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History. Lect. 1, 03.03.2016

De Giorgi, 1957

$$Lu = \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) \quad \text{in } \Omega \text{ in } \mathbb{R}^n.$$

$a_{ij}(x)$ sym. matrix, unif. elliptic:

$$\cancel{\Lambda}^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \cancel{\Lambda}|\xi|^2$$

meas in x. One considers $Lu=0$ in a weak sense, where solutions are from $W^{1,2}_{loc}$: $\int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}=0$ $\forall \varphi \in C_0^\infty(\Omega)$

Theorem. $u \in C^\alpha(K)$ for $\alpha=\alpha(\lambda,n)>0$. $\forall K \subset\subset \Omega$. $\Leftrightarrow C^\alpha_{loc}(\Omega)$.

J. Nash 1958: the same for parabolic PDE

$$\frac{\partial u}{\partial t} = Lu.$$

J. Moser 1960-61 Harnack inequality for $Lu=0$:

if $u>0$ in B_R, then $\operatorname{ess\,sup}_{B_{R/2}} u \le C \operatorname{ess\,inf}_{B_{R/2}} u$.

$\Rightarrow$ Hölder continuity

J. Moser 1964: Harnack for $\frac{\partial u}{\partial t}=Lu$.

$$Lu = a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j},$$

$Lu=0$ is understood in a strong sence: $u \in W^{2,p}_{loc}$

$1\le p\le\infty$, $Lu=0$ a.e..

1980. Krylov-Safonov: $u \in C^\alpha$ Moreover, the Moreover, also Harnack.

Same for $\frac{\partial u}{\partial t} = Lu$. Moreover, also Harnack.

Based on previous work of Landis.

Purpose of this short course: proof of Theorems of de Georgi - Moser - Krylov - Safonov for $Lu = 0$ ~~time~~
Using a unified approach of Landis.
This is rather historical topic but has become interesting again because of applicability to non-local operators L (

1. Hölder cont. for divergence form.

Start with mean value inequality.

Thm 1.1. Let $u \in W^{1,2}(B_R)$, $Lu \geq 0$ in a weak sense.

Then $\operatorname*{ess\,sup}_{B_{R/2}} u \leq \frac{C}{R^{n/2}} \|u_+\|_{L^2(B_R)}$

claim. Let $f \in W_0^{1,2}(\mathbb{R}^n)$ = closure of C_0^∞ in $W^{1,2}$.

~~Then~~ Let $f \geq 0$ and set $F = \{f > 0\}$.

Then $\|f\|_{L^2} \leq C_n |F|^{1/n} \|\nabla f\|_{L^2}$ (FK-inequality).

Proof. $v = f^2$. Use Sobolev inequality:

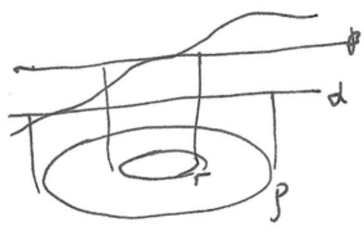
$$\|f\|_{L^2}^2 = \int_\Omega f^2 = \int_\Omega v = \int_G v \leq \Big(\int_{G \to \mathbb{R}^n} v^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \Big(\int_G 1\Big)^{1/n}$$

$$\leq C \int_{\mathbb{R}^n} |\nabla v| \cdot |F|^{1/n} = C \int |f| |\nabla f| \cdot |F|^{1/n}$$

$$\leq C \|f\|_{L^2} \cdot \|\nabla f\|_{L^2} |F|^{1/n} \quad \Rightarrow \text{the claim.}$$

For the proof of theorem, fix two values $0 < r < \rho < R$, $\beta > \alpha > 0$ and consider the quantities

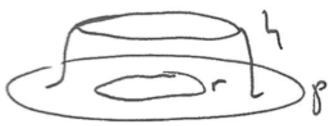
$$a = \int_{B_\rho} (u-\alpha)_+^2, \quad b = \int_{B_r} (u-\beta)_+^2$$



so that $b \leq a$.

Set $v = (u-\beta)_+$, $F = \{v > 0\} = \{u > \beta\}$.

Choose also a cuttoff function $\eta \in C_0^\infty(B_\rho)$, $\eta \equiv 1$ on B_r.



Then $v\eta^2 \in W_0^{1,2}(B_\rho)$ and $Lu \geq 0$

implies: $\left(\int Lu \cdot v\eta^2 \geq 0 \right)$

$$\int a_{ij}\, \partial_j u\, \partial_i (v\eta^2) \leq 0$$

$$\int a_{ij}\, \underbrace{\partial_j u}\, \underbrace{\partial_i v} \cdot \eta^2 \leq -\int a_{ij}\, \partial_j u\, v\, 2\eta\, \partial_i \eta$$

$\partial_j u\, \partial_i v = \partial_j v\, \partial_i v$ since on F $\quad v = u - \beta \Rightarrow \partial_i u = \partial_i v$

on F^c $\partial_i v = 0$.

$\partial_j u \cdot v = \partial_j v \cdot v$

$$\int a_{ij}\, \partial_j v\, \partial_i v\, \eta^2 \leq -2\int a_{ij}\, \partial_j v\, v\, \eta\, \partial_i \eta \leq 2\lambda \int |\nabla v||\nabla \eta|\, v\eta$$

$$\leq 2\lambda \left(\int |\nabla v|^2 \eta^2 \right)^{1/2} \cdot \left(\int v^2 |\nabla \eta|^2 \right)^{1/2}$$

$$\geq \lambda^{-1} \int |\nabla v|^2 \eta^2$$

$$\int |\nabla v|^2 \eta^2 \leq 4\lambda^4 \int v^2 |\nabla \eta|^2$$

$$\int |\nabla (v\eta)|^2 \leq 2\int |\nabla v|^2 \eta^2 + 2\int v^2 |\nabla \eta|^2$$

$$\leq (8\lambda^4 + 2) \int v^2 |\nabla \eta|^2 \leq \frac{C}{(\rho - r)^2} \int_{B_\rho} v^2$$

By FK: $\|v\|^2_{L^2(B_r)} \leq \|v\eta\|^2_{L^2(B_\rho)} \lesssim C|F|^{2/n} \cdot \|\nabla(v\eta)\|^2_{L^2(B_\rho)}$

$$b = \int_{B_r} (u-\beta)_+^2 = \int_{B_r} v^2 \le C|F|^{2/n} \cdot \frac{C}{(\rho-r)^2} \int_{B_\rho} v^2$$

$v = (u-\beta)_+ \le (u-\alpha)_+$.

$$\le \frac{C|F|^{2/n}}{(\rho-r)^2}\, a.$$

$$a = \int_{B_\rho} (u-\alpha)_+^2 \ge \int_F (u-\alpha)_+^2 \ge (\beta-\alpha)^2 |F|$$

$$\Rightarrow |F| \le \frac{a}{(\beta-\alpha)^2}$$

$$\Rightarrow \boxed{b \le \frac{C\, a^{1+2/n}}{(\rho-r)^2 \cdot (\beta-\alpha)^{4/n}}}$$

Renaming: $\boxed{\rho_k \rightsquigarrow R_k}$

Iteration procedure: Sequence ρ_k, s.t.

$\rho_0 = R$, $\rho_k = (\frac{1}{2} + \frac{1}{2^k})R$, $\rho_k \to \frac{1}{2}R$ $k \to \infty$. $\rho_k \downarrow$

$d_0 > 0$ fix, $d_k = (2 - \frac{1}{2^k})d$ $d_k \to 2d_0$ $k \to \infty$. $d_k \uparrow$

$a_k = \int_{B_{\rho_k}} (u - d_k)_+^2$, $a_k \downarrow$.

We show that $a_k \to 0 \Rightarrow \int_{B_{R/2}} (u - 2d_0)_+^2 = 0$

$\Rightarrow \operatorname{ess\,sup}_{B_{R/2}} u \le 2d_0 \Rightarrow$ mean value.

Setting $q = 1 + 2/n$, we have by the previous step

$$a_k \leq \frac{C\, a_{k-1}^{q}}{\underbrace{(r_{k-1} - r_k)}_{R/2^k}{}^{2}\,\underbrace{(d_k - d_{k-1})}_{d/2^k}{}^{4/n}}$$

$$\Rightarrow a_k \leq C^k \cdot \frac{a_{k-1}^{q}}{R^2\, d^{4/n}} \quad R^2 d^{4/n} =: M \qquad = \frac{C^k}{M}\, a_{k-1}^{q}.$$

$$a_k \leq \frac{C^k}{M} a_{k-1}^{q} \leq \frac{C^k}{M}\left(\frac{C^{k-1}}{M} a_{k-2}^{q}\right)^{q} = \frac{C^{k+q(k-1)}}{M^{1+q}} \cdot a_{k-2}^{q^2}$$

$$\leq \frac{C^{k+q(k-1)}}{M^{1+q}}\left(\frac{C^{k-2}}{M} \cdot a_{k-3}^{q}\right)^{q^2} = \frac{C^{k+q(k-1)+q^2(k-2)}}{M^{1+q+q^2}} \cdot a_{k-3}^{q^3}$$

$$\cdots \leq \frac{C^{k+q(k-1)+q^2(k-2)+\dots+q^{k-1}}}{M^{1+q\dots+q^{k-1}}}\, a_0^{q^k}$$

$$1 + q + \dots + q^{k-1} = \frac{q^k - 1}{q-1}$$

$$k + q(k-1) + \dots + q^{k-1} = \frac{q^{k+1} - (k+1)q + k}{(q-1)^2}$$

$$a_k \leq \frac{C^{\frac{q^{k+1} - (k+1)q + k}{(q-1)^2}}}{M^{\frac{q^k - 1}{q-1}}} \cdot a_0^{q^k} = \left[\frac{C^{\frac{q \dots}{(q-1)^2}}}{M^{\frac{1}{q-1} \dots}} \cdot a_0\right]^{q^k}$$

want: $\boxed{\dfrac{C^{\frac{q}{(q-1)^2}}\, a_0}{M^{\frac{1}{q-1}}} \leq 1/2} \Rightarrow a_k \to 0.$

How to achieve that? ~~Using~~ Choosing d!

