Path homologies of digraphs

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1 Paths in a finite set

Let V be a finite set. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V that will be denoted by $i_0...i_p$ or by $e_{i_0...i_p}$. A p-path over a field $\mathbb{K}$ is any formal $\mathbb{K}$-linear combinations of elementary p-paths, that is, any p-path has a form

$$u = \sum_{i_0, i_1, ..., i_p \in V} u^{i_0 i_1 ... i_p} e_{i_0 i_1 ... i_p}, \quad \text{where} \quad u^{i_0 i_1 ... i_p} \in \mathbb{K}.$$

Denote by $\Lambda_p = \Lambda_p(V)$ the $\mathbb{K}$-linear space of all p-paths. For example,

$$\begin{aligned} \Lambda_0 &= \operatorname{span}\{e_i : i \in V\} \\ \Lambda_1 &= \operatorname{span}\{e_{ij} : i, j \in V\} \\ \Lambda_2 &= \operatorname{span}\{e_{ijk} : i, j, k \in V\} \end{aligned}$$

Definition. Define for any $p \geq 1$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0...\widehat{i_q}...i_p},$$

where $\widehat{\ }$ means omission of the index. For $p = 0$ set $\partial e_i = 0$.

For example,

$$\partial e_{ij} = e_j - e_i \quad \text{and} \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}.$$

We claim that $\partial^2 = 0$. Indeed, for any $p \geq 2$ we have

$$\begin{aligned}
\partial^2 e_{i_0 \dots i_p} &= \sum_{q=0}^{p} (-1)^q \, \partial e_{i_0 \dots \widehat{i_q} \dots i_p} \\
&= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r \, e_{i_0 \dots \widehat{i_r} \dots \widehat{i_q} \dots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} \, e_{i_0 \dots \widehat{i_q} \dots \widehat{i_r} \dots i_p} \right) \\
&= \sum_{0 \leq r < q \leq p} (-1)^{q+r} \, e_{i_0 \dots \widehat{i_r} \dots \widehat{i_q} \dots i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} \, e_{i_0 \dots \widehat{i_q} \dots \widehat{i_r} \dots i_p}.
\end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0 \dots i_p} = 0$. This implies $\partial^2 u = 0$ for all $u \in \Lambda_p$.

Hence, we obtain a chain complex $\Lambda_*(V)$:

$$0 \leftarrow \Lambda_0 \stackrel{\partial}{\leftarrow} \Lambda_1 \stackrel{\partial}{\leftarrow} \dots \stackrel{\partial}{\leftarrow} \Lambda_{p-1} \stackrel{\partial}{\leftarrow} \Lambda_p \stackrel{\partial}{\leftarrow} \dots$$

Definition. An elementary p-path $e_{i_0...i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and irregular otherwise.

Let I_p be the subspace of Λ_p spanned by irregular $e_{i_0...i_p}$. We claim that $\partial I_p \subset I_{p-1}$. Indeed, if $e_{i_0...i_p}$ is irregular then $i_k = i_{k+1}$ for some k. We have

$$\begin{aligned} \partial e_{i_0...i_p} &= e_{i_1...i_p} - e_{i_0 i_2...i_p} + ... \\ &\quad + (-1)^k e_{i_0...i_{k-1} i_{k+1} i_{k+2}...i_p} + (-1)^{k+1} e_{i_0...i_{k-1} i_k i_{k+2}...i_p} \\ &\quad + ... + (-1)^p e_{i_0...i_{p-1}}. \end{aligned} \tag{1}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (1) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0...i_p} \in I_{p-1}$.

Hence, ∂ is well-defined on the quotient spaces $\mathcal{R}_p := \Lambda_p / I_p$, and we obtain the chain complex $\mathcal{R}_* (V)$:

$$0 \leftarrow \mathcal{R}_0 \stackrel{\partial}{\leftarrow} \mathcal{R}_1 \stackrel{\partial}{\leftarrow} \ldots \stackrel{\partial}{\leftarrow} \mathcal{R}_{p-1} \stackrel{\partial}{\leftarrow} \mathcal{R}_p \stackrel{\partial}{\leftarrow} \ldots$$

By setting all irregular p-paths to be equal to 0, we can identify $\mathcal{R}_p$ with the subspace of Λ_p spanned by all regular paths. For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}$$

because $e_{ii} = 0$.

2 Paths in a digraph

Definition. A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and a set $E \subset \{V \times V \setminus \mathrm{diag}\}$ of (directed) *edges*. If $(i, j) \in E$ then we write $i \to j$.

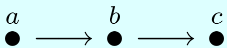
Definition. Let $G = (V, E)$ be a digraph. An elementary p-path $i_0...i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, ..., p - 1$, and *non-allowed* otherwise.

Let $\mathcal{A}_p = \mathcal{A}_p(G)$ be $\mathbb{K}$-linear space spanned by allowed elementary p-paths:

$$\mathcal{A}_p = \mathrm{span}\left\{e_{i_0...i_p} : i_0...i_p \text{ is allowed}\right\}.$$

The elements of $\mathcal{A}_p$ are called *allowed* p-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on subspaces $\mathcal{A}_p$ of $\mathcal{R}_p$. However, the spaces $\mathcal{A}_p$ are in general *not* invariant for ∂. For example, in the digraph



we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is not allowed.

Consider the following subspace of $\mathcal{A}_p$

$$\boxed{\Omega_p \equiv \Omega_p(G) := \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}}.$$

We claim that $\partial\Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial(\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

Definition. The elements of Ω_p are called *∂-invariant* p-paths or *currents*.

Hence, we obtain a chain complex $\Omega_* = \Omega_*(G)$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$.

Definition. *Path homologies* of G are defined as the homologies of the chain complex $\Omega_*(G)$:

$$H_p(G, \mathbb{K}) = H_p(G) := H_p(\Omega_*(G)) = \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

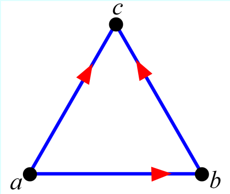
Betti numbers: $\beta_p(G) := \dim H_p(G)$. The Euler characteristic:

$$\chi(G) = \sum_{p=0}^{\infty} (-1)^p \beta_p(G) = \sum_{p=0}^{\infty} (-1)^p \dim \Omega_p(G).$$

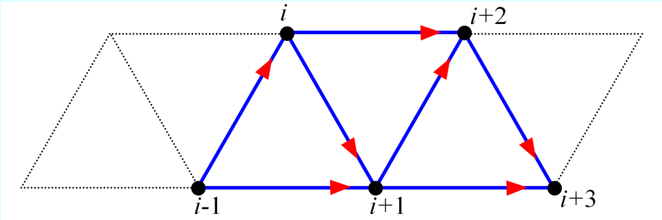
3 Examples of ∂-invariant paths

An 1-path e_{ab} is ∂-invariant if and only if it is allowed, that is, $a \to b$.

A *triangle* is a sequence of three vertices a, b, c such that $a \to b \to c$, $a \to c$
A triangle determines a 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$.



A *snake* of length $p \geq 2$ is a sequence of $p + 1$ vertices, say $0, 1, ..., p$, such that $i \to i + 1$ for all $i = 0, ..., p - 1$ and $i \to i + 2$ for all $i = 0, ..., p - 2$.



