Dirichlet heat kernel in the exterior of a compact set

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1 Introduction

Let M be a complete non-compact Riemannian manifold. For a compact set $K \subset M$, denote $\Omega = M \setminus K$ and consider the Dirichlet heat kernel $p_\Omega(t,x,y)$ in Ω. By definition, p_Ω as a function

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of t, x is a minimal positive solution of the following mixed problem in Ω:

$$\begin{cases} \partial_t u = \Delta u \\ u|_{\partial\Omega} = 0 \\ u|_{t=0} = \delta_y, \end{cases}$$

where Δ is the Laplace-Beltrami operator on M and δ_y is the Dirac function. The purpose of this work is to obtain estimates of p_Ω away from the boundary $\partial\Omega$. Surprisingly enough, the answer is non-trivial even if $M = \mathbb{R}^n$, $n > 1$, and K is the unit ball.

Let us denote by $p(t, x, y)$ the global heat kernel on M which is by definition the minimal positive fundamental solution to the heat equation on M. The comparison principle easily implies $p_\Omega \le p$. It is natural to ask whether p_Ω is substantially smaller than p or is comparable to p, when staying away from $\partial\Omega$. The answer depends on the property of the manifold to be parabolic or not. We say that a Riemannian manifold M is *parabolic* if the Brownian motion $(X_t)_{t\ge0}$ on M is recurrent, and *non-parabolic* if the Brownian motion is transient (for example, $\mathbb{R}^2$ is parabolic but $\mathbb{R}^3$ is not). If M is non-parabolic then there is a positive probability that $(X_t)_{t\ge0}$ will never hit $\partial\Omega$ started in Ω, which suggests that p and p_Ω should be comparable. If M is parabolic then $(X_t)_{t\ge0}$ hits $\partial\Omega$ with probability 1. Hence, one expects that the probability of getting from x to y without touching $\partial\Omega$ may be significantly smaller than in the absence of $\partial\Omega$, that is $p_\Omega \ll p$.

One of the main achievements of this work is to prove and quantify the validity of the above heuristic when M has non-negative Ricci curvature (and even under more general a priori assumptions). Denote by $d(x, y)$ the geodesic distance between points $x, y \in M$. Let $B(x, r)$ be the geodesic ball of radius r centered at x, and let $V(x, r)$ be its volume. Li and Yau [20] proved the following heat kernel estimate for complete manifolds having non-negative Ricci curvature:

$$p(t, x, y) \asymp \frac{1}{V(x, \sqrt{t})} e^{-d^2(x,y)/t}, \tag{1.1}$$

for all $t > 0$ and $x, y \in M$. Here the sign $\asymp$ means the following: we write

$$f(t, x, y) \asymp g(t, x, y) e^{-d^2(x,y)/t}$$

if, for some positive constants c_1, C_1, c_2, C_2,

$$c_1 g(t, x, y) e^{-C_1 d^2(x,y)/t} \le f(t, x, y) \le C_2 g(t, x, y) e^{-c_2 d^2(x,y)/t},$$

for all t, x, y in the specified domains (this is a rather informal notation, but the context should easy lead to the correct interpretation). It is easy to show that a manifold M admitting the estimate (1.1) is parabolic if and only if

$$\int^\infty \frac{dt}{V(x, \sqrt{t})} = +\infty. \tag{1.2}$$

Our main result in the non-parabolic case is as follows.

Theorem 1.1 *Let M be a complete non-compact Riemannian manifold having non-negative Ricci curvature. Let K be a compact set with non-empty interior and $\Omega = M \setminus K$. Assume that M is non-parabolic. Then*

$$p_\Omega(t, x, y) \asymp \frac{1}{V(x, \sqrt{t})} e^{-d^2(x,y)/t}, \tag{1.3}$$

for all $t > 0$ and all x, y which are far enough from K.

In other words, we can write in this case

$$cp(Ct,x,y) \le p_\Omega(t,x,y) \le p(t,x,y),$$

for some positive constants $c, C > 0$.

To state our result in the parabolic case, we need the following notation. For a given compact set K, fix a point $o \in K$ and denote

$$|x| := d(x,K) = \inf\{d(x,y) : y \in K\},$$

$$H(r) := 1 + \int_0^r \frac{se^{-1/s}}{V(o,s)} ds, \tag{1.4}$$

$$D(t,x,y) := \frac{H(|x|)H(|y|)}{(H(|x|)+H(\sqrt{t}))(H(|y|)+H(\sqrt{t}))}, \tag{1.5}$$

where $t > 0$ and $x, y \in M$. It is possible to prove that if M is non-parabolic then $H(r)$ stays between two positive constants, and hence so does $D(t,x,y)$. On the contrary, if M has non-negative Ricci curvature and is parabolic then by (1.2) $H(r)$ is unbounded which implies that $\inf D(t,x,y) = 0$.

Given a point $o \in M$, we say that a pointed manifold (M,o) satisfies the condition (RCA) if there exists $A > 1$ such that, for any $r > A^2$ and all $x, y \in \partial B(o,r)$, there exists a continuous path in $B(o,Ar) \setminus B(o,A^{-1}r)$ connecting x to y. Here (RCA) stands for *Relatively Connected Annuli.*

Theorem 1.2 *Let M be a complete non-compact Riemannian manifold with non-negative Ricci curvature. Let K be a compact set with non-empty interior and $\Omega = M \setminus K$. Assume that M is parabolic and satisfies* (RCA)*. Then*

$$p_\Omega(t,x,y) \asymp \frac{D(t,x,y)}{V(x,\sqrt{t})} e^{-d^2(x,y)/t}, \tag{1.6}$$

for all $t > 0$ and all x, y with large enough $|x|, |y|$.

Alternatively, we can write in this case

$$c_1 D(t,x,y) p(C_1 t,x,y) \le p_\Omega(t,x,y) \le C_2 D(t,x,y) p(c_2 t,x,y),$$

for some positive constants C_i, c_i. Since the heat kernel $p_\Omega(t,x,y)$ is symmetric in x, y, all the estimates (1.1), (1.3), (1.6) can be symmetrized. For example, the latter yields

$$p_\Omega(t,x,y) \asymp \frac{D(t,x,y)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} e^{-d^2(x,y)/t}. \tag{1.7}$$

Let us illustrate this estimate by examining two very concrete two-dimensional cases, the plane $\mathbb{R}^2$, and the half cylinder.

Example 1 Let $M = \mathbb{R}^2$, and denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^2$. Set $K = \overline{B(0,1)}$ and $\Omega = \mathbb{R}^2 \setminus K$, so that $\|x\| = 1 + |x|$. Since $V(o,r) = \pi r^2$, an easy computation shows that, for all $r > 0$,

$$H(r) \approx 1 + \log(1 + r)$$

(where $\approx$ means that the ratio of the left- and right-hand sides remains bounded both from above and below). Hence, for all x, y with large enough $\|x\|, \|y\|$ and all $t > 0$, (1.7) implies

$$p_\Omega(t,x,y) \asymp \frac{\log \|x\| \log \|y\|}{t(\log(1+\sqrt{t}) + \log \|x\|)(\log(1+\sqrt{t}) + \log \|y\|)} e^{-\|x-y\|^2/t}. \tag{1.8}$$

Let us recall for comparison the classical Gauss-Weierstrass formula for the heat kernel in $\mathbb{R}^2$:

$$p(t,x,y) = \frac{1}{4\pi t} \exp\left(-\frac{\|x-y\|^2}{4t}\right).$$

In the long time asymptotic regime when x and y remain fixed and $t \to +\infty$, we obtain

$$p_\Omega(t,x,y) \approx \frac{\log \|x\| \log \|y\|}{t \log^2 t}. \tag{1.9}$$

In the medium time asymptotic regime when $\|x\| \approx \|y\| \approx \sqrt{t}$ and $t \to +\infty$, (1.8) implies

$$p_\Omega(t,x,y) \approx t^{-1} \approx p(t,x,y). \tag{1.10}$$

Indeed, when the points x and y are at distance $\sqrt{t}$ from K, the Brownian path between x and y almost does not see the boundary before time t. On the contrary, the factor $\log^2 t$ in (1.9) quantifies the effect of the boundary in the long term.

We have not been able to find the estimate (1.8) in the literature although the question of estimating p_Ω in $\mathbb{R}^2$ is very natural and should have come up long ago. Of course, it is easy to expand p_Ω in terms of Bessel functions, using separation of variables, but this expansion does not suggest anything like (1.8). Furthermore, our method works also for all Riemannian metrics in $\mathbb{R}^2$, which are in finite ratio with the Euclidean metric and for which separation of variables is not possible.

The result closest to ours that we are aware of is the estimate of Murata [22, Theorem 4.1] for the heat kernel $q(t,x,y)$ of the Schrödinger operator $\Delta - Q(x)$ where $Q \geq 0$, $Q \not\equiv 0$, is a Hölder continuous function in $\mathbb{R}^2$ with a compact support. Murata proved the following asymptotic of q as $t \to +\infty$:

$$q(t,x,y) \sim \frac{\phi(x)\phi(y)}{t \log^2 t}, \tag{1.11}$$

where $\phi(x) \sim \frac{1}{2\pi} \log \frac{\|x\|}{2}$ as $x \to \infty$. It is plausible that (1.11) could be extended to the hard-core potential Q in K, that is $Q = +\infty$ in K and $Q = 0$ outside K, in which case $q = p_\Omega$. Then (1.11) would yield a sharper long time behavior for p_Ω than (1.9), but not the uniform estimate (1.8).

For simplicity of notation, the next example is presented in the context of manifolds with boundary. If manifold M has a boundary then we denote it by δM and impose the Neumann boundary condition on δM for p and p_Ω.

Example 2 Let M be the half-cylinder $\mathbb{S}^1 \times [0, +\infty)$ equipped with the product Riemannian structure. Let $K = \delta M = \mathbb{S}^1 \times \{0\}$ and, hence, $\Omega = \mathbb{S}^1 \times (0, +\infty)$. Then $V(o,r) \approx \min(r, r^2)$, $H(r) \approx 1 + r$, and the Dirichlet heat kernel on Ω satisfies

$$p_\Omega(t,x,y) \asymp \frac{|x||y| e^{-d^2(x,y)/t}}{\sqrt{t}\left(\sqrt{t} + |x|\right)\left(\sqrt{t} + |y|\right)}, \tag{1.12}$$

for all $t > 0$ and x, y with large enough $|x|$ and $|y|$. Here $|x| = d(x, \delta M)$ is the radial coordinate on the half-cylinder M. In particular, in the long time regime, we obtain $p_\Omega(t,x,y) \approx |x||y| t^{-3/2}$.

Of course, there is a well-known exact formula for p_Ω in the case of the semi-axis $(0, +\infty)$, and the behavior of p_Ω in this case is essentially the same as in the case of the half-cylinder (cf. the discussion in [4]).

Note that in Examples 1, 2 (as well as in Theorems 1.1, 1.2) we assume that x, y stay away from K. It is possible to show that the particular estimates (1.8) and (1.12) hold true up to the boundary ∂K. This requires additional arguments exploiting the regularity of the boundary of K, but we will not dwell on this in the present paper.

A simple interpretation of the above results is obtained if one considers the *heat content* of Ω at time t for a unit mass of heat originally concentrated at x, that is

$$\mathbf{C}_\Omega(t,x) := \int_\Omega p_\Omega(t,x,y)dy,$$

where dy stands for the Riemannian volume.

Theorem 1.3 *Let M be a complete non-compact Riemannian manifold with non-negative Ricci curvature. Let K be a compact set with non-empty interior and set $\Omega = M \setminus K$. The following estimates are true for all $t > 0$ and $x \in M$ with large enough $|x|$:*

(1) *If M is non-parabolic then $\mathbf{C}_\Omega(t,x) \approx 1$.*

(2) *If M is parabolic and satisfies the condition* (RCA) *then*

$$\mathbf{C}_\Omega(t,x) \approx \frac{H(|x|)}{H(|x|) + H(\sqrt{t})}.$$

As a consequence, we can rewrite the estimates of Theorems 1.1 and 1.2 in the following form:

$$p_\Omega(t,x,y) \asymp \frac{\mathbf{C}_\Omega(t,x)\mathbf{C}_\Omega(t,y)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} e^{-d^2(x,y)/t}. \tag{1.13}$$

The proofs presented in this paper do not directly use the Ricci curvature assumption. In fact, we will show that Theorems 1.1, 1.2 hold true for any manifold which is quasi-isometric (even roughly-isometric, under any reasonable bounded local geometry assumption) to a manifold with non-negative Ricci-curvature. In particular, the bounds of Examples 1, 2, hold true if the Laplace operator is replaced by a uniformly elliptic operator in divergence form. Thus, the bounds stated above are reasonably stable. The adequate hypothesis for our purpose is expressed in terms of a parabolic Harnack inequality or, equivalently, in terms of certain Poincaré inequality and volume growth (see below Section 2.2).

The present work originated from our desire to understand the behavior of the heat kernel on manifolds with more than one ends. Indeed, together with good estimates of certain hitting probabilities obtained in [14], the result presented here is one of the main building blocks in the proof of the sharp estimates for the heat kernel on manifolds with ends that have been announced in [11] and are proved in [12]. The following result complements Theorems 1.1, 1.2 in this direction.

Given a Riemannian manifold with k ends, let U be a relatively compact open set in M with smooth boundary such that $M \setminus U$ has exactly k unbounded connected components $E_1, \ldots, E_k$. Let $K_i = \partial U \cap E_i$, and consider E_i as a manifold with boundary $\delta E_i := K_i$. Denote by p_i the heat kernel on E_i and by p_{Ω_i} the Dirichlet heat kernel on $\Omega_i = E_i \setminus K_i$ (in other words, p_i satisfies the Neumann condition on K_i, whereas p_{Ω_i} satisfies the Dirichlet condition on K_i). Let also $V_i(x,t)$ be the volume function on E_i. For each end E_i, fix a point $o_i \in K_i$ and define the functions H_i, D_i relative to E_i by (1.4), (1.5), using V_i instead of V.

