/2-1} p(t, x, y) dt < \infty.$$

The comparison with (4.3) suggests that the transience of $\xi^{(\beta)}$ might be linked to the fact that n independent α-processes are asymptotically separated, where α, β and n are related by (4.4) or (4.6).

5 Random walks on infinite graphs

Let M be an infinite graph, that is a countable set of points such that some pairs of them are declared to be neighbors connected by an edge. If $x, y \in M$ are neighbors then we write $x \sim y$. Denote by $\rho(x, y)$ the combinatorial distance between $x, y \in M$ which, by definition, is the smallest number of edges in the paths connecting x to y. We always assume that the graph M is connected so that $\rho(x, y) < \infty$.

A random walk $\xi = \{\xi_x(k), \, k \in \mathbb{Z}_+\}$ on M is determined by transition probability $P(x, y)$ where $x, y \in M$. We denote by $\mathbb{P}_x$ the law of ξ_x. In the sequel, we always assume that ξ is a nearest neighborhood random walk, that is $P(x, y) = 0$ whenever $\rho(x, y) > 1$. The random walk is stochastically complete if, for all $x \in M$,

$$\sum_{y \sim x} P(x, y) = 1, \tag{5.1}$$

this is P is a Markov kernel. An analogue of the bounded geometry hypothesis is the assumption that, for some $\varepsilon_0 > 0$ and all $x \sim y$,

$$P(x, y) \geq \varepsilon_0. \tag{5.2}$$

This implies that the number d_x of the neighbors of any point x is uniformly bounded from above by ε_0^{-1}.

The random walk $\xi_x(k)$ started at x has, after k steps, the law $P_k(x, \cdot)$ where $P_k(x, y)$ is the (x, y)-entry of the matrix P^k – the kth convolution power of P. The Green kernel $G(x, y)$ of ξ is defined by

$$G(x, y) = \sum_{k=0}^{\infty} P_k(x, y).$$

The definitions of asymptotic separation and asymptotic proximity are simpler for random walks.

Definition 5.1 We say that n walks $\xi_{x_1}, \xi_{x_2}, ..., \xi_{x_n}$ are *asymptotically separated* if

$$\mathbb{P}_{\overline{x}}\left(\exists T \quad \forall k_1, ..., k_n > T: \quad \max_{1 \leq j,k \leq n} \rho(\xi_{x_i}(k_i),\, \xi_{x_j}(k_j)) > 0\right) = 1.$$

Otherwise, we say that $\xi_{x_1}, \xi_{x_2}, ..., \xi_{x_n}$ are *asymptotically close.*

The following is our main result for random walks.

Theorem 5.1 *Let M be a connected graph and ξ be a random walk on M satisfying (5.1) and (5.2). Assume that, for some point $x \in M$ and for an integer $n > 1$,*

$$\sum_{y \in M} G^n(x, y) < \infty. \tag{5.3}$$

Then any n independent random walks $\xi_{x_1}, \xi_{x_2}, \ldots, \xi_{x_n}$ are asymptotically separated.

The random walk ξ is called reversible with respect to a measure μ on M if

$$P(x, y)\mu(x) \equiv P(y, x)\mu(y). \tag{5.4}$$

Theorem 5.2 *Let M be a connected graph and ξ be a reversible random walk on M satisfying (5.1) and (5.2). Assume in addition that*

$$\inf_{x \in M} \mu(x) > 0. \tag{5.5}$$

If the following condition holds for $x = x_1$ and $x = x_2$

$$\sum_{k=1}^{\infty} k P_k(x, x) < \infty, \tag{5.6}$$

then the two independent walks ξ_{x_1} and ξ_{x_2} are asymptotically separated.

6 Abstract Markov processes

Let M be a metric space with a distance function $\rho(x,y)$. Assume that M is equipped with a Radon measure μ, that is a σ-additive measure defined on Borel subsets of M, such that μ is finite on compact sets. Let us denote by

$$B(x,r) = \{y \in M : \rho(x,y) < r\}$$

a metric ball in M, and by $V(x,r) := \mu(B(x,r))$ its measure.

We start with assumptions (A) and (B) on the state space M.

(A) Any ball $B(x,r) \subset M$ is precompact. Moreover, for all small enough $r > 0$, there exists a countable family $\mathfrak{B}_r$ of balls $B(y_i,r)$, $i = 1,2,3,...$, which covers all of M, and such that the family $\{B(y_i,2r)\}_{i\geq 1}$ of concentric balls of double radius has a uniformly finite multiplicity.

(B) For all $r > 0$, we have

$$\inf_{x\in M} V(x,r) > 0. \tag{6.1}$$

Let us observe that (A) and (B) imply that any ball $B(x,R)$ intersects only finitely many balls from $\mathfrak{B}_r$. Indeed, denote by $\mathfrak{I}$ the set of all balls from $\mathfrak{B}_r$ which intersect $B(x,R)$. Then $B(x,R+2r)$ contains all balls from $\mathfrak{I}$. Since the family $\mathfrak{B}_r$ has a finite multiplicity, we have

$$\sum_{B\in\mathfrak{I}} \mu(B) \leq CV(x,R+2r),$$

whence, by (6.1),

$$|\mathfrak{I}| \leq C'V(x,R+2r) < \infty.$$

Let a Markov process $\xi(t)$ be defined on M. The range $\mathcal{T}$ of time t is either $\mathbb{R}_+$ or $\mathbb{Z}_+$ (including 0). The time variable is always assumed to belong to $\mathcal{T}$. As usual, $\mathbb{P}_x$ and $\mathbb{E}_x$ are, respectively, the probability measure and the expectation associated with the process $\xi_x(t)$ started at the point $x \in M$. We assume throughout that the process ξ has a transition density $p(t,x,y)$ with respect to the measure μ, that is, for any open set U, any $x \in M$ and all $t > 0$,

$$\mathbb{P}_x(\xi(t) \in U) = \int_U p(t,x,y)d\mu(y).$$

Denote by $g(x,y)$ the Green kernel of $\xi(t)$ (possibly infinite) defined by

$$g(x,y) = \int_0^\infty p(t,x,y)\,dt.$$

Here dt is either the Lebesgue measure if $\mathcal{T} = \mathbb{R}_+$, or the counting measure if $\mathcal{T} = \mathbb{Z}_+$.

Definition 6.1 The process ξ is called *transient* if $g(x,y) < \infty$ for all $x \neq y$. Otherwise, ξ is called *recurrent.*

It is well known that transience of ξ is equivalent to fact that, for any precompact set $U \subset M$,

$$\mathbb{P}_x\{\forall T > 0\; \exists\, t \geq T \text{ such that } \xi(t) \in U\} = 0. \tag{6.2}$$

Definition 6.2 We say that the process ξ is *stochastically complete* if, for all $x \in M$ and $t \in \mathcal{T}$,

$$\int_M p(t,x,y)d\mu(y) = 1.$$

Stochastic completeness may fail at least for two reasons. There may be some killing conditions like for a process generated by an elliptic Schrödinger operator or for Brownian motion in a bounded region of $\mathbb{R}^d$ with absorbing boundary. On the other hand, Brownian motion on a geodesically complete manifold may escape to infinity in finite time for a geometric reason – see [3].

Let us fix some exhaustion $\{W_i\}$ of M, that is, an increasing sequence of precompact open subsets W_i. If M satisfies (A) then we may take $W_i = B(o,i)$ for some o. The following two definitions do not depend on the choice of $\{W_i\}$.

Definition 6.3 We say that the process ξ is *minimal* if, for any precompact open set $U \subset M$ and for all $x \in M$ and $t \in \mathcal{T}$,

$$\mathbb{P}_x(\xi(t) \in U) = \lim_{i\to\infty} \mathbb{P}_x\left(\xi(t) \in U \text{ and } \xi(s) \in W_i \text{ for all } s \in [0,t]\right).$$

In particular, if we denote by ξ^{W_i} the process ξ inside W_i with the killing condition on $M \setminus W_i$, then ξ^{W_i} converges to ξ in distribution as $i \to \infty$.

The diffusion on a manifold discussed in Section 2 is automatically minimal, by construction (see [14]). On the other hand, Brownian motion in a bounded open set in $\mathbb{R}^n$ with the reflecting boundary condition is not minimal. A random walk on a graph is minimal just because of its finite propagation speed.

Definition 6.4 We say that the process ξ is *stochastically compact* if, for all $x \in M$ and $T > 0$,

$$\lim_{i\to\infty} \mathbb{P}_x\left(\xi(t) \notin W_i \quad \text{for some } t \le T\right) = 0. \tag{6.3}$$

If the process ξ escapes to infinity in a finite time then it may be not stochastically compact. In the same way, Brownian motion in a bounded region in $\mathbb{R}^n$ with the reflecting boundary condition is not stochastically compact. However, the following is true.

