# Lectures on path homology theory of digraphs

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# Preface

This text is based on a series of lectures that I delivered at the online joint seminar of Tsinghua University and Bielefeld University in Spring 2022. The purpose of those lectures was to introduce to young researchers a new emerging area of research – the theory of *path homology* on digraphs.

There exists a number of ways to define the notion of homology for graphs and digraphs, for example, clique homology ([**16**], [**51**]) or singular homology ([**5**], [**51**], [**60**]). However, the notion of path homology has certain advantages as it enjoys the adequate functorial properties with respect to graph-theoretical operations, such as morphisms of digraphs, Cartesian products, joins, homotopy etc.

The concept of path homology is derived from the concept of a *path chain complex* that is non-trivial and highly interesting by itself as it encodes a lot of information about the underlying digraph. Based on the path chain complex, we define also the notions of *combinatorial curvature* of digraphs (that is analogous to the Gauss curvaature), as well as *Hodge Laplacians* acting on the chain spaces. The study of spectra of Hodge Laplacians on digraphs is a new large area of research.

The notions of path homology and path chain complex have rich mathematical content, and I hope that they will become useful tools in various areas of pure and applied mathematics.

I have tried to keep here the presentation style of the online seminar, which, in particular, featured a wealth of examples and open problems. I give here an overview of the already published results in this field, state and prove some new results, as well as pose some open questions and conjectures.

The material on the following topics is new:
- random digraphs;
- path cochain complexes on products;
- intersection form and signature;

while the rest of the material is based on [**31**], [**33**], [**34**], [**35**], [**39**], [**41**], [**42**], [**44**], [**43**], [**45**].

A complete list of the topics covered is shown in the table of contents. Most of the material of the book should be accessible for undergraduate and graduate students with a solid background in Linear Algebra and a basic knowledge of Homological Algebra.

For further reading on path homology theory, its applications, and the related topics I recommend the following literature: [**1**], [**3**], [**4**], [**6**], [**7**], [**8**], [**9**], [**11**], [**12**], [**13**], [**14**], [**15**], [**17**], [**18**], [**19**], [**20**], [**21**], [**22**], [**26**], [**25**], [**29**], [**30**], [**32**], [**36**], [**37**], [**38**], [**40**], [**46**], [**48**], [**49**], [**50**], [**53**], [**56**], [**57**], [**59**], [**61**], [**62**], [**63**].

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