Heat kernel and Harnack inequality on Riemannian manifolds

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Contents

1 Laplace operator and heat kernel

A weighted manifold is a couple (M, μ) where M is a Riemannian manifold and μ is a measure on M with smooth positive density with respect to the Riemannian measure μ_0; that is, $d\mu = h^2 d\mu_0$ where $h \in C^\infty(M)$, $h > 0$.

The weighted Laplace operator Δ on M is defined by

$$\Delta = \frac{1}{h^2} \operatorname{div}\left(h^2 \nabla\right).$$

Observe that Δ is a symmetric operator with respect to measure μ. Indeed, for all $f, g \in C_0^\infty(M)$,

$$\int_M (\Delta f)\, g d\mu = \int_M \operatorname{div}\left(h^2 \nabla f\right) g d\mu_0 = -\int_M h^2 \langle \nabla f, \nabla g \rangle\, d\mu_0$$

whence

$$\int_M (\Delta f)\, g d\mu = \int_M f \Delta g d\mu.$$

Furthermore, the operator Δ with the domain $C_0^\infty(M)$ is admits the Friedrichs extension to a self-adjoint operator in $L^2(M,\mu)$, which will also be denoted by Δ.

The heat semigroup $\{\exp(t\Delta)\}_{t\geq 0}$ is defined by means of spectral theory as a family of operators $\exp(t\Delta)$ in $L^2(M,\mu)$, and the *heat kernel* $p_t(x,y)$ is a unique smooth positive function of $t>0$ and $x,y\in M$ such that

$$e^{t\Delta} f(x) = \int_M p_t(x,y)\, f(y)\, d\mu(y),$$

for all $f\in L^2(M,\mu)$. It is known that $p_t(x,y)$ exists on any weighted manifold and coincides with the minimal positive fundamental solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u$ on $\mathbb{R}_+ \times M$. Besides, the heat kernel satisfies the following properties.

- symmetry: $p_t(x,y) = p_t(y,x)$.
- the semigroup identity:

$$p_{t+s}(x,y) = \int_M p_t(x,z)\, p_s(z,y)\, d\mu(z). \tag{1}$$

- $\int_M p_t(x,y)\, d\mu(y) \leq 1$.

Recall that in $\mathbb{R}^n$ with the Lebesgue measure μ, Δ is the classical Laplace operator $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$, and its heat kernel is given by the Gauss-Weierstrass formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Explicit formulas for the heat kernel exist also in hyperbolic spaces $\mathbb{H}^n$ (when μ is the Riemannian measure). For example in $\mathbb{H}^3$

$$p_t(x,y) = \frac{1}{(4\pi t)^{3/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right). \tag{2}$$

For arbitrary $\mathbb{H}^n$ the formula looks complicated, but it implies the following estimate, for all $t>0$ and $x,y\in\mathbb{H}^n$:

$$p_t(x,y) \simeq \frac{(1+r+t)^{\frac{n-3}{2}}(1+r)}{t^{n/2}} \exp\left(-\lambda t - \frac{r^2}{4t} - \sqrt{\lambda} r\right), \tag{3}$$

where $\lambda = \frac{(n-1)^2}{4}$ is the bottom of the spectrum of the Laplace operator on $\mathbb{H}^n$.

2 Uniform Faber-Krahn inequality

Let (M, μ) be a weighted manifold. For any relatively compact open set $\Omega \subset M$, denote by $\lambda_1(\Omega)$ the smallest eigenvalue of the (weak) Dirichlet problem for Δ in Ω.

Let Λ be a function on $(0, +\infty)$. We say that a weighted manifold (M, μ) satisfies the (*uniform*) *Faber-Krahn inequality* with function Λ if, for any non-empty relatively compact open set $\Omega \subset M$,

$$\lambda_1(\Omega) \geq \Lambda(\mu(\Omega)). \tag{4}$$

For example, $\mathbb{R}^n$ satisfies the Faber-Krahn inequality with function $\Lambda(v) = cv^{-2/n}$. Also, any Cartan-Hadamard manifold of dimension n satisfies the Faber-Krahn inequality with the same function Λ (but possibly with a different constant c). If K is a k-dimensional compact manifold then the Riemannian product $M = \mathbb{R}^m \times K$ satisfies the Faber-Krahn inequality with function

$$\Lambda(v) = c \begin{cases} v^{-2/n}, & v \leq 1, \\ v^{-2/m}, & v \geq 1, \end{cases} \tag{5}$$

where $n = \dim M = k + m$. Any n-dimensional manifold with bounded geometry satisfies the Faber-Krahn inequality with the function

$$\Lambda(v) = c \begin{cases} v^{-2/n}, & v \leq 1, \\ v^{-2}, & v \geq 1. \end{cases} \tag{6}$$