Boxed eq $\iff$ $M = (C\, a_0)^{q-1}$

$\iff$ $R^2 d^{4/n} = (C\, a_0)^{q-1} = (C\, a_0)^{2/n}$

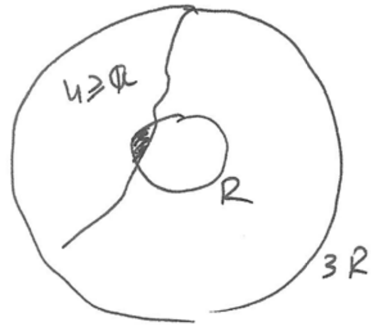
$$a_0 = \int_{B_R} (u-d)_+^2$$

Set $d := \dfrac{(C\, a_0)^{1/2}}{R^{n/2}} \leq \dfrac{C \cdot \|u_+\|_{L^2(B_R)}}{R^{n/2}}$.

Hence $\operatorname{ess\,sup}_{B_{R/2}} u \leq 2d \leq \dfrac{2C}{R^{n/2}} \|u_+\|_{L^2(B_R)}$, q.e.d.

Th 1.2. (weak Harnack inequality).
<=> lemma of growth of Landis.

$Lu = 0$ in B_{3R}, $u \geq 0$.
choose $a>0$



Set $E = \{u \geq a\} \cap B_R$

$\forall \varepsilon > 0 \ \exists \delta = \delta(n, \lambda, \varepsilon) > 0$ s.t.

if $\dfrac{|E|}{|B_R|} \geq \varepsilon$ then

$\operatorname{ess\,inf}_{B_R} u \geq \delta a$

For actual Harnack inequality $a = \operatorname{ess\,sup}_{B_R} u$

For actual Harnack inequality, $|E|$ could be zero,
s. that $|E|$ could be zero,
~~s. th~~ and δ does not depend on ε.

Proof. ~~We take $a = 1$.~~ ~~Also can~~
can assume that $\operatorname{ess\,inf}_{B_{3R}} u > 0$, since if $=0$,
then consider $u+m$, $m>0$, prove all for this
function and then let $m \to 0$.
Also can assume $a = 1$. Note: $u \geq 1 \Leftrightarrow v \leq 0$.
Use function $v = \log \frac{1}{u}$. This function
is bounded from above, and loc. bounded from
below (since u is loc. bounded from above
by Th 1.1).

Let us prove that $Lv \geq 0$.

<u>Idea</u>: If $L = \Delta$ ~~then~~ and $\Delta u = 0$ then

$\Delta v = |\nabla v|^2 \geq 0$.

We need to prove that for any $\eta \in C_0^\infty(B_{3R})$, $\eta \geq 0$,

$$-\int a_{ij}\, \partial_j v\, \partial_i \eta \geq 0.$$

Since $\partial_j v = -\frac{1}{u}\partial_j u$, ~~we have this is ≥~~

$$\partial_i\left(\frac{\eta}{u}\right) = \frac{\partial_i \eta}{u} - \frac{\partial_i \eta}{u^2}$$

$$\boxed{-\int a_{ij}\, \partial_j v\, \partial_i \eta} = \int a_{ij}\, \frac{\partial_j u\, \partial_i \eta}{u}$$

$$= \underbrace{\int a_{ij}\, \partial_j u\, \partial_i\left(\frac{\eta}{u}\right)}_{=0 \text{ since } Lu=0} + \int a_{ij}\, \partial_j u\, \eta\, \frac{\partial_i u}{u^2}$$

$$\boxed{= \int a_{ij}\, \underbrace{\partial_j v\, \partial_i v}_{\geq 0} \cdot \eta} \geq 0. \qquad (*)$$

Let us now use Poincaré inequality to function v in B_r, $r \leq 3R$: if $H = \{v \leq 0\} \cap B_r$ then

$$\int_{B_r} v_+^2 \leq C \frac{r^2 |B_r|}{|H|} \int_{B_r} |\nabla v_+|^2 \qquad \text{(see below)}$$

For $r = 2R$ use $|B_{2R}| \leq C|B_R|$ and

for $E = \{w \leq 0\} \cap B_R$ we have $|E| \leq |H|$

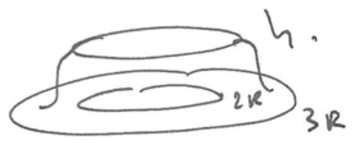
$$\Rightarrow \int_{B_{2R}} v_+^2 \leq C R^2 \frac{|B_R|}{|E|} \int_{B_{2R}} |\nabla v_+|^2$$

By the mean value inequality we have

$$\operatorname*{ess\,sup}_{B_R} v \le \frac{C}{R^{n/2}}\left(\int_{B_{2R}} v_+^2\right)^{1/2}$$

$$\le \frac{C}{R^{n/2}}\cdot\left(\frac{CR^2}{\varepsilon}\cdot\int_{B_{2R}}|\nabla v_+|^2\right)^{1/2} \qquad (**).$$

Let us estimate $\int_{B_{2R}}|\nabla v_+|^2$



Choose

$$\int|\nabla v|^2\eta^2 \le \lambda\int_{B_{3R}} a_{ij}\cdot\partial_i v\,\partial_j v\,\eta^2 \overset{(*)}{=} -\lambda\int a_{ij}\cdot\partial_j v\,\partial_i(\eta^2)$$

$$= -2\lambda\int a_{ij}\,\partial_j v\,\partial_i\eta\cdot\eta \le 2\lambda^2\int|\nabla v|\cdot|\nabla\eta|\cdot\eta$$

$$\le 2\lambda^2\left(\int|\nabla v|^2\eta^2\right)^{1/2}\left(\int|\nabla\eta|^2\right)^{1/2}$$

End of lect 1

$$\Rightarrow \int|\nabla v|^2\eta^2 \le 4\lambda^4\int|\nabla\eta|^2 \le CR^{n-2}.$$

$$\Rightarrow \int_{B_{2R}}|\nabla v|^2 \le CR^{n-2}.$$

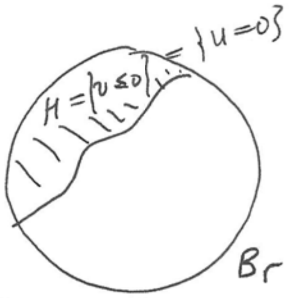
$(**) \Rightarrow$

$$\operatorname*{ess\,sup}_{B_R} v \le \frac{C}{R^{n/2}}\cdot\left(\frac{CR^2\cdot R^{n-2}}{\varepsilon}\right)^{1/2} \sim \varepsilon^{-1/2}$$

$$\Rightarrow \operatorname{ess\,inf} u \ge e^{-\varepsilon^{-1/2}}.$$

Poincaré inequality If $v \in W^{1,2}(B_{r+\varepsilon})$ then

$$\int_{B_r} v_+^2 \le C r^2 \frac{|B_r|}{|H|} \int_{B_r} |\nabla v_+|^2$$



where $H = \{v \le 0\} \cap B_r$

(if $H = \emptyset$ then there is no estimate for $v \equiv 1$).

Set $u = v_+$ and apply Classical Poincaré inequality to function u:

$$\int_{B_r} |\nabla u|^2 \ge \frac{C}{r^2} \int_{B_r} (u-\xi)^2 \, , \text{ where } \xi = \fint_{B_r} u .$$

$$\Rightarrow \int_{B_r} |\nabla u|^2 \ge \frac{C}{r^2} \int_H (\underset{0}{\underset{\shortparallel}{u}} - \xi)^2 = \frac{C}{r^2} |H| \cdot \xi^2 = \frac{C}{r^2} \frac{|H|}{|B_r|} \int_{B_r} \xi^2$$

Also: $$\int |\nabla u|^2 \ge \frac{C}{r^2} \frac{|H|}{|B_r|} \int (u-\xi)^2$$

$$\frac{1}{2}(a^2+b^2) \ge \left(\frac{a+b}{2}\right)^2$$

Adding up:

$$\int |\nabla u|^2 \ge \frac{C}{r^2} \frac{|H|}{|B_r|} \frac{1}{2} \int \underbrace{\left((u-\xi)^2 + \xi^2\right)}_{\ge \frac{1}{4} u^2}$$

$$\int |\nabla u|^2 \ge \frac{C}{r^2} \frac{|H|}{|B_r|} \frac{1}{4} \int u^2 \quad , \text{ q.e.d.}$$

Oscillation inequality

Th 1.3 $Lu = 0$ in B_{3R}. Then

$$\operatorname*{osc}_{B_R} u \le \gamma \operatorname*{osc}_{B_{3R}} u \quad \text{for some } \gamma < 1,$$

$\gamma = \gamma(\lambda, n)$.

Here $\operatorname*{osc}_A u = \operatorname*{ess\,sup}_A u - \operatorname*{ess\,inf}_A u$.

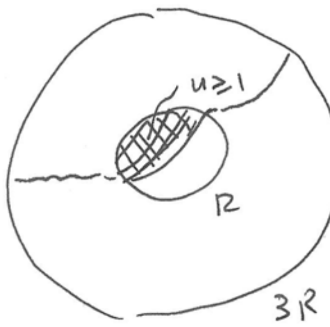
Proof. Without loss of generality assume:

$$\operatorname*{ess\,inf}_{B_{3R}} u = 0, \quad \operatorname*{ess\,sup}_{B_{3R}} u = 2.$$

Consider the sets

$$\{u \ge 1\} \cap B_R, \quad \{u \le 1\} \cap B_R.$$

One of them has measure $\ge \frac{1}{2} |B_R|$, Let it be $\{u \ge 1\} \cap B_R$ (otherwise replace u by $2-u$)..



~~Apply~~ By Th 1.2 with $\varepsilon = 1/2$,

The set $E = \{u \ge 1\} \cap B_R$

satisfies $\dfrac{|E|}{|B_R|} \ge \dfrac{1}{2} = \varepsilon$.

Therefore, $\operatorname*{ess\,inf}_{B_R} u \ge \delta = \delta(n, \lambda, \frac{1}{2}) > 0$

Then $\operatorname*{osc}_{B_R} u = \operatorname*{ess\,sup}_{B_R} u - \operatorname*{ess\,inf}_{B_R} u \le 2 - \delta$

$= \dfrac{2-\delta}{2} \cdot 2 = \gamma \cdot \operatorname*{osc}_{B_{3R}} u$, where $\gamma = \dfrac{2-\delta}{2} < 1$.