Lemma 6.1 *If ξ is stochastically complete and minimal, then ξ is stochastically compact.*

Proof. We have

$$\begin{aligned}\mathbb{P}_x\left(\exists t \le T : \xi(t) \notin W_i\right) &= 1 - \mathbb{P}_x\left(\forall t \le T : \xi(t) \in W_i\right)\\ &= 1 - \mathbb{P}_x\left(\forall t \le T : \xi^{W_i}(t) \in W_i\right)\\ &= 1 - \mathbb{P}_x\left(\xi^{W_i}(T) \in W_i\right).\end{aligned}$$

Take any precompact open set $U \subset M$, and let i be so large that $U \subset W_i$. Therefore,

$$\mathbb{P}_x\left(\exists t \le T : \xi(t) \notin W_i\right) \le 1 - \mathbb{P}_x\left(\xi^{W_i}(T) \in U\right).$$

By the minimality of ξ,

$$\lim_{i\to\infty} \mathbb{P}_x\left(\xi^{W_i}(T) \in U\right) = \mathbb{P}_x\left(\xi(T) \in U\right).$$

By letting $U \nearrow M$, we have

$$\lim_{U\nearrow M} \mathbb{P}_x\left(\xi(T) \in U\right) = 1.$$

Hence, for any $\varepsilon > 0$ there exist i and U such that

$$\mathbb{P}_x \left(\xi^{W_i}(T) \in U \right) > 1 - \varepsilon,$$

whence

$$\mathbb{P}_x \left(\exists t \le T : \xi(t) \notin W_i \right) < \varepsilon,$$

and (6.3) follows. ∎

The following conditions (C), (D) and (E) in general may be true or not.

(C) For some $a_0 > 0$,

$$\inf_{x,y \in M,\ \rho(x,y) \le a_0} g(x,y) > 0\,. \tag{6.4}$$

(D) (*A local Harnack inequality*) For any $a \in (0, a_0)$ and for all $x, y \in M$ such that $\rho(x,y) > 4a$, we have

$$\sup_{z \in B(y,2a)} g(x,z) \le C_H \inf_{z \in B(y,2a)} g(x,z)\,, \tag{6.5}$$

with a constant C_H which is independent of x, y, a.

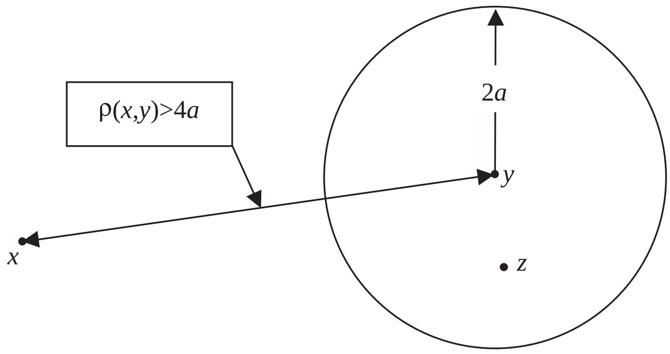

Figure 2: The ball $B(y, 2a)$ in which the Green kernel satisfies the Harnack inequality

(E) The process ξ is strong Markov, right continuous in t, minimal and stochastically complete.

The next statement is our main result for the abstract setting.

Theorem 6.2 *Assume that all the hypotheses (A)–(E) hold. Suppose also that, for some points* $x_1, x_2, ..., x_n \in M$ *and a number* $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{M \setminus \Omega_\varepsilon^n} g(x_1, y) g(x_2, y) ... g(x_n, y) d\mu(y) < \infty, \tag{6.6}$$

where Ω_ε^n *is the* ε*-neighborhood in* M *of the set* $\{x_1, x_2, ..., x_n\}$*. Then the independent processes* ξ_{x_1}, ξ_{x_2},... ξ_{x_n} *are asymptotically separated. Here* ε_0 *is a positive number which is determined by the constants in the hypotheses (A)–(D).*

In many cases, the condition (6.6) amounts to

$$\int_{M\setminus B(x,\varepsilon)} g^n(x,y)d\mu(y) < \infty, \tag{6.7}$$

for some/all $x \in M$. Let us note for comparison that, for Lévy processes in $\mathbb{R}^d$, the convergence of the integral (6.7) in a neighborhood of x is equivalent to the existence of n-multiple points - see [18] and [20].

For reversible processes and for $n = 2$ Theorem 6.2 can be simplified as follows.

Theorem 6.3 *Assume that all the hypotheses (A)–(E) hold, and in addition, that the process ξ is transient and reversible with respect to measure μ. Suppose also*

$$\int_1^\infty t\, p(t,x,x)dt < \infty, \tag{6.8}$$

for $x = x_1$ and $x = x_2$, where x_1 and x_2 are two distinct points on M. Then the processes ξ_{x_1} and ξ_{x_2} are asymptotically separated.

Normally in applications the rate of decay of the heat kernel $p(t,x,x)$ as $t \to \infty$ does not depend on the choice of x. In such cases, it suffices to assume (6.8) for *some* $x \in M$. Also, the transience of ξ normally follows from (6.8).

As we have seen in the previous sections, for diffusions and α-processes on manifolds, there are extensions of Theorem 6.3 to n processes. In the general case, it is possible to state such a theorem as well but the statement becomes very bulky because of numerous additional hypotheses. This is why we have chosen to state the case of n processes only for the manifold case.

Another setting when all the hypotheses (A)–(E) are satisfied is a Brownian motion on *fractals*. Uniform heat kernel and Green kernel estimates on fractals are also available – see [5], [4]. For certain unbounded Sierpinski carpets, one has the following properties, for some $\alpha > \beta \geq 2$:

1. a uniform volume growth

$$V(x,r) \simeq r^\alpha, \quad \forall x \in M, r > 0,$$

2. and a uniform Green kernel decay

$$g(x,y) \simeq \rho(x,y)^{-(\alpha-\beta)}, \quad \forall x \neq y.$$

Then the condition (6.6) for asymptotic separation of n processes easily amounts to

$$n > \frac{\alpha}{\alpha - \beta}. \tag{6.9}$$

This includes also the case $M = \mathbb{R}^d$ with $\alpha = d$ and $\beta = 2$ (cf. (2.8)).

7 Examples

Let us consider examples of spaces and processes satisfying the hypotheses (A)–(E). Note that in all the examples below, the process ξ will also be reversible. Therefore, application of one of the above theorems amounts to verifying one of the conditions (6.6) or (6.8).

7.1 Diffusions on manifolds

Let M be a manifold with bounded geometry and L be a uniformly elliptic operator on M defined in Section 2. The conditions (A) and (B) are known to hold on such a manifold (see, for example, [32], [34]). The local Harnack inequality (D) follows from the fact that the operator L can be written in a local chart as a uniformly elliptic operator for which the Harnack inequality was proved by Moser [42]. To verify (C), let us take $a_0 = r_0/4$ where r_0 is the injectivity radius, and consider the Green function g_U with the vanishing Dirichlet boundary value on ∂U where $U = B(x, 2a_0)$. Then g_U is the Green function of a uniformly elliptic operator in U, and by theorem of Littman, Stampaccia and Weinberger [38], $g_U(x, y)$ admits a positive uniform lower bound provided $y \in B(x, a_0)$. Since $g \geq g_U$, the inequality (6.4) in condition (C) follows. By definition, the L-diffusion on M is constructed as a minimal process - see [14]. On the manifold M with bounded geometry, the L-diffusion is stochastically complete (see [24], [48]). Finally, diffusion processes are strong Markov and have continuous paths. Hence, the condition (E) is also satisfied.

Let us discuss the notion of asymptotic proximity in the present context. As follows from Definition 1.1, the processes ξ_x and ξ_y are asymptotically close if, for any $a > 0$,

$$\mathbb{P}_{x,y}(\forall T > 0\ \exists t, s > T :\ \rho(\xi_x(t),\, \xi_y(s)) < a) > 0. \tag{7.1}$$

In particular, this condition is satisfied provided

$$\mathbb{P}_{x,y}(\exists \{t_k\}, \{s_k\} \to \infty :\ \rho(\xi_x(t_k),\, \xi_y(s_k)) \to 0) = 1 \tag{7.2}$$

(see Fig. 3).

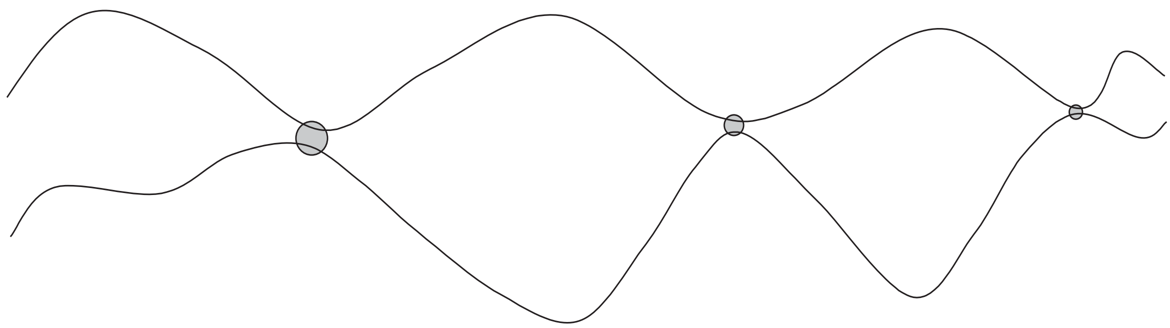

Figure 3: Two trajectories are asymptotically close provided with probability 1 they become arbitrarily close at a sequence of large times.

If a tail σ-algebra of the L-diffusion is trivial, then the probability in (7.1) is equal to 1, and by letting $a \to 0$, we obtain (7.2). Hence, in this case, (7.1) and (7.2) are equivalent. However, in general (7.1) does not imply (7.2). For example, let M be a connected sum of two copies of $\mathbb{R}^3$ (see Fig. 4). Then there is a positive probability that the independent processes ξ_x and ξ_y will escape to infinity along different sheets so that (7.2) is false (see [36]). Nevertheless, ξ_x and ξ_y are asymptotically close. Indeed, with a positive probability, both trajectories will eventually stay on the same sheet (see Fig. 4). Under this condition, they intersect infinitely many times with probability 1, as in $\mathbb{R}^3$.

