Math. Notes, 48 (1990), no. 5-6

# DIMENSION OF SPACES OF HARMONIC FUNCTIONS

A. A. Grigor'yan

Suppose M is a smooth connected non-compact Riemannian manifold. Let B(M) be the space of bounded harmonic functions on M and DB(M) the space of bounded harmonic functions whose Dirichlet integral is finite. In this article we study dimension of spaces B(M) and DB(M). Since constant functions belong to both of these spaces, they are at least one-dimensional. If B(M) is one-dimensional then the two-sided Liouville theorem holds, i.e., every bounded harmonic function on M is constant. If DB(M) is one-dimensional then the following so-called D-Liouville theorem holds: every harmonic function on M with a finite Dirichlet integral is constant [1]. If these Liouville theorems are not satisfied it is then natural to ask the question about dimension of spaces B(M) and DB(M).

Dimension of the space B(M) has been studied in numerous articles for various classes of manifolds. For example, Anderson [2] and Sullivan [3] proved that if M is a Cartan-Hadamard manifold then $\dim B(M) = \infty$. On the other hand, if M is a complete manifold with non-negative Ricci curvature outside a compact set then $\dim B(M) < \infty$ (see [4, 5]). A somewhat more general situation is discussed in [6].

In contrast to the mentioned articles (and many others), we do not restrict the manifold M a priori in any way. We define massive and D-massive subsets of M and prove that B(M) (respectively, $\dim DB(M)$) is equal to the maximal number of pairwise non-intersecting massive (respectively, D-massive) subsets of M.

To effectively use the stated theorem we need criteria of massivity and D-massivity of sets. We proved in [1] a criterion of D-massivity in terms of capacity (there we also proved a particular case of our main theorem, namely we cited conditions for which $\dim B = 1$, $\dim DB = 1$). In particular, it implies that the dimension of the space DB(M) is an invariant under quasi-isometric mappings.

At present there is no effective criterion of massivity.

We note that Lyons [7] recently proved that $\dim B(M)$ is not in general an invariant under quasi-isometries.

We now state the exact formulations. A harmonic function on M is called a smooth solution of an equation $\Delta u = 0$, where $\Delta$ is the Laplace oeprator associated with the Riemannian metric on the manifold M. If manifold M has a boundary then in the definition of a harmonic function we also require that Neuman's condition is satisfied on the boundary $\partial M$, i.e., $\partial u/\partial \omega|_{\partial M} = 0$, where $\nu$ is the normal to $\partial M$.

Volgograd State University. Translated from Matematicheskie Zametki, Vol. 48, No. 5, pp. 55-61, November, 1990. Article submitted April 18, 1988.

 0001-4346/90/4856-1114$12.50 © 1991 Plenum Publishing Corporation

A continuous function u defined on some open set $\Omega \subset M$ is called subharmonic (superharmonic) if, for every domain $G \Subset \Omega$ and a harmonic function, $v \in C(\bar{G})$, $u|_{\partial G} = v|_{\partial G}$ implies $u \le v$ in G (respectively, $u \ge v$).

Definition. An open proper subset $\Omega \subset M$ is called massive if there is a subharmonic function $u \in C(\bar{\Omega})$ such that $u|_{\partial\Omega} = 0$, $0 \le u \le 1$, $u \not\equiv 0$. Such function u is called an inner potential of the set $\Omega$. If the inner potential $u \in W_{2,loc}{}^{1}(\Omega)$ and

$$D(u) \equiv \int_{\Omega} |\nabla u|^2 < \infty,$$

then $\Omega$ is called D-massive.

Clearly, by applying the principle of maximum for subharmonic functions, we see that a massive set is not precompact.

We will need the following useful property of massive sets.

LEMMA 1. Suppose $\Omega_1 \subset \Omega_2$ are open proper subsets of M. Then

a) if $\Omega_1$ is massive (D-massive) then $\Omega_2$ is also massive (respectively, D-massive);

b) if $\Omega_2$ is massive (D-massive) and $\bar{\Omega}_2 \setminus \Omega_1$ compact then $\Omega_1$ is also massive (respectively, D-massive).

Proof. First of all, we note that if u is inner potential of an open set $\Omega$ then by extending u outside $\Omega$ with zero we obtain a subharmonic function on the entire manifold M, which will also be called inner potential of $\Omega$ and also denoted by u.

a) If u is an inner potential then u is also an inner potential of $\Omega_2$.

b) Suppose u is an inner potential of $\Omega_2$ such that $\sup u = 1$. The strict maximum principle then implies that

$$m \equiv \sup_{\bar{\Omega}_2 \setminus \Omega_1} u < 1.$$

Then a function $(u - m)_+$ is an inner potential of $\Omega_1$. Clearly, if $D(u) < \infty$, then $D((u - m)_+) < \infty$.