Let us say that manifold M has *asymptotically non-negative sectional curvature* if there exists a continuous decreasing non-negative function $\lambda(r)$ such that $\int^\infty r\lambda(r)dr < +\infty$ and a point $o \in M$ such that the sectional curvature of M at distance r from o is bounded below by $-\lambda(r)$. For example, one can take $\lambda(r) = r^{-(2+\varepsilon)}$ for any $\varepsilon > 0$.

Theorem 1.4 *Let M be a complete Riemannian manifold satisfying one of the following two conditions:*

1. *M has non-negative Ricci curvature outside a compact set and finite topological type.*
2. *M has asymptotically non-negative sectional curvature.*

Then M has finitely many ends. Furthermore, referring to the notation introduced above, we have the following estimates, for each end E_i:

(i) For all $t > 0$ and $x, y \in E_i$,

$$p_i(t,x,y) \asymp \frac{1}{V_i(x,\sqrt{t})} e^{-d^2(x,y)/t}.$$

(ii) For all $t > 0$ and $x, y \in E_i$ with large enough $|x|, |y|$,

$$p_{\Omega_i}(t,x,y) \asymp \frac{D_i(t,x,y)}{V_i(x,\sqrt{t})} e^{-d^2(x,y)/t}.$$

In Section 2, we introduce the necessary background. In Section 3, we prove the results for non-parabolic manifolds, including Theorem 1.1 (cf. Theorem 3.1). In Section 4, we introduce techniques based on h-transform, and prove the results for parabolic manifolds. In particular, Theorem 1.2 is covered by a more general Theorem 4.9. In Section 5, we give examples of applications of the above results. In particular, we prove there Theorems 1.3 and 1.4 (cf. Theorems 5.1 and 5.3, respectively).

2 Preliminaries

2.1 Weighted manifolds

Let M be a Riemannian manifold of dimension N. The manifold M may have a boundary δM. Given a smooth positive function σ on M, define a measure μ on M by

$$d\mu = \sigma^2 d\mathbf{v}$$

where $d\mathbf{v}$ is the Riemannian measure which is given in local coordinates $x_1, x_2, ..., x_N$ by $d\mathbf{v}(x) = \sqrt{g(x)}dx$. Here $g(x)$ is the determinant of the Riemannian tensor (g_{ij}). Similarly, let μ' be the measure with density σ^2 with respect to the Riemannian measure of codimension 1 on any smooth hypersurface, in particular, on δM. The pair (M, μ) is called *a weighted manifold*. The Riemannian metric induces the Riemannian distance $d(x,y)$, $x, y \in M$. The geodesic balls and their volume on (M, μ) are denoted by

$$B(x,r) = \{y \in M \mid d(x,y) < r\}, \quad V(x,r) := \mu(B(x,r)).$$

We say that M is *complete* if the metric space (M, d) is complete. Recall that M is complete if and only if all balls $B(x,r)$ are precompact, in which case $V(x,r)$ is finite.

For any smooth function f, let ∇f be the gradient of f defined by

$$(\nabla f)^i = \sum_{j=1}^{N} g^{ij} \frac{\partial f}{\partial x_j},$$

in any coordinate chart, where $\left(g^{ij}\right) = (g_{ij})^{-1}$. The *divergence* div_μ of the weighted manifold (M, μ) is defined in local coordinates by

$$\mathrm{div}_\mu F := \frac{1}{\sigma^2\sqrt{g}} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sigma^2 \sqrt{g} F^i\right),$$

where F is a smooth vector field. If $\sigma \equiv 1$ then $\mathrm{div}_\mu F = \mathrm{div}\, F$ is the Riemannian divergence. For general σ, we have

$$\mathrm{div}_\mu F = \sigma^{-2}\mathrm{div}(\sigma^2 F).$$

The *Laplace operator* Δ of weighted manifold (M, μ) is the second order differential operator on functions defined by

$$\Delta f := \mathrm{div}_\mu(\nabla f) = \sigma^{-2}\mathrm{div}(\sigma^2 \nabla f).$$

In particular, if $\sigma \equiv 1$ then Δ coincides with the Laplace-Beltrami operator $\mathrm{div} \circ \nabla$.

Consider the Dirichlet form

$$\mathcal{E}(u, v) = \int_M (\nabla u, \nabla v) d\mu \tag{2.1}$$

defined on $C_0^\infty(M)$, that is, on the set of all smooth functions on M with compact support. The form $\mathcal{E}$ is closable in $L^2(M, \mu)$ and is positive definite. Denote by $\bar{\Delta}$ its infinitesimal generator. By integration by parts, for all $u, v \in C_0^\infty(M)$,

$$\mathcal{E}(u, v) = \int\limits_M (\nabla u, \nabla v)\, d\mu = -\int_M v \Delta u d\mu - \int_{\delta M} v \frac{\partial u}{\partial \mathbf{n}} d\mu' \tag{2.2}$$

where $\mathbf{n}$ denotes the inward unit normal vector field on δM (note that u and v do not necessarily vanish on δM). If $u \in C^2 \cap Dom(\bar{\Delta})$ then $\frac{\partial u}{\partial \mathbf{n}} = 0$ on δM and $\bar{\Delta} u = \Delta u$. Hence, $\bar{\Delta}$ can be considered as the extension of Δ with the Neumann boundary condition on δM. We say that u is *harmonic* on M if $u \in C^2(M)$, $\Delta u = 0$ in $M \setminus \delta M$ and $\frac{\partial u}{\partial \mathbf{n}} = 0$ on δM.

The heat semigroup $P_t = e^{t\bar{\Delta}}$ possesses a positive, smooth, symmetric kernel $p(t, x, y)$, which is called the *heat kernel* of (M, μ). Alternatively, the heat kernel can be defined as the minimal positive solution $u(t, x) = p(t, x, y)$ of the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u \quad \text{on} \quad (0, \infty) \times M \\ u(0, x) = \delta_y(x) \\ \frac{\partial u}{\partial \mathbf{n}}\big|_{\delta M} = 0 \end{cases} \tag{2.3}$$

(see [2], [5], [24]). In addition, the heat kernel satisfies the semigroup identity

$$p(t, x, y) = \int_M p(s, x, z) p(t - s, z, y) d\mu(z), \tag{2.4}$$

for all $0 < s < t$ and $x, y \in M$, and the inequality

$$\int_M p(t, x, y) d\mu(y) \le 1. \tag{2.5}$$

The operator $\bar{\Delta}$ generates a diffusion process $(X_t)_{t\ge 0}$ on M (reflected at δM). Here, implicitly, we add to M a point at infinity to take into account the possibility of explosion in finite time (i.e., stochastic incompleteness). Denote by $\mathbb{P}_x$ the law of $(X_t)_{t\ge 0}$ given $X_0 = x \in M$, and by $\mathbb{E}_x$

the corresponding expectation. Then the heat kernel p is exactly the transition density of $(X_t)_{t\geq 0}$ with respect to measure μ, that is, for any Borel set $U \subset M$, we have

$$\mathbb{P}_x\left(X_t \in A\right) = \int_U p(t,x,y)d\mu(y).$$

The Green function is defined by

$$G(x,y) = \int_0^\infty p(t,x,y)d\mu(y). \tag{2.6}$$

Equivalently, $G(x,y)$ is the infimum of all positive fundamental solutions of the operator Δ (see [5]). It is known that either $G(x,y) \equiv \infty$ or $G(x,y) < \infty$ for all $x \neq y$. In the latter case, $G(x,y)$ is the integral kernel of the operator $\left(-\bar{\Delta}\right)^{-1}$ in $L^2(M,\mu)$.

Definition 2.1 We say that the weighted manifold (M,μ) is *parabolic* if the process $(X_t)_{t\geq 0}$ is recurrent, and *non-parabolic* otherwise.

Each of the following two properties is equivalent to the parabolicity (see, for example, [10]):

1. There is no finite positive Green function on M, that is $G(x,y) \equiv \infty$.
2. Any positive superharmonic function on M is constant.

For any open set $\Omega \subset M$, let us denote

$$\delta\Omega := \delta M \cap \Omega.$$

Then Ω can be regarded as a manifold with boundary $\delta\Omega$, and all the constructions above can be repeated on Ω. This yields the heat semigroup P_t^Ω with the kernel $p_\Omega(t,x,y)$ which satisfies the Neumann boundary condition on $\delta\Omega$ and the (weak) Dirichlet condition in $\partial\Omega$ (see Fig. 1).

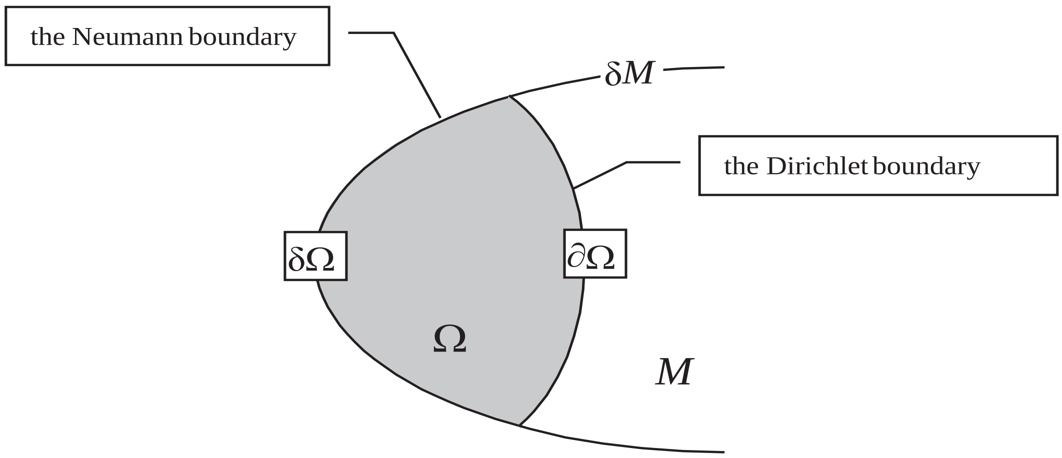

Figure 1 Set Ω and the boundaries $\partial\Omega$ and $\delta\Omega$

Let K be any closed subset of M. Denote by τ_K the first time the process X_t enters K, that is

$$\tau_K = \inf\{t \geq 0 : X_t \in K\}.$$

Since X_t has continuous paths and K is closed, τ_K is a stopping time (see e.g., [17, Ch. 1.]). Denote $\Omega := M \setminus K$ and consider the heat semigroup P_t^Ω and its kernel $p_\Omega(t,x,y)$ in Ω. Then, for any Borel set $U \subset \Omega$,

$$\int_U p_\Omega(t,x,y)d\mu(y) = \mathbb{P}_x\left(X_t \in U \text{ and } t < \tau_K\right) \tag{2.7}$$

is the probability that the process X_t is in U at time t without having visited K at any earlier time.

Another related quantity is

$$\psi_K(x) := \mathbb{P}_x(\tau_K < \infty),$$

which is the probability that X_t ever hits K starting from x. If (M,μ) is parabolic then $\psi_K \equiv 1$ whenever K has non-empty interior. If (M,μ) is non-parabolic then $\psi_K(x) < 1$ provided K is compact and $x \notin K$. In the latter case, the function $\psi_K(x)$ can be represented as follows

$$\psi_K(x) = \int_K G(x,y)de_K(y), \tag{2.8}$$

where e_K is the equilibrium measure of K, that is a measure of finite total mass $e_K(K)$ which is equal to the capacity of K (see e.g., [3],[14]).

2.2 Parabolic Harnack inequality

This paper focuses mainly on weighted manifolds satisfying a uniform *parabolic Harnack inequality.*

Definition 2.2 We say that a weighted manifold (M,μ) admits the (uniform) *parabolic Harnack inequality* (PHI) if there exists a constant C_0 such that any non-negative solution u of the heat equation $\partial_t u = \Delta u$ in any cylinder $Q := (\tau, \tau+T) \times B(x,r)$ with $T = r^2$ and $\tau \in (-\infty, +\infty)$, satisfies

$$\sup_{Q_-}\{u\} \le C_0 \inf_{Q_+}\{u\}\,, \tag{2.9}$$

where

$$Q_- := (\tau + T/4, \tau + T/2) \times B(x, r/2), \quad Q_+ := (\tau + 3T/4, \tau + T) \times B(x, r/2)$$

(see Fig. 2).

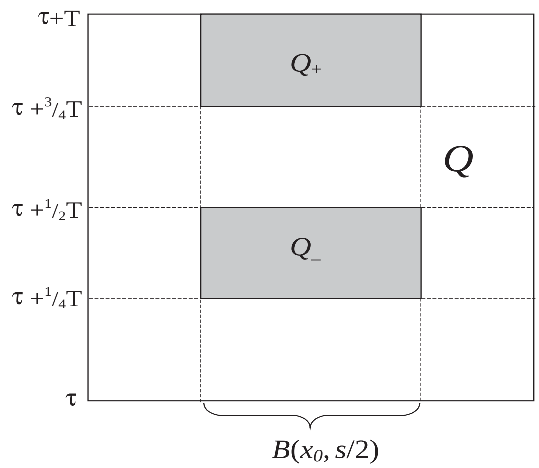

Figure 2 Cylinders Q_+ and Q_- in Q.

For example, (PHI) holds for complete Riemannian manifold M of non-negative Ricci curvature (see [20]). Moreover, (PHI) still holds if the weighted manifold M is quasi-isometric to a complete manifold having non-negative Ricci curvature and $\sigma \approx 1$ (see [9], [25]). Other examples are described in [13], [25], [27].