Let us consider another example showing that the condition (6.8) of Theorem 6.3 is not necessary for asymptotic separation of two processes. Let M be a connected sum of $\mathbb{R}^d$ (where

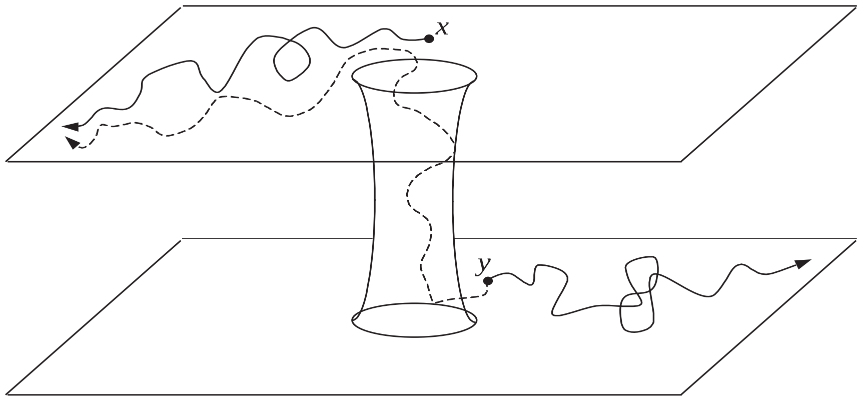

Figure 4: Two Brownian trajectories can escape to ∞ along the same sheet or along different sheets (both with positive probabilities).

d is large enough) and of the manifold $\mathbb{R}_+ \times K$, where K is a compact manifold of dimension $d-1$ (see Fig. 5). We claim that any two independent trajectories of Brownian motion on M are asymptotically separated, whereas the heat kernel's long time behavior is given by

$$p(t,x,x) \asymp t^{-3/2}, \quad t \to \infty \tag{7.3}$$

(the latter implies that the condition (6.6) fails).

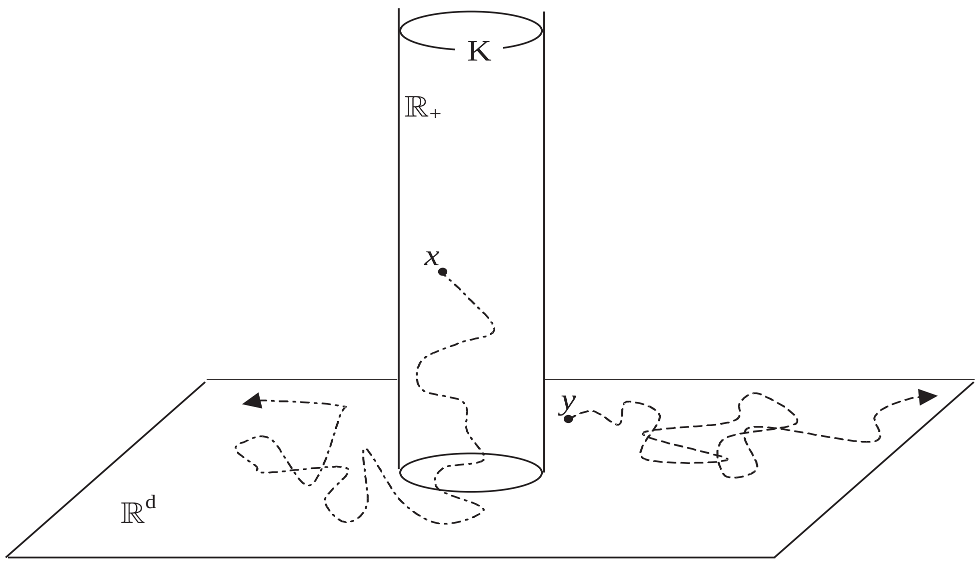

Figure 5: The connected sum of $\mathbb{R}^d$ and $\mathbb{R}_+ \times K$

Indeed, the asymptotic proximity of ξ_x and ξ_y would mean that, for any $a > 0$,

$$\mathbb{P}_{x,y}(\exists \{t_i\}, \{s_i\} \to \infty : \rho(\xi_x(t_i), \xi_y(s_i)) < a) > 0.$$

The sequence $\{\xi_x(t_i)\}$ cannot belong to $\mathbb{R}_+ \times K$ with positive probability, because $\mathbb{R}^d$ is transient and $\mathbb{R}_+ \times K$ is recurrent. However, if both $\xi_x(t_i)$ and $\xi_y(s_i)$ are in $\mathbb{R}^d$, then they are asymptotically separated provided $d > 4$.

The estimate (7.3) follows from the following estimate (see [31, Corollary 5]):

$$p(t,x,y) \asymp \left(\frac{1}{t^{d/2}} + \frac{|x|}{t^{3/2}\,|y|^{d-2}} \right) \exp\left(-\frac{\rho^2(x,y)}{ct} \right),$$

where $x \in \mathbb{R}_+ \times K$, $y \in \mathbb{R}^d$ and $|x| > 1$,$|y| > 1$ (see Fig. 5), and from the observation that $\frac{p(t,x,y)}{p(t,x,x)}$ remains bounded from above and below as $t \to \infty$.

7.2 The α-process

Let $\xi^{(\alpha)}$ be the α-process on a manifold M with bounded geometry. The Green kernel $g_\alpha(x,y)$ for $\xi^{(\alpha)}$ is given by

$$g_\alpha(x,y) = \int_0^\infty t^{\alpha/2-1} p(t,x,y)dt. \tag{7.4}$$

Let us emphasize that $p(t,x,y)$ is the heat kernel for the Brownian motion ξ on M, not for the α-process. For example, if $M = \mathbb{R}^d$ with the standard Lebesgue measure μ, then $\xi^{(\alpha)}$ is the α-stable Lévy process with the Green function

$$g_\alpha(x,y) = \frac{c_{\alpha,d}}{|x-y|^{d-\alpha}}. \tag{7.5}$$

In order to verify (C) and (D) for g_α, we will use the following properties of the heat kernel on manifolds of bounded geometry.

(i) A *local* parabolic Harnack inequality for the heat kernel $p(t,x,y)$. Let $a_0 > 0$ be a small fraction of the injectivity radius of M. Then, for all $r \le 2a_0$, $x, y \in M$ and $t \ge r^2$,

$$\sup_{z \in B(y,r)} p(t,x,z) \le C \inf_{z \in B(y,r)} p(t+r^2, x, z). \tag{7.6}$$

This follows from Moser's Harnack inequality [43] (see also [44]), since $p(t,x,y)$ locally satisfies a uniformly parabolic equation.

(ii) A lower bound of the heat kernel: for all $x, y \in M$ and $t > 0$,

$$p(t,x,y) \ge \frac{1}{Ct^{d/2}} \exp\left[-C\left(\frac{\rho^2}{t} + t \right) \right], \tag{7.7}$$

for some large constant $C > 0$, where $d = \dim M$ and $\rho = \rho(x,y)$ (see [10], [13]).

(iii) An upper bound of the heat kernel: for all $x, y \in M$ and $t > 0$,

$$p(t,x,y) \le \frac{C}{\min(a_0^d, t^{d/2})} \exp\left(-\frac{\rho^2}{Ct} \right) \tag{7.8}$$

(see [9], [25], [28]).

Note that all the properties (i)–(iii) hold also for the heat kernel associated with the uniform elliptic operator L given by (2.1).

To prove (C), let us assume that $\rho(x,y) \le a_0$, and integrate (7.7) in time. We obtain from (7.4)

$$g_\alpha(x,y) \ge \int_0^\infty \frac{t^{\alpha/2-1}}{Ct^{d/2}} \exp\left[-C\left(\frac{a_0^2}{t}+t\right)\right] dt = \mathrm{const} > 0$$

which implies (6.4).

Let us prove (D). We will verify that if $r \le 2a_0$, $z_1, z_2 \in B(y,r)$ and $\rho(x,y) > 2r$ then

$$g_\alpha(x,z_1) \le \mathrm{const}\, g_\alpha(x,z_2), \tag{7.9}$$

which is equivalent to (D) with $a = 2r$ (see also Fig. 6).

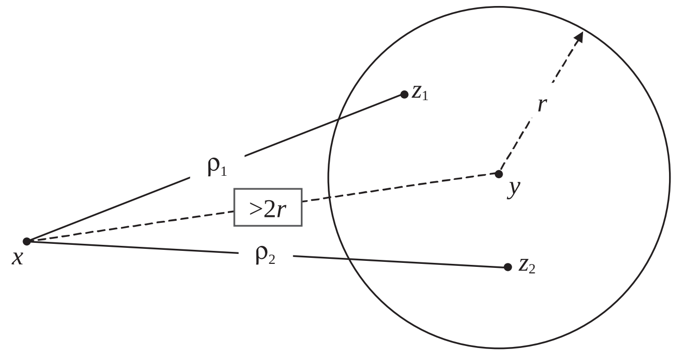

Figure 6: The ratio ρ_2/ρ_1 is bounded from above and below.