Theorem 1 *Assume that (M, μ) satisfies the Faber-Krahn inequality (4) with a positive continuous decreasing function Λ such that*

$$\int_0 \frac{dv}{v\Lambda(v)} < \infty. \tag{7}$$

Then, for all $t > 0$,

$$\sup_{x \in M} p_t(x, x) \leq \frac{4}{\gamma(t/2)}, \tag{8}$$

where the function γ is defined by the identity

$$t = \int_0^{\gamma(t)} \frac{dv}{v\Lambda(v)}. \tag{9}$$

Approach to the proof. Assuming that (4) holds, one deduces the following *Nash type* inequality: for any non-zero function $u \in \mathcal{D}$,

$$\int_M |\nabla u|^2 d\mu \geq (1 - \varepsilon) \|u\|_{L^2}^2 \Lambda\left(\frac{2}{\varepsilon}\frac{\|u\|_{L^1}^2}{\|u\|_{L^2}^2}\right), \tag{10}$$

for any $\varepsilon \in (0, 1)$. Then one applies Nash's argument: extending (10) to $u = p_t(x, \cdot)$ and noticing that

$$\int_M |\nabla u|^2 d\mu = -\int_M u\Delta u d\mu = -\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \tag{11}$$

and $\|u\|_{L^1} \leq 1$, one obtains from (10) and (11) a differential inequality for $\|u\|_{L^2}^2 = p_{2t}(x, x)$, whence the result follows. ■

3 Gaussian upper bounds

Fix some value $D \in (0, +\infty]$ and consider the following weighted integral of the heat kernel:

$$E(t,x) := \int_M p_t^2(x,z) \exp\left(\frac{d^2(x,z)}{Dt}\right) d\mu(z). \tag{12}$$

A priori, the value of $E(t,x)$ may be $+\infty$. For example, in $\mathbb{R}^n$ $E(t,x) = \infty$ if $D \le 2$.

Lemma 2 *For all $x, y \in M$, $t > 0$, the following inequality holds*

$$p_t(x,y) \le \sqrt{E(t/2,x)E(t/2,y)} \exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{13}$$

Proof. For any points $x, y, z \in M$, let us denote $\alpha = d(y,z)$, $\beta = d(x,z)$ and $\gamma = d(x,y)$. By the triangle inequality, $\alpha^2 + \beta^2 \ge \frac{1}{2}\gamma^2$.

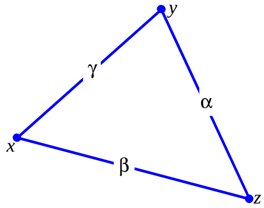


Figure 1:

We have then

$$\begin{aligned}
p_t(x,y) &= \int_M p_{t/2}(x,z) p_{t/2}(y,z) d\mu(z) \\
&\le \int_M p_{t/2}(x,z) e^{\frac{\beta^2}{Dt}} p_{t/2}(y,z) e^{\frac{\alpha^2}{Dt}} e^{-\frac{\gamma^2}{2Dt}} d\mu(z) \\
&\le \left(\int_M p_{t/2}^2(x,z) e^{\frac{2\beta^2}{Dt}} d\mu(z)\right)^{\frac{1}{2}} \left(\int_M p_{t/2}^2(y,z) e^{\frac{2\alpha^2}{Dt}} d\mu(y)\right)^{\frac{1}{2}} e^{-\frac{\gamma^2}{2Dt}} \\
&= \sqrt{E(t/2,x)E(t/2,y)} \exp\left(-\frac{d^2(x,y)}{2Dt}\right),
\end{aligned}$$

which was to be proved. ■

Lemma 3 *If $D \ge 2$ then for any $x \in M$, the function $E(t,x)$ is decreasing in t. In particular, if $E(t,x) < \infty$ for some $t = t_0$ then $E(t,x) < \infty$ for all $t > t_0$.*