<u>Thm 1.4</u> (de Giorgi) If $Lu=0$ in $\Omega\subset\mathbb{R}^n$, $u\in W^{1,2}_{loc}$ then $u\in C^\alpha(\Omega)$, where $\alpha=\alpha(n,\lambda)>0$. Moreover, for any compact ~~convex~~ set $K\subset\Omega$,

$$\|u\|_{C^\alpha(K)}\le C\|u\|_{L^2(\Omega)}$$

where $C=C(K,\Omega,n,\lambda)$,

$$\|u\|_{C^\alpha(K)}:=\sup_K|u|+\sup_{x,y\in K}\frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

<u>Rem.</u> Can assume $\|u\|_{L^2(\Omega)}<\infty$, otherwise replace Ω by $\Omega'\subset\Omega$.

<u>Proof</u>. Let $\rho=\operatorname{dist}(K,\partial\Omega)$, s.t. $\forall x\in K$ the ball $B_\rho(x)$ is contained in Ω. <u>Fix $z\in K$.</u>

<u>Step 1</u> Set $\rho_k=\rho 3^{-k}$. By T 1.3

$$\operatorname*{osc}_{B_{\rho_{k+1}}(z)}u\le\gamma\operatorname*{osc}_{B_{\rho_k}(z)}u$$

$$\Rightarrow\operatorname*{osc}_{B_{\rho_k}(z)}u\le\gamma^{k-1}\operatorname*{osc}_{B_{\rho_1}(z)}u\le2\gamma^{k-1}\operatorname*{ess\,sup}_{B_{\rho_1}(z)}|u|.$$

It follows from Th 1.1: $\operatorname*{ess\,sup}_{B_{\rho_1}(z)}|u|\le C\|u\|_{L^2(B_\rho(z))}$

$C=C(n,\lambda,\rho)$

$$\Rightarrow\operatorname*{osc}_{B_{\rho_k}(z)}u\le C\gamma^k\|u\|_{L^2(\Omega)}$$

~~For arbitrary $r\le\rho$, choose k s.t. $3^{-(k+1)}\le\frac{r}{\rho}\le3^{-k}$~~

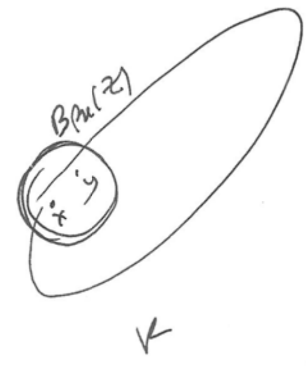
Step 2 Let us prove that for almost all $x, y \in K$ with $|x-y| \le \rho/2$

$$|u(x)-u(y)| \le C\,|x-y|^{\alpha}\,\|u\|_{L^2(\Omega)} \qquad (*)$$

where $\alpha = \log_3 \frac{1}{\gamma} > 0$.

~~Denote $r = |x-y|$~~ It suffices to prove $(*)$ under assumption $\frac{\rho}{2}\cdot 3^{-(k+1)} \le |x-y| \le \frac{\rho}{2}\cdot 3^{-k}$, $k = 0, 1, \dots$ The compact set K can be covered by a finite number of balls $B_{\frac{1}{2}\rho_k}(z_i)$ with $z_i \in K$. If x lies in $B_{\frac{1}{2}\rho_k}(z_i)$ then $y \in B_{\rho_k}(z_i)$. So, it suffices to prove $(*)$ for almost all $x, y \in B_{\rho_k}(z)$, where $z \in K$. ~~we have~~

For a.a. $x, y \in B_{\rho_k}(z)$ we have



$$|u(x)-u(y)| \le \operatorname*{osc}_{B_{\rho_k}(z)} u \le C\gamma^k \|u\|_{L^2}$$

Since $3^{k+1} \ge \frac{\rho}{2|x-y|}$, it follows that

$$k \ge \log_3 \frac{\rho}{2|x-y|} - 1$$

$$\gamma^k \le \gamma^{-1}\gamma^{\log_3 \frac{\rho}{2|x-y|}} = \gamma^{-1} 3^{\log_3\gamma \,\log_3 \frac{\rho}{2|x-y|}}$$

$$= \gamma^{-1}\left(\frac{\rho}{2|x-y|}\right)^{\log_3 \gamma} = \gamma^{-1}\left(\frac{2|x-y|}{\rho}\right)^{\log_3 \frac{1}{\gamma}}$$

$$= C\,|x-y|^{\alpha} \Rightarrow (*).$$

Step 3
Let us show that u has a C^α-version. ~~Suffices~~
Suffices to prove for $u|_K$.

This version will be:

$$\hat{u}(x) := \lim_{r\to 0} \fint_{B_r(x)} u .$$

~~(**)~~: We'll show that this limit exists $\forall x\in K$.
By Lebesgue theorem about points of density,
$\hat{u}(x) = u(x)$ a.e, s.t. $\hat{u}$ is a version of u.
Finally, we'll show that $\hat{u}$ satisfies $(*)$. $\forall x,y\in K$, $|x-y|<\rho/4$.

Denote $u_r(x) = \fint_{B_r(x)} u$.

To show that $\lim_{r\to 0} u_r(x)$ exists, suffices
to show that $\lim_{r,r'\to 0} (u_r(x) - u_{r'}(x)) = 0$.

We have $u_r(x) = \frac{1}{|B_r|}\int_{B_r(x)} u(\xi)d\xi = \frac{1}{|B_r||B_{r'}|}\iint_{B_r\times B_{r'}} u(\xi)d\xi d\eta$

$$u_{r'}(x) = \frac{1}{|B_r||B_{r'}|}\int_{B_r\times B_{r'}} u(\eta)d\xi d\eta$$

$$\Rightarrow u_r(x) - u_{r'}(x) = \frac{1}{|B_r||B_{r'}|}\int_{B_r\times B_{r'}} (u(\xi)-u(\eta))d\xi d\eta .$$

$\xi\in B_r$,
$\eta\in B_{r'}$

Assume $r'<r \Rightarrow$
$$|u(\xi)-u(\eta)| \le C|\xi-\eta|^\alpha \|u\|_{L^2}$$
$$\le C r^\alpha \|u\|_{L^2}$$

$\Rightarrow \quad |u_r(x) - u_{r'}(x)| \le C\, r^{\alpha} \|u\|_{L^2} \to 0$ as $r \to 0$.

Hence $\tilde{u}(x) := \lim\limits_{r \to 0} u_r(x)$ is well defined $\forall x$.

~~We~~ Let us ~~show~~ estimate $u_r(x) - u_r(y)$.

Using
$$u_r(x) = \frac{1}{|B_r||B_r|} \int_{B_r(x) \times B_r(y)} u(\xi)\, d\xi\, d\eta$$

$$u_r(y) = \frac{1}{|B_r||B_r|} \int_{B_r(x) \times B_r(y)} u(\eta)\, d\xi\, d\eta$$

$$\Rightarrow \quad |u_r(x) - u_r(y)| \le \frac{1}{|B_r|^2} \int_{B_r(x) \times B_r(y)} |u(\xi) - u(\eta)|\, d\xi\, d\eta$$

If ~~x~~ $|x-y| < \rho/4$ and $r < \rho/8$

$\Rightarrow \quad |\xi - y| < \rho/2 \quad \Rightarrow$ for a.a. ξ, η

$$|u(\xi) - u(\eta)| \le C\, |\xi - \eta|^{\alpha} \|u\|_{L^2}$$
$$\le C\, (|x-y| + 2r)^{\alpha} \|u\|_{L^2}$$

$$\Rightarrow \quad |u_r(x) - u_r(y)| \le C\, ((|x-y| + 2r)^{\alpha} \|u\|_{L^2}.$$

as $r \to 0$ we obtain, $\forall x, y \in K$, $|x-y| < \rho/4$,

$$|\tilde{u}(x) - \tilde{u}(y)| \le C\, |x-y|^{\alpha} \|u\|_{L^2}.$$

Hence $\tilde{u}$ is a Hölder version of $u|_K$.

Rename $u = \tilde{u}$.

Finally, the estimate of the Hölder norm:

$$\|u\|_{C^\alpha(K)} = \sup_K |u| + \sup_{x,y\in K} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

For any $x \in K$ we have by theorem 1.1

$$|u(x)| \le \sup_{B_{\rho/2}(x)} |u| \le C \cdot \|u\|_{L^2(B_\rho(x))} \le C\,\|u\|_{L^2(\Omega)}.$$

If we restrict $\sup\limits_{x,y\in K}$ to $|x-y| < \rho/2$ then

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C\,\|u\|_{L^2} \quad \text{by } (*).$$

If $|x-y| \ge \rho/2$ then

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le \frac{2\sup_K |u|}{(\rho/2)^{\alpha/2}} \le C\,\|u\|_{L^2},$$

which finishes the proof of Thm 1.4.

2. Hölder cont. for non-div. form

Lect 2 (cont.)

$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ in $\Omega \subset \mathbb{R}^n$,

$a_{ij} = a_{ji}(x)$ – meas, uniformly elliptic:

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \le \lambda |\xi|^2.$$

Strong solution: $u \in W^{2,p}_{loc}(\Omega)$, $Lu = 0$. a.e.

Classical solution: $u \in C^2(\Omega)$; $Lu = 0$ poinwise.

Th 2.1 (Krylov-Safonov) If $Lu = 0$ in Ω, u – strong solution, ~~then~~ from $W^{2,p}_{loc}$ then $u \in C^\alpha(\Omega)$, $\alpha = \alpha(n, \lambda) > 0$. Moreover, for any convex compact set $K \subset \Omega$, $\|u\|_{C^\alpha(K)} \le C \|u\|_{W^{2,p}(\Omega)}$.

Th 2.1′ Assume $a_{ij} \in C^\infty$, u is a classical solution. Then

$$\|u\|_{C^\alpha(K)} \le C \|u\|_{C(\Omega)}.$$

where $\alpha = \alpha(n, \lambda) > 0$, $C = C(K, \Omega, n, \lambda)$.

Thm 2.1 can be obtained from Th 2.1′ as follows. For any strong solution u $\exists$ a sequence $\{u_k\}$ of C^∞-functions, s.t. $u_k \to u$ in $W^{2,p}_{loc}(\Omega)$ and each u_k solves $L^{(k)} u_k = 0$, where $L^{(k)} = \sum a_{ij}^{(k)} \frac{\partial^2}{\partial x_i \partial x_j}$ operator

with C^∞-coefficients $a_{ij}^{(k)}$ with ellipticity constant $\le 2\lambda$.

