Overview of path homology theory of digraphs

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1 Path homology of digraphs

1.1 Paths in a finite set

Let V be a finite set. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V. Fix a field $\mathbb{K}$ and denote by $\Lambda_p = \Lambda_p(V, \mathbb{K})$ the $\mathbb{K}$-linear space that consists of all formal $\mathbb{K}$-linear combinations of elementary p-paths in V. Any element of Λ_p is called a p-path.

An elementary p-path $i_0, ..., i_p$ as an element of Λ_p will be denoted by $e_{i_0 \ldots i_p}$. For example, we have

$$\Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \qquad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle$$

Any p-path u can be written in a form $u = \sum_{i_0, i_1, \ldots, i_p \in V} u^{i_0 i_1 \ldots i_p} e_{i_0 i_1 \ldots i_p}$, where $u^{i_0 i_1 \ldots i_p} \in \mathbb{K}$.

Definition. Define for any $p \geq 1$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0 \ldots \widehat{i_q} \ldots i_p},$$

where $\widehat{}$ means omission of the index. For $p = 0$ set $\partial e_i = 0$.

For example, $\partial e_{ij} = e_j - e_i$ and $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}$.

Lemma 1.1 $\partial^2 = 0$.

Proof. Indeed, for any $p \geq 2$ we have

$$\begin{aligned}
\partial^2 e_{i_0 \dots i_p} &= \sum_{q=0}^{p} (-1)^q \, \partial e_{i_0 \dots \widehat{i_q} \dots i_p} \qquad (1.1) \\
&= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r \, e_{i_0 \dots \widehat{i_r} \dots \widehat{i_q} \dots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} \, e_{i_0 \dots \widehat{i_q} \dots \widehat{i_r} \dots i_p} \right) \\
&= \sum_{0 \leq r < q \leq p} (-1)^{q+r} \, e_{i_0 \dots \widehat{i_r} \dots \widehat{i_q} \dots i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} \, e_{i_0 \dots \widehat{i_q} \dots \widehat{i_r} \dots i_p}.
\end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0 \dots i_p} = 0$. This implies $\partial^2 u = 0$ for all $u \in \Lambda_p$. ■

Hence, we obtain a chain complex $\Lambda_* (V)$:

$$0 \leftarrow \Lambda_0 \overset{\partial}{\leftarrow} \Lambda_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Lambda_{p-1} \overset{\partial}{\leftarrow} \Lambda_p \overset{\partial}{\leftarrow} \dots$$

Definition. An elementary p-path $e_{i_0...i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and irregular otherwise.

Let I_p be the subspace of Λ_p spanned by irregular $e_{i_0...i_p}$. We claim that $\partial I_p \subset I_{p-1}$. Indeed, if $e_{i_0...i_p}$ is irregular then $i_k = i_{k+1}$ for some k. We have

$$\begin{aligned} \partial e_{i_0...i_p} &= e_{i_1...i_p} - e_{i_0 i_2...i_p} + ... \\ &\quad + (-1)^k e_{i_0...i_{k-1} i_{k+1} i_{k+2}...i_p} + (-1)^{k+1} e_{i_0...i_{k-1} i_k i_{k+2}...i_p} \\ &\quad + ... + (-1)^p e_{i_0...i_{p-1}}. \end{aligned} \tag{1.2}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (1.2) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0...i_p} \in I_{p-1}$.

Hence, ∂ is well-defined on the quotient spaces $\mathcal{R}_p := \Lambda_p / I_p$, and we obtain the chain complex $\mathcal{R}_* (V)$:

$$0 \leftarrow \mathcal{R}_0 \overset{\partial}{\leftarrow} \mathcal{R}_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \dots$$

By setting all irregular p-paths to be equal to 0, we can identify $\mathcal{R}_p$ with the subspace of Λ_p spanned by all regular paths. For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}$$

because $e_{ii} = 0$.

1.2 Chain complex and path homology of a digraph

Definition. A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and a set $E \subset \{V \times V \setminus \mathrm{diag}\}$ of arrows (directed edges). If $(i, j) \in E$ then we write $i \to j$.

Definition. Let $G = (V, E)$ be a digraph. An elementary p-path $i_0...i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, ..., p-1$, and *non-allowed* otherwise.