Approach to the proof. The proof of the monotonicity of $E(t,x)$ amounts to verifying that its time derivative is non-positive. It is essential for the proof that $|\nabla d(x,\cdot)| \le 1$, which implies that the function $\xi(t,x) = \frac{d^2(x,y)}{Dt}$ satisfies

$$\frac{\partial \xi}{\partial t} + \frac{D}{4} |\nabla \xi|^2 \le 0. \tag{14}$$

■

Theorem 4 *Assume that, for some $x \in M$ and for all $t > 0$,*

$$p_t(x,x) \le \frac{C}{\gamma(t)}, \tag{15}$$

where $\gamma(t)$ is an increasing positive function on $\mathbb{R}_+$ satisfying the doubling condition:

$$\gamma(2t) \le A\gamma(t) \quad \text{for all } t > 0 \tag{16}$$

for some constant $A > 1$. Then, for any $D > 2$ and all $t > 0$,

$$E(t,x) \le \frac{C'}{\gamma(\varepsilon t)}, \tag{17}$$

for some $\varepsilon = \varepsilon(D) > 0$ and $C' = C'(A, C, D)$.

By putting together Theorem 4 and Lemma 2, we obtain the following result.

Corollary 5 *Assume that, for some points $x, y \in M$ and for all $t > 0$,*

$$p_t(x,x) \le \frac{C}{\gamma_1(t)} \quad \text{and} \quad p_t(y,y) \le \frac{C}{\gamma_2(t)}, \tag{18}$$

where γ_1 and γ_2 are increasing positive function on $\mathbb{R}_+$ both satisfying (16). Then, for any $D > 2$ and all $t > 0$,

$$p_t(x,y) \le \frac{C'}{\sqrt{\gamma_1(\varepsilon t)\gamma_2(\varepsilon t)}} \exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{19}$$

Corollary 6 *On any weighted manifold (M, μ) and for any $D > 2$, $E(t,x)$ is finite for all $t > 0$, $x \in M$. Moreover, the function $t \mapsto E(t,x)$ is continuous and monotone decreasing.*

Sketch of proof. By Theorem 4 and Lemma 3, it suffices to prove that for any $x \in M$ there exist positive constants C and T such that

$$p_t(x,x) \le Ct^{-n/2}, \quad \text{for all } 0 < t < T, \tag{20}$$

where $n = \dim M$.

Fix a small relatively compact open set Ω containing the point x. By compactness argument, the weighted manifold (Ω, μ) satisfies the following Faber-Krahn inequality: for all open sets $U \subset \Omega$, such that $\mu(U) \le \frac{1}{2}\mu(\Omega)$,

$$\lambda_1(U) \ge c\mu(U)^{-2/n},$$

where $c > 0$ depends on Ω. Hence, by a slight modification of Theorem 1, we obtain that the heat kernel p_t^Ω of (Ω, μ) satisfies the estimate

$$p_t^\Omega(x,x) \le Ct^{-n/2}, \quad \text{for all } t \in (0,T),$$

where C and T depend on Ω.

Consider the function

$$u(t,y) = p_t(x,y) - p_t^{\Omega}(x,y)$$

and extend it to $t \leq 0$ by setting $u(t,y) \equiv 0$. This function satisfies in $\mathbb{R} \times \Omega$ the equation $\frac{\partial u}{\partial t} = \Delta u$ and hence it is C^{∞}-smooth in $\mathbb{R} \times \Omega$. In particular, the function $t \mapsto u(t,x)$ is bounded on $[0,T]$, say $u(t,x) \leq C$. Then we obtain, for all $0 < t < T$,

$$p_t(x,x) = p_t^{\Omega}(x,x) + u(t,x) \leq Ct^{-n/2} + C,$$

whence (20) follows. ∎

Theorem 7 *On any weighted manifold (M,μ) and for any $D > 2$, there exists a positive continuous function $\Phi(t,x)$ on $\mathbb{R}_+ \times M$, which is decreasing in t and such that the following inequality holds*

$$p_t(x,y) \leq \Phi(t,x)\Phi(t,y) \exp\left(-\lambda_{\min} t - \frac{d^2(x,y)}{2Dt}\right), \tag{21}$$

for all $x, y \in M$ and $t > 0$, where $\lambda_{\min}$ is the bottom of the spectrum of $-\Delta$ on M.