The parabolic Harnack inequality (2.9) can be characterized in terms of the volume growth and the Poincaré inequality as stated below by Theorem 2.8.

Definition 2.3 We say that (M, μ) admits *the volume doubling property* (VD) if there exists a constant C such that, for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq CV(x, r). \tag{2.10}$$

The following lemma is a folklore fact the proof of which can be found in many places (see, e.g., [9]).

Lemma 2.4 *If (M, μ) is a complete non-compact manifold satisfying* (VD) *then there exist positive constants c, C, α, β such that, for all $x, y \in M$ and for any positive $r \leq R$*

$$c\left(\frac{R}{r + d(x, y)}\right)^{\beta} \leq \frac{V(x, R)}{V(y, r)} \leq C\left(\frac{R + d(x, y)}{r}\right)^{\alpha}. \tag{2.11}$$

The upper bound in (2.11) holds also if M is compact, whereas the lower bound requires completeness and non-compactness.

Definition 2.5 We say that (M, μ) admits *the Poincaré inequality* (PI) if there exist positive constants c and $\kappa \leq 1$ such that, for any ball $B(x, r) \subset M$ and for any function $f \in C^1(B(x, r))$,

$$\int_{B(x,r)} |\nabla f|^2 \, d\mu \geq \frac{c}{r^2} \inf_{\xi \in \mathbb{R}} \int_{B(x,\kappa r)} (f - \xi)^2 \, d\mu \,. \tag{2.12}$$

If $\kappa = 1$ then (2.12) is equivalent to the fact that the first non-zero Neumann eigenvalue of Δ in the ball $B(x, r)$ is bounded below by cr^{-2}.

Definition 2.6 We say that (M, μ) admits the *two-sided heat kernel estimate* (TSE) if, for all $x, y \in M,\ t > 0$,

$$p(t, x, y) \asymp \frac{1}{V(x, \sqrt{t})} e^{-d^2(x,y)/t} \,. \tag{2.13}$$

Remark 2.7 If (VD) holds then by taking $R = r = \sqrt{t}$ in (2.11) we obtain, for any $\varepsilon > 0$,

$$\frac{V(x, \sqrt{t})}{V(y, \sqrt{t})} \leq C_\varepsilon \exp\left(\varepsilon \frac{d^2(x, y)}{t}\right). \tag{2.14}$$

This and the symmetry of $p(t, x, y)$ imply that the factor $\frac{1}{V(x,\sqrt{t})}$ in (2.13) can be replaced by either

$$\frac{1}{V(y, \sqrt{t})} \quad \text{or} \quad \frac{1}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}},$$

by adjusting the constant factors in the Gaussian term.

The following theorem contains a combined result of [9] and [25] (see also [29]).

Theorem 2.8 *Let (M, μ) be a complete weighted manifold with boundary. Then the following equivalences hold:*

$$\text{(VD)+(PI)} \iff \text{(PHI)} \iff \text{(TSE)}.$$

Let us mention some consequences of (PHI) on complete non-compact manifold (M, μ).

1. The manifold (M, μ) is stochastically complete; that is, for all $x \in M$ and $t > 0$,

$$\int_M p(t, x, y) d\mu(y) \equiv 1. \tag{2.15}$$

Indeed, (VD) implies that the volume function $V(x, r)$ has at most polynomial growth in r, which together with the completeness of the metric space (M, d) ensures stochastic completeness (see [7]).

2. The Green function admits the estimate

$$G(x, y) \approx \int_{d^2(x,y)}^{\infty} \frac{ds}{V(x, \sqrt{s})}, \tag{2.16}$$

which follows by integrating (TSE) in t (see [20], [25]). In particular, (M, μ) is parabolic if and only if

$$\int^{\infty} \frac{ds}{V(x, \sqrt{s})} = +\infty,$$

for some or all $x \in M$ (see [20], [27], [31]).

3. Let x, y be two points in M connected by a curve γ of length d. Assume that the function u is a positive solution to the heat equation $\partial_t u = \Delta u$ in

$$(0, \infty) \times (\rho\text{-neighborhood of } \gamma). \tag{2.17}$$

Then, for all $0 < s < t$,

$$u(s, x) \le u(t, y) \exp\left(C\left(\frac{t}{s} + \frac{d^2}{t-s} + \frac{t-s}{\rho^2}\right)\right) \tag{2.18}$$

with the constant $C > 0$ depending only on the Harnack inequality constant in (2.9) (see [21], [13]). Moreover, under the same hypotheses, there exists $s \in [\frac{t}{2}, t)$ such that

$$u(s, x) \le u(t, y) \exp\left(C\left(1 + \frac{d^2}{t} + \frac{d}{\rho}\right)\right) \tag{2.19}$$

Indeed, (2.19) follows from (2.18) for

$$s = \begin{cases} t/2, & \text{if } t < 2\rho d, \\ t - \rho d, & \text{if } t \ge 2\rho d. \end{cases}$$

Lemma 2.9 *Let (M, μ) be a complete, non-compact, non-parabolic manifold that admits* (PHI). *Then, for any compact set K, the hitting probability $\psi_K(x) := \mathbb{P}_x(\tau_K < \infty)$ satisfies $\psi_K(x) \to 0$ as $x \to \infty$.*

Proof. Since (M, μ) is non–parabolic, applying (2.16) with swapped x, y we see that $G(x, y) \to 0$ as $x \to \infty$ locally uniformly in y. Using the representation (2.8) for ψ_K, we conclude that $\psi_K(x) \to 0$ as $x \to \infty$. ∎

We will need one more result concerning a stability property of (PHI) under changes of the weight. If a point o is chosen on a manifold M then we refer to the pair (M, o) as a *pointed manifold.*

Definition 2.10 We say that a pointed Riemannian manifold (M, o) satisfies the *relatively connected annuli* condition (RCA) if there exists $A > 1$ such that, for any $r > A^2$ and all x, y with $d(o, x) = d(o, y) = r$, there exists a continuous path $\gamma : [0, 1] \to M$ with $\gamma(0) = x$, $\gamma(1) = y$ whose image is contained in $B(o, Ar) \setminus B(o, A^{-1}r)$ (see Fig. 3).

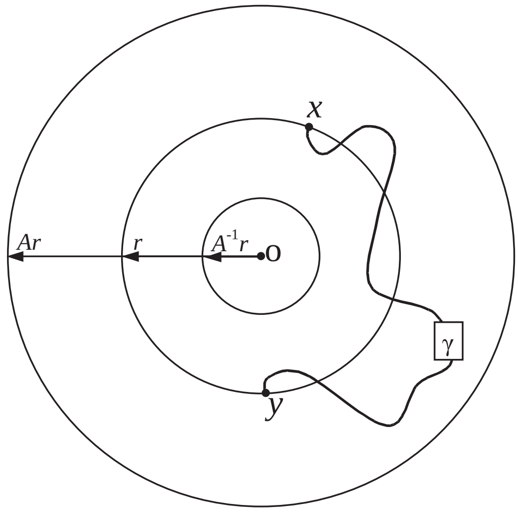

Figure 3 Condition (RCA): a path γ connects the points x and y staying in $B(o, Ar) \setminus B(o, A^{-1}r)$.

Theorem 2.11 *Let (M, μ) be a complete, non-compact, weighted manifold satisfying* (PHI)*. Let $o \in M$ and assume that the pointed manifold (M, o) satisfies* (RCA)*. Let h be a positive smooth function on M such that*

$$\sup_{B(o,2r)} h \le C \inf_{M \setminus B(o,r)} h, \tag{2.20}$$

for some constant C and for all $r \ge A$. Define a new measure ν on M by $d\nu = h^2 d\mu$. Then the weighted manifold (M, ν) also satisfies the parabolic Harnack inequality (PHI).

This theorem is proved in [13] where other related results of this sort are discussed. The connectedness property of annuli cannot be dropped here. For instance, the conclusion is false if $M = \mathbb{R}$, $o = 0$ and $h(x) = (1 + |x|)^\alpha$, $\alpha \ge 1$, despite the fact that this function satisfies (2.20).

3 The non-parabolic case

Throughout this section, we fix a compact set $K \subset M$ and denote by K_r the r-neighborhood of K, that is

$$K_r = \{x \in M : d(x, K) < r\}.$$

The main result of this section is the following theorem.

Theorem 3.1 *Let (M, μ) be a complete non-parabolic weighted manifold. Assume that the parabolic Harnack inequality* (PHI) *holds on (M, μ). Let $K \subset M$ be a compact set and denote $\Omega := M \setminus K$. Then there exist positive constants C, c, and δ such that, for all $t > 0$ and all $x, y \notin K_\delta$,*

$$cp(Ct, x, y) \le p_\Omega(t, x, y) \le p(t, x, y)\,. \tag{3.1}$$

Remark 3.2 This theorem contains Theorem 1.1 from Introduction because (PHI) holds on complete Riemannian manifolds of non-negative Ricci curvature, by the result of Li and Yau [20].

Note that the upper bound in (3.1) trivially follows from the maximum principle and is true always. Hence, the main difficulty is to prove the lower bound for p_Ω for which non-parabolicity of (M, μ) is essential. If (M, μ) is parabolic then p_Ω may be substantially smaller than p (see Theorem 4.9 in Section 4 below). Theorem 3.1 will be derived in the end of this section (see Corollary 3.5) from the following theorem containing a slightly weaker lower bound for p_Ω.

Theorem 3.3 *Under the hypotheses and notation of* Theorem 3.1, *there exists* $\delta > 0$ *and, for each* $t_0 > 0$, *there exist positive constants* C *and* c *such that, for all* $t > t_0$ *and all* $x, y \notin K_\delta$,

$$p_\Omega(t, x, y) \geq \frac{c}{V(x, \sqrt{t})} \exp\left(-C\frac{d^2(x, y)}{t}\right). \tag{3.2}$$

Recall that (PHI) implies the bound (2.13) for $p(t, x, y)$. Thus, (3.2) is equivalent to the lower bound of $p_\Omega(t, x, y)$ in (3.1), although only for $t > t_0$.

Corollary 3.4 *Let* (M, μ) *be a complete non-parabolic weighted manifold satisfying* (PHI). *Then* M *has only one end. Moreover, for any compact* $K \subset M$, *there exists* $\delta > 0$ *such that* $M \setminus K_\delta$ *has only one connected component.*

Proof. By Theorem 3.3, for any non-empty compact K and $\Omega = M \setminus K$, the heat kernel $p_\Omega(t, x, y)$ is strictly positive for all $x, y \in \Omega' := M \setminus K_\delta$ and $t > t_0$. Therefore, Ω' is connected, which means in particular that M cannot have more than one end. ■

For Riemannian manifolds of non-negative Ricci curvature this result was proved in [1] (cf. [30]). A similar but slightly different result can be found in [26]. Note that if (M, μ) is parabolic then the conclusion of Corollary 3.4 is not true. For example, $\mathbb{R}$ has two ends and satisfies (PHI). The number of ends of a manifold M satisfying (PHI) is finite since (PHI) implies (VD) but, for parabolic manifolds, no universal bound can be given on the number of ends of M.

Proof of Theorem 3.3. It will be convenient to split the proof into eight steps.

STEP 1. Fix a constant $A \geq 1$ to be chosen later. Let us show that, for all $t > 0$, $x \in M$ and $y \in B(x, 2A\sqrt{t})$,

$$\int_{B(x,\sqrt{t})} p(t, y, z) d\mu(z) \geq \varepsilon, \tag{3.3}$$

with some positive constant $\varepsilon = \varepsilon(A)$. Fix $x \in M$, $r > 0$ and consider the function

$$u(t, y) = u_{r,x}(t, y) := \int_{B(x,r)} p(t, y, z) d\mu(z) = P_t \mathbf{1}_{B(x,r)}(y).$$

Let us extend $u(t, y)$ to negative t by setting $u(t, y) = \mathbf{1}_{B(x,r)}(y)$ if $t < 0$. Then $u(t, y)$ is a smooth non-negative solution to the heat equation $u_t = \Delta u$ in $(-\infty, \infty) \times B(x, r)$. Applying the Harnack inequality (2.9) to u in the cylinder $(-\frac{r^2}{2}, \frac{r^2}{2}) \times B(x, r)$, we obtain

$$1 = u(0, x) \leq Cu(r^2/4, x)$$

or

$$u(r^2/4, x) \geq \eta \tag{3.4}$$

where $\eta := C^{-1} > 0$ (see Fig. 4).

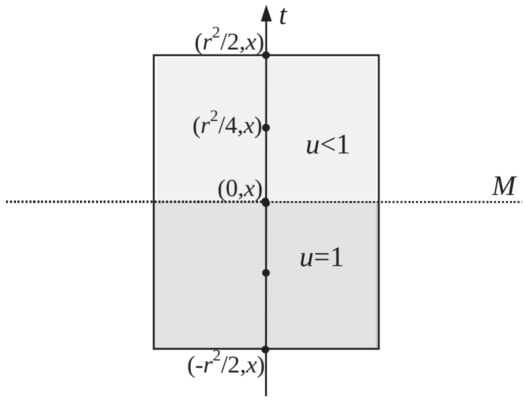

Figure 4 Function u in the cylinder $(-\frac{r^2}{2}, \frac{r^2}{2}) \times B(x, r)$

Since $u(t, y)$ is a positive solution of the heat equation in $(0, \infty) \times M$, it satisfies (2.18) for all $0 < s < t$ and with an arbitrarily large ρ. Given $t > 0$, choose $r = \sqrt{t}$ and $s = t/4$. Then (3.4) and (2.18) yields

$$u(t, y) \geq \eta \exp\left(-C\frac{d^2(x, y)}{t}\right). \tag{3.5}$$

For $y \in B(x, 2A\sqrt{t})$, the right hand side of (3.5) is bounded below by a constant $\varepsilon = \varepsilon(A) > 0$ whence (3.3) follows.