Then a p-path $u = e_{01...p}$ is ∂-invariant, because $u \in \mathcal{A}_p$ and

$$\partial u = \sum_{q=0}^{p} (-1)^q \, e_{0...(q-1)\widehat{q}(q+1)...p} \in \mathcal{A}_{p-1}, \quad \text{since } q - 1 \to q + 1.$$

A p-*simplex* is a sequence of $p+1$ vertices, say, $0, 1, ..., p$ such that $i \to j$ for all $i < j$. Equivalently, a p-simplex is a directed *clique*. A p-simplex contains a snake so that the p-path $e_{01...p}$ is ∂-invariant. Since

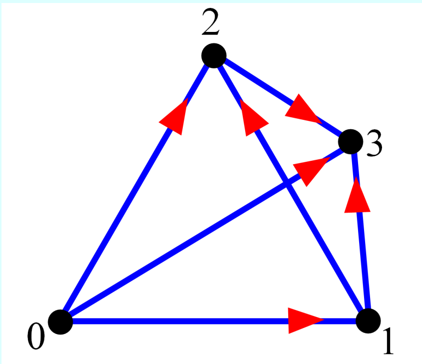
$$\partial e_{012...p} = e_{12...p} - e_{02...p} + ... + (-1)^p e_{01...(p-1)},$$

the boundary of p-simplex is an alternating sum of $(p-1)$-simplexes.

An 1-simplex is any arrow $a \to b$.

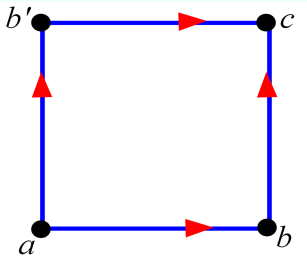
A 2-simplex is a triangle as above.

A 3-simplex is shown here:



A *square* is a sequence of four vertices a, b, b', c such that $a \to b$, $b \to c$, $a \to b'$, $b' \to c$.
A square determines a 2-path $u := e_{abc} - e_{ab'c} \in \Omega_2$ because $u \in \mathcal{A}_2$ and

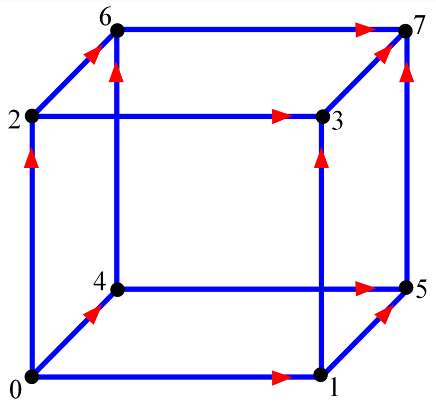
$$\partial u = \left(e_{bc} - \underline{e_{ac}} + e_{ab}\right) - \left(e_{b'c} - \underline{e_{ac}} + e_{ab'}\right)$$
$$= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1$$



A 3-*cube* is a sequence of 8 vertices, say, $0, 1, 2, 3, 4, 5, 6, 7$, connected by arrows as here.

A 3-cube determines a ∂-invariant 3-path

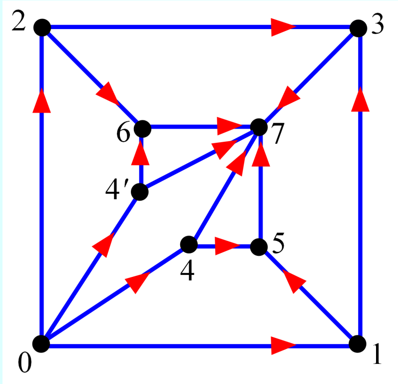
$$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}$$



Indeed, $u \in \mathcal{A}_3$ and

$$\begin{aligned} \partial u &= (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267}) \\ &\quad - (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2. \end{aligned}$$

An *exotic cube* is this subgraph containing 9 vertices and 15 edges. It is obtained from 3-cube by "splitting" the vertex 4 into 4, 4′ and adding the edges $4 \to 7$, $4' \to 7$.



The exotic cube determines the following ∂-invariant 3-path:

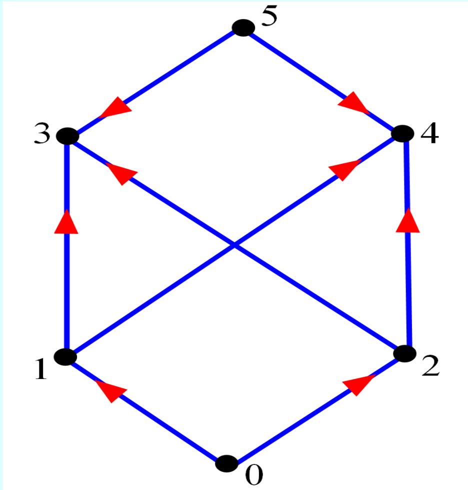
$$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{04'67} - e_{0267}.$$

Indeed, we have $u \in \mathcal{A}_3$ and

$$\begin{aligned} \partial u \;=\; & e_{237} - \underline{e_{037}} + \underline{e_{027}} - e_{023} \\ & - e_{137} + \underline{e_{037}} - \underline{e_{017}} + e_{013} \\ & + e_{157} - \underline{e_{057}} + \underline{e_{017}} - e_{015} \\ & - e_{457} + \underline{e_{057}} - e_{047} + e_{045} \\ & + e_{4'67} - \underline{e_{067}} + e_{04'7} - e_{04'6} \\ & - e_{267} + \underline{e_{067}} - \underline{e_{027}} + e_{026} \in \mathcal{A}_2. \end{aligned}$$

4 Examples of digraphs and spaces Ω_p

Consider the following digraph with 6 vertices and 8 edges:



$\Omega_0 = \mathcal{A}_0 = \mathrm{span}\{e_0, e_1, e_2, e_3, e_4, e_5\}$,
$\Omega_1 = \mathcal{A}_1 = \mathrm{span}\{e_{01}, e_{02}, e_{13}, e_{14}, e_{23}, e_{24}, e_{53}, e_{54}\}$
Hence, $\dim \Omega_0 = 6$ and $\dim \Omega_1 = 8$

$\mathcal{A}_2 = \mathrm{span}\{e_{013}, e_{014}, e_{023}, e_{024}\}$, $\dim \mathcal{A}_2 = 4$
However, none of these 2-paths is ∂-invariant.

Ω_2 is spanned by two squares:
$\Omega_2 = \mathrm{span}\{e_{013} - e_{023},\ e_{014} - e_{024}\}$, $\dim \Omega_2 = 2$.

There are no allowed p-paths for any $p \geq 3$.
Hence, $\Omega_p = \mathcal{A}_p = \{0\}$ for all $p \geq 3$.

One computes $\dim H_0 = \dim H_1 = 1$ and $\dim H_p = 0$ for $p \geq 2$.

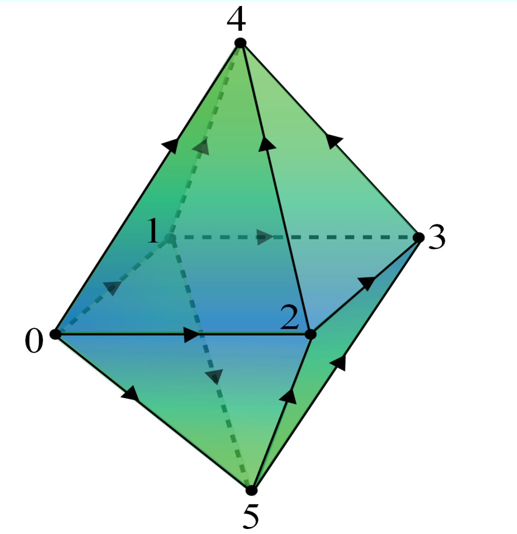
In fact, $H_0 = \mathrm{span}\{e_0\}$, $H_1 = \mathrm{span}\{e_{13} - e_{53} + e_{54} - e_{14}\}$.

The Euler characteristic: $\chi = \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2 = 6 - 8 + 2 = 0$.