Then each u_k satisfies

$$\|u_k\|_{C^\alpha(K)} \le C \|u_k\|_{C(\Omega')} \le C \|u_k\|_{W^{2,p}(\Omega)}$$

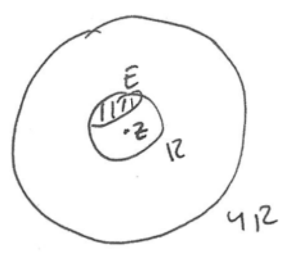
at least in the case $p > \frac{n}{2}$ by Sobolev embedding.

(case $p < \frac{n}{2}$ requires add. argument).

passing to limit as $k \to \infty$, we obtain the same for u.

We are going to prove Thm 2.1'. In fact, it will be sufficient to prove weak Harnack inequality. Then Hölder cont. follows in the same way, as in Ch 1 (since we use $\|u\|_{C(\Omega)}$ instead of $\|u\|_{L^2(\Omega)}$) the mean-value inequality is not needed here).

Thm 2.2 Let $Lu = 0$ in B_{4R}, $u \geq 0$. (all classical).



Set $E_a = \{u \geq a\} \cap B_R$,

If ~~ε =~~ $\frac{|E_a|}{|B_R|} \geq \varepsilon$,

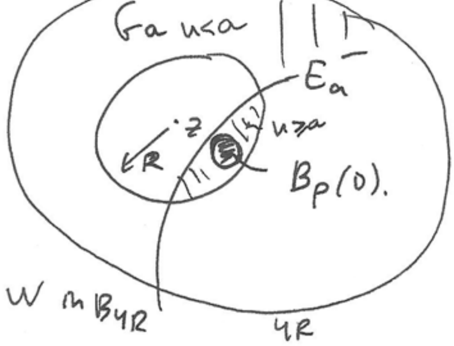
Then

$$\inf_{B_R} u \geq \delta a, \text{ where } \delta = \delta(n, \lambda, \varepsilon) > 0.$$

The proof in the sequence of Lemmas.

<u>Lemma 1</u>. If E_a contains a ball B_ρ of rad $=\rho$ then

$$\inf_{B_R} u \geq c\,a\cdot\left(\frac{\rho}{R}\right)^s, \quad s = s(n,\lambda),\ c = c(n,\lambda) > 0.$$



Let Ball B_R have center z,

Ball B_ρ have center 0.

Consider the set $G_a = \{u < a\}$ in B_{4R}

and construct a barrier function w in B_{4R}

s.t. $Lw \geq 0$ in B_{4R}, $w|_{\partial B_{4R}} \leq 0$, $w|_{\partial G_a} \leq a$.

If so, then in G_a we have $Lu = 0$, $u|_{\partial B_{4R}} \geq 0$,

$u|_{\partial G} = a \Rightarrow u \geq w$ in G_a

$\Rightarrow u \geq w$ on $G_a \cap B_R$,

which will give us lower bound for u.

Observe that

$$L\frac{1}{|x|^s} = s|x|^{-s-2}\left((s+2)\sum_{i,j=1}^{n} a_{ij}\frac{x_i x_j}{|x|^2} - \sum_{i=1}^{n} a_{ii}\right)$$

Since $\dfrac{a_{ij}x_i x_j}{|x|^2} \geq \lambda^{-1}$, $\sum a_{ii} \leq n\lambda$

$\Rightarrow L\frac{1}{|x|^s} > 0$ if ~~s is~~ $(s+2)\lambda^{-1} > n\lambda$, $s > n\lambda^2$.

~~Set $w = a\frac{\rho^s}{|x|^s}$. Then on outside $B_\rho(0)$, in particular, on G_a, we have $w \leq a$.~~

Now take

$$w(x) = a\rho^s\left(\frac{1}{|x|^s} - \frac{1}{(3R)^s}\right).$$

Clearly, $Lw > 0$. On ∂B_{4R} we have $|x| \geq 3R$

$\Rightarrow w(x) \leq 0$.

On G_a we have $|x| \geq \rho \Rightarrow w(x) \leq a$.

Hence, w satisfies all conditions, and we obtain on $G_a \cap B_R$

$$u(x) \geq w(x).$$

Since in B_R $|x| \leq 2R$, it follows that

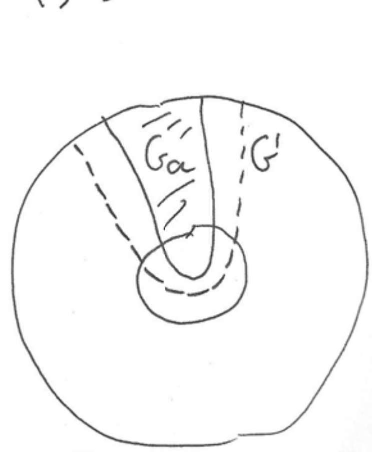
$$w(x) \geq a\rho^s\left(\frac{1}{(2R)^s} - \frac{1}{(3R)^s}\right) = c_s a\left(\frac{\rho}{R}\right)^s$$

$$\Rightarrow u(x) \geq c_s a\left(\frac{\rho}{R}\right)^s$$

in ~~$B_R \setminus G_a$~~ In $B_R \setminus G_a$ we have $u \geq a$, q.e.d.

<u>Lemma 2</u> (Key Lemma). If $\frac{|G_a|}{|B_{4R}|} < \varepsilon$ ~~is small enou~~

where $\varepsilon = \varepsilon(\lambda, n)$, then $\inf_{B_R} u \geq \frac{1}{2}a$.



Let G' be an open set around G_a,

s.t. $\frac{|G'|}{|B_{4R}|} < \varepsilon$; $f \in C^\infty(\overline{B_{4R}})$,

s.t. $f = 1$ in G_a, ~~f~~ $\operatorname{supp} f \subset \overline{G'}$.

We solve D. problem

$$\begin{cases} Lv = f & \text{in } B_{4R} \\ v = 0 & \text{on } \partial B_{4R} \end{cases}$$

Since $Lv \le 0 \Rightarrow v \ge 0$ in B_{4R}.

By Alexandrov–Pucci: $\sup v \le CR \cdot \|f\|_{L^n(B_{4R})}$

$$\le CR \cdot |G'|^{1/n}$$

$$\le CR\, \varepsilon^{1/n} R = CR^2 \varepsilon^{1/n}.$$

end of lect 2.

Consider function

$$w = c_1 - c_2 |x|^2 - c_3 v \quad (\text{make } z \to 0 \text{ here}).$$

c_i to be chosen.

Want: $Lw \ge 0$ in G, $w|_{\partial B_{4R}} \le 0$, $w|_G \le 1$

$$Lw = -2c_2 \sum a_{ii} + c_3 f$$

① On G_a we have $Lw = -2c_2 \sum a_{ii} + c_3 \overset{?}{\geq} 0$.

end of lect 2

$\boxed{c_3 \ge 2n c_2 \lambda.}$

② $w|_{\partial B_{4R}} \overset{?}{\le} 0$ on B_{4R} $w \le c_1 - c_2 (4R)^2 \overset{??}{\le} 0$ $\boxed{c_1 \le c_2 (4R)^2}$

③ $w|_{\partial_{int} G_a} \overset{?}{\le} a \Leftarrow \boxed{c_1 \le a}$: $c_1 = a$, $c_2 = \frac{a}{(4R)^2}$, $c_3 = n\lambda \frac{a}{8R^2}$.

Under this choice of c_1, c_2, c_3 we have

$Lw \ge 0$ in G_a, on ∂B_{4R} $w \le 0$, on $\partial_{int} G_a$ $w \le a$. Comparing with u, we obtain

$w \le u$ in $G_a \Rightarrow$ on $G_a \cap B_R$

$$u \ge \inf_{G_a \cap B_R} w \ge \inf_{B_R} w \ge c_1 - c_2 |R|^2 - c_3 v.$$

Lect 3 17.03.2016

<u>Recall</u>: $L = \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ in a domain $\Omega \subset \mathbb{R}^n$, where $a_{ij}(x) = a_{ji}(x)$, uniformly elliptic:

$$\lambda^{-1}|\xi|^2 \le \sum a_{ij}(x)\xi_i\xi_j \le \lambda |\xi|^2$$

and. $a_{ij}(x) \in C^\infty(\Omega)$.

Consider solution $u \in C^2$: $Lu = 0$ in Ω. Then we are proving Hölder estimate: $\forall K \Subset \Omega$ | Th 2.1'

$$\|u\|_{C^\alpha(K)} \le C \|u\|_{C(\Omega)}$$

$\alpha = \alpha(n,\lambda) > 0$, $C = C(K, \Omega, n, \lambda)$

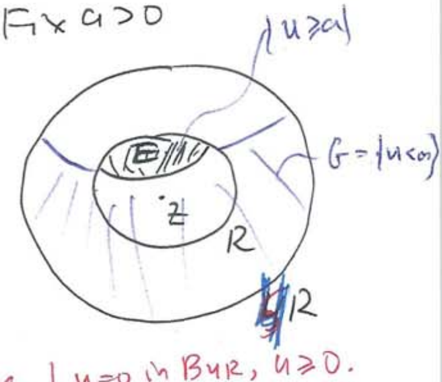
It suffices to prove weak Harnack inequality:

<u>Th 2.2</u>. Let $Lu = 0$ in B_{4R}, $u \ge 0$. Fix $a > 0$

Set $E = \{u \ge a\} \cap B_R$.

If $\frac{|E|}{|B_R|} \ge \theta \Rightarrow \inf_{B_R} u \ge \delta a$

where $\delta = \delta(n, \lambda, \theta) > 0$.



Next take $a \equiv 1$. In next lemmas, $Lu = 0$ in B_{4R}, $u \ge 0$.

<u>Lemma 1</u>. If E contains a ball of radius ρ,

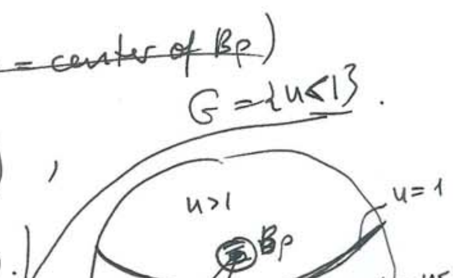
then $\inf_{B_R} u \ge c\left(\frac{\rho}{R}\right)^s$ In this case $\delta = c\,\theta^{s/n}$.