The Harnack inequality (7.6) implies, for all $t \ge r^2$,

$$p(t,x,z_1) \le Cp(t+r^2,x,z_2).$$

By integrating this in t from r^2 to ∞, we obtain

$$\begin{aligned}
\int_{r^2}^\infty t^{\alpha/2-1} p(t,x,z_1)dt &\le C\int_{r^2}^\infty t^{\alpha/2-1} p(t+r^2,x,z_2)dt \\
&= C\int_{2r^2}^\infty \left(t-r^2\right)^{\alpha/2-1} p(t,x,z_2)dt \\
&\le C\int_0^\infty (t/2)^{\alpha/2-1} p(t,x,z_2)dt \\
&= C' g_\alpha(x,z_2).
\end{aligned} \tag{7.10}$$

Let us show that

$$\int_0^{r^2} t^{\alpha/2-1} p(t,x,z_1)dt \le C'' g_\alpha(x,z_2). \tag{7.11}$$

Denote $\rho_i = \rho(x,z_i)$, $i=1,2$. By (7.8), we have

$$\int_0^{r^2} t^{\alpha/2-1} p(t,x,z_1)dt \le \int_0^{r^2} \frac{Ct^{\alpha/2-1}}{t^{d/2}} \exp\left(-\frac{\rho_1^2}{Ct}\right) dt.$$

On the other hand, by (7.7) and for some (large) K,

$$\begin{aligned} g_\alpha(x,z_2) &= \int_0^\infty t^{\alpha/2-1} p(t,x,z_2)dt \\ &\geq \int_0^{Kr^2} \frac{t^{\alpha/2-1}}{Ct^{d/2}} \exp\left[-C\left(\frac{\rho_2^2}{t}+t\right)\right] dt \\ &= K^{\frac{\alpha-d}{2}} \int_0^{r^2} \frac{t^{\alpha/2-1}}{Ct^{d/2}} \exp\left[-C\left(\frac{\rho_2^2}{Kt}+Kt\right)\right] dt \end{aligned}$$

By taking K large enough, we can ensure that

$$C\frac{\rho_2^2}{Kt} \leq \frac{\rho_1^2}{Ct},$$

since the ratio ρ_2/ρ_1 stays bounded (see Fig. 6). The term $\exp\left[-C(Kt)\right]$ is bounded from below because $t \leq r^2 \leq (2a_0)^2$. Therefore,

$$\exp\left(-\frac{\rho_1^2}{Ct}\right) \leq \operatorname{const} \exp\left[-C\left(\frac{\rho_2^2}{Kt}+Kt\right)\right],$$

whence (7.11) follows. Together with (7.10), this implies (7.9).

Finally, the α-process $\xi^{(\alpha)}$ is strong Markov, right continuous, minimal and stochastically complete (see, for example, [22] and [41]) so that (E) holds.

7.3 Random walks

Let M be a graph endowed with a Markov kernel$P(x,y)$ as was described in Section 5. Let us introduce a measure μ on M by setting $\mu(x) \equiv 1$, for any point $x \in M$. Assuming (5.1) and (5.2), the conditions (A) and (B) are trivially satisfied, because a ball $B(x,r)$ with radius $r < 1$ amounts to a singe point set $\{x\}$.

To verify (C), let us observe that, by (5.1) and (5.2),

$$\begin{aligned} G(x,x) &= \sum_{k=0}^{\infty} P_k(x,x) \geq P_2(x,x) \\ &= \sum_{y\sim x} P(x,y)P(y,x) \geq \varepsilon_0 \sum_{y\sim x} P(x,y) = \varepsilon_0. \end{aligned}$$

Note that the Green kernel $g(x,y)$ is defined by

$$g(x,y) = \frac{G(x,y)}{\mu(y)}.$$

Hence, (6.4) follows for $a_0 < 1$ by $\mu \equiv 1$.

The Harnack inequality (6.5) of the condition (D) follows trivially for $a_0 < 1/2$ since $z = y$. However, we will need (D) also for all $a_0 < 1$. This means that in (6.5), the point z is a neighbor of y. In this case, (6.5) follows from the following inequality

$$G(x,y) \geq \varepsilon_0 G(x,z), \tag{7.12}$$

for any two neighboring points $y, z \neq x$. To show (7.12), we use the fact that the Green function $u = G(x,\cdot)$ is harmonic outside x, whence

$$u(y) = \sum_{v\sim y} u(v)P(y,v) \geq \varepsilon_0 u(z).$$

The hypothesis (E) is obvious.

8 Proofs

We continue using notations introduced in Section 6.

8.1 Intersections of trajectories with covering balls

Denote by $[\xi_x]$ the set of points $\{\xi_x(t)\}_{t\in\mathcal{T}}$, and call it *the trace* of ξ_x. The following statement is one of the tools for proving Theorem 6.2.

Proposition 8.1 *Suppose that the space M satisfies the hypothesis (A). Assume also that the process ξ is transient, minimal and stochastically complete. Denote by N_b the number of the balls $B(y_i,b)\in\mathfrak{B}_b$ intersected by all traces $[\xi_{x_1}]$, $[\xi_{x_2}]$, ..., $[\xi_{x_n}]$. Then the processes ξ_{x_1}, ξ_{x_2}, ... ξ_{x_n} are asymptotically separated if and only if, for some $b>0$,*

$$N_b<\infty \quad \mathbb{P}_{\overline{x}}\text{-}a.s.$$

(see Fig. 7).

Remark: This statement does not make sense if ξ is recurrent since the recurrence already implies that ξ_{x_1}, ξ_{x_2}, ... ξ_{x_n} are asymptotically close. It is easy to see that the hypothesis of the minimality of ξ cannot be eliminated. Indeed, if ξ is the Brownian motion in a Euclidean open ball with a reflecting boundary condition (which obviously is not minimal) then the number N_a is always finite whereas the processes ξ_{x_1}, ξ_{x_2}, ... ξ_{x_n} are asymptotically close.

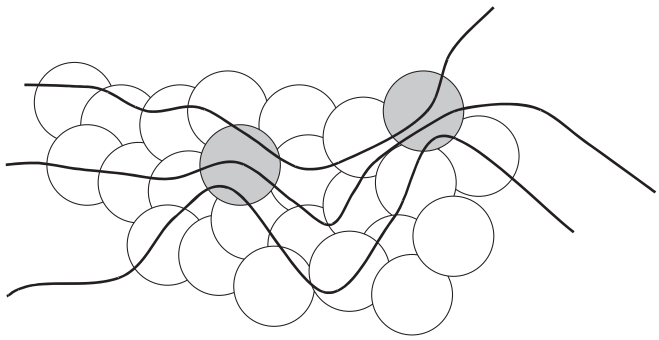

Figure 7: The trajectories are asymptotically separated if the number of balls from $\mathfrak{B}_b$ intersected by all of them, is finite with probability 1.

Proof. Fix some $a>0$ and, for any $T\in\mathcal{T}$, introduce the event $\mathcal{B}_T$ by

$$\mathcal{B}_T=\left\{\omega:\exists\, t_1,t_2,...,t_n>T \text{ such that } \max_{j,k}\rho(\xi_{x_j}(t_j),\,\xi_{x_k}(t_k))\le a\right\}.$$

Clearly, the processes $\xi_{x_1},\xi_{x_2},...,\xi_{x_n}$ are asymptotically separated if and only if, for some $a>0$,

$$\lim_{T\to\infty}\mathbb{P}_{\overline{x}}(\mathcal{B}_T)=0 \tag{8.1}$$

(cf. (1.4)). Let us assume that $\mathbb{P}_{\bar{x}}(N_b=\infty)=0$ and prove (8.1) for $a=b/2$, which will imply that the processes $\xi_{x_1},\xi_{x_2},...,\xi_{x_n}$ are asymptotically separated. Introduce another event, for $T\in\mathcal{T}$ and $i=1,2,...,$

$$\mathcal{A}_{i,T}=\left\{\omega:\exists\, t_1,t_2,...,t_n>T \text{ such that } \xi_{x_j}(t_j)\in B(y_i,2a)\right\},$$

where $B(y_i, 2a) \in \mathfrak{B}_{2a}$. In other words, $\mathcal{A}_{i,T}$ is the event that any trajectory ξ_{x_j} visits $B(y_i, 2a)$ after time T. We claim that

$$\mathcal{B}_T \subset \bigcup_{i=1}^{\infty} \mathcal{A}_{i,T}\,. \tag{8.2}$$

Indeed, assume that $\mathcal{B}_T$ is true. The point $\xi_{x_1}(t_1)$ belongs to one of the balls $B(y_i, a)$. Therefore, by the triangle inequality and by definition of $\mathcal{B}_T$, all $\xi_{x_j}(t_j)$, $j = 1, 2, ..., n$, belong to $B(y_i, 2a)$. Thus, $\mathcal{B}_T$ implies that one of $\mathcal{A}_{i,T}$ occurs, whence (8.2) follows.