Let $\mathcal{A}_p = \mathcal{A}_p(G)$ be $\mathbb{K}$-linear space spanned by allowed elementary p-paths:

$$\mathcal{A}_p = \langle e_{i_0...i_p} : i_0...i_p \text{ is allowed} \rangle.$$

The elements of $\mathcal{A}_p$ are called *allowed* p-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on subspaces $\mathcal{A}_p$ of $\mathcal{R}_p$. However, the spaces $\mathcal{A}_p$ are in general *not* invariant for ∂. For example, in the digraph

$$\overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \longrightarrow \overset{c}{\bullet}$$

we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is not allowed.

Consider the following subspace of $\mathcal{A}_p$

$$\boxed{\Omega_p \equiv \Omega_p(G) := \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}}.$$

We claim that $\partial\Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial(\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

Definition. The elements of Ω_p are called *∂-invariant* p-paths.

Hence, we obtain a chain complex $\Omega_* = \Omega_*(G)$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$.

Definition. *Path homologies* of G are defined as the homologies of the chain complex $\Omega_*(G)$:

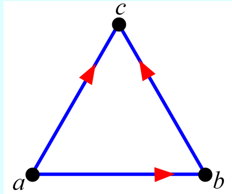
$$H_p(G) = \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

Betti numbers: $\beta_p(G) := \dim H_p(G)$.

It is easy to show: $\beta_0(G) = \#$of connected components of G.

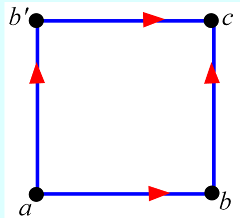
1.3 Examples of ∂-invariant paths

A *triangle* is a sequence of three vertices a, b, c such that $a \to b \to c$, $a \to c$.
It determines 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$.

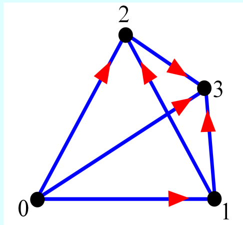


A *square* is a sequence of four vertices a, b, b', c such that $a \to b$, $b \to c$, $a \to b'$, $b' \to c$.
It determines a 2-path $u = e_{abc} - e_{ab'c} \in \Omega_2$ because $u \in \mathcal{A}_2$ and

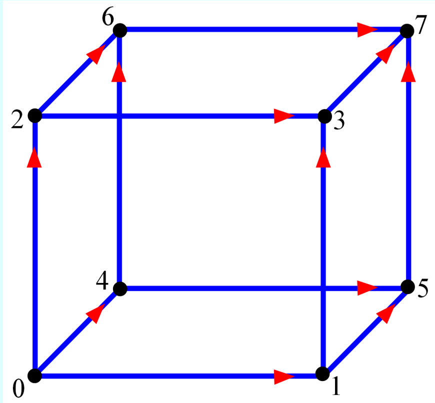
$$\partial u = \left(e_{bc} - \underline{e_{ac}} + e_{ab}\right) - \left(e_{b'c} - \underline{e_{ac}} + e_{ab'}\right) = e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1$$



A *p-simplex* (or p-clique) is a sequence of $p+1$ vertices, say, $0, 1, ..., p$, such that $i \to j$ for all $i < j$. It determines a p-path $e_{01\ldots p} \in \Omega_p$. Here is a 3-simplex:



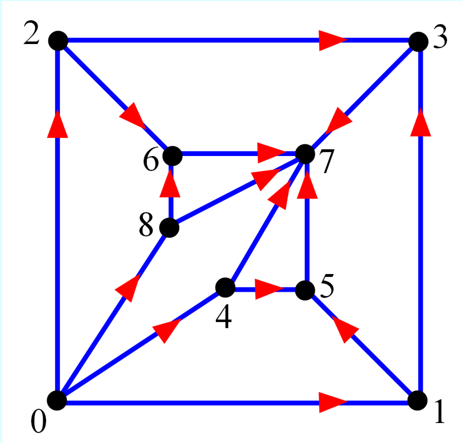
A 3-*cube* is a sequence of 8 vertices $0, 1, 2, 3, 4, 5, 6, 7$, connected by arrows as here.



A 3-cube determines a ∂-invariant 3-path
$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \Omega_3$
because $u \in \mathcal{A}_3$ and

$$\partial u = (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267})$$
$$- (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2$$

An *exotic cube* consists of 9 vertices connected by arrows as here.