Proof. Let us first set

$$\Phi(t,x) = \sqrt{E(\frac{1}{2}t,x)}. \tag{22}$$

By Corollary 6, this function is finite. By Lemma 3, the function $\Phi(t,x)$ is decreasing in t. By Lemma 2, we obtain

$$p_t(x,y) \leq \Phi(t,x)\Phi(t,y) \exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{23}$$

This estimates still does not match (21) because of the lack of the term $\lambda_{\min} t$. To handle it, let us find a positive smooth function h satisfying on M the equation

$$\Delta h + \lambda h = 0$$

where $\lambda = \lambda_{\min}$. Consider the measure $\widetilde{\mu}$ defined by $d\widetilde{\mu} = h^2 d\mu$ and the heat kernel $\widetilde{p}_t$ on the weighted manifold $(M,\widetilde{\mu})$. Applying (23) on $(M,\widetilde{\mu})$, we obtain that there exists a function $\widetilde{\Phi}(t,x)$ decreasing in t such that

$$\widetilde{p}_t(x,y) \leq \widetilde{\Phi}(t,x)\widetilde{\Phi}(t,y) \exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{24}$$

Using the relation between the heat kernels

$$p_t(x,y) = \widetilde{p}_t(x,y)h(x)h(y)e^{-\lambda t},$$

and (24), we obtain (21) with $\Phi(t,x) = \widetilde{\Phi}(t,x)h(x)$. ∎

Remark. As it follows from the construction of the function $\Phi(t,x)$ and from the proof of Corollary 6, for any compact set $K \subset M$ there exist positive constants C and T such that

$$\Phi(t,x) \leq Ct^{-n/4} \quad \text{for all } x \in K \text{ and } 0 < t < T. \tag{25}$$

4 Mean-value property

Here we present an alternative method of obtaining Gaussian upper bounds, which avoids using $E(t,x)$ and, instead, is based on a certain integral estimate of the heat kernel and the mean-value property.

The following theorem shows that the Gaussian exponential term appears naturally in the heat kernel upper estimates on arbitrary manifolds.

Theorem 8 *Let (M,μ) be a weighted manifold and let A and B be two μ-measurable sets on M. Then*

$$\int_A \int_B p_t(x,y)d\mu(x)d\mu(y) \le \sqrt{\mu(A)\mu(B)} \exp\left(-\frac{d^2(A,B)}{4t}\right), \tag{26}$$

where $d(A,B)$ is the geodesic distance between sets A and B.

To obtain pointwise bounds from (26) one needs to combine it with a *mean-value property.*

Definition. We say that the manifold M admits the mean-value property (MV) if, for all $t > \tau > 0$, $x \in M$ and for any positive solution $u(s,\eta)$ of the heat equation in the cylinder $(t-\tau, t] \times B(x, \sqrt{\tau})$, we have

$$u(t,x) \le \frac{C}{\tau V(x,\sqrt{\tau})} \int_{t-\tau}^{t} \int_{B(x,\sqrt{\tau})} u(s,z)d\mu(z)ds. \tag{27}$$

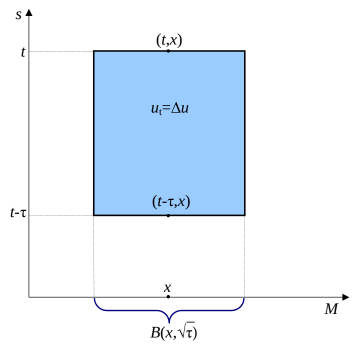


Figure 2:

Theorem 9 *Assume that the mean-value property (MV) holds on the manifold M. Then, for all $x \in M$ and $t > 0$,*

$$p_t(x,x) \le \frac{C}{V(x,\sqrt{t})}. \tag{28}$$

Moreover, for all $x, y \in M$, $t > 0$, $D > 2$,

$$p_t(x,y) \le \frac{C'}{\sqrt{V(x,\sqrt{t/2})V(y,\sqrt{t/2})}} \exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{29}$$

Theorem 9 admits a localized version. We say that the manifold M admits a *restricted* mean-value property $(MVxy\tau_0)$, for some $x, y \in M$ and $\tau_0 \in \mathbb{R}_+$, if the inequality (27) holds for all $\tau \in (0, \tau_0]$ in each of the balls $B(x, \sqrt{\tau})$ and $B(y, \sqrt{\tau})$. If M admits $(MVxy\tau_0)$ then similarly to the above theorem we obtain

$$p_t(x,y) \le \frac{C'}{\sqrt{V(x,\sqrt{\tau})V(y,\sqrt{\tau})}} \exp\left(-\frac{d^2(x,y)}{2Dt}\right) \tag{30}$$

where $\tau = \min(t/2, \tau_0)$.

Observe that the property $(MVxy\tau_0)$ holds on *any* manifold: for any given $x, y \in M$, there exists τ_0 such that $(MVxy\tau_0)$ is true. However, the constant C in the mean-value inequality (27) depends on the certain geometric properties of the balls $B(x, \sqrt{\tau_0})$ and $B(y, \sqrt{\tau_0})$.