We now prove our main result.

THEOREM. Let $m \ge 2$ be a natural number. The following statements are equivalent:

1) $\dim B(M) \ge m$ ($\dim DB(M) \ge m$);

2) there exist m pairwise non-intersection massive (respectively, D-massive) subsets of M.

COROLLARY 1. A manifold M satisfies the two-sided Liouville theorem (respectively, the D-Liouville theorem) if and only if every two massive (respectively, D-massive) subsets have a non-empty intersection.

This assertion is obtained from theorem 1 by letting $m = 2$. We proved it using a different method in [1].

COROLLARY 2. If manifolds $M_1$ and $M_2$ are such that if the exteriors of some compactums $K_1$ in $M_1$ and $K_2$ in $M_2$ are isometric then $\dim DB(M_1) = \dim DB(M_2)$, $\dim B(M_1) = \dim B(M_2)$.

Indeed, if $d_1 \equiv \dim B(M_1) \ge m$, then there exist m non-intersection massive sets $\Omega_1, \ldots, \Omega_m$ in $M_1$. Then Lemma 1 implies that sets $\Omega_i \setminus K_1$ are also massive. Their isometric images in $M_2 \setminus K_2$ are massive and do not intersect, so therefore $d_2 \equiv \dim B(M_2) \ge m$. Since this applies for all m, we have $d_2 \ge d_1$. We similarly prove the inequality the other way, obtaining $d_2 = d_1$. We similarly prove that $\dim DB(M_2) = \dim DB(M_1)$.

COROLLARY 3. The dimension of the space DB(M) does not change under quasi-isometric mappings of the manifold M.

Indeed, as shown in [1], the notion of D-massivity is an invariant under quasi-isometric mappings, which implies the above result.

Proof of Theorem. 2) ⇒ 1). Suppose $\Omega_1, \ldots, \Omega_m$ are pairwise non-intersection massive subsets of M with inner potentials $u_1, u_2, \ldots, u_m$, respectively. We prove that $\dim B(M) \geq m$, and if $\Omega_1, \ldots, \Omega_m$ are D-massive, then $\dim DB(M) \geq m$.

Let $\{B_k\}$ be an exhaustion of the manifold M by precompact domains with smooth boundaries (transversal to $\partial M$ if the boundary is not empty). We solve the following boundary value problems in $B_k$:

$$\Delta v_k^{(i)} = 0, \quad v_k^{(i)}|_{\partial B_k} = u_i, \quad \left.\frac{\partial v_k^{(i)}}{\partial \nu}\right|_{\partial M \cap B_k} = 0$$

(recall that $u_i = 0$ outside $\Omega_i$). Since $u_i$ is subharmonic, we have $v_k^{(i)} \geq u_i$ in $B_k$. Therefore, in $\partial B_k$ we have $v_{k+1}^{(i)} \geq v_k^{(i)} = u_i$, and the maximum principle implies that $v_{k+1}^{(i)} \geq v_k^{(i)}$ in $B_k$. Furthermore, $u_i \leq 1$ implies $v_k^{(i)} \leq 1$. Therefore, a sequence of harmonic functions $\{v_k^{(i)}\}$ $(k = 1, 2, \ldots)$ increases and is bounded. Consequently, a limit

$$v^{(i)} = \lim_{k\to\infty} v_k^{(i)},$$

exists and is a harmonic function in M. In addition, $1 \geq v^{(i)} \geq u_i \geq 0$. We can assume that $\sup u_i = 1$. Then we also have $\sup v^{(i)} = 1$. We prove that harmonic functions $v^{(1)}, v^{(2)}, \ldots, v^{(m)}$ are linearly independent, in which case $\dim B(M) \geq m$. To do this, we note that $\Omega_i \cap \Omega_j = \varnothing$ (for $i \neq j$) implies that $u_i + u_j \leq 1$. Therefore, $v_k^{(i)} + v_k^{(j)} \leq 1$ and

$$v^{(i)} + v^{(j)} \leqslant 1. \tag{1}$$

We now use (1) and the fact that $\sup v^{(i)} = 1$ to prove that $v^{(i)}$ $(i = 1, 2, \ldots, m)$ are linearly independent. Indeed, for every $\varepsilon > 0$ we can find a point $x_i \in M$ such that

$$v^{(i)}(x_i) > 1 - \varepsilon.$$

Inequality (1) then implies that $v^{(j)}(x_i) < \varepsilon$. Since we also have $v^{(j)}(x_i) \geq 0$, a matrix

$$\| v^{(i)}(x_j) \|_{i,j=1}^{m}$$

for sufficiently small $\varepsilon$ is non-degenerate (since the numbers on its diagonal are close to 1, and off-diagonal numbers are close to 0). Thus, functions $v^{(i)}$ $(i = 1, 2, \ldots, m)$ are linearly independent.