As a consequence of (3.3) and (2.5), we see that, for any $y \in B(x, 2A\sqrt{t})$,

$$\int_{M \setminus B(x,\sqrt{t})} p(t, y, z) d\mu(z) \leq 1 - \varepsilon. \tag{3.6}$$

STEP 2. Let $\varepsilon = \varepsilon(A)$ be given by Step 1. We claim that there exists $\delta = \delta(A) < +\infty$ such that, for any $t > 0$ and $y \notin K_\delta$,

$$\int_\Omega p_\Omega(t, y, z) d\mu(z) \geq 1 - \varepsilon/2 \tag{3.7}$$

The stochastic completeness (2.15) of (M, μ) implies

$$\int_\Omega p_\Omega(t, y, z) d\mu(z) \geq 1 - \psi_K(y).$$

Indeed, by (2.7) we have

$$1 - \int_\Omega p_\Omega(t, y, z) d\mu(z) = \mathbb{P}_y (\tau_K \leq t) \leq \mathbb{P}_y (\tau_K < \infty) = \psi_K(y). \tag{3.8}$$

By the non-parabolicity of M and Lemma 2.9, we have $\psi_K(y) \to 0$ as $y \to \infty$. Thus, there exists $\delta > 0$ such that $\psi_K(y) \leq \varepsilon/2$ outside K_δ, whence (3.7) follows.

STEP 3. We claim that, for all $x \in M$, $t > 0$ and $y \in B(x, 2A\sqrt{t}) \setminus K_\delta$,

$$\int_{B(x,\sqrt{t})} p_\Omega(t, y, z) d\mu(z) \geq \varepsilon/2 \tag{3.9}$$

(see Fig. 5). Here we follow the convention that p_Ω vanishes outside Ω, that is, on K.

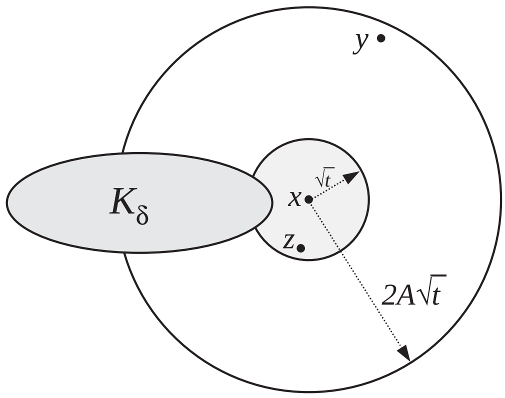

Figure 5 Illustration to Step 3.

The claim is trivially true if $B(x, 2A\sqrt{t}) \setminus K_\delta$ is empty. Otherwise, (3.6) and (3.7) imply

$$\begin{aligned}\int_{B(x,\sqrt{t})} p_\Omega(t,y,z)d\mu(z) &= \int_\Omega p_\Omega(t,y,z)d\mu(z) - \int_{\Omega\setminus B(x,\sqrt{t})} p_\Omega(t,y,z)d\mu(z) \\ &\geq \int_\Omega p_\Omega(t,y,z)d\mu(z) - \int_{M\setminus B(x,\sqrt{t})} p(t,y,z)d\mu(z) \\ &\geq (1-\varepsilon/2) - (1-\varepsilon) = \varepsilon/2\,,\end{aligned}$$

which proves (3.9).

As a consequence we obtain, that for all $x \notin K_\delta$ and $t > 0$

$$p_\Omega(t,x,x) \geq \frac{c}{V(x,\sqrt{t})}. \tag{3.10}$$

Indeed, we have

$$\begin{aligned}p_\Omega(2t,x,x) &\geq \int_{B(x,\sqrt{t})} p_\Omega^2(t,x,z)d\mu(z) \\ &\geq \frac{1}{V(x,\sqrt{t})}\left[\int_{B(x,\sqrt{t})} p_\Omega(t,x,z)d\mu(z)\right]^2 \geq \frac{\varepsilon^2}{4V(x,\sqrt{t})},\end{aligned}$$

where we have applied the semigroup identity (2.4), the Cauchy-Schwarz inequality, and (3.9) for $y = x$. Replacing $2t$ by t and using the doubling volume property (VD), we obtain (3.10).

STEP 4. For the rest of this section, we use the notation $|x| := d(x, K)$. Let us prove (3.2) assuming $t > 0$, $x, y \notin K_\delta$, and

$$d(x,y) \leq \frac{1}{2}\max(|x|, |y|)\,. \tag{3.11}$$

Assume first that $R := |x| \geq |y|$. Then the geodesic from x to y stays at distance at least $\frac{1}{2}R$ from K (see Fig. 6).

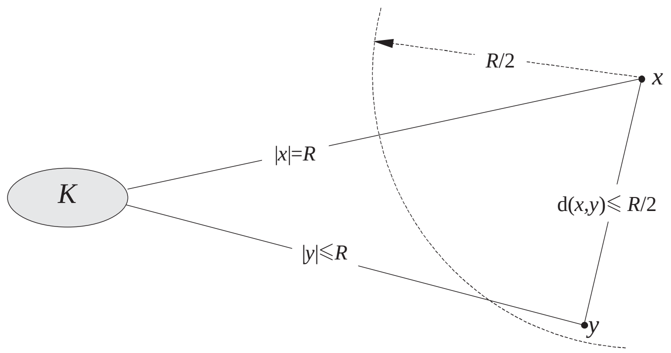

Figure 6 Illustration to Step 4: $d(x,y) \leq \frac{1}{2}\max(|x|, |y|)$

By (2.19), for some $s \in [t/2, t]$, we obtain

$$p_\Omega(t,x,y) \geq p_\Omega(s,x,x) \exp\left(-C\left(1 + \frac{d^2(x,y)}{t}\right)\right) \tag{3.12}$$

and, by (3.10),

$$p_\Omega(s,x,x) \geq \frac{c}{V(x,\sqrt{s})} \geq \frac{c}{V(x,\sqrt{t})}$$

whence (3.2) follows.

If instead $|x| < |y|$, then we obtain as above

$$p_\Omega(t,x,y) \geq \frac{c}{V(y,\sqrt{t})} \exp\left(-C\frac{d^2(x,y)}{t}\right). \tag{3.13}$$

Hence, (3.2) follows from (3.13) and (2.14) by adjusting the constant C in the Gaussian exponential.

STEP 5. Here we show that for all $t > 0$, $x \notin K_{2\sqrt{t}}$, $y \in B(x, A\sqrt{t}) \setminus K_\delta$, and $z \in B(x, \sqrt{t})$,

$$p_\Omega(t,y,z) \geq \frac{c}{V(x,\sqrt{t})} \tag{3.14}$$

with a positive constant $c = c(A)$ (see Fig. 7).

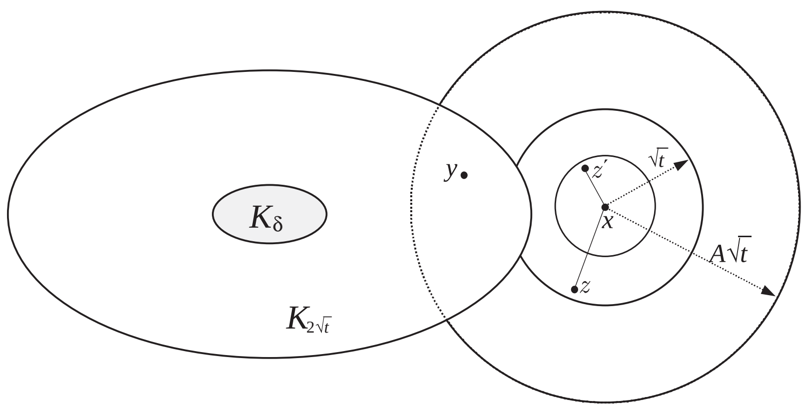

Figure 7 Illustration to Step 5.

Indeed, estimate (3.9) (with t replaced by $t/4$) implies that, for some $z' \in B(x, \frac{1}{2}\sqrt{t})$,

$$p_\Omega(t/4, y, z') \geq \frac{\varepsilon}{2V(x, \frac{1}{2}\sqrt{t})}.$$

Since the points z and z' can be connected through x by a curve of the length $\leq 2\sqrt{t}$ whose $\sqrt{t}$-neighborhood does not intersect K, it follows by (2.18) that

$$p_\Omega(t/4, y, z') \leq C p_\Omega(t, y, z).$$

Thus

$$p_\Omega(t, y, z) \geq \frac{\varepsilon}{2CV(x, \frac{1}{2}\sqrt{t})} \geq \frac{c}{V(x, \sqrt{t})}.$$

Of course, we can set $z = x$ in (3.14) and obtain (3.2) for the case $x \notin K_{2\sqrt{t}}$, $y \in B(x, A\sqrt{t}) \setminus K_\delta$. In the next steps, the range of the variables x, y for which (3.2) is true, will be extended by using various "chaining" arguments as well as (2.18), (2.19).

STEP 6. We can assume without loss of generality that δ found in Step 2, is large enough so that

$$r_0 := \operatorname{diam} K < \delta.$$

Let us show that if $t > \delta^2$ and

$$x, y \in K_{\delta+\sqrt{t}} \setminus K_\delta\,, \tag{3.15}$$

(see Fig. 8) then

$$p_\Omega(t, x, y) \geq \frac{c'}{V(x, \sqrt{t})}. \tag{3.16}$$

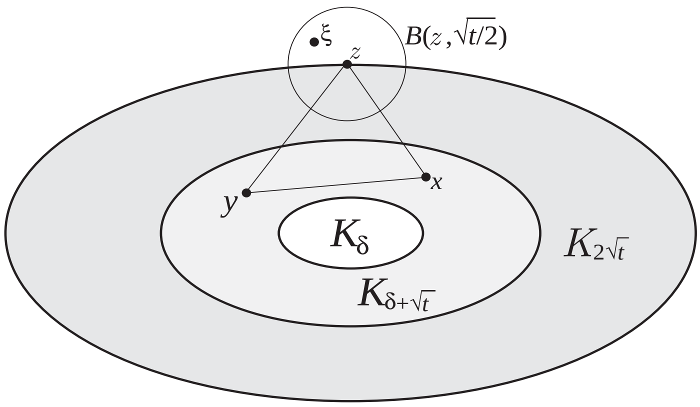

Figure 8 Illustration to Step 6: $x, y \in K_{\delta+\sqrt{t}} \setminus K_\delta$ and $t > \delta^2$

Inequalities $t > \delta^2 > r_0^2$ and (3.15) obviously imply

$$d(x, y) \leq r_0 + 2(\delta + \sqrt{t}) < 5\sqrt{t}.$$

Let z be any point at distance $2\sqrt{t}$ from K. Clearly, we have

$$d(x, z) < 5\sqrt{t} \quad \text{and} \quad d(y, z) < 5\sqrt{t}.$$

Since $x \in B(z, 5\sqrt{t}) \setminus K_\delta$, we can apply (3.9) to $t/2, z, x$ (instead of t, x, y) with, say, $A = 5$. This yields

$$\int_{B(z,\sqrt{t/2})} p_\Omega(t/2, x, \xi) d\mu(\xi) \geq \varepsilon/2. \tag{3.17}$$

Observe that $z \notin K_{2\sqrt{t/2}}$ and $y \in B(z, 5\sqrt{t}) \setminus K_\delta$. Thus, for any point $\xi \in B(z, \sqrt{t/2})$, we can apply (3.14) to $t/2, z, y, \xi$ (instead of t, x, y, z) with $A = 10$, to obtain

$$p_\Omega(t/2, y, \xi) \geq \frac{c}{V(z, \sqrt{t/2})}. \tag{3.18}$$

Thus, for the given x, y and $t > \delta^2$, (2.4), (3.17), (3.18) and (2.11) imply

$$\begin{aligned} p_\Omega(t,x,y) &= \int_\Omega p_\Omega(t/2,x,\xi)p_\Omega(t/2,\xi,y)d\mu(\xi) \\ &\geq \int_{B(z,\sqrt{t/2})} p_\Omega(t/2,x,\xi)p_\Omega(t/2,y,\xi)d\mu(\xi) \\ &\geq \frac{c\varepsilon}{2V(z,\sqrt{t/2})} \geq \frac{c'}{V(x,\sqrt{t})}, \end{aligned}$$

whence (3.16) follows.

As a consequence, we see that $M \setminus \overline{K_\delta}$ is a connected set. Indeed, for any $x, y \in M \setminus \overline{K_\delta}$, find t large enough so that (3.15) holds. Then (3.16) implies $p_\Omega(t,x,y) > 0$. Hence, x and y are at the same component of Ω whence the connectedness of $M \setminus \overline{K_\delta}$ follows. In particular, we deduce

$$p_\Omega(t,x,y) > 0, \quad \forall t > 0,\ \forall x, y \in M \setminus K_\delta \tag{3.19}$$

which follows for instance from a local version of (2.18).

STEP 7. Let us prove (3.16) provided $t_0 \leq t \leq \delta^2$ and still (3.15) is satisfied. Here $t_0 > 0$ is given by the statement of Theorem 3.3. Indeed, in this case (3.15) implies $x \in K_{2\delta}$ whence

$$V(x,\sqrt{t}) \geq \inf_{x \in K_{2\delta}} V(x,\sqrt{t_0}) = \text{const} > 0.$$

Hence, it suffices to show that, for all $x, y \in K_{2\delta} \setminus K_\delta$ and $t \in [t_0, \delta^2]$,

$$p_\Omega(t,x,y) \geq c. \tag{3.20}$$

This lower bound follows from the compactness of $[t_0,\delta^2] \times \overline{K_{2\delta} \setminus K_\delta}$ and from the continuity of the function $(t,x,y) \mapsto p_\Omega(t,x,y)$ on $(0,\infty) \times M \times M$, which, by (3.19), is strictly positive when $x, y \in M \setminus K_\delta$.