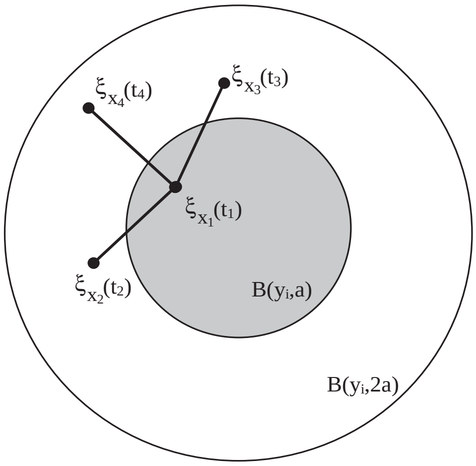

Figure 8: All points $\xi_{x_j}(t_j)$ belong to $B(y_i, 2a)$

The transience of the process ξ implies that

$$\mathbb{P}_{\overline{x}}\left(\bigcap_{T\in\mathcal{T}} A_{i,T}\right) = 0$$

(cf. (6.2)) whence we obtain

$$\lim_{T\to\infty} \mathbf{1}_{\mathcal{A}_{i,T}} = 0, \quad \mathbb{P}_{\overline{x}}\text{-a.s.}$$

Therefore,

$$\lim_{T\to\infty} \sum_{i=1}^{\infty} \mathbf{1}_{\mathcal{A}_{i,T}} = \sum_{i=1}^{\infty} \lim_{T\to\infty} \mathbf{1}_{\mathcal{A}_{i,T}} = 0, \quad \mathbb{P}_{\overline{x}}\text{-a.s.} \tag{8.3}$$

The interchanging of the summation and the limit is justified by the dominated convergence theorem, because

$$\mathbf{1}_{\mathcal{A}_{i,T}} \le \mathbf{1}_{\mathcal{A}_{i,0}}$$

and

$$\sum_{i=1}^{\infty} \mathbf{1}_{\mathcal{A}_{i,0}} = N_{2a} < \infty, \qquad \mathbb{P}_{\overline{x}}\text{-a.s.}$$

We obtain, from (8.2) and (8.3),

$$\lim_{T\to\infty} \mathbf{1}_{\mathcal{B}_T} \le \lim_{T\to\infty} \sum_{i=1}^{\infty} \mathbf{1}_{\mathcal{A}_{i,T}} = 0, \quad \mathbb{P}_{\overline{x}}\text{-a.s.}$$

whence (8.1) follows.

Now let us assume that the processes $\xi_{x_1}, \xi_{x_2}, ..., \xi_{x_n}$ are asymptotically separated and prove that $N_{a/2} < \infty$, $\mathbb{P}_{\overline{x}}$-a.s. Clearly, we have from (8.1)

$$\mathbb{P}_{\overline{x}}\left(\bigcap_{T\in\mathcal{T}} \mathcal{B}_T\right) = 0. \tag{8.4}$$

Hence, for $\mathbb{P}_{\overline{x}}$-almost all trajectories ω, there is $T(\omega) < \infty$ such that, for all $t_1, ..., t_n \geq T(\omega)$,

$$\max_{1\leq j,k\leq n} \rho(\xi_{x_j}(t_j),\, \xi_{x_k}(t_k)) > a.$$

By the triangle inequality, for all such t_j, $\xi_{x_j}(t_j)$ cannot be in the same ball $B(y_i, a/2) \in \mathfrak{B}_{a/2}$. Therefore $N_{a/2}(\omega)$ equals the number of balls from $\mathfrak{B}_{a/2}$ intersected by all trajectories $\xi_{x_j}(t)$ before time $T(\omega)$. Clearly, this number does not exceed the number of the balls intersected by *one* trajectory ξ_{x_1}. Hence, we are left to verify that (denote for simplicity $x_1 = z$)

$$\mathbb{P}_z\left(\xi \text{ intersects infinitely many balls from } \mathfrak{B}_{a/2} \text{ before time } T(\omega)\right) = 0. \tag{8.5}$$

For any $\theta \in (0, \infty)$, denote $T_\theta(\omega) = T(\omega) \wedge \theta$. By the Lebesgue monotone convergence theorem, (8.5) amounts to

$$\lim_{\theta\to\infty} \mathbb{P}_z\left(\xi \text{ intersects infinitely many balls from } \mathfrak{B}_{a/2} \text{ before time } T_\theta(\omega)\right) = 0 \tag{8.6}$$

which, in turn, will follow from

$$\mathbb{P}_z\left(\xi \text{ intersects infinitely many balls from } \mathfrak{B}_{a/2} \text{ before time } \theta\right) = 0, \quad \forall\theta < \infty. \tag{8.7}$$

Fix a point $o \in M$ and observe that, for any $R \in (0, \infty)$, the ball $B(o, R)$ intersects only finitely many balls from $\mathfrak{B}_{a/2}$. Therefore, ξ_z can intersect infinitely many balls from $\mathfrak{B}_{a/2}$ before θ only if $\xi_z(t)$ exits $B(o, R)$ before θ. Hence, (8.7) will follow from

$$\lim_{R\to\infty} \mathbb{P}_z\left(\xi \text{ exits } B(o, R) \text{ before time } \theta\right) = 0. \tag{8.8}$$

By Lemma 6.1, the process ξ is stochastically compact, whence (8.8) follows. ■

8.2 Hitting probability and Green kernel

The purpose of this section is to prove the estimate (8.11) for the hitting probability. Given a set $K \subset M$, denote by $\Psi(x, K)$ the $\mathbb{P}_x$-probability that $\xi(t)$ ever hits K, that is,

$$\Psi(x, K) = \mathbb{P}_x\left(\exists t \in \mathcal{T} : \xi(t) \in K\right).$$

Let τ_K be the first time the process $\xi(t)$ enters K, this is

$$\tau_K = \inf\left\{t \geq 0 : \xi(t) \in K\right\}.$$

For any $z \in M$, introduce the following measure on Borel subsets of M

$$\gamma_{z,K}(A) = \mathbb{P}_z(\xi(\tau_K) \in A),$$

which is called *a harmonic measure* of the set K (see Fig.9).

Clearly, if the trajectories of the process ξ are right continuous and if K is closed then the measure $\gamma_{z,K}$ sits on K. Moreover, its total mass $\gamma_{z,K}(K)$ is equal to the $\mathbb{P}_z$-probability of $\xi(t)$ ever hitting K whence

$$\gamma_{z,K}(K) = \Psi(z, K). \tag{8.9}$$

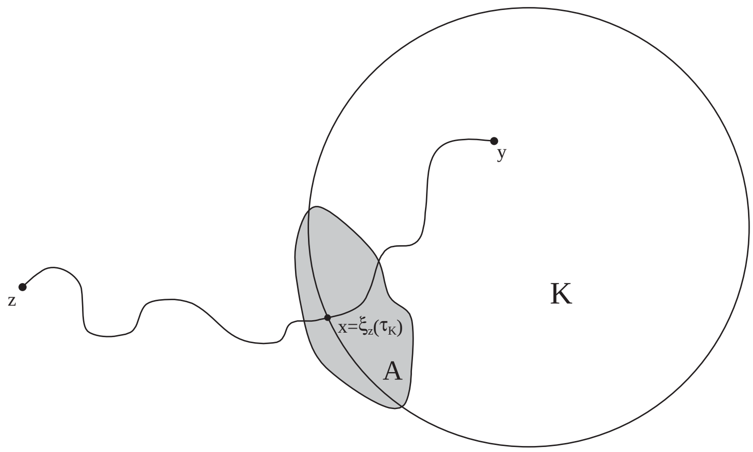

Figure 9: Entering the set K at the set A

Lemma 8.2 *Assume that the process ξ is strong Markov property, right continuous, and transient. Then, for any closed set $K \in M$ and for all $y \in K$ and $z \notin K$,*

$$g(z,y) = \int_K g(x,y) d\gamma_{z,K}(x). \tag{8.10}$$

Corollary 8.3 *Under the above hypotheses,*

$$\Psi(z,K) \le \frac{g(z,y)}{\inf_{x\in K} g(x,y)}. \tag{8.11}$$

Indeed, inequality (8.11) follows immediately from (8.10) and (8.9).

Proof of Lemma 8.2. Denote for simplicity $\tau = \tau_K$ and $\gamma = \gamma_{z,K}$. For any $y \in K$ and $z \notin K$, we have, by the strong Markov property,

$$\begin{aligned} p(t,z,y) &= \mathbb{E}_z \left(\mathbf{1}_{\{\tau \le t\}} \, p(t-\tau, \, \xi(\tau), \, y) \right) \\ &= \int_K \int_0^t p(t-s,x,y) \, d\gamma(s,x) \end{aligned}$$

where $\gamma(s,x)$ is a joint law of $(\tau, \xi_z(\tau))$. By integrating in t, we obtain

$$\begin{aligned} g(z,y) &= \int_0^\infty \int_K \int_0^t p(t-s,x,y) \, d\gamma(s,x) \, dt \\ &= \int_0^\infty \int_K \int_s^\infty p(t-s,x,y) \, dt \, d\gamma(s,x) \\ &= \int_0^\infty \int_K g(x,y) \, d\gamma(s,x) \\ &= \int_K g(x,y) \, d\gamma(x), \end{aligned}$$

which completes the proof. ■

8.3 Asymptotic separation in terms of the Green kernel

Here we prove Theorems 6.2, 2.1 and 5.1.