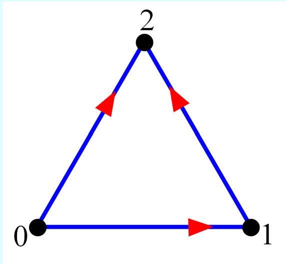
It determines a ∂-invariant 3-path

$$v = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0867} - e_{0267} \in \Omega_3$$

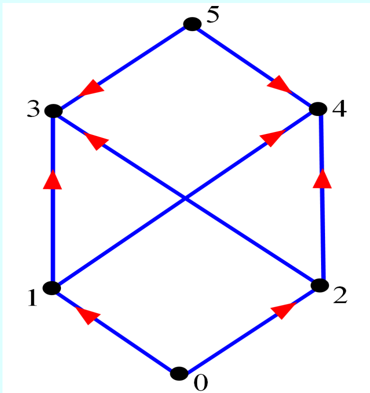
1.4 Examples of spaces Ω_p and H_p

For a vector space A over $\mathbb{K}$ we write $|A| = \dim_{\mathbb{K}} A$.

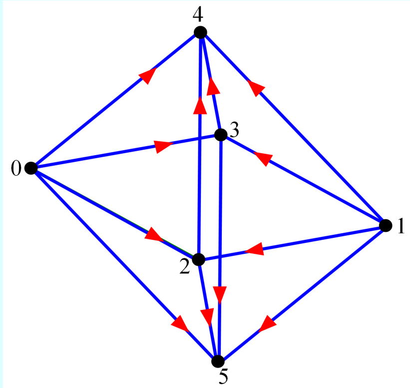
A triangle as a digraph:
$\Omega_1 = \langle e_{01}, e_{02}, e_{12} \rangle, \quad \Omega_2 = \langle e_{012} \rangle,$
$\Omega_p = \{0\}$ for $p \geq 3$
$\ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{12} \rangle$
but $e_{01} - e_{02} + e_{12} = \partial e_{012}$
so that $H_1 = \{0\}$.
$H_p = \{0\}$ for $p \geq 2$.



Hexagon with diagonals:
$|\Omega_0| = 6, \;\; |\Omega_1| = 8$
Ω_2 is spanned by 2 squares:
$\Omega_2 = \langle e_{013} - e_{023}, \;\; e_{014} - e_{024} \rangle,$
$\Omega_p = \{0\}$ for all $p \geq 3$
$H_1 = \langle e_{13} - e_{53} + e_{54} - e_{14} \rangle,$
$|H_1| = 1, \quad H_p = \{0\}$ for $p \geq 2$.



Octahedron: $|\Omega_0| = 6, \quad |\Omega_1| = 12$
Space Ω_2 is spanned by 8 triangles:
$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \rangle,$
$|\Omega_2| = 8, \ \Omega_p = \{0\}$ for all $p \geq 3$
$H_2 = \langle e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135} \rangle$
$|H_2| = 1, \quad |H_p| = 0$ for $p = 1$ and $p \geq 3$

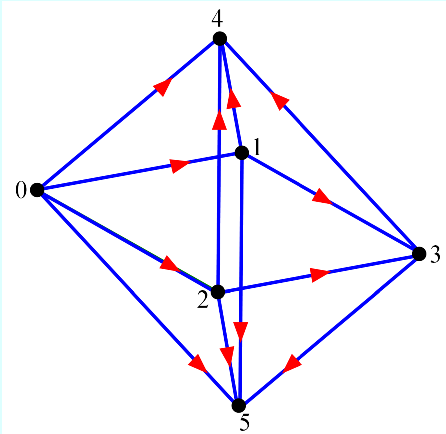


Octahedron with different orientation:
$\Omega_2 = \langle e_{024}, e_{025}, e_{014}, e_{015}, e_{234}, e_{235}, e_{134}, e_{135}, e_{013} - e_{023} \rangle$
$\Omega_3 = \langle e_{0234} - e_{0134}, \ e_{0235} - e_{0135} \rangle$
$|\Omega_2| = 9, \quad |\Omega_3| = 2, \quad \Omega_p = \{0\}$ for all $p \geq 4$.
$\ker \partial|_{\Omega_2} = \langle u, v \rangle$ where

$$u = e_{024} + e_{234} - e_{014} - e_{134} + (e_{013} - e_{023})$$
$$v = e_{025} + e_{235} - e_{015} - e_{135} + (e_{013} - e_{023})$$