Consider the following octahedral digraph with 6 vertices and 12 edges:

$\Omega_0 = \mathcal{A}_0 = \text{span}\{e_0, e_1, e_2, e_3, e_4, e_5\}$.
$\Omega_1 = \mathcal{A}_1 = \text{span}\{e_{01}, e_{02}, e_{04}, e_{05}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{34}, e_{52}, e_{53}\}$.
Hence, $\dim \Omega_0 = 6$, $\dim \Omega_1 = 12$.
$\mathcal{A}_2 = \text{span}\{e_{013}, e_{014}, e_{015}, e_{023}, e_{024}, e_{052}, e_{053}, e_{134}, e_{152}, e_{153}, e_{234}, e_{523}, e_{524}, e_{534}\}$.

Space Ω_2 is spanned by 8 triangles:
$e_{014}, e_{015}, e_{024}, e_{052}, e_{134}, e_{153}, e_{234}, e_{523}$
and 3 squares:
$e_{013} - e_{023}, \quad e_{013} - e_{053}, \quad e_{524} - e_{534}.$
Hence, $\dim \Omega_2 = 8 + 3 = 11$.



Space Ω_3 is spanned by five ∂-invariant 3-paths:
$e_{0153}, \quad e_{0523}, \quad e_{5234}, \quad e_{0134} - e_{0234}, \quad e_{0534} - e_{0134} - e_{0524}.$
Hence, $\dim \Omega_3 = 5$.

$\Omega_4 = \text{span}\{e_{05234}\}$. Hence, $\dim \Omega_4 = 1$.

There is only 1 allowed 5-path e_{015234} but it is not ∂-invariant. Hence, $\Omega_p = \{0\} \ \forall p \geq 5$.

The Euler characteristic is

$$\chi = \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2 - \dim \Omega_3 + \dim \Omega_4 = 6 - 12 + 11 - 5 + 1 = 1.$$

One can show that $\dim H_0 = 1$ and $\dim H_p = 0$ for all $p \geq 1$, which confirms $\chi = 1$.

Here is a verification of the ∂-invariance of five 3-paths and the 4-path:

$$\begin{aligned}
\partial e_{0153} &= e_{153} - e_{053} + e_{013} - e_{015} \in \mathcal{A}_2 \\
\partial e_{0523} &= e_{523} - e_{023} + e_{053} - e_{052} \in \mathcal{A}_2 \\
\partial e_{5234} &= e_{234} - e_{534} + e_{524} - e_{523} \in \mathcal{A}_2 \\
\partial \left(e_{0134} - e_{0234}\right) &= e_{134} - \underline{e_{034}} + e_{014} - e_{013} \\
&\quad - e_{234} + \underline{e_{034}} - e_{024} + e_{023} \\
&= e_{134} + e_{014} - e_{013} - e_{234} - e_{024} + e_{023} \in \mathcal{A}_2 \\
\partial \left(e_{0534} - e_{0134} - e_{0524}\right) &= e_{534} - \underline{e_{034}} + \underline{e_{054}} - e_{053} \\
&\quad - e_{134} + \underline{e_{034}} - e_{014} + e_{013} \\
&\quad - e_{524} + e_{024} - \underline{e_{054}} + e_{052} \\
&= e_{534} - e_{053} - e_{134} - e_{014} + e_{013} - e_{524} + e_{024} + e_{052} \in \mathcal{A}_2 \\
\partial e_{05234} &= e_{5234} - e_{0234} + e_{0534} - e_{0524} + e_{0523} \in \mathcal{A}_3
\end{aligned}$$

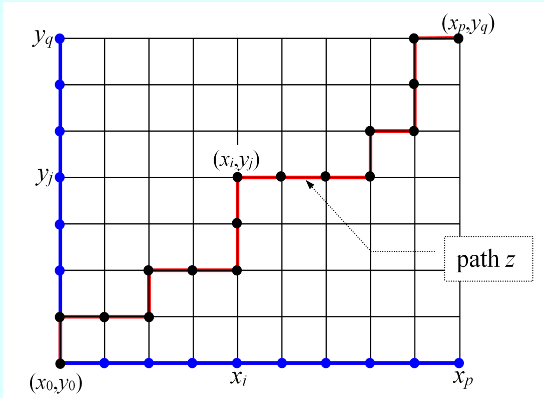
5 Cross product of paths

Given two finite sets X, Y, consider their product

$$Z = X \times Y = \{(a,b) : a \in X \text{ and } b \in Y\}.$$

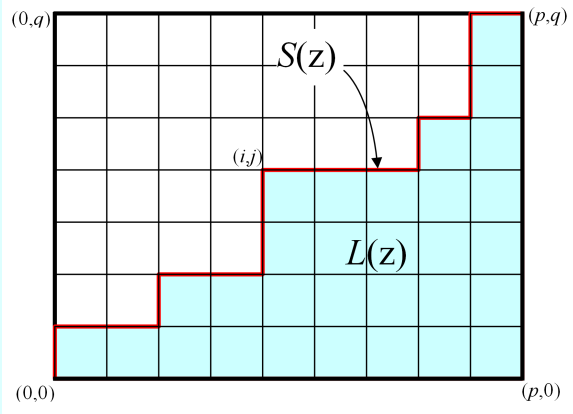
Let $z = z_0 z_1 ... z_r$ be a regular elementary r-path on Z, where $z_k = (a_k, b_k)$ with $a_k \in X$ and $b_k \in Y$. We say that z is *stair-like* if, for any $k = 1, ..., r$, either $a_{k-1} = a_k$ or $b_{k-1} = b_k$ is satisfied. That is, any couple $z_{k-1} z_k$ of consecutive vertices is either vertical (when $a_{k-1} = a_k$) or horizontal (when $b_{k-1} = b_k$).

Given a stair-like path z on Z, define its projection onto X as an elementary path x on X obtained from z by removing Y-components in all the vertices of z and then by collapsing in the resulting sequence any subsequence of repeated vertices to one vertex.
In the same way define projection of z onto Y and denote it by y.
Projections $x = x_0 ... x_p$ and $y = y_0 ... y_q$ are regular elementary paths, and $p + q = r$.



Every vertex (x_i, y_j) of path z can be represented as a point (i, j) of $\mathbb{Z}^2$ so that path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting points $(0, 0)$ and (p, q).

Define the *elevation* $L(z)$ of z as the number of cells in $\mathbb{Z}_+^2$ below the staircase $S(z)$.



For given elementary regular paths x on X and y on Y, denote by $\Sigma_{x,y}$ the set of all stair-like paths z on Z whose projections on X and Y are respectively x and y.

Definition. Define the *cross product* of the paths e_x and e_y as a path $e_x \times e_y$ on Z as follows:

$$e_x \times e_y = \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)} e_z. \tag{2}$$

Then extend the cross product by linearity to all paths $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ so that $u \times v \in \mathcal{R}_{p+q}(Z)$.