If $\Omega_i$ are D-massive then

$$\int_{\Omega_i} |\nabla u_i|^2 < \infty.$$

Dirichlet's principle implies that

$$\int_{B_k} |\nabla v_k^{(i)}|^2 \leqslant \int_{B_k} |\nabla u_i|^2.$$

Letting $k \to \infty$, we obtain

$$\int_M |\nabla v^{(i)}|^2 \leqslant \int_M |\nabla u_i|^2 = \int_{\Omega_i} |\nabla u_i|^2 < \infty,$$

i.e., $v^{(i)} \in DB(M)$, $\dim DB(M) \geq m$.

1) ⇒ 2). Suppose in M there are m linearly independent functions $u_i \in B(M)$. We prove that there are m pairwise non-intersection massive sets. Let $\hat{M}$ be the Cech compactification of the manifold M, i.e., $\hat{M}$ is a compact topological space such that M is an open, everywhere dense subset of $\hat{M}$ and every continuous bounded function on M can be continuously extended to $\hat{M}$. Let $\mu = \hat{M}\setminus M$, and extend functions $u_i$ to $\hat{M}$ by setting them equal to functions $f_i$ on $\mu$. Then $f_1, f_2, \ldots, f_m$ are continuous, linearly independent functions on $\mu$. Indeed, if $k_1f_1 + k_2f_2 + \ldots + k_mf_m = 0$ for some constants $k_1, k_2, \ldots, k_m$, then a harmonic function $u = k_1u_1 + k_2u_2 + \ldots + k_mu_m$ is equal to zero on $\mu$. The maximum principle implies that $u \equiv 0$ on M. The linear independence of functions $u_1, u_2, \ldots, u_m$ implies that $k_1 = k_2 = \ldots = k_m = 0$, i.e., $f_1, f_2, \ldots, f_m$ are linearly independent.

We could have chosen the desired massive sets as $\{x: u_i(x) > \sup u_i - \varepsilon\}$ $(i = 1, 2, \ldots, m)$, if for some $\varepsilon > 0$ they were pairwise non-intersection (note that a non-empty set $\{u_i > a\}$ is massive with an inner potential $(u_i - a)_+$). The latter is equivalent to a condition that the set sof points in $\mu$ at which $f_i(x) = \sup f_i$, are pairwise non-intersecting. However, this is not always the case. We circumvent this difficulty by using the following lemma.

LEMMA 2. Suppose $\mu$ is a compact topological space, $f_1, f_2, \ldots, f_m$ are linearly independent continuous functions on $\mu$. Then there exist functions $F_1, F_2, \ldots, F_m$, which are linear combinations of $f_1, f_2, \ldots, f_m$, such that sets $u_i = \{x \in \mu: F_i(x) = \max F_i\}$ are pairwise nonintersecting.

The proof of Lemma 2 is given below, after the completion of the proof of Theorem.

Since functions $F_i$ are linear combinations of functions $f_1, f_2, \ldots, f_m$, there exist functions $v_1, \ldots, v_m$, which are linear combinations of $u_1, u_2, \ldots, u_m$ such that $v_i|_\mu = F_i$. Clearly, $v_i \in B(M)$.

If, in addition, we have $D(u_i) < \infty$, then $D(v_i) < \infty$, i.e., $v_i \in DB(M)$.

Let $\Omega_i^\varepsilon = \{x \in M: v_i(x) > \max F_i - \varepsilon\}$. Clearly, for every $\varepsilon > 0$ the set $\Omega_i^\varepsilon$ is massive (and if $v_i \in DB(M)$, then it is D-massive).

We prove that for sufficiently small $\varepsilon > 0$ these sets are pairwise non-intersecting. Assuming the opposite, we have $\Omega_i^\varepsilon \cap \Omega_j^\varepsilon \neq \emptyset$ for some $i \neq j$ and $\varepsilon = \varepsilon_k$ $(k = 1, 2, \ldots)$, where the sequence $\{\varepsilon_k\}$ tends to zero as $k \to \infty$. Let $x_k$ be a point in $\Omega_i^{\varepsilon_k} \cap \Omega_j^{\varepsilon_k}$. As $k \to \infty$, the sequence $\{x_k\}$ has a limit point $x_0 \in \hat{M}$. Clearly, $v_l(x_0) = \max F_l = \sup v_l$, $l = i, j$. If $x_0 \in M$ then the strict maximum principle implies that $v_i = \text{const}$, $v_j = \text{const}$, which in turn implies that $F_i = \text{const}$, $F_j = \text{const}$, which contradicts the fact that functions $F_i$ and $F_j$ do not have common maximum points. If $x_0 \in \mu$, then $x_0$ is a common maximum point of functions $F_i$ and $F_j$, which again contradicts their choice.