Definition. We say that a weighted manifold (M, μ) satisfies the *volume doubling* property if, for all $x \in M$ and $r > 0$,

$$V(x, 2r) \le CV(x, r).$$

It is known that the volume doubling condition implies the volume comparison condition: for all $0 < r < R$ and $x \in M$,

$$\frac{V(x,R)}{V(x,r)} \le C\left(\frac{R}{r}\right)^N, \tag{31}$$

with some $N > 0$. Assuming (31), one can improve the estimate (29) as follows.

Theorem 10 *Assume that (M, μ) satisfies the mean-value property (MV) and the volume comparison (31). Then, for all $x, y \in M$ and $t > 0$,*

$$p_t(x,y) \le \frac{C'}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \left(1 + \frac{d}{\sqrt{t}}\right)^{N-1} \exp\left(-\frac{d^2}{4t}\right) \tag{32}$$

where $d = d(x, y)$.

Note for comparison that on $\mathbb{S}^n$ the heat kernel at the poles x, y admits the estimate

$$p_t(x,y) \sim \frac{c}{t^{n/2}} \left(1 + \frac{d}{\sqrt{t}}\right)^{n-1} \exp\left(-\frac{d^2}{4t}\right), \quad t \to 0,$$

which shows the sharpness of (32).

5 Relative Faber-Krahn inequality

Let (M, μ) be a geodesically complete weighted manifold.

Definition. We say that (M, μ) satisfies the *relative Faber-Krahn inequality* if there exist positive constants δ, c such that, for any geodesic ball $B(x, r)$ on M and for any non-empty relatively compact open set $\Omega \subset B(x, r)$,

$$\lambda_1(\Omega) \geq \frac{c}{r^2} \left(\frac{V(x, r)}{\mu(\Omega)} \right)^{\delta}. \tag{33}$$

For example, the relative Faber-Krahn inequality holds in $\mathbb{R}^n$ with $\delta = 2/n$ since $V(x, r) = cr^n$ and hence (33) amounts to the uniform Faber-Krahn inequality (4) with $\Lambda(v) = cv^{-2/n}$.

Theorem 11 *If M has non-negative Ricci curvature and μ is the Riemannian volume then (M, μ) satisfies the relative Faber-Krahn inequality.*

Approach to the proof. The following key property of manifolds of non-negative Ricci curvature is used in the proof of this theorem. For any $x, y \in M$ let $\gamma_{x,y} : [0, L] \to M$ be a shortest geodesic between x and y, where $L = d(x, y)$, $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(L) = y$. For any $x \in M$ and $0 < s < 1$, define a homothety $\Gamma_s^x : M \to M$ by $\Gamma_s^x(y) = \gamma_{x,y}(sL)$.

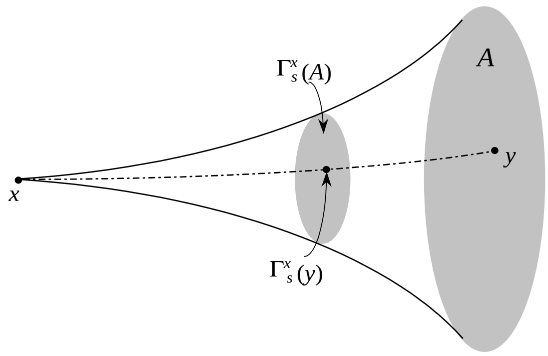


Figure 3: Homothety Γ_s^x

Then there exists $c > 0$ such that for any $\frac{1}{2} \leq s \leq 1$ and any Borel set $A \subset M$,

$$\mu(\Gamma_s^x(A)) \geq c\mu(A). \tag{34}$$

■

Furthermore, if a homothety with the property (34) can be defined on some manifold (M, μ) then this manifold satisfies the relative Faber-Krahn inequality.

The class of weighted manifolds with the relative Faber-Krahn inequality is much wider than those with non-negative Ricci curvature. In particular, this class is stable under quasi-isometry. Another example of stability: a connected sum of k copies of the same manifold satisfying the relative Faber-Krahn inequality also satisfies the relative Faber-Krahn inequality.

Theorem 12 *The relative Faber-Krahn inequality implies the volume doubling and the mean value property.*

Combining with Theorem 9 or 10, we obtain heat kernel bounds under the relative Faber-Krahn inequality.