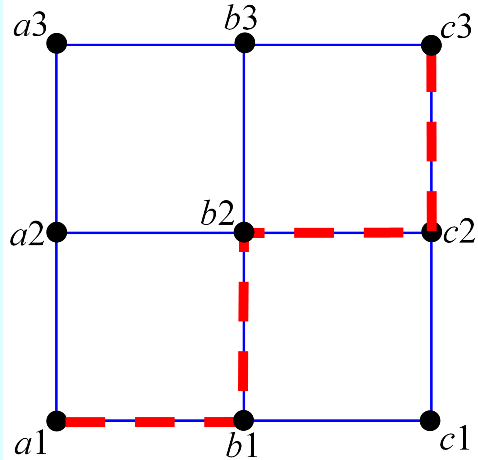
Example. Let us denote the vertices on X by letters a, b, c etc and the vertices on Y by integers $1, 2, 3$, etc so that the vertices on Z can be denoted as $a1, b2$ etc as the fields on the chessboard. Then we have

$$e_a \times e_{12} = e_{a1a2}, \quad e_{ab} \times e_1 = e_{a1b1}$$

$$e_{ab} \times e_{12} = e_{a1b1b2} - e_{a1a2b2}$$

$$e_{ab} \times e_{123} = e_{a1b1b2b3} - e_{a1a2b2b3} + e_{a1a2a3b3}$$

$$e_{abc} \times e_{123} = e_{a1b1c1c2c3} - e_{a1b1b2c2c3} + e_{a1b1b2b3c3}$$
$$+ e_{a1a2b2c2c3} - e_{a1a2b2b3c3} + e_{a1a2a3b3c3}$$



Proposition 1 *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

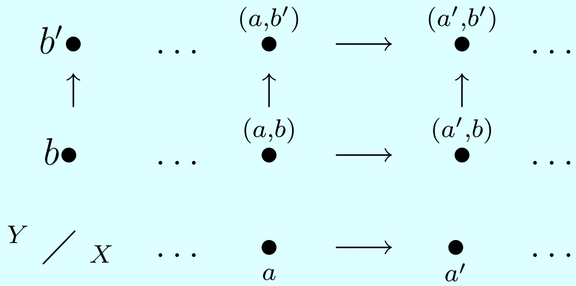
$$\partial(u \times v) = (\partial u) \times v + (-1)^p \, u \times (\partial v).$$

6 Cartesian product of digraphs

Denote a digraph and its set of vertices by the same letters to simplify notation. Given two digraphs X and Y, define there Cartesian product as a digraph $Z = X \square Y$ as follows:

- the set of vertices of Z is $X \times Y$, that is, the vertices of Z are the couples (a, b) where $a \in X$ and $b \in Y$;

- the edges in Z are of two types: $(a, b) \to (a', b)$ where $a \to a'$ (a *horizontal* edge) and $(a, b) \to (a, b')$ where $b \to b'$ (a *vertical* edge):



It follows that any allowed elementary path in Z is stair-like.

Moreover, any regular elementary path on Z is allowed if and only if it is stair-like and its projections onto X and Y are allowed.

It follows from definition (2) of the cross product that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \quad \Rightarrow \quad u \times v \in \mathcal{A}_{p+q}(Z). \tag{3}$$

Furthermore, the following is true.

Proposition 2 *If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_{p+q}(Z)$.*

Proof. $u \times v$ is allowed by (3). Since ∂u and ∂v are allowed, by (3) also $\partial u \times v$ and $u \times \partial v$ are allowed. By the product rule, $\partial(u \times v)$ is also allowed. Hence, $u \times v \in \Omega_{p+q}(Z)$. ■

Theorem 3 (Main Theorem) *Then any ∂-invariant path w on $Z = X \Box Y$ admits a representation in the form*

$$w = \sum_{i=1}^{k} u_i \times v_i$$

for some finite k, where u_i and v_i are ∂-invariant paths on X and Y, respectively.

Theorem 4 (Künneth formula) *Let X, Y be two finite digraphs and $Z = X \Box Y$. Then we have the following isomorphism of the chain complexes:*

$$\Omega_* (Z) \cong \Omega_* (X) \otimes \Omega_* (Y) . \tag{4}$$

It is given by the map $u \otimes v \mapsto u \times v$ with $u \in \Omega_* (X)$ and $v \in \Omega_* (Y)$.

A more detailed version of (4) is the following: for any $r \geq 0$,

$$\Omega_r (Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} (\Omega_p (X) \otimes \Omega_q (Y)) . \tag{5}$$

By an abstract theorem of Künneth, we obtain from (4)

$$H_* (Z) \cong H_* (X) \otimes H_* (Y) ,$$

that is, for any $r \geq 0$,

$$H_r (Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} (H_p (X) \otimes H_q (Y)) . \tag{6}$$

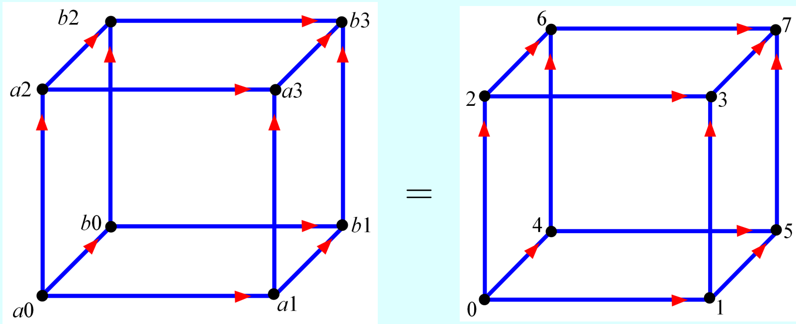
Consequently, $\beta_r (Z) = \sum_{\{p,q \geq 0 : p+q=r\}} \beta_p (X) \beta_q (Y)$.

Example. Consider the digraph $Z = X \Box Y$ where X is an interval and Y is a square:

$$X = {}^{a}\bullet \longrightarrow \bullet^{b} \quad \text{and} \quad Y = \begin{array}{ccc} {}_2\bullet & \longrightarrow & \bullet_3 \\ \uparrow & & \uparrow \\ {}_0\bullet & \longrightarrow & \bullet_1 \end{array}$$

Z has 8 vertices (i, j) where $i = a, b$, $j = 0, 1, 2, 3$. Let us enumerate them: $(a, i) \equiv i$ and $(b, i) \equiv i + 4$.

We see that Z is a 3-cube:



We have:

$$\Omega_1(X) = \operatorname{span}\{e_{ab}\}, \quad \Omega_p(X) = 0 \text{ for } p \geq 2,$$
$$\Omega_1(Y) = \operatorname{span}\{e_{01}, e_{13}, e_{23}, e_{02}\}, \; \Omega_2(Y) = \operatorname{span}\{e_{013} - e_{023}\}, \quad \Omega_q(Y) = 0 \text{ for } q \geq 3.$$

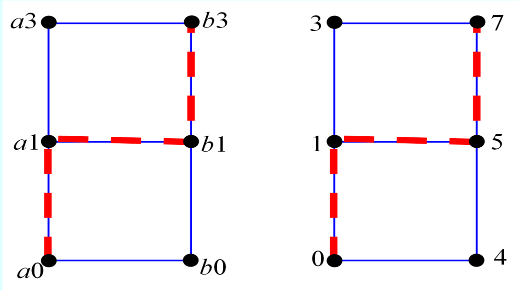
By (5) we obtain

$$\Omega_3(Z) \cong \Omega_1(X) \otimes \Omega_2(Y) = \operatorname{span}\{e_{ab} \times e_{013} - e_{ab} \times e_{023}\}.$$

$$e_{ab} \times e_{013} = e_{a0b0b1b3} - e_{a0a1b1b3} + e_{a0a1a3b3}$$
$$= e_{0457} - e_{0157} + e_{0137}$$

and

$$e_{ab} \times e_{023} = e_{0467} - e_{0267} + e_{0237}$$



Hence, we obtain

$$\Omega_3\left(Z\right) = \operatorname{span}\left\{e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}\right\}$$

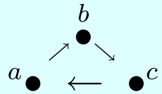
that is the ∂-invariant 3-path associated with 3-cube.