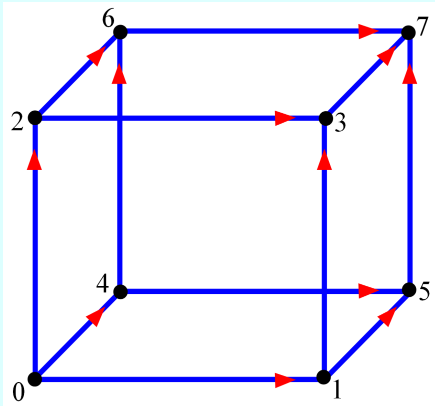
but $H_2 = \{0\}$ because

$$u = \partial (e_{0234} - e_{0134}) \quad \text{and} \quad v = \partial (e_{0235} - e_{0135})$$



A 3-cube:

We have $|\Omega_0| = 8, \quad |\Omega_1| = 12.$
Space Ω_2 is spanned by 6 squares:



$$\Omega_2 = \langle e_{013} - e_{023},\ e_{015} - e_{045},\ e_{026} - e_{046},\\ e_{137} - e_{157},\ e_{237} - e_{267},\ e_{457} - e_{467}\rangle$$

hence, $|\Omega_2| = 6$.

Space Ω_3 is spanned by one 3-cube:

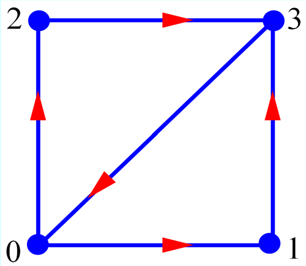
$$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}\rangle$$

hence, $|\Omega_3| = 1$.

$|\Omega_p| = 0$ for all $p \geq 4$ and $|H_p| = 0$ for all $p \geq 1$.

1.5 An example of computation of Ω_p and H_p

Consider the following digraph with 4 vertices and 5 arrows (square with a diagonal):



$\Omega_0 = \mathcal{A}_0 = \langle e_0, e_1, e_2, e_3 \rangle, \qquad |\Omega_0| = 4,$
$\Omega_1 = \mathcal{A}_1 = \langle e_{01}, e_{02}, e_{13}, e_{23}, e_{30} \rangle, \qquad |\Omega_1| = 5,$
$\mathcal{A}_2 = \langle e_{013}, e_{023}, e_{130}, e_{230}, e_{301}, e_{302} \rangle \qquad |\mathcal{A}_2| = 6.$

To find Ω_2, let us first compute $\partial|_{\mathcal{A}_2} \bmod \mathcal{A}_1$:

$$
\begin{aligned}
\partial e_{013} &= e_{13} - e_{03} + e_{01} = -e_{03} \mod \mathcal{A}_1 \\
\partial e_{023} &= e_{23} - e_{03} + e_{02} = -e_{03} \mod \mathcal{A}_1 \\
\partial e_{130} &= e_{30} - e_{10} + e_{13} = -e_{10} \mod \mathcal{A}_1 \\
\partial e_{230} &= e_{30} - e_{20} + e_{23} = -e_{20} \mod \mathcal{A}_1 \\
\partial e_{301} &= e_{01} - e_{31} + e_{30} = -e_{31} \mod \mathcal{A}_1 \\
\partial e_{302} &= e_{02} - e_{32} + e_{30} = -e_{32} \mod \mathcal{A}_1
\end{aligned}
$$

Hence,

$$\text{matrix of } \partial|_{\mathcal{A}_2} \bmod \mathcal{A}_1 = \begin{array}{c} \\ e_{03} \\ e_{10} \\ e_{20} \\ e_{31} \\ e_{32} \end{array} \begin{array}{cccccc} e_{013} & e_{023} & e_{130} & e_{230} & e_{301} & e_{302} \end{array} \hspace{-0pt} \begin{pmatrix} -1 & -1 & & & & 0 \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix} := D$$

$$\Omega_2 = \ker \partial|_{\mathcal{A}_2} \bmod \mathcal{A}_1 = \operatorname{nullspace} D = \langle e_{013} - e_{023} \rangle.$$

One can show that $|\Omega_p| = 0$ for all $p \geq 3$ and, hence, $|H_p| = 0$ for all $p \geq 3$.