$s = s(n,\lambda) > 0$, $c = c(n,\lambda) > 0$.

~~Proof uses Barrier function~~ $\cancel{(\text{center } 0 = \text{center of } B_\rho)}$ $G = \{u < 1\}$.

$$w(x) = \rho^s\left(\frac{1}{|x|^s} - \frac{1}{(3R)^s}\right),$$

~~where s is chosen so big that~~ $Lw \ge 0$.



On ~~[illegible]~~ ∂B_{4R} we have $|x| \ge \rho$, on ∂B_{4R}:

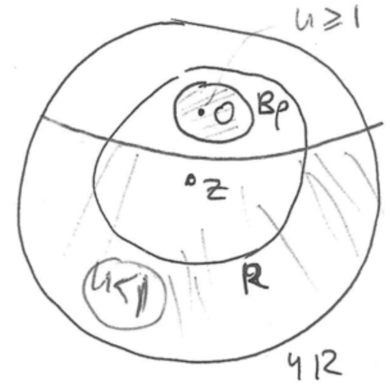
On $\partial G \Rightarrow w \le u$ in G

~~$|x| \ge 3R \Rightarrow w \le u$~~

$\Rightarrow$ the claim. | $w|_{\partial G} \le 1 = u$, $w|_{\partial B_{4R}} \le 0 \le u$.

Proof. Consider the set $G = \{u<1\}$ in $B_{4R}(z)$

Put the origin at the center of the ball of radius ρ.



Consider the barrier function

$$w(x) = \rho^s \left(\frac{1}{|x|^s} - \frac{1}{(3R)^s} \right).$$

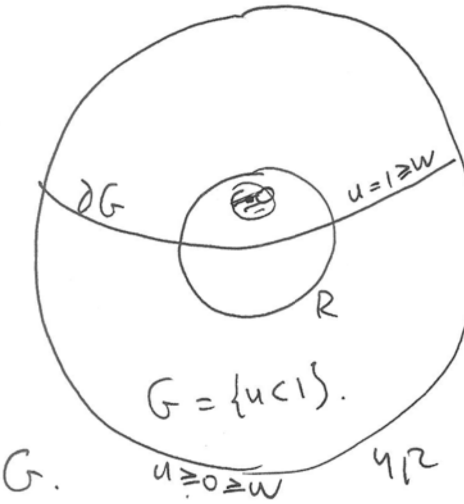
We have $Lw = \rho^s L \frac{1}{|x|^s}$.

Just by computation using explicit form of L, one finds that $L \frac{1}{|x|^s} > 0$ if $s > n\lambda^2$. Fix s.

Then $Lw > 0$ in B_{4R},

On ∂B_{4R} we have $|x| \geq 4R - R = 3R \Rightarrow w \leq 0$.

On G: $|x| \geq \rho \Rightarrow w \leq 1$.



Hence, in G: $Lw \geq Lu$,

On $\partial G \cap B_{4R}$: $u = 1 \geq w$

On ∂B_{4R}: $u \geq 0 \geq w$.

By max. principle, $u \geq w$ in ~~B_R~~ G.

Then $\inf_{B_R} u = \inf_{B_R \cap G} u \geq \inf_{B_R \cap G} w \geq \inf_{B_R} w$.

Since ~~$w \leq \rho^s \ldots$~~ on B_R $|x| \leq 2R$, $\Rightarrow$ on B_R

$$w(x) \geq \rho^s \left(\frac{1}{(2R)^s} - \frac{1}{(3R)^s} \right) = c_s \left(\frac{\rho}{R} \right)^s \Rightarrow \inf_{B_R} u \geq c \left(\frac{\rho}{R} \right)^s.$$

<u>Lemma 2</u> (Key Lemma) If $\frac{|G|}{|B_{4R}|} < \varepsilon$,

where $\varepsilon = \varepsilon(\lambda, n) > 0$, then

$$\inf_{B_R} u \geq \frac{1}{2}.$$

Let G' be an open set around G
such that $\frac{|G'|}{|B_{4R}|} < \varepsilon$. $(*)$

Choose $f \in C^\infty(\overline{B}_{4R})$,
s.t. $f = 1$ on G and $f = 0$ outside G'; <u>$0 \leq f \leq 1$.</u>

Solve the Dirichlet problem:

$$\begin{cases} Lv = -f & \text{in } B_{4R} \\ v = 0 & \text{on } \partial B_{4R}. \end{cases}$$

Since $Lv \leq 0 \Rightarrow v \geq 0$ in B_{4R}.

<u>Alexandrov–Pucci</u>:

$$\sup v \leq CR \|f\|_{L^n(B_{4R})}$$

$$\leq CR|G'|^{1/n} \qquad \underline{\text{using } (*)}.$$

$$\leq CR^2 \varepsilon^{1/n}.$$

Consider function (center at the center of B_R)

$$\boxed{w(x) = C_1 - C_2|x|^2 - C_3 v(x)},$$

where $C_1, C_2, C_3 > 0$ to be chosen.

<u>Want</u>. $Lw \geq 0$ in $\cancel{B_{4R}}G$ | $\Rightarrow \begin{cases} Lw \geq Lu \\ w \leq u \text{ on } \partial B_{4R} \\ w \leq u \text{ on } \partial G \cap B_{4R} \end{cases}$

$w|_{\partial B_{4R}} \leq 0$

$w|_G \leq 1$ | $\Rightarrow w \leq u$ in G.

① $Lw = -2c_2 \sum_{i=1}^n a_{ii} + c_3 f \geq c_3 f - 2\lambda n c_2 \underset{\text{on } G}{\geq} c_3 - 2\lambda n c_2$.

$\boxed{c_3 \geq 2\lambda n c_2}$

② on ∂B_{4R} : $|x| = 4R$,

$w(x) \leq c_1 - c_2(4R)^2$

$\boxed{c_1 \leq c_2(4R)^2}$

Need to satisfy:

③ in G: $w(x) \leq c_1$

$\boxed{c_1 \leq 1}$

Hence, take $c_1 = 1$, $c_2 = \frac{1}{(4R)^2}$, $c_3 = \frac{2\lambda n}{(4R)^2} = \frac{\lambda n}{8R^2}$.

Under this choice we have $u \geq w$ in G,

in particular,

$$\inf_{B_R} u = \inf_{G \cap B_R} u \geq \inf_{G \cap B_R} w \geq \inf_{B_R} w \geq$$

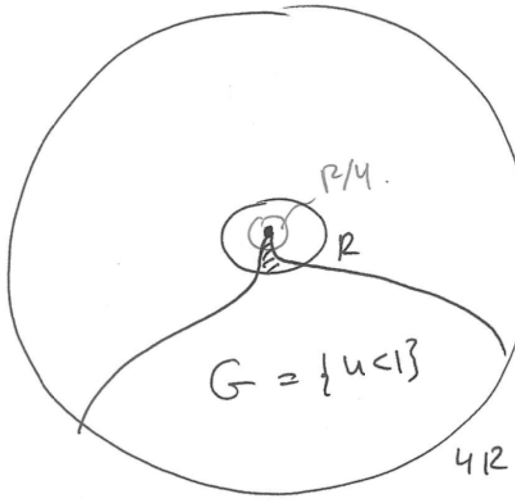
$$\geq c_1 - c_2 R^2 - c_3 \sup v$$

$$\geq c_1 - c_2 R^2 - c_3 c R^2 \varepsilon^{\gamma_n}$$

$$= 1 - \frac{1}{16} - \frac{\lambda n}{8} c \varepsilon^{\gamma_n}.$$

Clearly, if ε is small enough, $\varepsilon = \varepsilon(\lambda, n)$,

then $\inf_{B_R} u \geq \frac{1}{2}$, q.e.d.



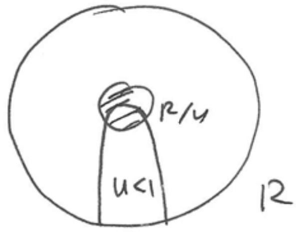
<u>Lemma 3</u>. If

$$\frac{|G \cap B_R|}{|B_R|} < \varepsilon := \varepsilon(n,\lambda)$$

then $\inf_{B_R} u \geq \gamma = \gamma(n,\lambda)$.

<u>Proof</u>. Let ε be from L. 2

Applying L 2 in Balls $B_{R/4}$, B_R,



we obtain

$$\inf_{B_{R/4}} u \geq \frac{1}{2}.$$

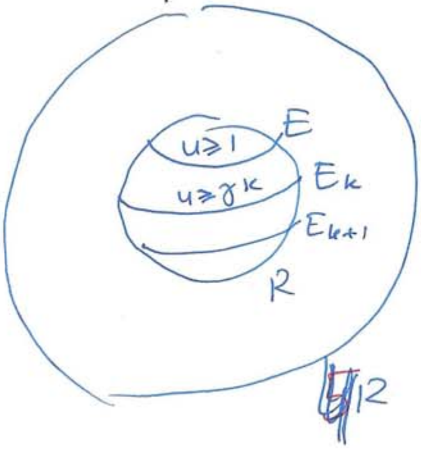
Now the set $\{u \geq \frac{1}{2}\} \cap B_R$ contains ball of radius $R/4$.

By Lemma 1 $\inf_{B_R} u \geq c\left(\frac{R/4}{R}\right)^{s} \cdot \frac{1}{2} =: \gamma$ does not depend on R.

Proof of Th 2.2 (weak Harnack):

$Lu=0$ in B_{4R}, $u \geq 0$. Set $E = \{u \geq 1\} \cap B_R$.

If $\frac{|E|}{|B_R|} \geq \theta \Rightarrow \inf_{B_R} u \geq \delta := \delta(n,\lambda,\theta) > 0$.



Consider the sets:

$$E_k = \{u \geq \gamma^k\} \cap B_R, \; k=0,1,2\ldots$$

$E_0 = E$, $E_k \uparrow B_R$ $k \to \infty$.

γ is from L.3, $\gamma = \gamma(n,\lambda) < 1$.

Main claim.

For any $k = 0, 1, 2 \ldots$ we have

either $\quad |E_{k+1}| \geq (1+\beta)|E_k| \qquad (1)$

$\beta = \beta(n,\lambda) > 0$,

or $\quad E_{k+l} = B_R \qquad (2)$

where $l = l(n,\lambda,\theta)$ - positive integer.

Suppose that we know already this claim.