Theorem 13 *The following conditions are equivalent:*

(a) (M, μ) *satisfies the relative Faber-Krahn inequality.*

(b) (M, μ) *satisfies the volume doubling property and the heat kernel on* (M, μ) *admits the estimate*

$$p_t(x, x) \leq \frac{C}{V\left(x, \sqrt{t}\right)}, \tag{35}$$

for all $x \in M$ *and* $t > 0$.

(c) (M, μ) *satisfies the volume doubling property and the heat kernel on* (M, μ) *admits the estimate*

$$p_t(x, y) \leq \frac{C}{V\left(x, \sqrt{t}\right)} \exp\left(-c\frac{d^2(x, y)}{t}\right), \tag{36}$$

for all $x, y \in M$ *and* $t > 0$.

The constant c in the exponential in (36) can be made arbitrarily close to $\frac{1}{4}$. In fact, it can be taken exactly $\frac{1}{4}$ at the expense of additional factors as in the following statement.

Theorem 14 *Let* (M, μ) *satisfy the relative Faber-Krahn inequality. Assume in addition that for some* $N > 0$ *and for all* $0 < r < R$ *and* $x \in M$,

$$\frac{V(x, R)}{V(x, r)} \leq C\left(\frac{R}{r}\right)^N. \tag{37}$$

Then, for all $x, y \in M$ *and* $t > 0$,

$$p_t(x, y) \leq \frac{C}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}}\left(1 + \frac{d(x, y)}{\sqrt{t}}\right)^{N-1} \exp\left(-\frac{d(x, y)^2}{4t}\right). \tag{38}$$

Note that the volume comparison (37) follows from the relative Faber-Krahn inequality (33) with $N = \frac{2}{\delta}$. However, N in (37) does not have to be $\frac{2}{\delta}$.

6 On-diagonal lower estimate

The following result allows to obtain an on-diagonal lower bound from an upper bound.

Theorem 15 *Assume that for some point $x \in M$,*

$$V(x, 2r) \leq CV(x, r) \quad \text{for all } r > 0$$

and

$$p_t(x, x) \leq \frac{C}{V\left(x, \sqrt{t}\right)}, \quad \text{for all } t > 0.$$

Then, for all $t > 0$,

$$p_t(x, x) \geq \frac{c}{V\left(x, \sqrt{t}\right)}. \tag{39}$$

Corollary 16 *The following conditions are equivalent:*

(*a*) *(M, μ) satisfies the relative Faber-Krahn inequality.*

(*b*) *The heat kernel on (M, μ) satisfies the estimates*

$$p_t(x, y) \leq \frac{C}{V\left(x, \sqrt{t}\right)} \exp\left(-c\frac{d^2(x, y)}{t}\right). \tag{40}$$

for all $x, y \in M$ and $t > 0$, and

$$p_t(x, x) \geq \frac{c}{V\left(x, \sqrt{t}/2\right)}, \tag{41}$$

for all $x \in M$ and $t > 0$.

Corollary 17 *If M has non-negative Ricci curvature and μ is the Riemannian volume then the heat kernel on (M, μ) satisfies the upper bounds (36), (38) and the lower bound (39).*

7 Harnack inequality and Li-Yau estimate

In this section we assume that (M, μ) is a geodesically complete weighted manifold. We say that the heat kernel on (M, μ) satisfies the *Li-Yau estimate* if, for all $x, y \in M$ and $t > 0$,

$$p_t(x, y) \asymp \frac{C}{V\left(x, \sqrt{t}\right)} \exp\left(-c\frac{d^2(x, y)}{t}\right). \tag{42}$$

P. Li and S.-T. Yau proved this estimate on geodesically complete Riemannian manifolds with non-negative Ricci curvature using the gradient estimates (see Theorem 22 below).

We say that (M,μ) satisfies the (*uniform parabolic*) *Harnack inequality* if, for any ball $B(z,r)$ on M and for any positive solution $u(t,x)$ of the heat equation in the cylinder $\mathcal{C}=(0,r^2)\times B(z,r)$, the following holds:

$$\sup_{\mathcal{C}_-} u(t,x) \le C \inf_{\mathcal{C}_+} u(t,x)$$

where $\mathcal{C}_-=\left(\frac{1}{4}r^2,\frac{1}{2}r^2\right)\times B\left(z,\frac{1}{2}r\right)$ and $\mathcal{C}_+=\left(\frac{3}{4}r^2,r^2\right)\times B\left(z,\frac{1}{2}r\right)$.