Let us compute H_1 and H_2. We have for the basis in Ω_1:

$$\begin{aligned} \partial e_{01} &= -e_0 + e_1 \\ \partial e_{02} &= -e_0 + e_2 \\ \partial e_{13} &= -e_1 + e_3 \\ \partial e_{23} &= -e_2 + e_3 \\ \partial e_{30} &= e_0 - e_3 \end{aligned}$$

Hence,

$$\text{matrix of } \partial|_{\Omega_1} = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccccc} e_{01} & e_{02} & e_{13} & e_{23} & e_{30} \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array} =: D$$

and

$$\ker \partial|_{\Omega_1} = \operatorname{nullspace} D = \langle e_{01} + e_{13} - e_{02} - e_{23},\ e_{01} + e_{13} + e_{30} \rangle.$$

Similarly, for the basis in Ω_2 we have

$$\partial\left(e_{013} - e_{023}\right) = \left(e_{13} - e_{03} + e_{01}\right) - \left(e_{23} - e_{03} + e_{02}\right) = e_{01} + e_{13} - e_{02} - e_{23}$$

whence

$$\operatorname{Im} \partial|_{\Omega_2} = \langle e_{01} + e_{13} - e_{02} - e_{23} \rangle \quad \text{and} \quad \ker \partial|_{\Omega_2} = \{0\}\,.$$

It follows that $H_2 = \{0\}$ and

$$H_1 = \ker \partial|_{\Omega_1} / \operatorname{Im} \partial|_{\Omega_2} = \langle e_{01} + e_{13} + e_{30} \rangle.$$

As we have seen, computation of the spaces $\Omega_p\left(G\right)$ and $H_p\left(G\right)$ amounts to computing ranks and null-spaces of large matrices. We currently use for numerical computation of $H_p\left(G, \mathbb{F}_2\right)$ a C++ program written by Chao Chen in 2012.

1.6 Structure of Ω_p

As we know, $\Omega_0 = \langle e_i \rangle$ consists of all vertices and $\Omega_1 = \{e_{ij} : i \to j\}$ consists of all arrows.

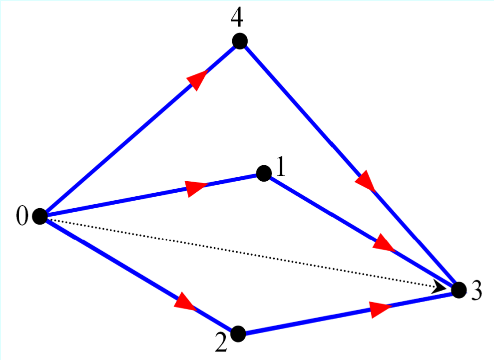
Proposition 1.2 *(a) The space Ω_2 is spanned by all triangles e_{abc}, squares $e_{abc} - e_{ab'c}$ and double arrows e_{aba}.*

(b) $|\Omega_2| = |\mathcal{A}_2| - s$ where s is the number of semi-arrows, that is, pairs of vertices (x, y) such that $x \not\to y$ but $x \to z \to y$ for some vertex z.

The triangles and double arrows are always linearly independent but the squares can be dependent.

For example, on this digraph we have three squares:
$e_{013} - e_{023}, \quad e_{043} - e_{013}, \quad e_{023} - e_{043}$
but their sum is 0.