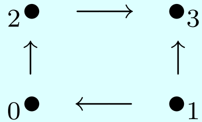
Define n-cube as follows:

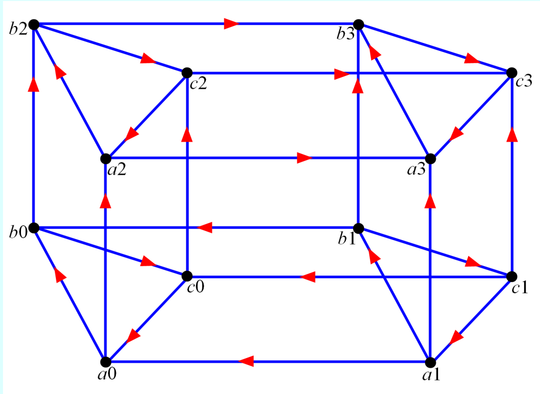
$$n\text{- cube} = \underbrace{I\Box I\Box ...\Box I}_{n},$$

where $I = {}^{a}\bullet \longrightarrow \bullet^{b}$. Similarly one shows that $\Omega_n\left(n\text{- cube}\right)$ is spanned by a single n-path that is an alternating sum of $n!$ elementary n-paths connecting the vertices 0 and $2^n - 1$. This corresponds to partitioning of a solid n-dim cube into $n!$ simplexes.

Example. Consider the digraph $Z = X \Box Y$ where

$X =$ 

$Y =$ 

$Z =$ 

One can show that

$$\begin{aligned} H_1(X) &= \operatorname{span}\{e_{ab} + e_{bc} + e_{ca}\}, & H_p(X) &= 0 \text{ for } p \geq 2 \\ H_1(Y) &= \operatorname{span}\{-e_{10} + e_{02} + e_{23} - e_{13}\}, & H_q(Y) &= 0 \text{ for all } q \geq 2 \end{aligned}$$

By (6) we obtain $H_1(Z) = H_0(X) \otimes H_1(Y) + H_1(X) \otimes H_0(Y) \cong \mathbb{K}^2$,

$$H_2(Z) \cong H_1(X) \otimes H_1(Y) = \operatorname{span}\{(e_{ab} + e_{bc} + e_{ca}) \times (-e_{10} + e_{02} + e_{23} - e_{13})\} \cong \mathbb{K},$$

and $H_r(Z) = 0$ for all $r \geq 2$.

7 Homotopy of digraphs

For vertices a, b of a digraph, write $a \,\overrightarrow{=}\, b$ if either $a \to b$ or $a = b$. Let X and Y be two digraphs.

Definition. A *mapping* $f : X \to Y$ called a *digraph map (or morphism)* if

$$a \to b \text{ on } X \quad \Rightarrow \quad f(a) \,\overrightarrow{=}\, f(b) \text{ on } Y.$$

Any digraph map $f : X \to Y$ induces a linear map

$$f_* \colon \mathcal{A}_p(X) \to \mathcal{A}_p(Y), \qquad f_*\left(e_{i_0 \ldots i_p}\right) = e_{f(i_0) \ldots f(i_p)}.$$

It is easy to check that $f_* \partial = \partial f_*$, which implies that f_* provides a morphism of chain complexes $f_* : \Omega_p(X) \to \Omega_p(Y)$ and, consequently, a homomorphism of homology groups $f_* : H_p(X) \to H_p(Y)$.

Definition. For any $n \geq 1$ define a *line digraph* I_n as any digraph with $n+1$ vertices $\{0, 1, \ldots, n\}$ and such that, for any $i = 0, ..., n-1$ holds either $i \to (i+1)$ or $(i+1) \to i$, and there is no other arrow.

Definition. Let X, Y be two digraphs. Two digraph maps $f, g \colon X \to Y$ are called *homotopic* if there exists a line digraph I_n and a digraph map $\Phi \colon X \Box I_n \to Y$ such that

$$\Phi|_{X \times \{0\}} = f \quad \text{and} \quad \Phi|_{X \times \{n\}} = g.$$

In this case we write $f \simeq g$. The map Φ is called a *homotopy* between f and g.

Definition. Two digraphs X and Y are called *homotopy equivalent* if there exist digraph maps

$$f : X \to Y, \quad g : Y \to X \tag{7}$$

such that

$$f \circ g \simeq \mathrm{id}_Y, \qquad g \circ f \simeq \mathrm{id}_X \,. \tag{8}$$

In this case we write $X \simeq Y$.

Theorem 5 *(i) Let $f, g \colon X \to Y$ be two digraph maps. If $f \simeq g$ then they induce the identical maps of homology groups:*

$$f_* : H_p(X) \to H_p(Y) \quad \textit{and} \quad g_* : H_p(X) \to H_p(Y).$$

(ii) If the digraphs X and Y are homotopy equivalent, then $H_*(X) \cong H_*(Y)$.

In particular, if a digraph X is contractible, that is, if $X \simeq \{*\}$, then all the homology groups of X are trivial except for H_0.

We say that a digraph Y is a *subgraph* of X if the set of vertices of Y is a subset of that of X and the arrows of Y are all those arrows of X whose adjacent vertices belong to Y.

Definition. Let X be a digraph and Y be its subgraph. A *retraction* of X onto Y is a digraph map $r : X \to Y$ such that $r|_Y = \mathrm{id}_Y$.

Theorem 6 *Let $r : X \to Y$ be a retraction of a digraph X onto a subgraph Y. Assume that*

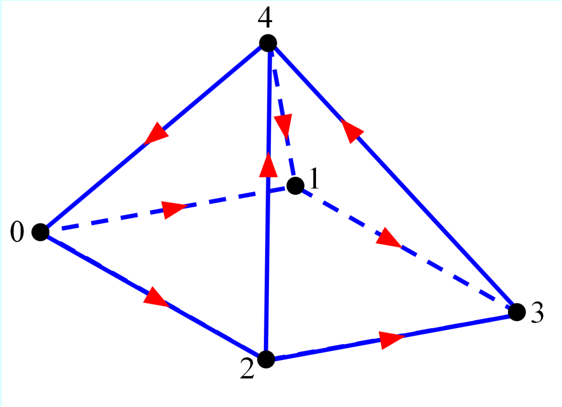
$$\text{either } x \overrightarrow{=} r(x) \text{ for all } x \in X \quad \text{or} \quad r(x) \overrightarrow{=} x \text{ for all } x \in X. \tag{9}$$

Then $X \simeq Y$ and, consequently, $H_*(X) \cong H_*(Y)$.

A retraction that satisfies (9) is called a *deformation retraction*.

Example. Let us show that n-cube is contractible. Indeed, a natural projection of n-cube onto $(n-1)$-cube is a deformation retraction. Hence, by induction we obtain n-cube $\simeq \{*\}$.

Example. Consider the digraph X as here.



Let Y be its subgraph with the vertex set $\{1, 3, 4\}$. Consider a retraction $r : X \to Y$ given by $r(0) = 1$, $r(2) = 3$. It is easy to see that r is a deformation retraction, whence $X \simeq Y$. Then we obtain

$$H_1(X) \cong H_1(Y) = \operatorname{span}\{e_{13} + e_{34} + e_{41}\} \cong \mathbb{K}$$

and $H_p(X) = \{0\}$ for $p \geq 2$.

8 Summary

Fix a finite set V and a field $\mathbb{K}$. For any $p \geq 0$, set $\mathcal{R}_p = \operatorname{span}_{\mathbb{K}} \left\{ e_{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is regular} \right\}$, where "regular" means that $i_k \neq i_{k+1}$ for all k. There is a boundary operator $\partial : \mathcal{R}_p \to \mathcal{R}_{p-1}$ such that $\partial^2 = 0$.

Let $G = (V, E)$ be a digraph. Set $\mathcal{A}_p = \operatorname{span}_{\mathbb{K}} \{ e_{i_0 \ldots i_n} : i_0 \ldots i_p \text{ is allowed} \} \subset \mathcal{R}_p$, where "allowed" means that $i_k \to i_{k+1}$ for all k.