Then weak Harnack is proved as follows.

~~Case 1. Assume that (1) does not hold for k=0,~~

~~then (2) holds, that is~~ $\cancel{E_l = B_R}$

$\cancel{\Rightarrow \inf_{B_R} u \geq \gamma^l =: \delta}$

Since (1) cannot hold for all k, there is a minimal $k = N$ s.t. (1) does not hold.

Then (1) holds for $k = 0, \dots N-1$, so that

$$|B_R| \geq |E_N| \geq (1+\beta)|E_{N-1}| \geq \dots \geq (1+\beta)^N |E_0|$$

$$\Rightarrow \quad (1+\beta)^N \leq \frac{|B_R|}{|E|} \leq \frac{1}{\theta}$$

$$\Rightarrow \quad N \leq \frac{\log \frac{1}{\theta}}{\log(1+\beta)}.$$

On the other hand, for $k = N$ we have (2),

that is $\quad E_{N+\ell} = B_R,$

$$\inf_{B_R} u \geq \gamma^{N+\ell} \geq \gamma^{\ell + \frac{\log \frac{1}{\theta}}{\log(1+\beta)}} =: \delta.$$

Proof of the main claim.

We prove it for $k = 0$:

either $\quad |E_1| \geq (1+\beta)|E_0|$

or $\quad E_\ell = B_R.$

For general k consider function $v = u/\gamma^k$.

Then $E_k = \underbrace{\{v \geq 1\} \cap B_R}_{E_0 \text{ for } v}$, $\quad E_{k+1} = \underbrace{\{v \geq \gamma\} \cap B_R}_{E_1 \text{ for } v}$

$E_{k+\ell} = \underbrace{\{v \geq \gamma^\ell\} \cap B_R}_{E_\ell \text{ for } v}$, and $\dfrac{|E_0(v)|}{|B_R|} = \dfrac{|E_k|}{|B_R|} \geq \dfrac{|E_0|}{|B_R|} \geq \theta.$

Hence, general k reduces to $k=0$.

Reformulate the claim again:

either $|E_1| \geq (1+\beta)|E_0|$

or $\inf_{B_R} u \geq \delta \quad (=\gamma^\ell)$

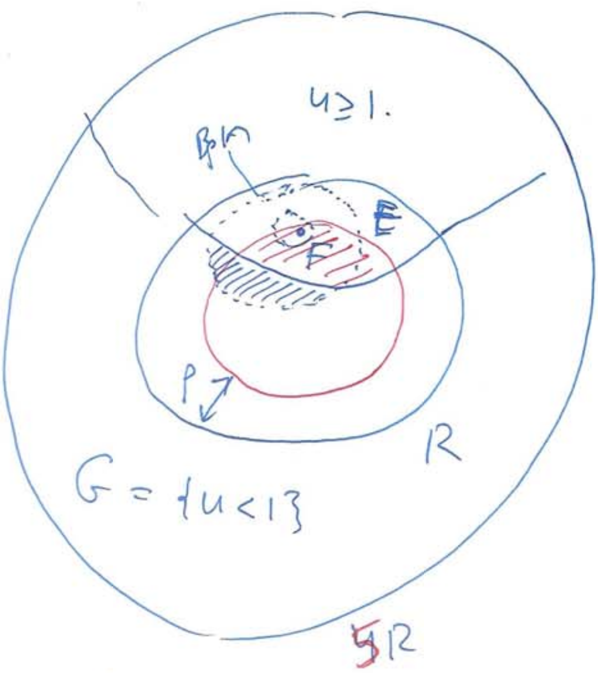
Choose $\rho < R$ s.t.

$$|E \cap B_{R-\rho}| = \frac{1}{2}|E|$$

and set $F = E \cap B_{R-\rho}$.

that is: $F = \{u \geq 1\} \cap B_{R-\rho}$

We consider two cases.



Case 1. Let $\exists x \in F$ s.t.

$$|G \cap B_\rho(x)| \leq \varepsilon |B_\rho(x)|$$

where ε is from L. 2.

By L 2 $\quad \inf_{B_{\rho/4}(x)} u \geq \frac{1}{2}$.

Note that $B_{\rho/4}(x) \subset B_R$, s.t. in B_R there is a ball of radius $\rho/4$ where $u \geq \frac{1}{2}$.

By L. 1: $\quad \inf_{B_R} u \geq \frac{c}{2}\left(\frac{\rho/4}{R}\right)^s$.

Let us estimate ρ from below:

$$|B_R| - |B_{R-\rho}| \geq \frac{1}{2}|E| \geq \frac{1}{2}\theta |B_R|$$

$$\Rightarrow \quad 1 - \left(\frac{R-\rho}{R}\right)^n \geq \frac{1}{2}\theta$$

$$\left(1-\frac{\rho}{R}\right)^n \leq 1 - \frac{1}{2}\theta$$

$$\frac{\rho}{R} \geq 1 - \sqrt[n]{\left(1-\frac{1}{2}\theta\right)}$$

$$\Rightarrow \quad \inf_{B_R} u \geq \delta = \delta(n, \lambda, \theta).$$

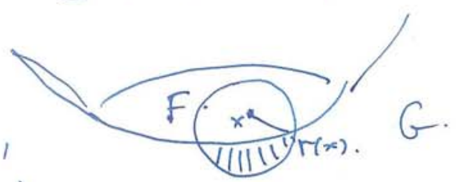
<u>Case 2</u> (main) Assume $\forall x \in F$

$$|G \cap B_\rho(x)| \geq \varepsilon \, |B_\rho(x)|.$$

For any $x \in F$ and $r > 0$, consider the quotient:

$$\frac{|G \cap B_r(x)|}{|B_r(x)|}.$$

As $r \to 0$, this $\to 0$ for almost all $x \in F$ (because $F \subset G^c$). On the other hand, for $r = \rho$ this is $\geq \varepsilon$. Therefore, for almost all $x \in F$ $\exists\, r(x) \in (0, \rho)$, s.t. this quotient $= \varepsilon$.

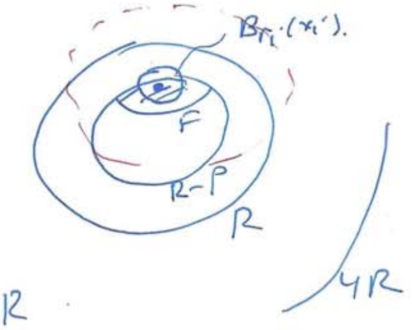


Denote this set of points x by F', s.t. $F' \subset F$, $|F'| = |F|$. Choose a compact subset $K \subset F'$ s.t. $|K| \geq \frac{1}{2}|F'|$.

Then $\{B_{r(x)}(x)\}$ is an open covering of K.

Choose a finite subcover $\{B_{r_i}(x_i)\}$, $r_i = r(x_i)$.

By standard ball covering argument, we can further select a subsequence of the balls, s.t. $\{B_{r_i}(x_i)\}$ is disjoint, while $\{B_{3r_i}(x_i)\}$ cover K.



Observe that $x_i \in B_{R-\rho}$,

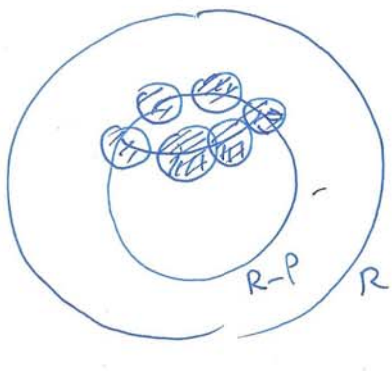
$$|x_i| \le R-\rho$$

$$|x_i| + 4r_i \le R-\rho+4r_i \le R+3\rho \le R+3R \le 4R$$

$$\Rightarrow B_{4r_i}(x_i) \subset B_{4R}.$$

We apply in $B_{4r_i}(x_i)$ Lemma 3, because

$$\frac{|\overset{=\{u<1\}}{G} \cap B_{r_i}(x_i)|}{|B_{r_i}(x_i)|} \overset{(\le)}{=} \varepsilon.$$

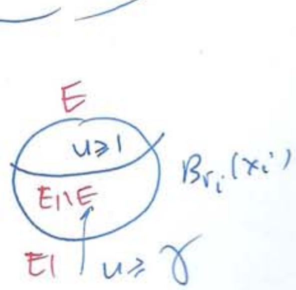


Hence, by Lemma 3

$$\inf_{B_{r_i}(x_i)} u \ge \gamma$$

All balls $B_{r_i}(x_i)$ lie in B_R

$$\Rightarrow B_{r_i}(x_i) \subset E_1 = \{u \ge \gamma\} \cap B_R.$$



$$(E_1 \setminus E) \cap B_{r_i}(x_i) = \{u<1\} \cap B_{r_i}(x_i)$$

$$|(E_1 \setminus E) \cap B_{r_i}(x_i)| = \varepsilon \, |B_{r_i}(x_i)|$$

$$|E_1 \setminus E| \geq \sum_i \varepsilon\, |B_{r_i}(x_i)| \geq \varepsilon \cdot c \sum_i |B_{3r_i}(x_i)|$$

$$\geq \varepsilon \cdot c\, |K| \geq \varepsilon \tfrac{c}{2} |F|$$

$$\geq \varepsilon \tfrac{c}{4} |E|$$

$$\Rightarrow \quad |E_1| \geq \left(1 + \varepsilon \tfrac{c}{4}\right) |E|.$$

which finishes the proof.

Lect. 4 24.03 2016

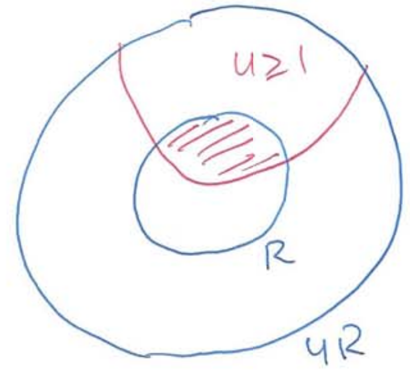
3. Harnack inequality.

Let L be one of the operators $\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ or $\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$, let L be uniformly elliptic with the ellipticity constant $=\lambda$.

We have proved the following w. Harnack inequality:

If $Lu=0$ in B_{4R} and $u\geq 0$ then $\forall \theta>0$

$$\frac{|\{u\geq 1\}\cap B_R|}{|B_R|}\geq \theta \Rightarrow \inf_{B_R} u \geq \delta=\delta(\theta,n,\lambda)>0.$$



Note also that if $Lu=0$, then also $L(au+b)=0$ for arbitrary $a, b\in\mathbb{R}$.