In this case $|\Omega_2| = 2 \quad (= |\mathcal{A}_2| - s = 3 - 1)$

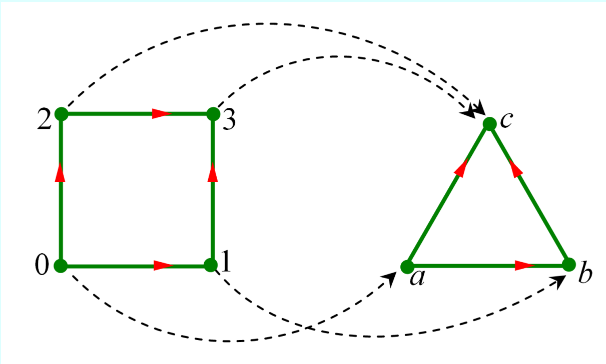


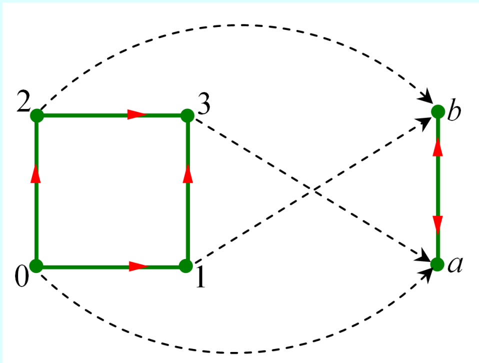
Let X, Y be two digraphs. A map $f : X \to Y$ is called a *morphism of digraphs* if for any arrow $a \to b$ in X we have either $f(a) \to f(b)$ or $f(a) = f(b)$ (that is, the image of an arrow is either an arrow or a vertex). Define images of paths by

$$f\left(e_{i_0 \dots i_p}\right) = e_{f(i_0) \dots f(i_p)}$$

so that the image of an allowed path is either allowed or zero (that is also allowed). It is easy to see that $f \circ \partial = \partial \circ f$ so that the morphism images of ∂-invariant paths are again ∂-invariant.

A triangle e_{abc} and a double arrow e_{aba} are morphism images of a square $e_{013} - e_{023}$ as on these pictures:



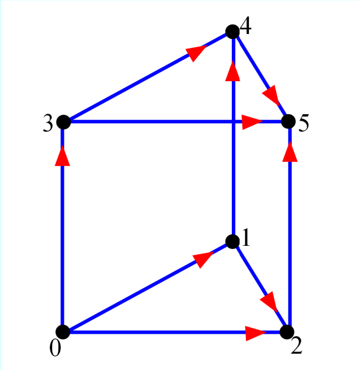


Hence, we can rephrase Proposition 1.2 as follows: Ω_2 is spanned by squares and their morphism images. Or: squares are *basic shapes* of Ω_2.

Problem 1.3 *Describe all basic shapes in* Ω_3 *(as well as in* Ω_p *for* $p > 3$*).*

One basic shape is obvious: a 3-cube. For example, a 3-simplex is a morphism image of a 3-cube.

Another morphism image of a 3-cube is a prism:



However, an exotic cube (p. 14) is also a ∂-invariant 3-path, but it is *not* a morphism image of a 3-cube.

1.7 Dependence on the field $\mathbb{K}$

The dimensions $|\Omega_0| = |V|$ and $|\Omega_1| = |E|$ do not depend on the choice of a field $\mathbb{K}$. By using a geometric characterization of Ω_2 in Prop. 1.2, we see that $|\Omega_2|$ is also independent of $\mathbb{K}$.

Conjecture 1.4 *$|\Omega_p|$ is independent of $\mathbb{K}$ for any p (a priori $|\Omega_p|(G, \mathbb{Q}) \leq |\Omega_p|(G, \mathbb{F}_q)$).*

Let us turn to $|H_p|$. It is easy to show that $|H_0| = c$, where c is the number of connected components of G and, hence, is independent of $\mathbb{K}$.

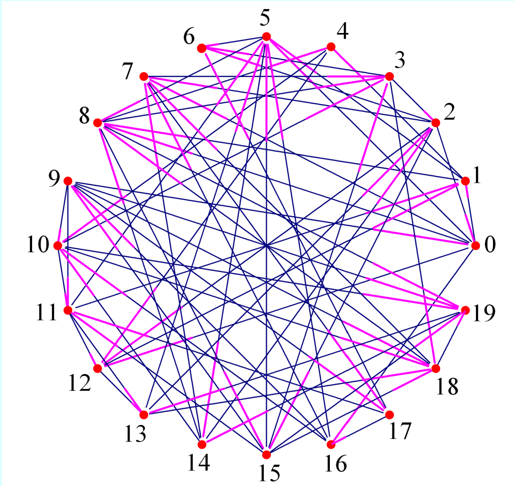
Conjecture 1.5 *$|H_1|$ is independent of $\mathbb{K}$.*

Approach to the proof: $|H_1| = |\Omega_1| - |\partial\Omega_1| - |\partial\Omega_2|$. Since $|\Omega_1| = |E|$, $|\partial\Omega_1| = |V| - c$, it remains to verify that $|\partial\Omega_2|$ is independent of $\mathbb{K}$.