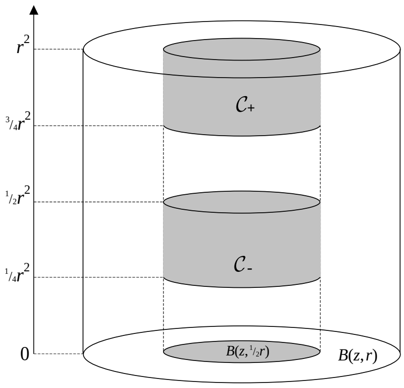


Figure 4: Cylinders $\mathcal{C}_+$ and $\mathcal{C}_-$

It is well known that the Harnack inequality holds for uniformly parabolic equations in $\mathbb{R}^n$. The relation to heat kernels is given by the following statement.

Theorem 18 *A manifold (M,μ) satisfies the Li-Yau estimate if and only if it satisfies the uniform Harnack inequality.*

To characterize manifolds with the Harnack inequality, we need one more notion. We say that a weighted manifold satisfies the (*weak*) *Poincaré inequality* if there exists $\varepsilon\in(0,1)$ such that for any ball $B(z,r)$ and for any function $u\in C^1(B(z,r))$,

$$\inf_{s\in\mathbb{R}}\int_{B(z,\varepsilon r)}(u-s)^2\,d\mu \le Cr^2\int_{B(z,r)}|\nabla u|^2\,d\mu. \tag{43}$$

(The term "weak" refers here to the factor $\varepsilon<1$).

Theorem 19 *A manifold (M,μ) satisfies the Harnack inequality if and only if it satisfies the doubling volume property and the Poincaré inequality.*

Hence, the Li-Yau estimate holds if and only if the doubling volume property and the Poincaré inequality hold.

Theorem 19 admits a localized version: Harnack inequality holds in all balls of radii $\le R$ if and only if the doubling volume property and Poincaré inequality hold in all balls of radii $\le R'$.

The proof of Theorem 19 uses the following result, which is of its own interest.

Theorem 20 *If (M, μ) satisfies the Poincaré inequality and the volume doubling property then it satisfies the relative Faber-Krahn inequality.*

Note that the converse to Theorem 20 is not true: it is possible to show that a connected sum of two copies of $\mathbb{R}^n$, $n \geq 3$, satisfies the relative Faber-Krahn inequality but not the Poincaré inequality.

Using Corollary 16, we obtain that the Poincaré inequality and the doubling volume property imply the upper bound and the on-diagonal lower bound in (42). The off-diagonal lower bound requires additional tools, which we do not touch here and which are similar to Moser's original proof of the Harnack inequality in $\mathbb{R}^n$.

Connection to the Ricci curvature comes from the following statement.

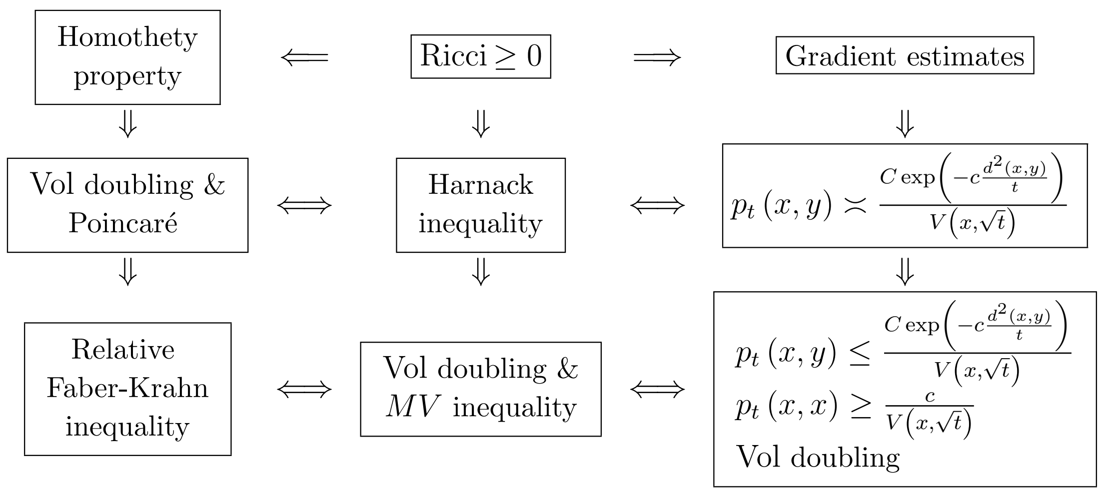
Theorem 21 *If M has non-negative Ricci curvature and μ is the Riemannian volume then (M, μ) satisfies the Poincaré inequality and the volume doubling property.*

In fact, both Poincaré inequality and volume doubling property come from the property (34) of the homothety on such manifolds. Clearly, Theorems 21 and 20 imply Theorem 11.