We already have used this to derive from w. Harnack, that all solutions are Hölder continuous.

Now we use w. Harnack to prove the full Harnack inequality.

 Harnack inequality:

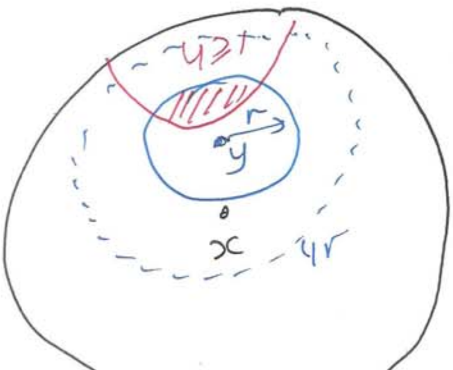
Theorem 3.1 If $Lu=0$ in B_{2R} and $u \geq 0$

then $\sup_{B_R} u \leq C \inf_{B_R} u$, $C = C(n,\lambda)$.

Lemma 1. Let $Lu=0$, $u \geq 0$ in $B_R(x)$

Consider a ball $B_r(y) \subset B_R(x)$

where $y \in B_{\frac{1}{9}R}(x)$ and $r \leq \frac{2}{9}R$.



If $\dfrac{|\{u \geq 1\} \cap B_r(y)|}{|B_r(y)|} \geq \theta$

then $u(x) \geq \left(\dfrac{r}{R}\right)^{s} \delta$

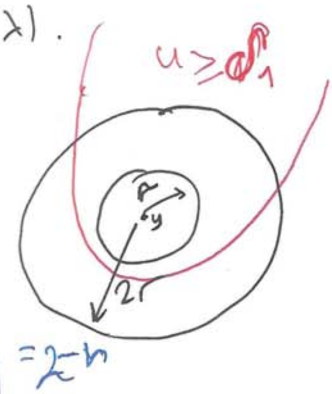
where $s = s(n,\lambda) > 0$ $\delta = \delta(\theta, n, \lambda) > 0$.

Proof. We have $B_{4r}(y) \subset B_R(x)$

because $|x-y| + 4r < \frac{1}{9}R + \frac{8}{9}R = R$.

Applying w. Harnack in $B_r(y)$, we obtain:

$$\inf_{B_r(y)} u \geq \delta_1 := \delta(\theta, n, \lambda).$$



It follows that

$$\frac{|\{u \geq \delta_1\} \cap B_{2r}(y)|}{|B_{2r}(y)|} \geq \frac{|B_r|}{|B_{2r}|} = 2^{-n}$$

If $B_{8r}(y) \subset B_R(x)$ then using w. Harnack for $\frac{u}{\delta_1}$, we obtain:

$$\inf_{B_{2r}(y)} u \geq \delta_1 \cdot \underbrace{\delta(2^{-n}, n, \lambda)}_{=:\,\varepsilon} = \varepsilon \delta_1 .$$

Hence,

$$\frac{|\{u \geq \varepsilon\delta_1\} \cap B_{4r}(y)|}{|B_{4r}(y)|} \geq \frac{|B_{2r}|}{|B_{4r}|} = 2^{-n},$$

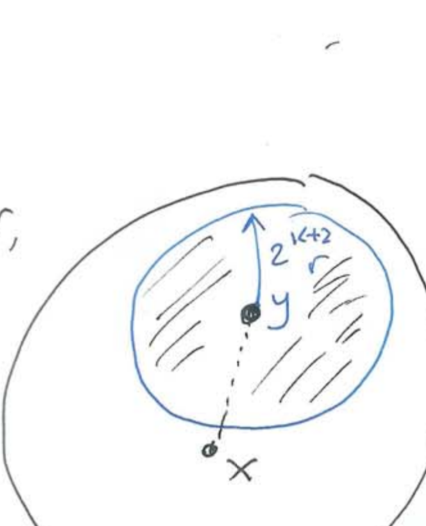
and if $B_{16r}(y) \subset B_R(x)$, then we obtain by w. Harnack

$$\inf_{B_{4r}(y)} u \geq (\varepsilon\delta_1)\,\varepsilon = \varepsilon^2 \delta_1 .$$

Continue by induction, we see that as long as $B_{2^{k+2}r}(y) \subset B_R(x)$, we have

$$(*) \qquad \inf_{B_{2^k r}(y)} u \geq \varepsilon^k \delta_1 .$$

Let $k \geq 0$ be the maximal number, s.t. $B_{2^{k+2}r}(y) \subset B_R(x)$.



Then $|x-y| + 2^{k+2}r \leq R$

while $|x-y| + 2^{k+3}r > R$.

It follows that $2^{k+3} r > R - |x-y|$

$$2^k r > \frac{R-|x-y|}{8} \geq |x-y|$$

because of $|x-y| < \frac{1}{9} R$. It follows, that for this k $x \in B_{2^k r}(y)$. Then $(*)$ implies

$$u(x) \geq \varepsilon^k \delta_1 .$$

On the other hand, we have

$$2^k r < |x-y| + 2^{k+2} r \leq R$$

$$\Rightarrow \quad k \leq \log_2 \frac{R}{r}$$

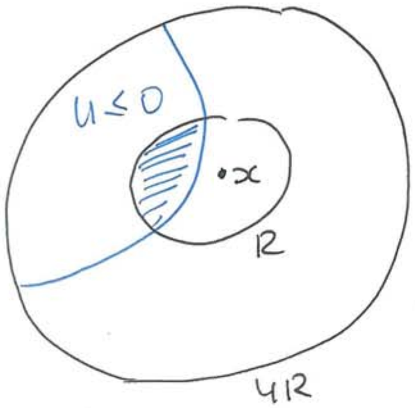
$$\Rightarrow \quad u(x) \geq \varepsilon^{\log_2 \frac{R}{r}} \delta_1 = \delta_1 \left(\frac{R}{r}\right)^{\log_2 \varepsilon} = \delta_1 \cdot \left(\frac{r}{R}\right)^{\log_2 \frac{1}{\varepsilon}}$$

$$\Rightarrow \quad u(x) \geq \delta_1 \left(\frac{r}{R}\right)^{s} \text{ with } s = \log_2 \frac{1}{\varepsilon}$$

Next Lemma is a reformulation of w. Harnack inequality.

<u>Lemma 2</u>. Let u be a solution of $Lu=0$ in $B_{4R}(x)$. If $\frac{|\{u\leq 0\}\cap B_R|}{|B_R|}\geq \theta>0$

then $\sup_{B_{4R}} u \geq (1+\delta)u(x)$, where $\delta=\delta(\theta,n,\lambda)>0$.



<u>Proof</u> If $u(x)\leq 0$ then nothing to prove, since $(1+\delta)u(x)\leq u(x)$.
Assume $u(x)>0$. By rescaling, we assume $\sup_{B_{4R}} u = 1$.
Consider function $v=1-u$, that is nonnegative in B_{4R} and $Lv=0$.
Observe that $u\leq 0 \iff v\geq 1$. Hence, we have $\frac{|\{v\geq 1\}\cap B_R|}{|B_R|}\geq \theta$,

which by w. Harnack gives $v(x)\geq\delta$
$(\Leftarrow \inf_{B_R} v\geq \delta)$. It follows:

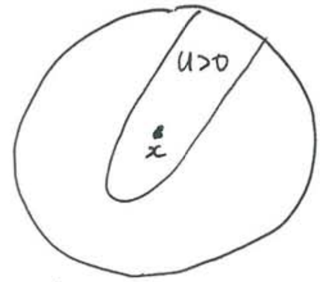
$u(x)\leq 1-\delta < \frac{1}{1+\delta} \Rightarrow$

$\sup_{B_{4R}} u = 1 \geq (1+\delta)u(x)$, q.e.d.

Lemma 3 (Lemma of growth)

$\exists\, \varepsilon>0$, $\varepsilon = \varepsilon(n,\lambda)$, such that, for any solution $Lu=0$ in a ~~any~~ ball $B_R(x)$,

if $\frac{|\{u>0\}\cap B_R|}{|B_R|} \leq \varepsilon \implies \sup\limits_{B_R} u \geq 4u(x)$



Remark. In the case of L of non-divergence form, this Lemma coinsides with L.2 of Sect 2. Indeed, assume $\sup\limits_{B_R} u = 1$ and consider $v = 1-u$. Then $v\geq 0$, $Lv=0$ in B_R, and $\frac{|\{v<1\}\cap B_R|}{|B_R|} \leq \varepsilon$

$\implies$ by Lemma 2 from Sect 2, that

$$\inf_{B_{\frac{1}{4}R}} v \geq \frac{1}{2} \implies v(x) \geq \frac{1}{2}$$

$\implies u(x) \leq \frac{1}{2}$, $\sup\limits_{B_R} u \geq 2u(x)$.

By modifying the proof, one can make $4u(x)$.

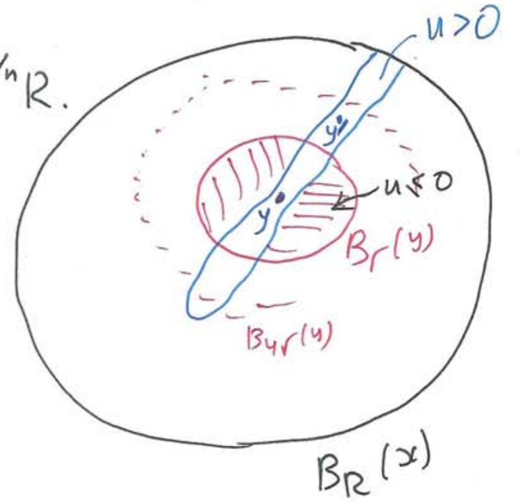
Corollary. If $\frac{|\{u>a\}\cap B_R|}{|B_R|} \leq \varepsilon$

$\implies \sup\limits_{B_R} u \geq a + 4(u(x)-a)$

Just use L.3 for $u-a$.

Proof of L 3 (needed only for div. form).
The value of $\varepsilon > 0$ will be determined later.
Consider some ball $B_r(y) \subset B_R(x)$,

s.t. $\dfrac{|B_r|}{|B_R|} = 2\varepsilon \Leftrightarrow r = (2\varepsilon)^{1/n} R$.