STEP 8. Assume $t \geq t_0$ and $x, y \notin K_\delta$ as required by the statement of Theorem 3.3. So far we have verified the lower bound (3.2) either if $d(x,y) \leq \frac{1}{2}\max(|x|,|y|)$ (Step 4) or if $x, y \in K_{\delta+\sqrt{t}}$ (Steps 6,7). Let us finally treat the remaining case when one of the points x, y does not belong to $K_{\delta+\sqrt{t}}$ and

$$d(x,y) > \frac{1}{2}\max(|x|,|y|). \tag{3.21}$$

It suffices to consider the case $|x| \geq |y|$ (the symmetric case $|x| \leq |y|$ is handled by (2.14)), so that $x \notin K_{\delta+\sqrt{t}}$. Let x' be the point at distance $\delta+\sqrt{t}$ from K lying on the shortest geodesic from x to K (see Fig. 9). As the $\sqrt{t}$-neighborhood of the shortest geodesic from x to x' is contained in Ω, we can apply (2.19), which yields, for some $t' \in [\frac{t}{2}, t)$,

$$\begin{aligned} p_\Omega(t,x,y) &\geq p_\Omega(t',x',y)\exp\left\{-C\left(1+\frac{d^2(x,x')}{t}+\frac{d(x,x')}{\sqrt{t}}\right)\right\} \\ &\geq p_\Omega(t',x',y)\exp\left(-C'\frac{|x|^2}{t}\right). \end{aligned}$$

If $y \in K_{\delta+\sqrt{t}}$ then, by Steps 6-7,

$$p_\Omega(t',x',y) \geq \frac{c}{V(x',\sqrt{t'})}.$$

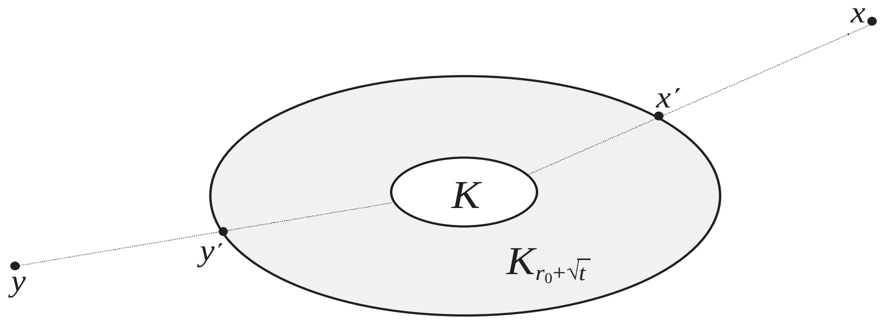

Figure 9 Illustration to Step 8: $d(x,y) > \frac{1}{2}\max(|x|, |y|)$ and x, y are outside $K_{\delta+\sqrt{t}}$

If $y \notin K_{\delta+\sqrt{t}}$ then consider the point y' at distance $\delta + \sqrt{t}$ from K lying on the shortest geodesic from y to K (see Fig. 9). Then, for some $t'' \in [\frac{t'}{2}, t') \subset [\frac{t}{4}, t)$,

$$p_\Omega(t', x', y) \geq p_\Omega(t'', x', y') \exp\left(-C' \frac{|y|^2}{t}\right) \geq \frac{c}{V(x', \sqrt{t''})} \exp\left(-C' \frac{|y|^2}{t}\right).$$

In both case, we have

$$\begin{aligned} p_\Omega(t, x, y) &\geq \frac{c}{V(x', \sqrt{t})} \exp\left\{-C'\left(\frac{|x|^2}{t} + \frac{|y|^2}{t}\right)\right\} \\ &\geq \frac{c'}{V(x, \sqrt{t})} \exp\left\{-C'' \frac{d^2(x, y)}{t}\right\}, \end{aligned}$$

where we have used the hypothesis (3.21) and the inequality (2.14).

This finishes the proof of Theorem 3.3. ■

The following corollary extends the estimate (3.2) of Theorem 3.3 to *all* $t > 0$, which concludes the proof of Theorem 3.1.

Corollary 3.5 *Assume that the hypotheses of Theorem 3.1 are satisfied, and let $a > 0$ be such that $M \setminus \overline{K_a}$ has no compact components. Then the lower bound (3.2) holds for all $x, y \notin \overline{K_a}$ and $t > 0$ with constants $c, C > 0$ depending on a but not on x, y, t.*

Proof. First observe that the doubling volume property (VD) shows that $M \setminus \overline{K_a}$ has only finitely many components, for any fixed $a > 0$. Let δ be given by Theorem 3.3. Without loss of generality, we can assume that $\delta > a$.

Since $M \setminus \overline{K_a}$ has only finitely many connected components and no compact component, any point $x \in K_\delta \setminus \overline{K_a}$ can be connected to a point $x' \notin K_\delta$ by a curve in $M \setminus \overline{K_a}$ of length uniformly bounded by $R_1(a)$. For the same reason, any two points in $K_\delta \setminus \overline{K_a}$ can be joined by a curve in $M \setminus \overline{K_a}$ of length uniformly bounded by $R_2(a)$.

Assume that $x, y \in M \setminus \overline{K_a}$ and $t > a^2$. Then one can apply (2.18) with $s = t - a^2/4$, $\rho = a$ and $d = R_1(a)$ to reduce (3.2) to the case where $x, y \notin K_\delta, t > a^2/2$ which is covered by Theorem 3.3.

Assume now that $x, y \in M \setminus \overline{K_a}$ and $t \leq a^2$. Observe that (PHI) easily yields in this case

$$p_\Omega(t, x, x) \geq \frac{c}{V(x, \sqrt{t})}. \tag{3.22}$$

Let γ be a geodesic in M joining x to y. Consider two cases depending on whether or not $\gamma \subset M \setminus K_{a/2}$.

If $\gamma \subset M \setminus K_{a/2}$, we can apply (2.19) with $\rho = a/2$ and $d = d(x,y)$ to obtain

$$\begin{aligned} p_\Omega(t,x,y) &\geq p_\Omega(s,x,x)\exp\left(-C\left(1+\frac{d^2}{t}+\frac{2d}{a}\right)\right) \\ &\geq p_\Omega(t,x,x)\exp\left(-C_a\left(1+\frac{d^2}{t}\right)\right). \end{aligned}$$

Here, $s \in [t/2, t)$ and we have used the fact that $t \mapsto p_\Omega(t,x,x)$ is non-increasing and that $t \leq a^2$. The desired inequality (3.2) now follows from (3.22).

If γ intersects $M \setminus K_{a/2}$ then $d(x,y) \geq a$ because $x, y \in M \setminus \overline{K_a}$. Moreover, there is a curve in $M \setminus \overline{K_a}$ of length at most $d(x,y) + R_2(a)$ joining x to y. Applying (2.19) to x, y and the curve γ with $\rho = a$ and $d = d(x,y) + R_2(a)$, we obtain as above

$$p_\Omega(t,x,y) \geq p_\Omega(t,x,x)\exp\left(-C'_a(1+\frac{d^2}{t})\right),$$

for some constant $C'_a > 0$. Again, the desired lower bound (3.2) follows from (3.22). ■

4 The parabolic case

4.1 Generalities on h-transform

Any positive smooth function h on the weighted manifold (M,μ) induces a new weighted manifold (M,ν) with measure ν defined by

$$d\nu = h^2 d\mu.$$

The Laplace operator Δ^h of (M,ν) is then given by

$$\Delta^h f = h^{-2}\mathrm{div}_\mu(h^2\nabla f) = (h\sigma)^{-2}\mathrm{div}((h\sigma)^2\nabla f). \tag{4.1}$$

Denote by p^h the heat kernel on (M,ν). The well-known Doob's h-transform technique is based on the observation that if h is harmonic on (M,μ) then there is a tight connection between the objects relative to (M,μ) and those relative to (M,ν), for example $p^h(t,x,y) = h(x)h(y)p(t,x,y)$. In the present work, we will consider the case where h is harmonic only in a subset of the manifold M.

Lemma 4.1 *Assume that U is an open set in which $\Delta h = 0$. Then, for any function smooth function f in U,*

$$\Delta^h f = h^{-1}\Delta(hf).$$

Proof. We have

$$\Delta^h f = h^{-2}\mathrm{div}_\mu(h^2\nabla f) = \Delta f + 2h^{-1}(\nabla h, \nabla f)$$

and

$$h^{-1}\Delta(hf) = \Delta f + h^{-1} f\Delta h + 2h^{-1}(\nabla h, \nabla f).$$

As $\Delta h = 0$ in U, the claim follows. ■

Proposition 4.2 *If h is harmonic in an open set $U \subset M$ then the Dirichlet heat kernels p_U and p^h_U in U associated with Δ and Δ^h are related by*

$$p_U(t,x,y) = h(x)h(y)p^h_U(t,x,y), \tag{4.2}$$

for all $t > 0$, $x, y \in U$.

Proof. Denote

$$\mathcal{E}_U(u,v) = \int_U (\nabla u, \nabla v)\, d\mu, \quad \mathcal{E}_U^h(u,v) = \int_U (\nabla u, \nabla v)\, d\nu.$$

The transform $Tf := hf$ is obviously an isometry from $L^2(U,\nu)$ to $L^2(U,\mu)$. By (2.2), we have, for all $u, v \in C_0^\infty(U)$,

$$\begin{aligned}
\mathcal{E}_U(Tu, Tv) &= \int_M (\nabla(hu), \nabla(hv))\, d\mu \\
&= -\int_M u[h^{-1}\Delta(hv)]h^2 d\mu - \int_{\delta M} \frac{\partial(hv)}{\partial \mathbf{n}} uh\, d\mu' \\
&= -\int_M u[\Delta_\nu v]h^2 d\mu - \int_{\delta M} \frac{\partial v}{\partial \mathbf{n}} uh^2\, d\mu' \\
&= -\int_M u[\Delta_\nu v] d\nu - \int_{\delta M} \frac{\partial v}{\partial \mathbf{n}} u\, d\nu' = \int_M (\nabla u, \nabla v)\, d\nu = \mathcal{E}_U^h(u,v).
\end{aligned}$$

Of course, the equality $\mathcal{E}_U(u,v) = \mathcal{E}_U^h(u,v)$ extends to the domains of these Dirichlet forms. It follows that the associated semigroups P_t^U, $P_t^{U,h}$ satisfy $TP_t^{U,h}T^{-1} = P_t^U$. By definition of the heat kernels, we have

$$P_t^U f(x) = \int_U p_U(t,x,\cdot) f d\mu.$$

Therefore,

$$TP_t^{U,h}T^{-1}f(x) = h(x)\int_U p_U^h(t,x,\cdot)\left(h^{-1}f\right)h^2 d\mu = \int_U h(x)p_U^h(t,x,y)h(y)f(y)d\mu(y),$$

whence (4.2) follows. ∎

We now briefly explain how we will use h-transform to estimate the Dirichlet heat kernel p_Ω in the complement Ω of a compact K in a parabolic manifold. It turns out that, for any parabolic weighted manifold (M,μ) and for a "nice" compact set $K \subset M$, there always exists a harmonic function h in Ω such that (M,ν) is *non-parabolic.* Our strategy is then to obtain estimates of the Dirichlet kernel p_Ω using estimates of p_Ω^h and (4.2). This however will require additional hypotheses on (M,μ) so that we can apply Theorem 3.1 on (M,ν). This program is realized in Section 4.4, whereas in the next Sections 4.2, 4.3 we introduce the necessary tools for that.

4.2 Capacities

Given a precompact set $K \subset M$ and an open set U containing $\overline{K}$, define the *capacity* of the capacitor (K,U) by

$$\operatorname{cap}(K,U) = \inf_{\phi \in \mathcal{T}(K,U)} \int_U |\nabla\phi|^2\, d\mu,$$

where $\mathcal{T}(K,U)$ is the set of test functions defined by

$$\mathcal{T}(K,U) = \left\{\phi \in C_0^\infty(U) : \phi|_K = 1\right\}.$$

If U is precompact then we define *the equilibrium potential* φ of (K,U) as the weak solution to the following boundary value problem in $U \setminus \overline{K}$

$$\begin{cases}
\Delta\varphi = 0 \\
\varphi|_{\partial K} = 1 \\
\varphi|_{\partial U} = 0 \\
\left.\frac{\partial\varphi}{\partial\mathbf{n}}\right|_{\delta(U\setminus\overline{K})} = 0.
\end{cases}$$

The following identities are always fulfilled, even so φ may be not in $\mathcal{T}(K,U)$:

$$\operatorname{cap}(K,U) = \int_{U\setminus\overline{K}} |\nabla\varphi|^2\, d\mu = -\text{flux}\,(\varphi)\,. \tag{4.3}$$

Here, for any function u which is harmonic outside $\overline{K}$, flux$\,(u)$ is defined by

$$\text{flux}(u) := \int_{\partial W} \frac{\partial u}{\partial \mathbf{n}} d\mu',$$

where W is any open region in the domain of u, with a smooth precompact boundary, such that $\overline{K} \subset W$, and $\mathbf{n}$ is the outward normal unit vector field on ∂W. By the Green formula (2.2) and the harmonicity of u, flux(u) does not depend on the choice of W.