Recall that for manifolds $|H_p|$ may depend on $\mathbb{K}$, for example,

$$|H_2|\left(\mathbb{RP}^2, \mathbb{Q}\right) = 0 < 1 = |H_2|\left(\mathbb{RP}^2, \mathbb{F}_2\right).$$

Example. The following digraph G is a candidate for $|H_2|(G,\mathbb{Q}) < |H_2|(G,\mathbb{F}_2)$.



For this digraph we have

$$|V| = 20, \quad |E| = 69, \quad \dim \Omega_2 = 71$$

$$|H_1|(G,\mathbb{F}_2) = |H_1|(G,\mathbb{Q}) = 2$$

and

$$|H_2|(G,\mathbb{F}_2) = 5.$$

Conjecture 1.6 *For this digraph* $|H_2|(G,\mathbb{Q}) = 4$.

A motivation for this conjecture is as follows. One of five generators of $H_2(G,\mathbb{F}_2)$ is

$$\begin{aligned} u = {} & (e_{8\,3\,18} + e_{8\,15\,18}) + e_{8\,15\,19} + e_{9\,10\,18} + e_{9\,10\,19} + e_{10\,3\,18} \\ & + e_{14\,8\,3} + (e_{14\,8\,19} + e_{14\,10\,19}) + e_{14\,10\,3} + e_{15\,9\,18} + e_{15\,9\,19} \end{aligned}$$

By changing the signs of the terms appropriately, we obtain the following element of $H_2(G,\mathbb{Q})$:

$$\begin{aligned}\widetilde{u} = &(e_{8\,3\,18} - e_{8\,15\,18}) + e_{8\,15\,19} - e_{9\,10\,18} + e_{9\,10\,19} - e_{10\,3\,18}\\ &- e_{14\,8\,3} + (e_{14\,8\,19} - e_{14\,10\,19}) + e_{14\,10\,3} - e_{15\,9\,18} + e_{15\,9\,19}.\end{aligned}$$

The same method works for 4 out of 5 generators of $H_2(G,\mathbb{F}_2)$. The fifth generator is

$$\begin{aligned}&e_{0\,7\,3} + e_{0\,8\,3} + e_{3\,2\,6} + e_{3\,2\,7} + e_{3\,18\,7} + e_{5\,14\,8} + e_{8\,15\,3} + e_{8\,15\,19} + (e_{9\,0\,7} + e_{9\,18\,7}) + e_{14\,2\,7} + e_{14\,7\,3}\\ &+ e_{9\,10\,18} + (e_{9\,0\,5} + e_{9\,11\,5}) + (e_{9\,11\,13} + e_{9\,19\,13}) + e_{9\,10\,19} + e_{10\,3\,18} + e_{11\,13\,5} + (e_{13\,2\,6} + e_{13\,5\,6})\\ &+ e_{0\,5\,8} + e_{14\,10\,3} + (e_{14\,8\,19} + e_{14\,10\,19}) + (e_{15\,3\,6} + e_{15\,5\,6}) + (e_{15\,5\,14} + e_{15\,19\,14}) + (e_{19\,13\,2} + e_{19\,14\,2})\end{aligned}$$

but for this generator changing of the signs does not work.

Conjecture 1.7 *It is always possible to choose bases in $\Omega_p(G,\mathbb{Q})$ and $H_p(G,\mathbb{Q})$ so that each element of the basis has the form $\sum \omega^{i_0\ldots i_p} e_{i_0\ldots i_p}$ with $\omega^{i_0\ldots i_p} \in \{\pm 1, 0\}$.*

Conjecture 1.8 *A basis in $\Omega_p(G,\mathbb{F}_3)$ (resp. $H_p(G,\mathbb{F}_3)$) is also a basis in $\Omega_p(G,\mathbb{Q})$ (resp. $H_p(G,\mathbb{Q})$). In particular, the Betti numbers over $\mathbb{F}_3$ and $\mathbb{Q}$ are the same.*

2 Connection to simplexes

2.1 Path complex

The notion of *path complex* unifies digraphs and simplicial complexes.