Then
$$\frac{|\{u>0\} \cap B_r(y)|}{|B_r|} \leq \frac{|\{u>0\} \cap B_R(x)|}{|B_R|} \times$$
$$\times \frac{|B_R|}{|B_r|} \leq \varepsilon \cdot \frac{1}{2\varepsilon} = \frac{1}{2}.$$

Hence, $\dfrac{|\{u \leq 0\} \cap B_r(y)|}{B_r(y)} \geq \dfrac{1}{2}$.

By Lemma 2:
$$\sup_{B_{4r}(y)} u \geq (1+\delta) u(y)$$
provided $B_{4r}(y) \subset B_R(x)$.
We obtain the following claim:

Claim: If $B_{4r}(y) \subset B_R(x)$ where $r = (2\varepsilon)^{1/n} R$,
then $\exists\, y' \in \overline{B_{4r}}(y)$ s.t.
$$u(y') \geq (1+\delta) u(y)$$
↑ slightly reduce δ

$\delta = \delta(n, \lambda) > 0$.

Let us apply this claim for $y = x$.

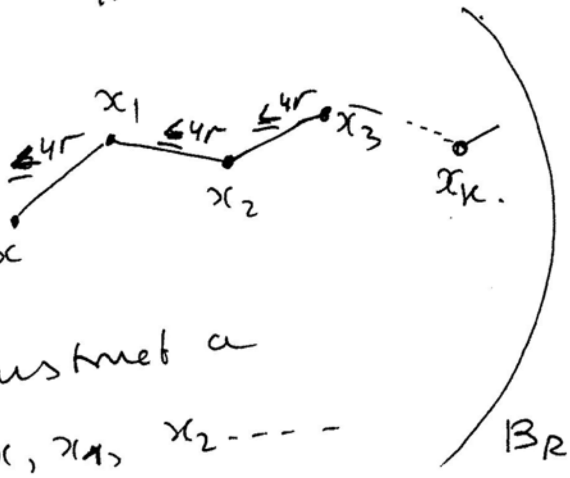
We obtain that $\exists\, x_1 \in \bar{B}_{4r}(x)$, s.t. $u(x_1) \geq (1+\delta)u(x)$.

Applying claim to $y = x_1$, obtain

$\exists\, x_2 \in \bar{B}_{4r}(x_1)$, s.t. $u(x_2) \geq (1+\delta)u(x_1)$,

etc. We can continue this procedure,

as long as $B_{4r}(x_k) \subset B_R(x)$.



Then we construct a

sequence $x_0 = x, x_1, x_2 \ldots$

s.t. $|x_{k+1} - x_k| \leq 4r$

and $u(x_{k+1}) \geq (1+\delta)u(x_k)$,

provided $B_{4r}(x_k) \subset B_R(x)$.

It follows, that $|x_k - x| < 4rK$

and $u(x_k) \geq (1+\delta)^k u(x)$.

Cleary, if $4rK < R$ then x_k exists in this sequence. Choose max K with this property. Then $4r(K+1) \geq R$, $K \geq \frac{R}{4r} - 1$

$$K \geq \frac{1}{4(2\varepsilon)^{1/n}} - 1.$$

We obtain

$$\sup_{B_R(x)} u \geq u(x_k) \geq (1+\delta)^k u(x)$$

$$\geq (1+\delta)^{\frac{1}{4(2\varepsilon)^{1/n}} - 1} u(x).$$

Choosing $\varepsilon > 0$ small enough, we obtain

$$\sup_{B_R(x)} u \geq 4\, u(x).$$

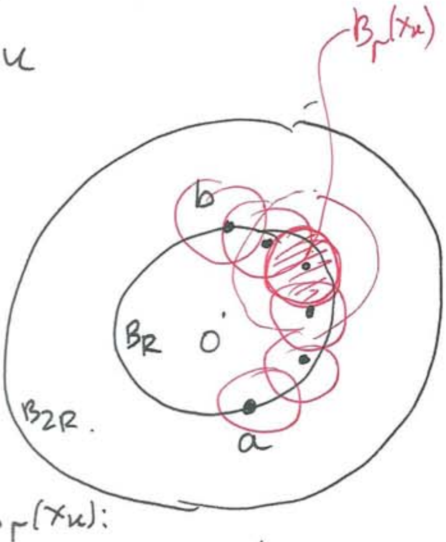
<u>Proof of theorem 3.1</u>. We prove equivalent form: if $u \geq 0$, $Lu = 0$ in $B_{KR}(x)$, where K is large enough, then

$$\sup_{B_R(x)} u \leq C\, u(x)$$

If one has $u \geq 0$, $Lu = 0$ in $B_{2R}(0)$ then choose points $a, b \in \overline{B_R}(0)$, where

$$u(a) = \sup_{B_R} u, \quad u(b) = \inf_{B_R} u$$

and construct a sequence $\{x_k\}_{k=0}^{N}$ of points in $B_R(0)$, s.t. $x_0 = a$, $x_N = b$, $|x_{k+1} - x_k| \leq r := \frac{R}{K}$.



Then $B_{Kr}(x_k) = B_R(x_k) \subset B_{2R}(0)$ and by the above version in $B_r(x_k)$:

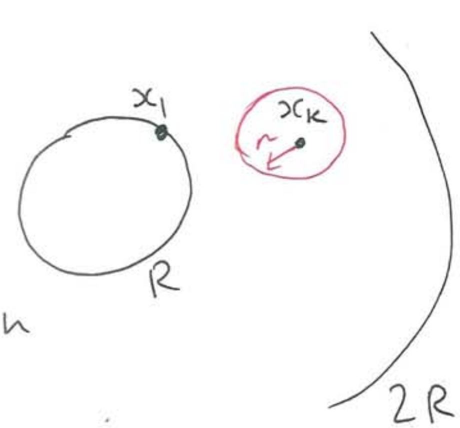
$$u(x_{k+1}) \leq C u(x_k) \Rightarrow u(a) \leq C^N u(b).$$

Since $N = N(K, n)$, $\Rightarrow$ we obtain full Harnack.

Now let us prove the K-form of Harnack.
Assume without loss of generality that
$\sup_{B_R} u = 2$, and prove that $u(x) \geq c = c(n,\lambda) > 0$.

For that let us construct a sequence
of points $\{x_k\}_{k\geq 1}$ s.t. $u(x_k) = 2^k$, $x_k \in B_{2R}(x)$.

A point x_1 with $u(x_1) = 2$ exists in $\overline{B}_R(x)$
by assumption $\sup_{B_R} u = 2$



Let $x_k \in B_{2R}$ with
$u(x_k) = 2^k$ be already
constructed. For small enough
$r > 0$, we have
$$\sup_{B_r(x_k)} u \leq 2^{k+1}.$$

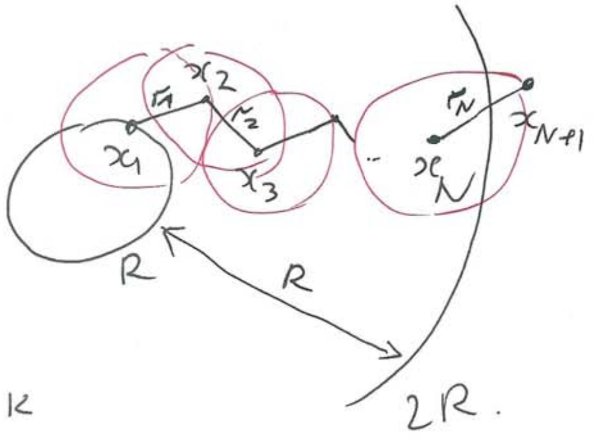
Set
$$r_k = \sup\left\{ r \in (0,R] : \sup_{B_r(x_k)} u \leq 2^{k+1} \right\}.$$

If $r_k = R$ then we stop inductive process
without constructing x_{k+1}. If $r_k < R$
then we have
$$\sup_{B_{r_k}(x_k)} u = 2^{k+1}.$$

Then $\exists\, x_{k+1} \in \overline{B}_{r_k}(x_k)$ s.t. $u(x_{k+1}) = 2^{k+1}$.

However, if x_{k+1} then we disregard x_{k+1} and stop the process.

Hence, we obtain a sequence of balls $\{B_{r_k}(x_k)\}$, s.t. $r_k \le R$,



$x_k \in B_{2R}(x)$, $\quad u(x_k) = 2^k$

and $\quad \sup\limits_{B_{r_k}(x_k)} u \le 2^{k+1}$. $\quad$ Moreover, $|x_{k+1} - x_k| \le r_k$.

This sequence cannot be infinite because $u(x_k) \to \infty$. Let N be the last k in this sequence. Then either $r_N = R$ or $x_{N+1} \notin B_{2R}$. In the both cases we have

$$\boxed{r_1 + r_2 + \dots + r_N \ge R. \qquad (*)}$$

In any ball $B_{r_k}(x_k)$ we have

$$\sup_{B_{r_k}(x_k)} u \le 2^{k+1} < \underbrace{2^{k-1}}_{a} + 4(\underbrace{2^k}_{u(x)} - \underbrace{2^{k-1}}_{a})$$

$$= a + 4(u(x) - a).$$

Corollary of

By Lemma 3: $\quad \dfrac{|\{u > a\} \cap B_{r_k}(x_k)|}{|B_{r_k}|} > \varepsilon$

that is, $\dfrac{|\{u \geq 2^{k-1}\} \cap B_{r_k}(x_k)|}{|B_{r_k}|} \geq \varepsilon$

$\Rightarrow$ Lemma 1 $\quad u(x) \geq \left(\dfrac{r_k}{R}\right)^s \delta \cdot 2^{k-1}$.

where $\delta = \delta(\varepsilon, n, \lambda)$, $s = s(n, \lambda)$.

How to ~~set~~ estimate $r_k^s \cdot 2^{k-1}$ from below?

<u>Trick of Landis</u>:

$(*) \Rightarrow \exists\, k$ s.t. $\quad r_k \geq \dfrac{R}{k(k+1)}$

because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.

Then for this k we obtain

$$u(x) \geq \frac{\delta\, 2^{k-1}}{[k(k+1)]^s} \geq \delta \underbrace{\left(\inf_{k \geq 1} \frac{2^{k-1}}{[k(k+1)]^s}\right)}_{>0\,!} =: c;$$

which finishes the proof.