Fix a point $o \in M$ and, for simplicity, set

$$B_r = B(o,r) \quad \text{and} \quad V(r) = V(o,r). \tag{4.4}$$

If M is complete then the following universal estimate of capacity is true, for all $0 < r < R$:

$$\operatorname{cap}(B_r, B_R)^{-1} \geq \frac{1}{2}\int_r^R \frac{s\,ds}{V(s)} \tag{4.5}$$

(see [23], [28] as well as [10, p.174]). The purpose of this section is to prove an opposite estimate, under certain additional assumptions on (M,μ), as stated in the following lemma.

Lemma 4.3 *Let (M,μ) be a complete non-compact weighted manifold. Fix a point $o \in M$ and assume that the following hypotheses are satisfied:*

(i) The heat kernel upper bound

$$p(t,o,x) \leq \frac{C}{V(\sqrt{t})}\exp\left(-\frac{d^2(o,x)}{Ct}\right), \tag{4.6}$$

for all $x \in M$ and $t > 0$.

(ii) The central doubling volume property: $V(2r) \leq CV(r)$, for all $r > 0$.

Then, for all r and R such that $0 < r \leq \frac{1}{2}R$,

$$\operatorname{cap}(B_r, B_R)^{-1} \leq C_1 \int_r^R \frac{s\,ds}{V(s)}, \tag{4.7}$$

where the constant C_1 in (4.7) depends only on the constants C in (i) and (ii).

Remark 4.4 This lemma solves positively Problem 21 from [10, p.240].

Proof. Let $G_R(x,y)$ be the Dirichlet Green kernel in the ball B_R. We will use the following general inequality

$$\operatorname{cap}(B_r, B_R)^{-1} \leq \max_{x\in\partial B_r} G_R(o,x) \tag{4.8}$$

which was proved in [6] (see also [10, p. 154]). Hence, it suffices to show that

$$\max_{x\in\partial B_r} G_R(o,x) \leq C_1 \int_r^R \frac{s\,ds}{V(s)} \tag{4.9}$$

provided $r \leq R/2$. To prove (4.9), recall that

$$G_R(o,x) = \int_0^\infty p_R(t,o,x)dt, \tag{4.10}$$

where $p_R := p_{B_R}$ is the Dirichlet heat kernel in B_R. Since $p_R \leq p$, the upper bound (4.6) applies also for p_R. However, this upper bound alone is not good enough for us. We will use it together with another estimate for p_R, which follows from the hypotheses (i) and (ii)

$$p_R(t,o,x) \leq \frac{C}{V(R)} \exp\left(-\frac{t}{CR^2}\right), \quad \forall t \geq R^2 \tag{4.11}$$

(see [15] for the proof). Putting together (4.10), (4.6), (4.11) and using elementary estimates of the integrals, we obtain, for any $x \in \partial B_r$,

$$\begin{aligned} G_R(o,x) &\leq \int_0^{R^2} p(t,o,x)dt + \int_{R^2}^\infty p_R(t,o,x)dt \\ &\leq \int_0^{R^2} \frac{C}{V(\sqrt{t})} \exp\left(-\frac{r^2}{Ct}\right) dt + \int_{R^2}^\infty \frac{C}{V(R)} \exp\left(-\frac{t}{CR^2}\right) dt \\ &\leq C \int_{r^2}^{R^2} \frac{dt}{V(\sqrt{t})} + \frac{CR^2}{V(R)} \\ &\leq C \int_r^R \frac{s\,ds}{V(s)}, \end{aligned}$$

which was to be proved. ■

4.3 Unbounded harmonic functions

In this section, we prove the existence of a positive harmonic function with certain properties which can be used for h-transforms on parabolic manifolds. We say that a precompact set K has *locally positive capacity* if, for some precompact open set U containing $\overline{K}$, $\operatorname{cap}(K,U) > 0$. It is not difficult to show that if $\operatorname{cap}(K,U) > 0$ for some U then it is true for all U containing $\overline{K}$.

The following Lemma plays a central role (we keep using notation (4.4)).

Lemma 4.5 *Assume that (M,μ) is a complete non-compact parabolic weighted manifold. Fix a point $o \in M$ and assume that the following hypotheses are satisfied:*

(i) *The heat kernel upper bound*

$$p(t,o,x) \leq \frac{C}{V(\sqrt{t})} \exp\left(-\frac{d^2(o,x)}{Ct}\right), \tag{4.12}$$

for all $x \in M$ and $t > 0$.

(ii) *The central doubling volume property: $V(2r) \leq CV(r)$, for all $r > 0$.*

(iii) *The elliptic Harnack inequality in annuli: for some $A > 1$, $R_0 > 0$ and for any positive harmonic function u in $B_{AR} \setminus B_{A^{-1}R}$ with $R \geq R_0$,*

$$\sup_{\partial B_R} u \leq C \inf_{\partial B_R} u\,. \tag{4.13}$$

Let K be a compact set of locally positive capacity, which is contained in some ball B_{r_0}. Then there exists a non-negative harmonic function v in $\Omega := M \setminus K$ such that, for all r large enough and all $x \in \partial B_r$,

$$v(x) \approx \int_{r_0}^{r} \frac{s\,ds}{V(s)}. \tag{4.14}$$

Moreover, the constants which bound the ratio of the right- and left-hand sides of (4.14) depend only on the constant C in conditions (i)-(iii).

Remark 4.6 If (M, μ) is non-parabolic then the statement is trivially true. Indeed, take $v \equiv 1$ then (4.14) is satisfied because the non-parabolicity implies

$$\int^{\infty} \frac{s\,ds}{V(s)} < \infty.$$

On the contrary, in the parabolic case, under the hypothesis (4.12), the right hand side of (4.14) is unbounded as $r \to +\infty$ which means that the function $v(x)$ is also unbounded.

Note also that (4.14) can be rewritten as

$$v(x) \approx \int_{1}^{r} \frac{s\,ds}{V(s)}. \tag{4.15}$$

Proof of Lemma 4.5. For each $R > r_0$, let φ_R be the equilibrium potential of the capacitor (K, B_R). By (4.3), we have

$$\mathrm{cap}(K, B_R) = -\mathrm{flux}\,(\varphi_R)\,.$$

By the hypothesis, we have $\mathrm{cap}(K, B_R) > 0$. Consider the following function

$$v_R = \frac{1 - \varphi_R}{\mathrm{cap}(K, B_R)},$$

for which $\mathrm{flux}(v_R) = 1$. This, together with the fact that v_R vanishes at the regular points of $\partial\Omega = \partial K$, allows us to prove, by compactness argument, that a subsequence of v_R converges as $R \to \infty$ to a non-negative harmonic function v in Ω, also satisfying

$$\mathrm{flux}(v) = 1. \tag{4.16}$$

See [8], [18], [30] for the details of this construction. The parabolicity of (M, μ) implies that v is unbounded. Indeed, let us extend v to K by 0 so that v becomes subharmonic on M. By (4.16), v is non-constant. However, a non-constant bounded subharmonic function can exist only on non-parabolic manifolds. Note that so far we have not used any of the hypotheses (i)-(iii). As was proved in [8, Lemma 2], in the presence of condition(iii), a non-negative harmonic function v in Ω is *uniquely* determined by the facts that v vanishes on $\partial\Omega$ and has flux 1.

Let us show that the function v satisfies (4.14). By the condition (iii), we have

$$\max_{\partial B_r} v \le C \min_{\partial B_r} v, \tag{4.17}$$

provided $r \ge \max(R_0, A r_0)$. Together with the maximum principle and the fact that v is unbounded, this implies $v(x) \to +\infty$ as $x \to \infty$. On the other hand, we claim that the following estimate holds, for all $r > r_0$:

$$\min_{\partial B_r} v \le \mathrm{cap}(K, B_r)^{-1} \le \max_{\partial B_r} v. \tag{4.18}$$

Indeed, denote $m_r = \min_{\partial B_r} v$. By the minimum principle, the set $U_r := \{x \in M : v(x) < m_r\}$ is inside B_r and contains K (see Fig. 10; recall that $v = 0$ on K).

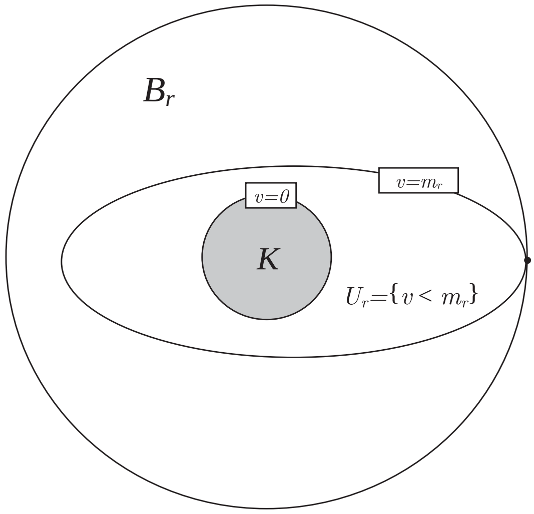

Figure 10 Set U_r.

The function $1-\frac{v}{m_r}$ is the equilibrium potential for the capacitor (K,U_r) whence

$$\operatorname{cap}(K,B_r)\le \operatorname{cap}(K,U_r)=\operatorname{flux}\left(\frac{v}{m_r}\right)=\frac{1}{m_r},$$

which yields the left hand side inequality of (4.18). The right hand side inequality is proved in the same way, considering $\max_{\partial B_r} v$.

Together, (4.17) and (4.18) imply

$$v(x)\approx \operatorname{cap}(K,B_r)^{-1},$$

for all $x\in\partial B_r$ and r large enough. Therefore, we are left to verify the following capacity estimate

$$\operatorname{cap}(K,B_r)^{-1}\approx\int_{r_0}^{r}\frac{s\,ds}{V(s)}. \tag{4.19}$$

The lower bound in (4.19) follows from (4.5) and $K\subset B_{r_0}$ because

$$\operatorname{cap}(K,B_r)^{-1}\ge \operatorname{cap}(B_{r_0},B_r)^{-1}\ge\frac{1}{2}\int_{r_0}^{r}\frac{s\,ds}{V(s)}.$$

Let us prove the upper bound

$$\operatorname{cap}(K,B_r)^{-1}\le C\int_{r_0}^{r}\frac{s\,ds}{V(s)}, \tag{4.20}$$

for r large enough. By the hypotheses (i), (ii) and by Lemma 4.3, we have

$$\operatorname{cap}(B_{r_0},B_r)^{-1}\le C\int_{r_0}^{r}\frac{s\,ds}{V(s)}.$$

Hence, it suffices to verify that, for r large enough,

$$\operatorname{cap}(B_{r_0},B_r)\le C\operatorname{cap}(K,B_r). \tag{4.21}$$

Indeed, if $r\to\infty$ then the equilibrium potential φ_r of the capacitor (K,B_r) increases and tends to 1, due to the parabolicity of (M,μ). Therefore, for large r, $\varphi_r\ge\frac{1}{2}$ on B_{r_0}. Let $\tilde\varphi_r$ be equilibrium potential of (B_{r_0},B_r). Then, by the maximum principle, we obtain $\tilde\varphi_r\le 2\varphi_r$ in $B_r\setminus B_{r_0}$ (see Fig. 11).

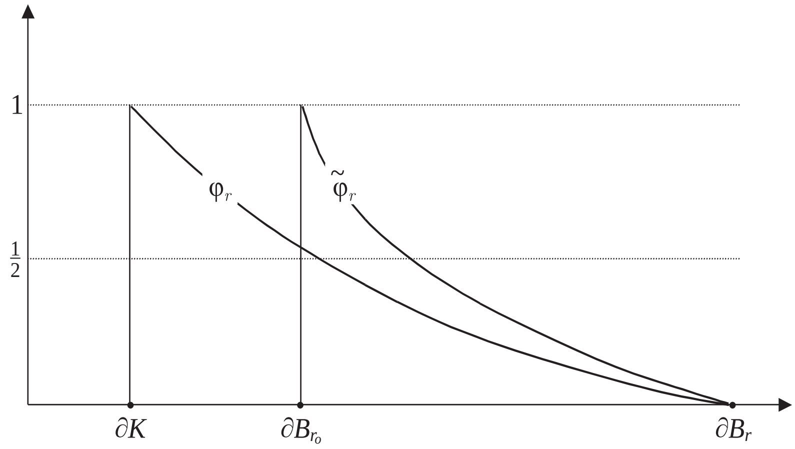

Figure 11 The equilibrium potentials satisfy the inequality $\tilde{\varphi}_r \leq 2\varphi_r$ in $B_r \setminus B_{r_0}$

Since both φ_r and $\tilde{\varphi}_r$ vanish on ∂B_r, we see that on ∂B_r

$$\frac{\partial \tilde{\varphi}_r}{\partial \mathbf{n}_-} \leq 2\frac{\partial \varphi_r}{\partial \mathbf{n}_-},$$

where $\mathbf{n}_-$ is the normal vector field on ∂B_r inward with respect to B_r. This implies

$$\operatorname{cap}(B_{r_0}, B_r) = \operatorname{flux}(-\tilde{\varphi}_r) \leq 2 \operatorname{flux}(-\varphi_r) = 2\operatorname{cap}(K, B_r),$$

which was to be proved. ■

4.4 h-transforms on parabolic manifolds and the Dirichlet heat kernel

We say that a subset $K \subset M$ is *admissible* if K is a non-empty compact set, and one of the following two conditions holds:

(1) Either $K \subset M \setminus \delta M$ and K is the closure of a non-empty open set in M, or

(2) $K = \delta M$ (in which case δM is compact).

Clearly, if K is admissible then K has locally positive capacity.