Spaces of ∂-invariant paths: $\Omega_p = \left\{ u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1} \right\}$.

Chain complex $\Omega_* (G)$: $0 \leftarrow \Omega_0 \stackrel{\partial}{\leftarrow} \Omega_1 \stackrel{\partial}{\leftarrow} \ldots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \ldots$.

Path homology: $H_p (G) = \ker \partial|_{\Omega_p} / \operatorname{Im} \partial|_{\Omega_{p+1}}$.

Theorem 4 $\Omega_* (X \Box Y) \cong \Omega_* (X) \otimes \Omega_* (Y)$ *and* $H_* (X \Box Y) \cong H_* (X) \otimes H_* (Y)$

A mapping $f : X \to Y$ is called a digraph map if $a \to b$ in X implies $f (a) \overrightarrow{=} f (b)$ in Y.

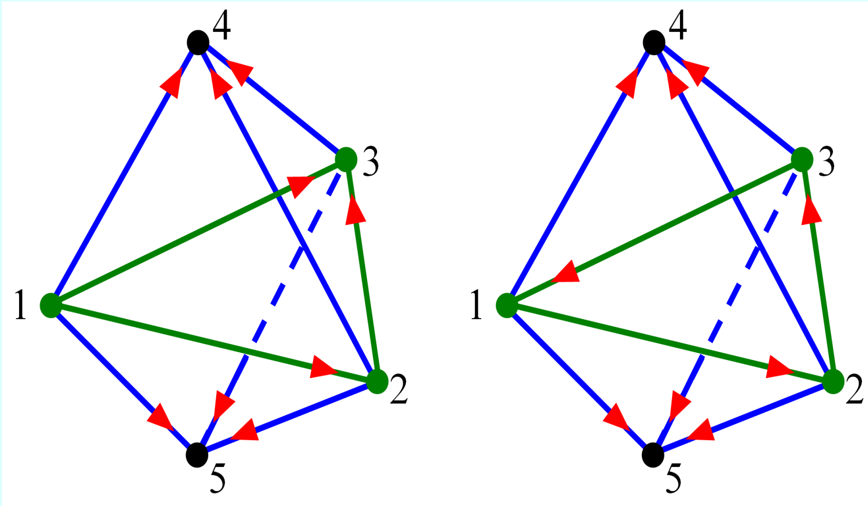
We have also defined *homotopy equivalence* $X \simeq Y$ of two digraphs.

Theorem 5 *If* $X \simeq Y$ *then* $H_* (X) \cong H_* (Y)$.

Theorem 6 *If Y is a subgraph of X then $X \simeq Y$ provided there exists a deformation retraction $r : X \to Y$, that is:*

(i) $r|_Y = \mathrm{id}$;
(ii) *r is a digraph map;*
(iii) *either $x \overrightarrow{=} r(x)$ for all $x \in X$ or $r(x) \overrightarrow{=} x$ for all $x \in X$.*

For example, consider digraphs:
The left hand side digraph is contractible as there is a sequence of two deformation retractions reducing it to $\{*\}$:
$r_1(4) = r_1(5) = 3$
$r_2(1) = r_2(2) = 3$



The right hand side digraph differs only by one arrow $3 \to 1$, but it is not contractible because $H_2 \neq \{0\}$

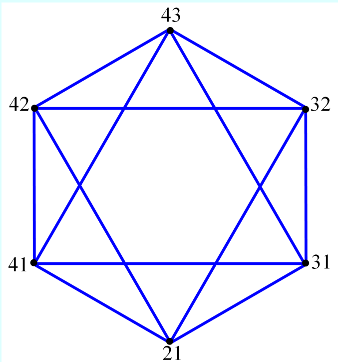
$$H_2 = \mathrm{span}\{e_{124} + e_{234} + e_{314} - e_{125} - e_{235} - e_{315}\}$$

9 Undirected graphs

If $G = (V, E)$ is an undirected graph then it can be turned into a digraph by allowing both arrows $x \to y$ and $y \to x$ whenever $x \sim y$. All the above results can be reformulated for undirected graphs in an obvious way.

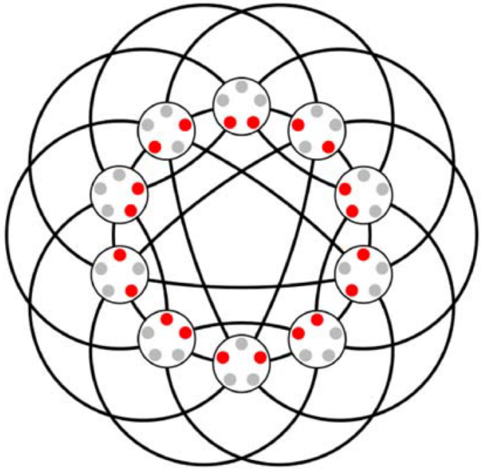
Example. Fix integers $1 \le k \le n$ and a set S of n elements. The *Johnson graph* $J(n, k)$ is the graph whose vertices are k-subsets of S, and the edges are defined as follows: two k-subsets are connected by an edge if their intersection contains $k - 1$ elements of S.

Let us describe $J(4, 2)$. Taking $S = \{1, 2, 3, 4\}$, we see that the vertices of $J(4, 2)$ are the pairs 43, 42, 41, 32, 31, 31. The graph $J(4, 2)$ has twelve edges:



Johnson graph

Johnson graphs are a special class of undirected graphs defined from systems of sets. The vertices of the Johnson graph $J(n, k)$ are the k-element subsets of an n-element set; two vertices are adjacent when the intersection of the two vertices (subsets) contains $(k-1)$-elements.[1] Both Johnson graphs and the closely related Johnson scheme are named after Selmer M. Johnson.

Johnson graph	
	
The Johnson graph $J(5,2)$	
Named after	Selmer M. Johnson
Vertices	$\binom{n}{k}$
Edges	$\frac{k(n-k)}{2}\binom{n}{k}$

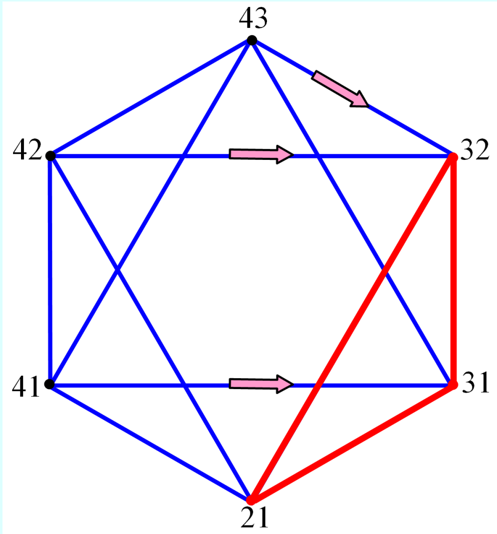
Contents

Special cases

Proposition 7 *For all $n > k \geq 1$ we have $J(n,k) \simeq J(n-1,k)$.*

Consequently, $J(n,k) \simeq J(n-1,k) \simeq ... \simeq J(k,k) = \{*\}$, and all the homology groups of $J(n,k)$ are trivial.

For the proof, assume that $J(n,k)$ is constructed over the set $S = \{1,...,n-1,n\}$, so that graph $J(n-1,k)$ is a subgraph of $J(n,k)$. Then there exists a deformation retraction $r : J(n,k) \to J(n-1,k)$. Here is a deformation retraction $r : J(4,2) \to J(3,2)$:



In general, we construct r as follows. Any vertex a of $J(n,k)$ is represented by a monotone decreasing sequence $a=a_1a_2...a_k$ of integers from $\{1,...,n\}$: $n\geq a_1>a_2>...>a_k\geq 1$. Define $r(a)=a'=a_1'...a_k'$ where

$$a_1'=\min(a_1,n-1), \quad a_2'=\min(a_2,n-2)\ , \ ... \ a_k'=\min(a_k,n-k).$$

Then $n-1\geq a_1'>a_2'>...>a_k'\geq 1$, so that a' is a vertex of $J(n-1,k)$. We claim that $r:J(n,k)\to J(n-1,k)$ is a deformation retraction.