Proof of Theorem 6.2. By Corollary 8.3, we have, for any positive a and all distinct points $x, y \in M$ such that $\rho(x,y) > a$,

$$\Psi(x, B(y,a)) \leq \frac{g(x,y)}{\inf_{v \in \overline{B(y,a)}} g(v,y)}. \tag{8.12}$$

Let us denote

$$C_a := \sup_{y,v \in M,\ \rho(y,v) \leq a} \frac{1}{g(v,y)}.$$

The hypothesis (C) implies that $C_a < \infty$, for all a small enough, and we can rewrite (8.12) as follows

$$\Psi(x, B(y,a)) \leq C_a g(x,y). \tag{8.13}$$

Let us fix $a > 0$ to be small enough. By the hypothesis (A), the metric space M can be covered by a countable family of balls $B(y_i, a) \in \mathfrak{B}_a$ so that the family of double balls $\{B(y_i, 2a)\}$ has a uniformly finite multiplicity. Let us denote $U_i = B(y_i, 2a)$, and introduce the events

$$\mathcal{A}_i = \left\{\omega : \forall j = 1, 2, ..., n \quad \exists\, t_j(\omega) \text{ such that } \xi_{x_j}(t_j) \in U_i\right\}.$$

In other words, $\mathcal{A}_i$ is the event that all traces $[\xi_{x_i}]$ intersect U_i. Let $N = N_{2a}$ be the number of sets U_i which intersect all traces $[\xi_{x_j}]$ for $j = 1, 2, ..., n$. Clearly,

$$N = \sum_i \mathbf{1}_{\mathcal{A}_i}.$$

Let us prove that $\mathbb{E}_{\overline{x}}(N) < \infty$. Since the processes ξ_{x_i} are independent, we have

$$\mathbb{P}_{\overline{x}}(\mathcal{A}_i) = \prod_{j=1}^{n} \mathbb{P}_{x_j}([\xi_{x_j}] \cap U_i \neq \emptyset) = \prod_{j=1}^{n} \Psi(x_j, U_i).$$

Therefore, by (8.13),

$$\begin{aligned}
\mathbb{E}_{\overline{x}}(N) &= \mathbb{E}_{\overline{x}}\left(\sum_i \mathbf{1}_{\mathcal{A}_i}\right) = \sum_i \mathbb{P}_{\overline{x}}(\mathcal{A}_i) = \sum_i \prod_{j=1}^{n} \Psi(x_j, U_i) \\
&= \sum_i \frac{\prod_{j=1}^{n} \Psi(x_j, U_i)}{\mu(U_i)} \mu(U_i) \\
&\leq C_a^n \sum_i \frac{\prod_{j=1}^{n} g(x_j, y_i)}{\mu(U_i)} \mu(U_i) \\
&\leq \frac{C_a^n}{\inf_{y \in M} V(y, 2a)} \sum_i \prod_{j=1}^{n} g(x_j, y_i) \mu(U_i). \qquad (8.14)
\end{aligned}$$

By the hypothesis (B), we have

$$\inf_{y \in M} V(y, 2a) > 0.$$

Next, we claim that

$$\sum_i \prod_{j=1}^n g(x_j, y_i)\mu(U_i) < \infty. \tag{8.15}$$

As follows from the hypothesis (A), the number of points y_i which are located at a distance $\le 4a$ from some of x_j, is finite. Therefore, it suffices to restrict the summation in (8.15) to those i for which $\rho(y_i, x_j) > 4a$, for any $j = 1, 2, ...n$. For such i, we have, by the Harnack inequality (6.5) of the hypothesis (D),

$$g(x_j, y_i) \le C_H \inf_{y \in U_j} g(x_j, y).$$

Therefore,

$$\prod_{j=1}^n g(x_j, y_i)\mu(U_i) \le C_H \prod_{j=1}^n \int_{U_i} g(x_j, y)\, d\mu(y).$$

Let us denote by C_M the maximal multiplicity of the cover set $\{U_i\}$. Then we have

$$\begin{aligned}\sum_i \prod_{j=1}^n g(x_j, y_i)\mu(U_i) &\le C_H \sum_i \int_{U_i} \prod_{j=1}^n g(x_j, y)\, d\mu(y) \\ &\le C_H C_M \int_{M\setminus\Omega_{2a}} \prod_{j=1}^n g(x_j, y)\, d\mu(y),\end{aligned}$$

which is finite by (6.6).

Hence, we have proved (8.15) and thus, by (8.14), $\mathbb{E}_{\overline{x}}(N) < \infty$. This implies immediately $N < \infty$, $\mathbb{P}_{\overline{x}}$-a.s. By Proposition 8.1, the processes ξ_{x_j}, $j = 1, 2, ..., n$, are asymptotically separated. ∎

Proof of Theorem 2.1. We will reduce this theorem to Theorem 6.2. As was mentioned in Sections 7.1 all hypotheses (A)-(E) are satisfied for the manifold M with bounded geometry, and for the process ξ generated by the uniformly elliptic operator L given by (2.1). We are left to verify that (6.6) follows from (2.3). Assuming that (2.3) holds, we have also

$$\int_{M\setminus B(x,\varepsilon/2)} g^n(x, y) d\mu(y) < \infty, \tag{8.16}$$

for some $\varepsilon > 0$ and $x \in M$.

Let us connect x with every x_j by a finite set of covering balls $B(y_i, \varepsilon/2) \in \mathfrak{B}_{\varepsilon/2}$, $i = 1, 2, ..., m$, with $\varepsilon > 0$ small enough. Denote by K the closure of the union of the balls $B(y_i, \varepsilon)$ over all $i = 1, 2, ..., m$. For any point y outside Ω^n_ε, we have by the local Harnack inequality (6.5) and by the symmetry of the Green function,

$$g(x_j, y) \le C_H^m g(x, y).$$

Thus,

$$\begin{aligned}\int_{M\setminus(\Omega^n_\varepsilon \cup K)} g(x_1, y)g(x_2, y)...g(x_n, y)\, d\mu(y) &\le C_H^{nm} \int_{M\setminus(\Omega^n_\varepsilon \cup K)} g^n(x, y)\, d\mu(y) \\ &\le C_H^{nm} \int_{M\setminus B(x,\varepsilon/2)} g^n(x, y)\, d\mu(y),\end{aligned}$$

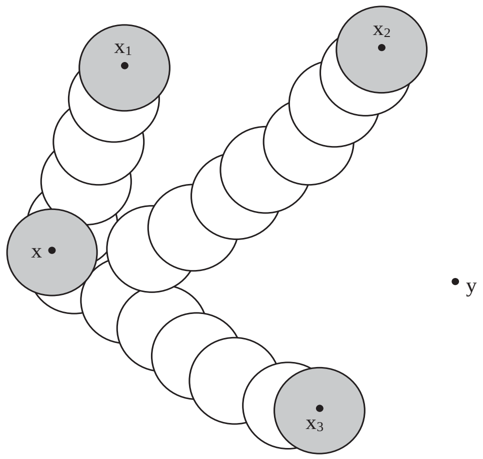

Figure 10: Point y is disjoint from K, which is the union of the white balls

which is finite by (8.16). We have used the fact that $K \supset B(x, \varepsilon/2)$.

Finally, we are left to observe that

$$\int_{K \setminus \Omega_\varepsilon^n} g(x_1, y) g(x_2, y) ... g(x_n, y)\, d\mu(y) < \infty\,,$$

due to the continuity of the Green function outside Ω_ε^n. ■

Proof of Theorem 5.1. This theorem also follows from Theorem 6.2. As was mentioned in Section 7.3, all hypotheses (A)-(E) are satisfied. Clearly, (6.6) follows from (5.3) by (7.12). ■

8.4 Asymptotic separation for two trajectories

Here we prove the results of asymptotic separation of two independent processes in terms of the heat kernel decay, that is Theorems 6.3, 2.2, 4.1 and 5.2.

Proof of Theorems 6.3, 2.2 and 4.1. Theorem 2.2 is clearly a particular case of Theorem 6.3, due to the following remark. The starting points x_1 and x_2 are assumed to be different in Theorem 6.3, whereas they are arbitrary in Theorem 2.2. However, if $x_1 = x_2$, then it suffices to consider the processes started at the random points $y_1 = \xi_1(\varepsilon)$ and $y_2 = \xi_2(\varepsilon)$, for some $\varepsilon > 0$, because $y_1 \neq y_2$ almost surely.

In what follows, we will simultaneously prove Theorems 6.3 and 4.1. By the same argument as above, we can assume $x_1 \neq x_2$.

In the setting of Theorem 4.1, we set $\xi = \xi^{(\alpha)}$, i.e. ξ is the α-process on a Riemannian manifold M. As before, $p(t, x, y)$ denotes the heat kernel of the Brownian motion on M. Recall that the Green kernel g_α of $\xi^{(\alpha)}$ is given by

$$g_\alpha(x, y) = \int_0^\infty t^{\alpha/2-1} p(t, x, y) dt. \tag{8.17}$$

In the setting of Theorem 6.3, ξ is a reversible Markov process on the space M (satisfying the hypotheses of Theorem 6.3), $p(t, x, y)$ is the heat kernel of ξ and $g(x, y)$ is the Green kernel of

ξ, that is,

$$g(x,y)=\int_0^\infty p(t,x,y)dt \tag{8.18}$$

(here t ranges either in $\mathbb{R}_+$ or in $\mathbb{Z}_+$; in the latter case dt means the counting measure).

Note that in both theorems in question, the heat kernel $p(t,x,y)$ is symmetric in x and y, and (8.18) is formally a particular case of (8.17) for $\alpha=2$. So, we can use (8.17) in all computations, assuming that, in the case of Theorem 6.3, $\alpha=2$. Also, both hypotheses (6.8) and (4.2) formally look the same:

$$\int_1^\infty t^{\alpha-1}\,p(t,x,x)\,dt<\infty. \tag{8.19}$$

This allows us to conduct the proof of both theorems simultaneously.

In both cases, we will apply Theorem 6.2. All the hypotheses of this theorem, except for (6.6), are satisfied (see Section 7.2 for the case of α-processes), so we are left to verify that each of the hypotheses (6.8) and (4.2) implies (6.6). In fact, we will prove even more:

$$\int_M g_\alpha(x_1,y)g_\alpha(x_2,y)d\mu(y)<\infty. \tag{8.20}$$

The next lemma is related to continuous time processes.

Lemma 8.4 *For all $\alpha,\beta>0$, we have the identity*

$$\int_M g_\alpha(x_1,y)g_\beta(x_2,y)d\mu(y)=c_{\alpha\beta}g_{\alpha+\beta}(x_1,x_2) \tag{8.21}$$

where $c_{\alpha\beta}\in(0,\infty)$.