Fix a point $o \in M$ and denote as above $B_r = B(o,r)$ and $V(r) = V(o,r)$. Consider the following function

$$H(r) = 1 + \int_0^r \frac{se^{-1/s}}{V(s)} ds. \tag{4.22}$$

Observe that $H(r) \geq 1$ for all r, and

$$H(r) \approx 1 + \left(\int_1^r \frac{s\,ds}{V(s)} \right)_+ = 1 + \frac{1}{2} \left(\int_1^{r^2} \frac{dt}{V(\sqrt{t})} \right)_+ . \tag{4.23}$$

Denote also

$$D(t,x,y) = \frac{H(|x|)H(|y|)}{(H(|x|) + H(\sqrt{t}))(H(|y|) + H(\sqrt{t}))} \tag{4.24}$$

where $|x| = d(x,K)$. Recall that, if $\lim_{r\to\infty} H(r) = +\infty$, then (M,μ) is parabolic. Thus, if (M,μ) is non-parabolic, then $H(r) \approx 1$ and $D(t,x,y) \approx 1$.

The following lemma will be crucial for us. It was proved in [16, Proposition 2.19 and Theorem 2.24], using certain ideas from [19]. Here we give an independent proof based on Lemma 4.5.

Lemma 4.7 *Assume that (M, μ) is a complete non-compact parabolic weighted manifold satisfying* (PHI). *Assume also that the pointed manifold (M, o) satisfies the condition* (RCA), *and $K \subset M$ is an admissible set. Then there exists a positive smooth function h on M which is harmonic in $\Omega := M \setminus K$ and admits the estimate*

$$h(x) \approx H(|x|), \tag{4.25}$$

for all $x \in M$. Moreover, this function h satisfies the hypotheses (2.20) *of* Theorem 2.11.

Proof. Let us verify that all conditions (i)-(iii) of Lemma 4.5 hold under the hypotheses of Lemma 4.7. Indeed, by Theorem 2.8, (PHI) implies (i) and (ii). The annuli Harnack inequality (iii) is implied by (PHI) and (RCA) as follows. Connect any two points $x, y \in \partial B_R$ by a path as is guaranteed by (RCA). Then the doubling property (VD) implies that this path can be covered by a fixed number of ball of the radius comparable to R. Applying the elliptic Harnack inequality in each ball, we obtain (4.13).

If K has non-empty interior then let K_0 be a closed ball contained in the interior of K. By Lemma 4.5, we construct outside K_0 the harmonic function v. By changing v inside K, we can extend it in K so that v becomes a positive smooth function on M and still harmonic in Ω. In the case $K = \delta M$, the function v obtained by Lemma 4.5, is already smooth on M.

By Lemma 4.5 and (4.23), the function $h(x) = 1 + v(x)$ satisfies the estimate $h(x) \approx H(|x|)$ for large $|x|$ since $|x| \sim d(x, o)$. The condition (2.20) follows from the properties of H. Indeed, H is increasing and, for $r > 1$,

$$\begin{aligned} H(2r) &= 1 + \int_0^{2r} \frac{se^{-1/s}}{V(s)} ds = 1 + 2\int_0^r \frac{2\eta e^{-1/(2\eta)}}{V(2\eta)} d\eta \\ &\le C\left(1 + \int_1^r \frac{\eta\, d\eta}{V(\eta)}\right) \approx H(r). \end{aligned}$$

For small $|x|$, we have $h(x) \approx H(|x|)$ just because both functions restricted to small $|x|$ are bounded and separated from 0. ■

Lemma 4.8 *Referring to the setting and notation of* Lemma 4.7, *the weighted manifold (M, ν) with $d\nu = h^2 d\mu$ satisfies* (PHI) *and is non-parabolic. Moreover, $V^h(x, r) := \nu(B(x, r))$ satisfies the estimate*

$$V^h(x, r) \approx V(x, r)(H(|x|) + H(r))^2, \tag{4.26}$$

for all $x \in M$ and $r > 0$.

Proof. Lemma 4.7 and Theorem 2.11 immediately yield that (M, ν) satisfies (PHI). The non-parabolicity of (M, ν) follows from the fact that h is unbounded. Indeed, outside K we have, by Lemma 4.1,

$$\Delta^h(\frac{1}{h}) = \frac{1}{h}\Delta 1 = 0.$$

Hence, the function $w(x) = \frac{1}{h(x)}$ is Δ^h-harmonic outside a compact set and goes to 0 as $x \to \infty$. For any $r > 0$, denote

$$m_r = \min_{x \in \partial B_r} w(x).$$

As $m_r \to 0$, the minimum principle implies $w(x) \ge m_r$ in B_r, for r large enough. Fix such r. Then the function $x \mapsto \min(w(x), m_r)$ is Δ^h-superharmonic on the whole of M, positive and non-constant, which implies that (M, ν) is non-parabolic.

Finally, let us prove the volume estimate (4.26). Consider first the case $x \in K$. By (4.25), we obtain the upper bound for $V^h(x,r)$:

$$V^h(x,r) = \int_{B(x,r)} h^2 d\mu \leq V(x,r) \sup_{B(x,r)} h^2 \approx V(x,r) H^2(r).$$

For the lower bound, we first observe that, by (2.11), there exists $\varepsilon > 0$ such that

$$V(x, \varepsilon r) \leq \frac{1}{2} V(x,r),$$

for all $x \in M$ and $r > 0$. Hence,

$$V^h(x,r) \geq \int_{B(x,r) \setminus B(x,\varepsilon r)} h^2 d\mu \geq (V(x,r) - V(x,\varepsilon r)) \inf_{M \setminus B(x,\varepsilon r)} h^2 \approx V(x,r) H^2(r),$$

and (4.26) follows.

Assume now that $|x| \leq 2r$. Let $x' \in K$ be the nearest point to x so that $d(x,x') = |x|$. Then we apply (2.11) to V and V^h (observe that V^h is doubling by Theorem 2.11) and obtain

$$V(x,r) \approx V(x',r) \quad \text{and} \quad V^h(x,r) \approx V^h(x',r).$$

By by the previous case, we know (4.26) for x' whence we obtain the same for x.

In the remaining case $|x| > 2r$, all points of the ball $B(x,r)$ lie at distance $\approx |x|$ from K, whence

$$V^h(x,r) = \int_{B(x,r)} h^2 d\mu \approx V(x,r) H^2(|x|),$$

which was to be proved. ∎

We can now state and prove the main result of this section.

Theorem 4.9 *Assume that (M,μ) is a complete non-compact parabolic weighted manifold satisfying* (PHI). *Assume also that the pointed manifold (M,o) satisfies the condition* (RCA), *and $K \subset M$ is an admissible set. Then there exist positive numbers δ and c_i, C_i, $i = 1,2$, such that, for all $x, y \in M \setminus K_\delta$ and all $t > 0$,*

$$c_1 D(t,x,y) p(C_1 t, x, y) \leq p_\Omega(t,x,y) \leq C_2 D(t,x,y) p(c_2 t, x, y). \tag{4.27}$$

Remark 4.10 This theorem obviously contains Theorem 1.2 from Introduction.

Proof. Since (M,μ) satisfies (PHI), Theorem 2.8 yields the following estimate

$$p(t,x,y) \asymp \frac{1}{\sqrt{V(x,\sqrt{t}) V(y,\sqrt{t})}} e^{-d^2(x,y)/t},$$

which is the symmetric form of (TSE) (cf. Remark 2.7). In the view of that, it suffices to prove that

$$p_\Omega(t,x,y) \asymp \frac{D(t,x,y)}{\sqrt{V(x,\sqrt{t}) V(y,\sqrt{t})}} e^{-d^2(x,y)/t}. \tag{4.28}$$

Let $h(x)$ be the function on M obtained by Lemma 4.7. Define measure ν by $d\nu = h^2 d\mu$. By Lemma 4.8, the manifold (M,ν) satisfies (PHI) and is non-parabolic. Applying Theorem 2.8 to (M,ν), we obtain the following estimate for p^h:

$$p^h(t,x,y) \asymp \frac{1}{\sqrt{V^h(x,r) V^h(y,r)}} e^{-d^2(x,y)/t},$$

for all $t > 0$ and $x, y \in M$. Since (M, ν) is non-parabolic, Theorem 3.1 implies a similar estimate for p_Ω^h :

$$p_\Omega^h(t, x, y) \asymp \frac{1}{\sqrt{V^h(x, r)V^h(y, r)}} e^{-d^2(x,y)/t}, \tag{4.29}$$

for some $\delta > 0$ and all $t > 0$, $x, y \in M \setminus K_\delta$. By Proposition 4.2, we have

$$p_\Omega(t, x, y) = h(x)h(y)p_\Omega^h(t, x, y),$$

for all $t > 0$ and all $x, y \in \Omega$. Substituting here (4.29), using the estimate (4.26) for V^h, (4.25) for h and the definition (4.24) of function D, we obtain (4.28). ■

5 Examples and applications

5.1 The heat content

Keeping the same notation as above, consider as in the introduction the *heat content* of a set $\Omega \subset M$ at time t for a unit mass of heat originally concentrated at x, that is

$$\mathbf{C}_\Omega(t, x) = \int_\Omega p_\Omega(t, x, y)d\mu(y).$$

The following statement contains Theorem 1.3 from the Introduction.

Theorem 5.1 *Let (M, μ) be a complete non-compact weighted manifold satisfying* (PHI). *Let K be an admissible set, and set $\Omega = M \setminus K$. Then there exists $\delta > 0$ such that the following is true:*

(*i*) *If M is non-parabolic then, for all x with $|x| \geq \delta$,*

$$\mathbf{C}_\Omega(t, x) \approx 1. \tag{5.1}$$

(*ii*) *If M is parabolic and in addition satisfies* (RCA) *then*

$$\mathbf{C}_\Omega(t, x) \approx \frac{H(|x|)}{H(|x|) + H(\sqrt{t})}, \tag{5.2}$$

for all x with $|x| \geq \delta$, where H is given by (4.22).

Remark 5.2 Comparing (5.1), (5.2) with the definition (4.24) of the function D and with the estimates of Theorem 3.1 and 4.9, we see that those estimates can be written in the following way:

$$p_\Omega(t, x, y) \asymp \frac{\mathbf{C}_\Omega(t, x)\mathbf{C}_\Omega(t, y)}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}} e^{-d^2(x,y)/t}. \tag{5.3}$$

Proof. In the non-parabolic case, the estimate (3.7) together with (2.5) yields (5.1). Let (M, μ) be parabolic. By the upper bound of Theorem 4.9 and (2.5), we obtain

$$\mathbf{C}_\Omega(t, x) \leq C \int_\Omega D(t, x, y)p(ct, x, y)d\mu(y) \leq C \sup_{y \in \Omega} D(t, x, y) \approx \frac{H(|x|)}{H(|x|) + H(\sqrt{t})}.$$

The lower bound in (5.2) can be derived by direct integration of the estimate (4.27). Here we give a shorter proof using a fragment from the proof of Theorem 3.3. Following the notation introduced of the proof of Theorem 4.9, we have

$$\begin{aligned}\mathbf{C}_\Omega(t,x) &= \int_\Omega p_\Omega^h(t,x,y)h(x)h(y)d\mu(y) = \int_\Omega p_\Omega^h(t,x,y)\frac{h(x)}{h(y)}d\nu(y) \\ &\geq \frac{h(x)}{\sup_{B(x,\sqrt{t})} h(y)} \int_{B(x,\sqrt{t})} p_\Omega^h(t,x,y)d\nu(y). \end{aligned} \tag{5.4}$$

Since the manifold (M,ν) is non-parabolic and satisfies (PHI), we can apply to p_Ω^h the estimate (3.9) from the proof of Theorem 3.3. Assuming that $|x|>\delta$ with δ from that proof, we obtain

$$\int_{B(x,\sqrt{t})} p_\Omega^h(t,x,y)d\nu(y) \approx 1.$$

Finally, substituting into (5.4) the estimates $h(x) \approx H(|x|)$ and

$$h(y) \approx H(|y|) \leq H(|x|+\sqrt{t}) \leq C\left(H(|x|)+H(\sqrt{t})\right),$$

we obtain the lower bound in (5.2). ■

The non-parabolic case in Theorem 5.1 admits the following alternative proof, without using the Dirichlet heat kernel estimates. Denote by $\psi_K(t,x)$ the probability that the Brownian motion in (M,μ) started at the point x, first hits K before time t, that is

$$\psi_K(t,x) = \mathbb{P}_x(\tau_K \leq t).$$

As (M,μ) is stochastically complete, we have by (2.7) $\mathbf{C}_\Omega(t,x) = 1-\psi_K(t,x)$. This implies that $\mathbf{C}_\Omega(t,x)$ is decreasing in t, and

$$\lim_{t\to+\infty} \mathbf{C}_\Omega(t,x) = 1-\psi_K(x). \tag{5.5}$$

In the parabolic case, $\psi_K(x)\equiv 1$, and (5.5) is not useful. In the non-parabolic case, Lemma 2.9 says that $\psi_K(x)\to 0$ as $x\to\infty$ which immediately implies that $\mathbf{C}_\Omega(t,x)\approx 1$ as long as x stays away from K.

5.2 Surfaces of revolution

Consider the polar coordinates $x=(r,\theta)$ around the origin in $\mathbb{R}^2$ and the following Riemannian metric

$$dr^2 + f^2(r)d\theta^2$$

where $f(r)$ is a smooth positive function on $(0,+\infty)$. Let $M=\mathbb{R}^2\setminus B(0,1)$ be the manifold with boundary, equipped with this metric, and μ be the Riemannian measure on M. In this section, we will show how to obtain heat kernel estimates on such manifolds.