(i) If $a\in J(n-1,k)$ then $r(a)=a$ because $a_1\leq n-1$, $a_2\leq n-2$,..., $a_k\leq n-k$, which implies $a_i'=a_i$.

(ii) If $a\sim b$ in $J(n,k)$ then $r(a)\sim r(b)$ or $r(a)=r(b)$ because sequences $a_1...a_k$ and $b_1...b_k$ have $k-1$ common elements, whence it follows that a' and b' have at least $k-1$ common elements.

(iii) If $a\in J(n,k)\setminus J(n-1,k)$ then $r(a)\sim a$. In this case $a_1=n$. Assume $a_2\leq n-2$. Then $a_3\leq n-3,....,a_k\leq n-k$, which implies

$$a_1'=n-1,\ a_2'=a_2,...,\ a_k'=a_k$$

that is, $r(a)=(n-1)\,a_2...a_k$ and $r(a)\sim a$. The case $a_2=n-1$ is a bit more involved.

10 C-homotopy of loops

For any digraph G and a vertex $*$ of G, denote by G^* a *based digraph.*

Definition. A *loop* on G^* is a digraph map $\varphi : I_n \to G$ such that $\varphi(0) = \varphi(n) = *$.

Here I_n is any line digraph with any $n \geq 0$.

Definition. Consider in G^* two loops $\varphi\colon I_n \to G$ and $\psi\colon I_m \to G$. An *one-step direct C-homotopy* from φ to ψ is a digraph map $h : I_n \to I_m$ such that

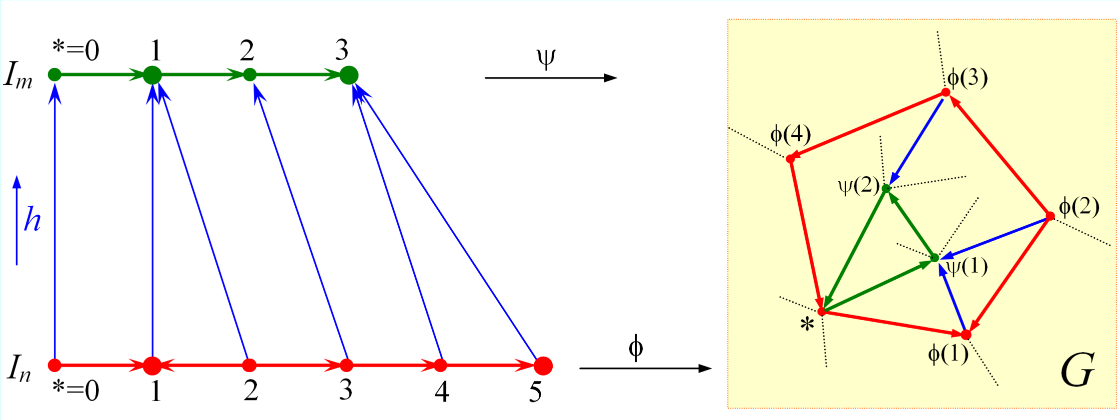
(a) $h(0) = 0$, $h(n) = m$ and $h(i) \leq h(j)$ whenever $i \leq j$;

(b) $\varphi(i) \overset{\rightarrow}{=} \psi(h(i))$ for all $i \in I_n$.

If in (b) holds $\varphi(i) \overset{\leftarrow}{=} \psi(h(i))$ for all $i \in I_n$ then h is called an one-step *inverse* C-homotopy.

We denote an one-step direct C-homotopy with $\varphi \overset{C}{\to} \psi$ and the one-step inverse C-homotopy with $\varphi \overset{C}{\leftarrow} \psi$.

Example. On the next diagram we have $\varphi \overset{C}{\to} \psi$.



Condition (b) means that φ and ψ provide a digraph map from the digraph on the left panel to G.

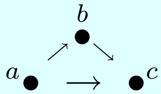
Definition. We call two loops φ, ψ *C-homotopic* and write $\varphi \overset{C}{\simeq} \psi$ if there exists a finite sequence $\{\varphi_k\}_{k=0}^m$ of loops in G^* such that $\varphi_0 = \varphi$, $\varphi_m = \psi$ and, for any $k = 0, ..., m-1$, holds $\varphi_k \overset{C}{\to} \varphi_{k+1}$ or $\varphi_k \overset{C}{\leftarrow} \varphi_{k+1}$.

Obviously, C-homotopy is an equivalence relation. A loop φ is called contractible if $\varphi \overset{C}{\simeq} e$ where $e : I_0 \to G$ is a trivial loop.

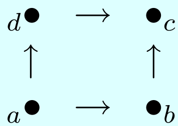
The following theorem gives an efficient way of verifying if two loops are C-homotopic.

Any loop $\varphi\colon I_n \to G$ defines a sequence $\theta_\varphi = \{\varphi(i)\}_{i=0}^{n}$ of vertices of G. We consider θ_φ as a word over the alphabet V.

Theorem 8 *Two loops $\varphi : I_n \to G$ and $\psi : I_m \to G$ are C-homotopic if and only if θ_ψ can be obtained from θ_φ by a finite sequence of the following word transformations (or inverses to them):*

(i) ...abc... $\mapsto$ ...ac... where a, b, c is a triangle 

in G or any permutation of a triangle.

(ii) ...abc... $\mapsto$ ...adc... where a, b, c, d is a square 

in G or any cyclic permutation of a square or an inverse cyclic permutation of a square.

(iii) ...abcd... $\mapsto$ ...ad... where a, b, c, d is as in (ii).

(iv) ...aba... $\to$ ...a... if $a \to b$ or $b \to a$.

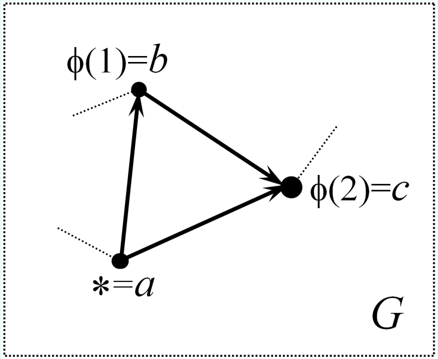
(v) ...aa... $\mapsto$ ...a...