Proof. Using (8.17) and the Markov property

$$\int_M p(t,x,y)p(s,y,z)d\mu(y)=p(t+s,x,z),$$

we obtain

$$\begin{aligned}
\int_M g_\alpha(x_1,y)g_\beta(x_2,y)d\mu(y) &= \int_M\int_0^\infty\int_0^\infty t^{\alpha/2-1}s^{\beta/2-1}p(t,x_1,y)p(s,x_2,y)\,dt\,ds\,d\mu(y)\\
&= \int_0^\infty\int_0^\infty t^{\alpha/2-1}s^{\beta/2-1}\int_M p(t,x_1,y)p(s,x_2,y)\,d\mu(y)\,dt\,ds\\
&= \int_0^\infty\int_0^\infty t^{\alpha/2-1}s^{\beta/2-1}p(t+s,x_1,x_2)\,dt\,ds\\
&= \int_0^\infty\int_s^\infty (t-s)^{\alpha/2-1}s^{\beta/2-1}p(t,x_1,x_2)\,dt\,ds\\
&= \int_0^\infty\left(\int_0^t (t-s)^{\alpha/2-1}s^{\beta/2-1}ds\right)p(t,x_1,x_2)\,dt.
\end{aligned}$$

Clearly, we have

$$\int_0^t (t-s)^{\alpha/2-1}s^{\beta/2-1}ds=t^{\alpha/2+\beta/2-1}\int_0^1(1-u)^{\alpha/2-1}u^{\beta/2-1}du=c_{\alpha\beta}t^{\alpha/2+\beta/2-1} \tag{8.22}$$

whence (8.21) follows. ■

By Lemma 8.4, we obtain[2]

$$\int_M g_\alpha(x_1,y)g_\alpha(x_2,y)d\mu(y) = c_\alpha \int_0^\infty t^{\alpha-1}p(t,x_1,x_2)dt$$
$$= c_\alpha \left[\int_0^1 + \int_1^\infty\right] t^{\alpha-1}p(t,x_1,x_2)dt. \tag{8.23}$$

The first integral in (8.23) is finite by transience because

$$\int_0^1 t^{\alpha-1}p(t,x_1,x_2)\,dt \le \int_0^1 t^{\alpha/2-1}p(t,x_1,x_2)\,dt \le g(x_1,x_2) < \infty. \tag{8.24}$$

The finiteness of the second integral in (8.23) follows from the hypothesis (8.19). Indeed, by the semigroup identity, the symmetry of heat kernel and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} p(t,x_1,x_2) &= \int_M p(\frac{t}{2},x_1,z)p(\frac{t}{2},z,x_2)\,d\mu(z) \\ &\le \left[\int_M p^2(\frac{t}{2},x_1,z)\,d\mu(z)\right]^{1/2}\left[\int_M p^2(\frac{t}{2},x_2,z)\,d\mu(z)\right]^{1/2} \\ &= \left[p(t,x_1,x_1)p(t,x_2,x_2)\right]^{1/2}. \end{aligned} \tag{8.25}$$

Therefore, by (8.25) and (8.19),

$$\int_1^\infty t^{\alpha-1}\,p(t,x_1,x_2)\,dt \le \left[\int_1^\infty t^{\alpha-1}\,p(t,x_1,x_1)\,dt\right]^{1/2}\left[\int_1^\infty t^{\alpha-1}\,p(t,x_2,x_2)\,dt\right]^{1/2} < \infty. \tag{8.26}$$

Thus, (8.24) and (8.26) imply that the right-hand side in (8.23) is finite, whence (8.20) follows. ■

Proof of Theorem 5.2. The heat kernel is defined by

$$p(k,x,y) = \frac{P_k(x,y)}{\mu(y)}.$$

The reversibility of the random walk ξ was defined by (5.4). Clearly, this is equivalent to the symmetry of the heat kernel $p(k,x,y)$ in x and y. The hypotheses (5.6) and (5.5) imply

$$\sum_k kp(k,x,x) < \infty,$$

and the rest follows by Theorem 6.3. ■

8.5 Asymptotic separation for n trajectories

Here we prove Theorems 2.3 and 4.2, which contain sufficient conditions for the asymptotic separation of n trajectories in terms of the heat kernel decay.

Proof of Theorem 2.3. We shall prove that the hypotheses of Theorem 2.3 imply those of Theorem 2.1, so that the latter can be applied. We only have to verify that (2.5) implies (2.3).

[2]Strictly speaking, the reference to Lemma 8.4 is illegal if the time t is discrete. However, most of the proof of Lemma 8.4 goes through in this case too, except for the change in the integral (8.22). However, in the discrete case we need this Lemma only for $\alpha = \beta = 2$, in which case the integral in (8.22) is obviously equal to t.

Let us first show that (2.5) implies

$$\sup_{y\in M\setminus B(x,\varepsilon)} \int_0^\infty \phi(t)\, p(t,x,y)\, dt < \infty, \tag{8.27}$$

where the function $\phi(t)$ is defined by

$$\phi(t) = \begin{cases} \theta(1), & 0 < t \le 1, \\ t^{\frac{1}{n-1}}\theta(t), & t > 1. \end{cases} \tag{8.28}$$

Indeed, we have

$$\int_0^\infty \phi(t)\, p(t,x,y)\, dt = \theta(1)\int_0^1 p(t,x,y)\, dt + \int_1^\infty t^{\frac{1}{n-1}}\, p(t,x,y)\,\theta(t)\, dt. \tag{8.29}$$

The first integral in (8.29) is uniformly bounded from above. Indeed, the heat kernel of L admits the upper bound (7.8). Integrating (7.8) from 0 to t and using $\rho(x,y) \ge \varepsilon$, we obtain

$$\int_0^1 p(t,x,y)\, dt \le \text{const}.$$

The second integral in (8.29) is uniformly (in y) bounded from above by the hypothesis (2.5).

Let us show (2.3). By the definition of the Green function, (2.3) is equivalent to

$$\int_{M'} \left(\int_0^\infty p(t,x,y)dt\right)^n d\mu(y) < \infty, \tag{8.30}$$

where $M' := M \setminus B(x,\varepsilon)$. Let us apply the Hölder inequality

$$\int fg \le \left(\int f^a\right)^{1/a} \left(\int g^b\right)^{1/b}$$

with $a = \frac{n}{n-1}$, $b = n$, $f = (p\phi)^{\frac{n-1}{n}}$ and $g = p^{\frac{1}{n}}\phi^{-\frac{n-1}{n}}$. We obtain

$$\left(\int_0^\infty p(t,x,y)dt\right)^n \le \left(\int_0^\infty p(t,x,y)\phi(t)dt\right)^{n-1} \left(\int_0^\infty p(t,x,y)\frac{dt}{\phi^{n-1}(t)}\right). \tag{8.31}$$

The first integral in the right-hand side of (8.31) is uniformly bounded from above by (8.27). Therefore, the integral in (8.30) is majorized up to a constant factor by

$$\begin{aligned} \int_{M'}\int_0^\infty p(t,x,y)\frac{dt}{\phi^{n-1}(t)}d\mu(y) &= \int_0^\infty \int_{M'} p(t,x,y)d\mu(y)\frac{dt}{\phi^{n-1}(t)} \\ &\le \int_0^\infty \frac{dt}{\phi^{n-1}(t)} \\ &= \text{const} + \int_1^\infty \frac{dt}{t\theta^{n-1}(t)} < \infty, \end{aligned} \tag{8.32}$$

which completes the proof. Here we have used the general property of the heat kernel

$$\int_M p(t,x,y)d\mu(y) \le 1,$$

and the hypothesis (2.6) which yields the last inequality in (8.32). ∎

Remark: If we take $\theta \equiv 1$ in (8.28) then the integral (8.32) diverges at ∞. This is the reason why we have to introduce the function θ satisfying (2.6).

Proof of Theorem 4.2. The proof uses Theorem 6.2 and follows the same line as the proofs of Theorem 2.1 and 2.3. All the hypotheses of Theorem 6.2, except for (6.6), were verified in this setting in Section 7.2. Let us prove that (6.6) follows from the hypothesis (4.3) of Theorem 4.2. The Green kernel $g_\alpha(x,y)$ for the α-process ξ is given by

$$g_\alpha(x,y) = \int_0^\infty t^{\alpha/2-1} p(t,x,y)dt. \tag{8.33}$$

Let us emphasize that $p(t,x,y)$ is the heat kernel for the Brownian motion on M, not for the α-process.

Clearly, the condition (6.6) of Theorem 6.2 follows from

$$\int_{M\setminus\Omega_\varepsilon} g_\alpha^n(x,y)d\mu(y) < \infty \tag{8.34}$$

in the same way as in the proof of Theorem 2.1. Let us deduce (8.34) from the hypothesis (4.3). In view of (8.33), this amounts to

$$\int_{M'} \left(\int_0^\infty t^{\alpha/2-1} p(t,x,y)dt\right)^n d\mu(y) < \infty \tag{8.35}$$

where $M' = M \setminus B(x,\varepsilon)$. Define $\phi(t)$ by

$$\phi(t) = \begin{cases} \theta(1), & 0 < t \le 1, \\ t^{\beta/2-1}\theta(t), & t > 1. \end{cases} \tag{8.36}$$

Then (8.27) is true again. By the Hölder inequality, we have

$$\left(\int_0^\infty t^{\alpha/2-1} p(t,x,y)dt\right)^n \le \left(\int_0^\infty p(t,x,y)\phi(t)dt\right)^{n-1} \left(\int_0^\infty p(t,x,y)\frac{t^{n(\alpha/2-1)}dt}{\phi^{n-1}(t)}\right).$$

The first integral on the right-hand side is uniformly bounded from above, which follows from (8.27). Hence, the integral in (8.35) is majorized by

$$\int_0^\infty \int_{M'} p(t,x,y)d\mu(y)\frac{t^{n(\alpha/2-1)}dt}{\phi^{n-1}(t)} \le \text{const} + \int_1^\infty \frac{dt}{t\theta^{n-1}(t)} < \infty,$$

where we have used $\phi(t) = t^{\beta/2-1}\theta(t)$ and

$$(n-1)(\beta/2-1) - n(\alpha/2-1) = 1,$$

which is a consequence of the definition (4.4) of β. ∎

References

[1] **Aizenman M.**, The intersection of Brownian paths as a case study of a renormalization group method for quantum field theory, *Commun. Math. Phys.*, **97** (1985) 91-110.