Obviously, (M,o) satisfies (RCA), for any point $o\in M$. The main difficulty lies in proving (PHI), so we have to restrict ourselves to those f for which (PHI) is known. It is proved in [13] that the parabolic Harnack inequality (PHI) holds on M for the following two classes of f:

(1) $f(r)=r^\alpha$ with $\alpha\in(-1,1]$

(2) $f(r)=r(1+\log r)^{-\beta}$ with $\beta>0$.

We assume in the sequel that f is one of the functions in (1) and (2). Note that $r \geq 1$ on M and $f(1) = 1$. It is easy to verify that the volume function on M admits the following estimate, for all $x = (r, \theta) \in M$ and $\tau > 0$,

$$V(x,\tau) \approx \begin{cases} \tau^2, & \text{if } \tau \leq f(r), \\ \tau f(r), & \text{if } f(r) \leq \tau \leq r, \\ \tau f(\tau), & \text{if } \tau \geq r, \end{cases}$$

which can be expressed in a more compact form as follows

$$V(x,\tau) \approx \tau \min\left[\tau, f\left(\max(\tau, r)\right)\right] \approx \frac{\tau^2}{1 + \frac{\tau}{f(\tau+r)}}.$$

Since $V(x,\tau) \leq C\tau^2$ for all τ, we conclude that (M,μ) is parabolic. Since (M,μ) satisfies (PHI), we obtain by Theorem 2.8 the following estimate of the global heat kernel on (M,μ):

$$p(t,x,y) \asymp \frac{1}{t}\left(1 + \frac{\sqrt{t}}{f(|x| + \sqrt{t})}\right)^{\frac{1}{2}}\left(1 + \frac{\sqrt{t}}{f(|y| + \sqrt{t})}\right)^{\frac{1}{2}} e^{-d^2(x,y)/t},$$

where we set $|x| := r$ for $x = (r, \theta)$.

Let $K = \delta M = \{(r,\theta) : r = 1\}$ and $\Omega = M \setminus K$. For any point $o \in K$ and $s \geq 1$, we have $V(o,s) \approx s f(s)$ so that we obtain by (4.23)

$$H(\tau) \approx 1 + \left(\int_1^\tau \frac{ds}{f(s)}\right)_+.$$

Hence, computing the integral, we obtain

$$H(\tau) \approx \begin{cases} 1 + \tau^{1-\alpha}, & \text{if } f(r) = r^\alpha \text{ and } \alpha \in (-1,1), \\ 1 + \log_+ \tau, & \text{if } f(r) = r, \\ (1 + \log_+ \tau)^{1+\beta}, & \text{if } f(r) = r(1 + \log r)^{-\beta} \text{ and } \beta > 0. \end{cases}$$

Here we use the following notation: $\log_+ \tau = \log \tau$ if $\tau > 1$, and $\log_+ \tau = 0$ if $\tau \leq 1$.

By (5.2), the heat content admits the following estimates corresponding to the above cases, for all $t > 0$ and large enough $|x|$:

$$\mathbf{C}_\Omega(t,x) \approx \begin{cases} \left(\frac{|x|}{|x| + \sqrt{t}}\right)^{1-\alpha} & \text{if } \alpha \in (-1,1), \\ \frac{\log|x|}{\log|x| + \log_+ t} & \text{if } \alpha = 1, \\ \left(\frac{\log|x|}{\log|x| + \log_+ t}\right)^{1+\beta} & \text{if } \beta > 0. \end{cases}$$

Combining the bounds for $C_\Omega(t,x)$ and $V(x, \sqrt{t})$, we obtain by (5.3) the estimate of the Dirichlet heat kernel p_Ω . For example, in the case $\alpha \in (-1,1)$, it has the form

$$p_\Omega(t,x,y) \asymp \frac{1}{t} \frac{(|x|\,|y|)^{1-\alpha}\left[1 + \frac{\sqrt{t}}{(|x| + \sqrt{t})^\alpha}\right]^{1/2}\left[1 + \frac{\sqrt{t}}{(|y| + \sqrt{t})^\alpha}\right]^{1/2}}{\left((|x| + \sqrt{t})(|y| + \sqrt{t})\right)^{1-\alpha}} e^{-d^2(x,y)/t}.$$

The estimate in the case $\alpha = 1$ is the same as in the Example 1.

5.3 Bodies of revolution

Let (r, u, v) be Cartesian coordinates in $\mathbb{R}^3$. Given a smooth positive function $f(r)$ on $(0, +\infty)$, consider the following domain of revolution in $\mathbb{R}^3$ (see Fig. 12):

$$M = \{(r, u, v) \in \mathbb{R}^3 : r \geq 0, \ \sqrt{u^2 + v^2} \leq f(r)\}.$$

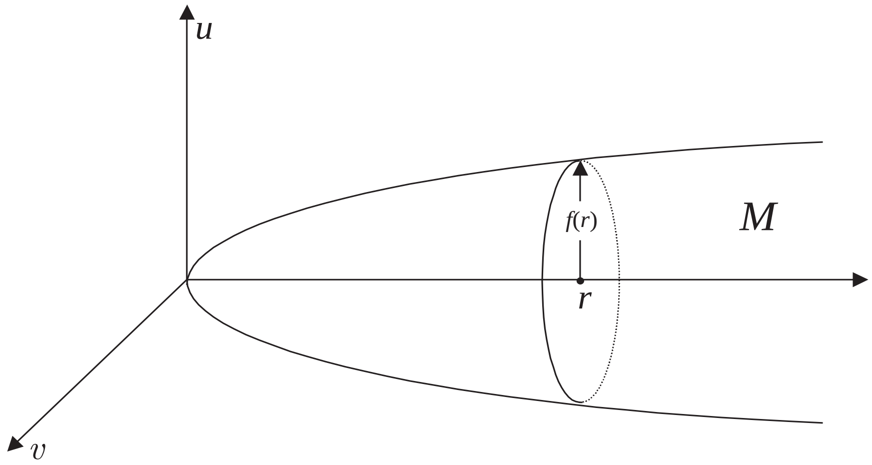

Figure 12 The domain of revolution.

If f possesses certain regularity at $r = 0$ (in particular, $f(0) = 0$) then M can be regarded as a manifold with boundary. Let us endow M with the Euclidean metric and the Lebesgue measure μ. Assume in the sequel that f is concave, that is $f'' \leq 0$. Then M is convex as a subset of $\mathbb{R}^3$, and the result of [20] and [9] implies that M satisfies (PHI). Clearly, (M, o) satisfies (RCA) for any $o \in M$. Computing the volume function on M yields, for any $x = (r, u, v)$ and $\tau > 0$,

$$V(x, \tau) \approx \begin{cases} \tau^3, & \text{if } \tau \leq f(r), \\ \tau f^2(r), & \text{if } f(r) \leq \tau \leq r, \\ \tau f^2(\tau), & \text{if } \tau \geq r, \end{cases}$$

that is

$$V(x, \tau) \approx \tau \left(\min\left[\tau, f\left(\max(\tau, r)\right)\right]\right)^2 \approx \frac{\tau^3}{1 + \frac{\tau^2}{f^2(r+\tau)}}.$$

By Theorem 2.8, we obtain, for all $t > 0$ and $x, y \in M$,

$$p(t, x, y) \asymp \frac{1}{t^{3/2}} \left(1 + \frac{t}{f^2(|x| + \sqrt{t})}\right)^{1/2} \left(1 + \frac{t}{f^2(|y| + \sqrt{t})}\right)^{1/2} e^{-d^2(x,y)/t},$$

where $|x| = r$ if $x = (r, u, v)$. In particular, this implies that M is parabolic if and only if

$$\int^{\infty} \frac{ds}{f^2(s)} = \infty.$$

Let us now specify the function $f(r)$ for $r \geq 1$ as follows: $f(r) = \sqrt{r(1 + \log r)^{\alpha}}$. Then M is parabolic if and only if $\alpha \leq 1$ which will be assumed in the sequel. Set $K = \{(r, u, v) \in M : r \leq 1\}$, $\Omega = M \setminus K$ and fix a point $o \in \partial K$. Then, for any $s > 1$, we have $V(o, s) \approx s^2 (1 + \log s)^{\alpha}$, so that (4.23) yields

$$H(\tau) \approx 1 + \left(\int_1^{\tau} \frac{ds}{s(1 + \log s)^{\alpha}}\right)_+ \approx \begin{cases} (1 + \log_+ \tau)^{1-\alpha}, & \text{if } \alpha < 1 \\ 1 + \log_+ \log_+ \tau, & \text{if } \alpha = 1. \end{cases}$$

The estimate (5.2) of the heat content reads, for all $t > 0$ and large enough $|x|$,

$$\mathbf{C}_\Omega(t,x) \approx \begin{cases} \left(\frac{\log|x|}{\log|x|+\log_+ t}\right)^{1-\alpha} & \alpha < 1, \\ \frac{\log_+\log_+|x|}{\log_+\log_+|x|+\log_+\log_+ t} & \alpha = 1. \end{cases}$$

The estimate of p_Ω follows by (5.3).

5.4 Manifolds with ends

Theorem 3.1 and Corollary 3.4 show that non-parabolic manifolds satisfying (PHI) must have only one end. In general, manifolds satisfying (PHI) must have only finitely many ends. Obviously, the connectedness hypothesis (RCA) implies that M has only one end. In view of Theorem 4.9, it is then natural to ask what happens if M satisfies (PHI), is parabolic, but has more than one end. This is a rather subtle question and we will only give a partial answer.

Given a weighted Riemannian manifold (M,μ) with k ends, let U be a relatively compact open set in M with smooth boundary such that $M \setminus U$ has n unbounded connected components $E_1,\ldots,E_n$, $n \le k$ which we call the ends of M relative to U (see Fig. 13). We assume that either U does not intersect δM or U contains δM.

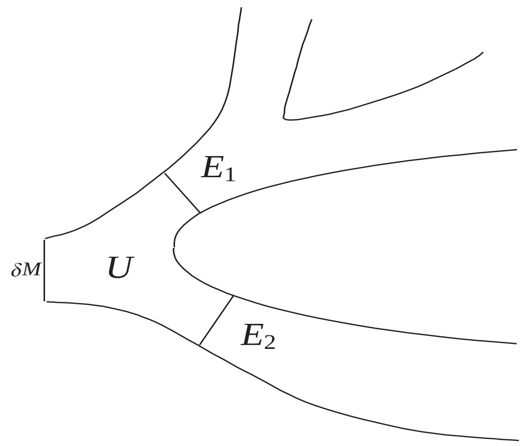

Figure 13 Manifold with ends ($k = 3$, $n = 2$).

We can regard each $(E_i,\mu) = (E_i, \mu|_{E_i})$ as a weighted manifold with boundary. Note that, if (M,μ) is complete and non-compact then each E_i is complete and non-compact. It is also well-known (see [10]) that (M,μ) is parabolic if and only if all (E_i,μ) are parabolic.

Let $K_i = \partial U \cap E_i$ and $\Omega_i = E_i \setminus K_i$. Denote by p_i the heat kernel on E_i and by p_{Ω_i} the Dirichlet heat kernel in Ω_i. For each E_i, fix a point $o_i \in K_i$ and define the function H_i, D_i relative to (E_i,μ) and $K_i \subset E_i$ by (1.4) and (1.5). Let also $V_i(x,t)$ be the volume function on E_i. The following theorem is combination of the results of this paper and [13].

Theorem 5.3 *Assume that (M,μ) is a complete non-compact weighted manifold satisfying* (PHI)*. Assume that (M,μ) is parabolic. Referring to the notation introduced above, let E_i be one of the ends relative to U and assume that (E_i,o_i) satisfies the connectedness condition* (RCA)*. Then (E_i,μ) satisfies* (PHI)*. In particular, the heat kernel p_i of (E_i,μ) satisfies*

$$p_i(t,x,y) \asymp \frac{1}{V_i(x,\sqrt{t})} e^{-d^2(x,y)/t},$$

for all $t > 0$ and $x, y \in E_i$, and the Dirichlet heat kernel on Ω_i satisfies

$$p_{\Omega_i} \asymp \frac{D_i(t,x,y)}{V_i(x,\sqrt{t})} e^{-d^2(x,y)/t}, \tag{5.6}$$

for all $t > 0$ and all $x, y \in \Omega_i$ with large enough $|x|, |y|$.

Proof. The fact that the weighted manifold (E_i, μ) satisfies (PHI) is proved in [13]. Then the estimate (5.6) holds true by Theorem 4.9 applied to the manifold (E_i, μ). ■

We caution the reader that the above result cannot be extended to the case where E_i does not satisfies (RCA). For instance, consider the infinite cylinder $M = \mathbb{R}^1 \times \mathbb{S}^1$. Let U be a small ball around $(0,0) \in \mathbb{R}^1 \times \mathbb{S}^1$ so that $M \setminus U$ is connected. Then it is shown in [12] that p_{Ω_i} satisfies a different estimate which is not compatible with (5.6).

We end this section by the proof of Theorem 1.4. It resembles the one of Theorem 5.3 and also relies on results of [13]. However, under the hypotheses of Theorem 1.4, the Riemannian manifold M with the Riemannian measure μ does not necessarily satisfy (PHI), and is not necessarily parabolic. Using the notation above, we assume as in Theorem 1.4 that M as exactly k ends $E_1, \ldots, E_k$, that is $n = k$. Then it follows from [13] that each (E_i, μ) satisfies (PHI) and (RCA). Thus, Theorem 1.4 follows from Theorems 3.1 and 4.9. The curvature assumptions of conditions 1 and 2 in Theorem 1.4 are used in [13] in three crucial different ways:

(1) They provide some parabolic Harnack inequalities for balls that are far away from K_i in each E_i,

(2) they imply that each E_i satisfies condition (RCA), and

(3) they provide a certain control of the volume growth of each end E_i.

One of the main results of [13] says that these three ingredients imply the parabolic Harnack inequality (PHI) on E_i.

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