Examples

1. Consider a triangular loop
$\varphi : (0 \to 1 \to 2 \leftarrow 3) \to G$

It is contractible because

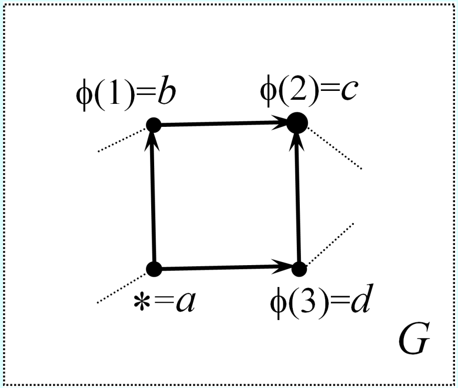
$\theta_\varphi = abca \overset{(i)}{\sim} aca \overset{(iv)}{\sim} a.$



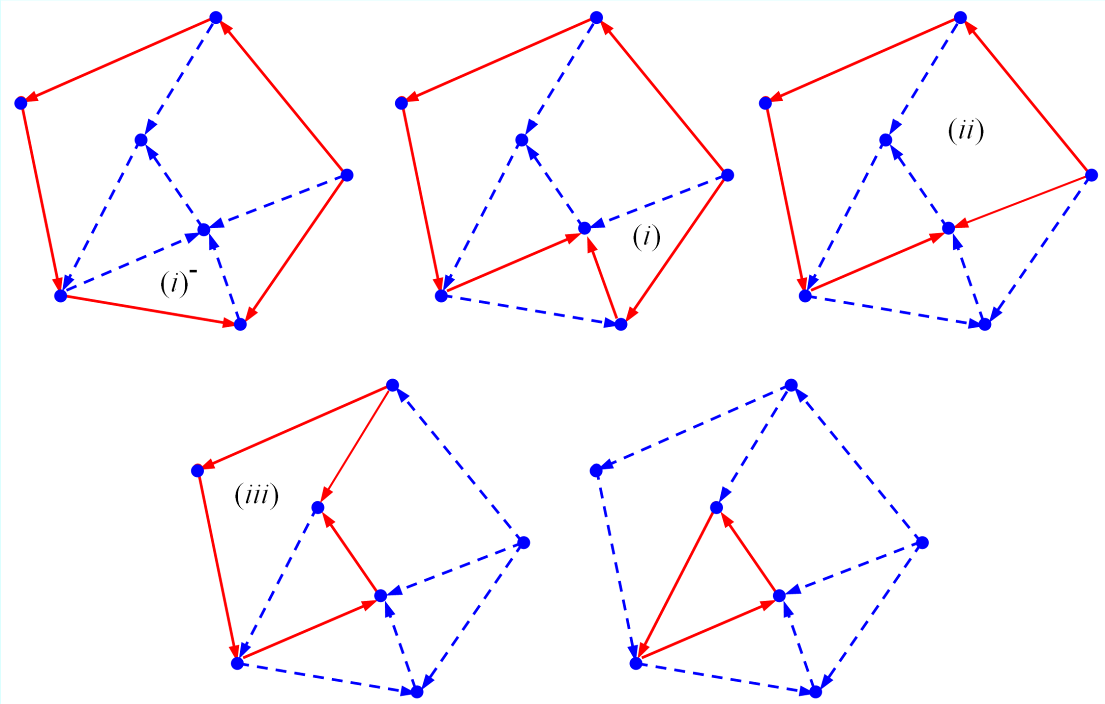
2. Consider a square loop
$\varphi : (0 \to 1 \to 2 \leftarrow 3 \leftarrow 4) \to G$

It is contractible because

$\theta_\varphi = abcda \overset{(iii)}{\sim} ada \overset{(iv)}{\sim} a.$



3. Consider the loops $\varphi : I_5 \to G$ and $\psi : I_3 \to G$ as on p.33. It is shown here how to transform θ_φ to θ_ψ by means of Theorem 8: using successively transformations $(i)^-$, (i), (ii) and (iii).



11 Fundamental group

The C-homotopy equivalence class of a loop $\varphi : I_n \to G$ will be denoted by $[\varphi]$. For any two loops $\varphi : I_n \to G$ and $\psi : I_m \to G$ define their concatenation $\varphi \vee \psi : I_{n+m} \to G$ by

$$\varphi \vee \psi(i) = \begin{cases} \varphi(i), & 0 \leq i \leq n \\ \psi(i-n), & n \leq i \leq n+m. \end{cases}$$

Then the product $[\varphi] \cdot [\psi] := [\varphi \vee \psi]$ of equivalence classes is then well-defined.

Theorem 9 (*a*) *The set of all equivalence classes* $[\varphi]$ *with the above product is a group with the neutral element* $[e]$. *It is denoted by* $\pi_1(G^*)$.

(*b*) *Any based digraph map* $f : X^* \to Y^*$ *induces a group homomorphism*

$$\pi_1(f) : \pi_1(X^*) \to \pi_1(Y^*), \quad (\pi_1(f))\,[\phi] = [f \circ \phi].$$

(*c*) *If* $f, g : X^* \to Y^*$ *are two digraph maps then* $f \simeq g$ *implies* $\pi_1(f) = \pi_1(g)$.

(*d*) *If* X, Y *are connected and* $X \simeq Y$ *then* $\pi_1(X^*) \cong \pi_1(Y^*)$.

Theorem 10 *For any based connected digraph G^* we have an isomorphism*

$$\pi_1(G^*) / [\pi_1(G^*), \pi_1(G^*)] \cong H_1(G, \mathbb{Z}),$$

where $[\pi_1(G^*), \pi_1(G^*)]$ is a commutator subgroup.

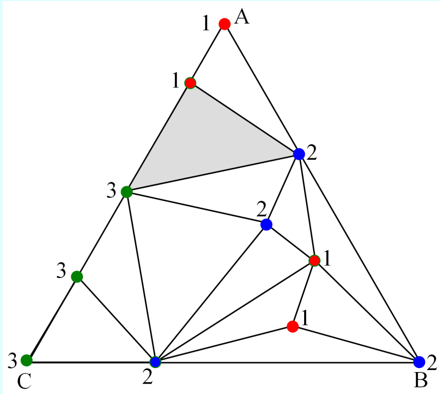
12 Application to graph coloring

An an illustration of the theory of digraph homotopy, we give here a new proof of the classical lemma of Sperner, using the notion the fundamental group of digraphs.

Consider a triangle ABC on the plane $\mathbb{R}^2$ and its triangulation T. Assume that the set of vertices of T is colored in three colors $1, 2, 3$ so that:

- the vertex A in colored in 1, B – in 2, C – in 3;
- each vertex on the side AB is colored in 1 or 2, on the side AC – in 1 or 3, on the side BC – in 2 or 3.

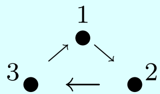
Lemma of Sperner. Under the above conditions, there exists in T a 3-color triangle, that is, a triangle, whose vertices are colored with three different colors.



Let us first modify the triangulation T so that there are no vertices on the sides AB, AC, BC except for A, B, C. If $X \in AB$ then move X a bit inside of ABC. A new triangle XYZ arises, where Y, Z are former neighbors of X on AB. However, since X, Y, Z are colored in two colors, no 3-color triangle emerges after that move. By induction, we remove all the vertices from all sides of ABC.

Consider the triangulation T as a graph and make it into a digraph G as follows. If a, b are two vertices on T and $a \sim b$ then choose direction between a, b using the colors of a, b and the following rule:

$$1 \to 2, \quad 2 \to 3, \;\; 3 \to 1$$
$$1 \leftrightarrows 1, \quad 2 \leftrightarrows 2, \;\; 3 \leftrightarrows 3$$

Denote by S the following colored digraph 

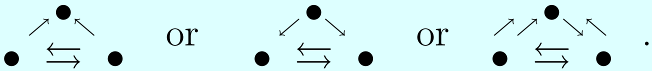
and define a mapping $f : G \to S$ to preserve colors of vertices. Then f is a digraph map by the choice of arrows in G.

Consider a 3-loop φ on G^* (with $* = A$) with the word

$$\theta_\varphi = ABCA.$$

For the loop $f \circ \varphi$ on S we have $\theta_{f \circ \varphi} = 1231$. This loop is not contractible because none of the transformations of Theorem 8 can be applied to the word 1231. By Theorem 9(b), the loop φ is also not contractible and, hence, $\pi_1(G^*) \neq \{0\}$.

Assume now that there is no 3-color triangle in T. Then each triangle from T looks in G like



In particular, each of them contains a triangle in the sense of Theorem 8. Using the partition of G into the triangles and transformations (ii) and (iv) of Theorem 8, we contract any loop on G to the empty word, which contradicts to $\pi_1(G) \neq \{0\}$.