Successive application of Theorems 21, 19, and 18 yields the following result.

Theorem 22 *If M has non-negative Ricci curvature and μ is the Riemannian volume then the heat kernel on (M, μ) satisfies the Li-Yau estimate (42).*

The above results are schematically presented on the diagram:



8 Estimates of derivatives of the heat kernel

Fix some $D \in (2, +\infty]$ and consider again the quantity

$$E(t,x) = \int_M p_t^2(x,y) \exp\left(\frac{d^2(x,y)}{Dt}\right) d\mu(y),$$

as well as

$$E_1(t,x) = \int_M |\nabla p_t|^2 (x,y) \exp\left(\frac{d^2(x,y)}{Dt}\right) d\mu(y)$$

and

$$E_2(t,x)=\int_M |\Delta p_t|^2(x,y)\exp\left(\frac{d^2(x,y)}{Dt}\right)d\mu(y)$$

where both operators ∇ and Δ act on the variable y. More generally, set

$$E_n(t,x)=\int_M |\nabla^n p_t|^2(x,y)\exp\left(\frac{d^2(x,y)}{Dt}\right)d\mu(y)$$

where

$$\nabla^n=\begin{cases}\Delta^{n/2}, & n \text{ even},\\ \nabla\Delta^{\frac{n-1}{2}}, & n \text{ odd}.\end{cases}$$

The quantity E_n can be used to obtain pointwise estimates for the time derivative of the heat kernel. The following inequality is similar to Lemma 2.

Lemma 23 *For all $x,y\in M$ and $t>0$, we have*

$$\left|\frac{\partial^n p_t(x,y)}{\partial t^n}\right|\leq\sqrt{E_{2n}(t/2,x)E(t/2,y)}\exp\left(-\frac{d^2(x,y)}{2Dt}\right).$$

Approach to proof. Using the semigroup identity

$$p_t(x,y)=\int_M p_{t-s}(x,z)p_s(y,z)d\mu(z),$$

we obtain

$$\begin{aligned}\frac{\partial^n}{\partial t^n}p_t(x,y) &= \int_M \frac{\partial^n}{\partial t^n}p_{t-s}(x,z)p_s(y,z)d\mu(z)\\ &= \int_M \Delta_z^n p_{t-s}(x,z)p_s(y,z)d\mu(z).\end{aligned}$$

Taking $s=t/2$ and using Cauchy-Schwarz inequality as in Lemma 2, we finish the proof. ■

This method does not work for estimating $|\nabla_x p_t(x,y)|$ as we would need ∇_z under the integral. However, if one knows a priori that

$$|\nabla_x p_t(x,z)|\simeq|\nabla_z p_t(x,z)|$$

then one can obtain in the same way a pointwise estimate for $|\nabla_x p_t(x,y)|$.

Our next purpose is to obtain the estimates for $E_n(t,x)$.

Theorem 24 *For any $x\in M$, the function $t\mapsto E_n(x,t)$ is a finite, continuous, and decreasing (as long as $D>2$).*

Theorem 25 *Assume that, for some $x\in M$ and all $t\in(0,T)$*

$$E(x,t)\leq\frac{1}{f(t)}$$

where $f(t)$ is some positive L^1_{loc} function on $(0,T)$. Define a function $f_n(t)$ on $(0,T)$ by induction as follows:

$$f_0 = f, \quad f_{n+1}(t) = \int_0^t f_n(\tau)\, d\tau.$$

Then, for all $t \in (0,T)$,

$$E_n(x,t) \leq \frac{c^{-n}}{f_n(t)}$$

where $c = \frac{D-2}{D/2+8}$.

In fact, f_n can be defined explicitly by

$$f_n(t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)\, d\tau.$$

For example, if $f(t) = t^\alpha$ then $f_n(t) = C_n t^{\alpha+n}$.

Corollary 26 *If (M,μ) satisfies the relative Faber-Krahn inequality (33) then for all $x, y \in M$ and $t > 0$*

$$\left|\frac{\partial^n}{\partial t^n} p_t\right|(x,y) \leq \frac{C\left(1 + \frac{d(x,y)}{\sqrt{t}}\right)^{N+n+1}}{t^n \sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp\left(-\frac{d(x,y)^2}{4t}\right),$$

where N is the exponent of the volume comparison condition (37).