Thus, for some $\varepsilon > 0$, sets $\Omega_i^\varepsilon$ $(i = 1, 2, \ldots, m)$ are pairwise non-intersecting and massive (D-massive), as required.

Proof of Lemma 2: Define a mapping $I: \mu \to \mathbb{R}^m$ as follows:

$$I(x) = (f_1(x), f_2(x), \ldots, f_m(x)).$$

Since I is a continuous mapping, its image $K = I(\mu)$ is compact in $\mathbb{R}^m$. We show that K is not contained in any $(m - 2)$-dimensional plane in $\mathbb{R}^m$. If that is not the case then K and the origin are contained in a hyperplane defined by

$$a_1X_1 + a_2X_2 + \ldots + a_mX_m = 0,$$

where $X_1, \ldots, X_m$ are moving coordinates in $\mathbb{R}^m$. In particular, for every $x \in \mu$ we have

$$a_1f_1(x) + a_2f_2(x) + \ldots + a_mf_m(x) = 0,$$

which contradicts the linear independence of functions $f_1, \ldots, f_m$.

A point $z \in K$ is called a support point if there exists a strictly supporting hyperplane P containing the point z, i.e., a hyperplane such that $K\backslash\{z\}$ lies strictly to one side of P. It is known that every compactum in $\mathbb{R}^m$, is contained in a closed convex envelope of its support points (see [8]). Therefore, K has at least m support points. Indeed, if there are no more than $m - 1$ support points, then their closed convex envelope, along with the compactum K, is contained in some $(m - 2)$-dimensional plane, which contradicts the above results. Thus, there are m different support points in K, say $z_1, z_2, \ldots, z_m$. Let $P_1, P_2, \ldots, P_m$ be the corresponding strictly supporting hyperplanes. Suppose $P_i$ is defined by an equation $l_i(X) = c_i$, where $l_i(X)$ is a linear function in $\mathbb{R}^m$ and $c_i = \text{const}$. The signs of $l_i$ and $c_i$ are chosen such that over K we have $l_i(X) \le c_i$. We assert that functions $F_i = l_i \circ I$ are the desired ones on $\mu$. Indeed, functions $l_i$ are linear combinations of coordinate functions $X_1, \ldots, X_m$, so therefore $F_i$ are linear combinations of functions $X_j \circ I = f_j$ on $\mu$. Furthermore, since $z_i$ is a support point, it is the only maximum point of the function $l_i$ on K. The maximum points of $F_i$ on $\mu$ are preimages $I^{-1}(z_i)$, which clearly are pairwise non-intersection for $i = 1, 2, \ldots, m$. Q.E.D.

Example. Suppose M is an unbounded closed region in $\mathbf{R}^n$ ($n \geq 3$) with a smooth boundary (regarded as a manifold with a boundary). Let

$$F = \{x \in \mathbf{R}^n : x_n > 0, \sqrt{x_1^2 + \ldots + x_{n-1}^2} < f(x_n)\},$$

where the continuous function f on $[0, +\infty)$ is such that

$$\int^{\infty} f(x)^{n-3}\,dx < \infty, \quad n > 3;$$

$$\int^{\infty} \frac{dx}{\ln(1 - x/f(x))} < \infty, \quad n = 3.$$

Suppose a set M\F has m connected components $\Omega_1, \ldots, \Omega_m$, each of which contains an infinite cone.† Then every uniformly elliptic equation

$$\sum_{i,j=1}^{n} \partial/\partial x_i (a_{ij}(x)\,\partial u/\partial x_j) = 0 \tag{2}$$

with smooth coefficients has at least m linearly independent bounded solutions in M which have a finite Dirichlet integral and satisfy Neuman's condition on the conormal on $\partial M$.

Indeed, sets $\Omega_1, \ldots, \Omega_m$ are D-massive in the manifold M with the Euclidean metric of $\mathbf{R}^n$ [1]. Let M* be a manifold equal to M as a set with a Riemannian metric such that Eq. (2) is Laplace's equation. Since (2) is uniformly elliptic, manifolds M and M* are quasi-isometric. Our theorem dictates that $\dim DB(M) \geq m$, so therefore Corollary 3 implies that $\dim DB(M^*) \geq m$, as desired.

In conclusion, we would like to thank E. M. Landis and N. S. Nadirashvili for their useful discussion of problems addressed in this article.

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†We mean a one-sided cone with a directing (n − 1)-sphere.