[2] **Albeverio S., Zhou X.Y.**, Intersections of random walks and Wiener sausages in four dimensions, preprint

[3] **Azencott R.**, Behavior of diffusion semi-groups at infinity, *Bull. Soc. Math. (France)*, **102** (1974) 193-240.

[4] **Barlow M.T.**, Diffusions on fractals, *in:* "Lectures on Probability Theory and Statistics, Ecole d'été de Probabilités de Saint-Flour XXV - 1995", Lecture Notes Math. 1690, Springer, 1998. 1-121.

[5] **Barlow M.T., Bass R.F.**, Brownian motion and harmonic analysis on Sierpinski carpets, to appear in *Canad. J. Math.*

[6] **Bishop R., O'Neill B.**, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969) 1-49.

[7] **Carlen E.A., Kusuoka S., Stroock D.W.**, Upper bounds for symmetric Markov transition functions, *Ann. Inst. H. Poincaré, Prob. et Stat.*, (1987) suppl. auno.2, 245-287.

[8] **Carron G.**, Inégalités isopérimétriques de Faber-Krahn et conséquences, *in:* "Actes de la table ronde de géométrie différentielle (Luminy, 1992)", Collection SMF Séminaires et Congrès 1, 1996. 205–232.

[9] **Chavel I., Feldman E.A.**, Modified isoperimetric constants, and large time heat diffusion in Riemannian manifolds, *Duke Math. J.*, **64** (1991) no.3, 473-499.

[10] **Cheeger J., Yau S.-T.**, A lower bound for the heat kernel, *Comm. Pure Appl. Math.*, **34** (1981) 465-480.

[11] **Cheng S.Y., Yau S.-T.**, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.*, **28** (1975) 333-354.

[12] **Davies E.B.**, Non-Gaussian aspects of heat kernel behaviour, *J. London Math. Soc.*, **55** (1997) no.1, 105-125.

[13] **Davies E.B., Mandouvalos N.**, Heat kernel bounds on hyperbolic space and Kleinian groups, *Proc. London Math. Soc.(3)*, **52** (1988) no.1, 182-208.

[14] **Dodziuk J.**, Maximum principle for parabolic inequalities and the heat flow on open manifolds, *Indiana Univ. Math. J.*, **32** (1983) no.5, 703-716.

[15] **Dvoretzky A., Erdös P., Kakutani S.**, Double points of paths of Brownian motion in n-space, *Acta Sci. Math. Szeged*, **12** (1950) 75-81.

[16] **Dvoretzky A., Erdös P., Kakutani S.**, Multiple points of paths of Brownian motion in the plane, *Bull. Res. Coun. Israel*, **3** (1954) 364-371.

[17] **Dvoretzky A., Erdös P., Kakutani S., Taylor S.J.**, Triple points of Brownian paths in 3-space, *Proc. Cambr. Phil. Soc.*, **53** (1957) 856-862.

[18] **Evans S.N.**, Multiple points in the sample paths of a Lévy process, *Probab. Th. Rel. Fields*, **76** (1987) 359-367.

[19] **Fernández R., Fröhlich J., Sokal A.**, "Random walks, critical phenomena, and triviality in quantum field theory" Springer, 1992.

[20] **Fitzsimmons P.J., Salisbury T.S.**, Capacity and energy for multiparameter Markov processes, *Ann. Inst. Henri Poincaré*, Probabilités et Statistiques, **25** no.3, (1989) 325-350.

[21] **Fukushima M.**, Basic properties of Brownian motion and a capacity on Wiener space, *J. Math. Soc. Japan*, **36** (1984) 161-176.

[22] **Fukushima M., Oshima Y., Takeda M.**, "Dirichlet forms and symmetric Markov processes" Studies in Mathematics 19, De Gruyter, 1994.

[23] **Grigor'yan A.**, On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds, (in Russian) *Matem. Sbornik*, **128** (1985) no.3, 354-363. Engl. transl. *Math. USSR Sb.*, **56** (1987) 349-358.

[24] **Grigor'yan A.**, On stochastically complete manifolds, (in Russian) *DAN SSSR*, **290** (1986) no.3, 534-537. Engl. transl. *Soviet Math. Dokl.*, **34** (1987) no.2, 310-313.

[25] **Grigor'yan A.**, On the fundamental solution of the heat equation on an arbitrary Riemannian manifold, (in Russian) *Mat. Zametki*, **41** (1987) no.3, 687-692. Engl. transl. *Math. Notes*, **41** (1987) no.5-6, 386-389.

[26] **Grigor'yan A.**, Stochastically complete manifolds and summable harmonic functions, (in Russian) *Izv. AN SSSR, ser. matem.*, **52** no.5, (1988) 1102-1108. Engl. transl. *Math. USSR Izvestiya*, **33** no.2, (1989) 425-432.

[27] **Grigor'yan A.**, Heat kernel upper bounds on a complete non-compact manifold, *Revista Mathemática Iberoamericana*, **10** (1994) no.2, 395-452.

[28] **Grigor'yan A.**, Heat kernel on a manifold with a local Harnack inequality, *Comm. Anal. Geom.*, **2** (1994) no.1, 111-138.

[29] **Grigor'yan A.**, Isoperimetric inequalities and capacities on Riemannian manifolds, *Operator Theory: Advances and Applications*, **109** (1999) 139-153.

[30] **Grigor'yan A.**, Estimates of heat kernels on Riemannian manifolds, *in:* "Spectral Theory and Geometry. ICMS Instructional Conference, Edinburgh 1998", ed. B.Davies and Yu.Safarov, London Math. Soc. Lecture Note Series 273, Cambridge Univ. Press, 1999. 140-225.

[31] **Grigor'yan A., Saloff-Coste L.**, Heat kernel on connected sums of Riemannian manifolds, *Math. Research Letters*, **6** (1999) no.3-4, 307-321.

[32] **Gromov M.**, "Structures métriques pour les variétés Riemannienes" Paris: Cedic/Ferdnand Nathan, 1981.

[33] **Kakutani S.**, On Brownian motion in n-space, *Proc. Japan Acad.*, **20** (1944) 648-652.

[34] **Kanai M.**, Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds, *J. Math. Soc. Japan*, **37** (1985) 391-413.

[35] **Karp L., Li P.**, The heat equation on complete Riemannian manifolds, unpublished.

[36] **Kuz'menko Yu.T., Molchanov S.A.**, Counterexamples to Liouville-type theorems, (in Russian) *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (1979) no.6, 39-43. Engl. transl. *Moscow Univ. Math. Bull.*, **34** (1979) 35-39.

[37] **Li P., Yau S.-T.**, On the parabolic kernel of the Schrödinger operator, *Acta Math.*, **156** (1986) no.3-4, 153-201.

[38] **Littman N., Stampaccia G., Weinberger H.F.**, Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Sup. Pisa (3)*, **17** (1963) 43-77.

[39] **Lyons T.**, The critical dimension at which quasi-every Brownian path is self-avoiding, *Advances Appl. Prob.*, (1986) , suppl. 87–99.

[40] **Maz'ya V.G.**, "Sobolev spaces" (in Russian) Izdat. Leningrad Gos. Univ. Leningrad, 1985. Engl. transl. Springer, 1985.

[41] **McGillivray I.**, A recurrence condition for some subordinated strongly local Dirichlet forms, *Forum Math.*, **9** (1997) 229-246.

[42] **Moser J.**, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.*, **14** (1961) 577-591.

[43] **Moser J.**, A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.*, **17** (1964) 101-134.

[44] **Porper F.O., Eidel'man S.D.**, Two-side estimates of fundamental solutions of second-order parabolic equations and some applications, (in Russian) *Uspekhi Matem. Nauk*, **39** (1984) no.3, 101-156. Engl. transl. *Russian Math. Surveys*, **39** (1984) no.3, 119-178.

[45] **Taylor S.J.**, Multiple points for the sample paths of the symmetric stable process, *Z. Wahrscheinlichkeitstheorie verw.*, **5** (1966) 247-264.

[46] **Varopoulos N.Th.**, Potential theory and diffusion of Riemannian manifolds, *in:* "Conference on Harmonic Analysis in honor of Antoni Zygmund. Vol I, II.", Wadsworth Math. Ser., Wadsworth, Belmont, Calif., 1983. 821-837.

[47] **Varopoulos N.Th.**, Hardy-Littlewood theory for semigroups, *J. Funct. Anal.*, **63** (1985) no.2, 240-260.

[48] **Yau S.-T.**, On the heat kernel of a complete Riemannian manifold, *J. Math. Pures Appl., ser. 9*, **57** (1978) 191-201.