Definition. A path complex on a finite set V is a collection $\mathcal{P}$ of elementary paths on V such that if $i_0 i_1 ... i_{p-1} i_p \in \mathcal{P}$ then also $i_1 ... i_p$ and $i_0 ... i_{p-1}$ belong to $\mathcal{P}$.

For example, each digraph $G = (V, E)$ gives rise to a path complex $\mathcal{P}$ that consists of all allowed elementary paths, that is, of the paths $i_0 \to i_1 \to ... \to i_p$. In general, all paths in a path complex $\mathcal{P}$ are also called allowed.

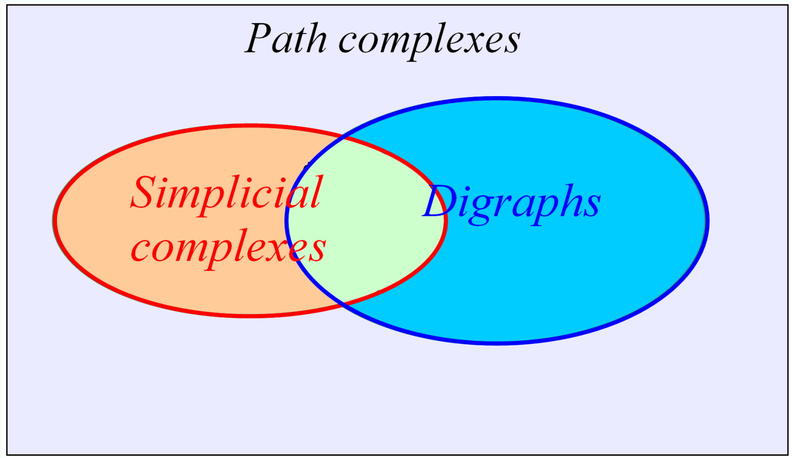
The above definitions of ∂-invariant paths, spaces Ω_p and H_p go through without any change to general path complexes in place of digraphs because they are based on the notion of allowed paths only.

For comparison let us recall the definition of an abstract simplicial complex.

Definition. A simplicial complex with the set of vertices V is a collections $\mathcal{S}$ of subsets of V such that if $\sigma \in \mathcal{S}$ then any subset of σ is also an element of $\mathcal{S}$.

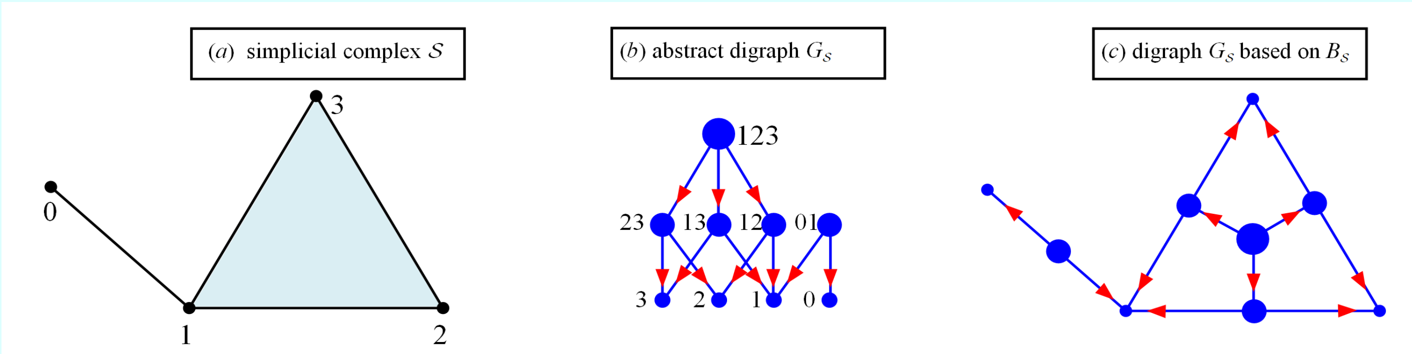
Let us enumerate all elements of V so that any subset σ of V can be regarded as a path $i_0...i_p$ with $i_0 < i_1 < .... < i_p$. The above definition means that if $i_0...i_p \in \mathcal{S}$ then also any sub-path $i_{k_0}...i_{k_q}$ with $0 \leq k_0 < k_1 < ... < k_q \leq p$ belongs to $\mathcal{S}$. Hence, a simplicial complex $\mathcal{S}$ is a path complex, and the theory of path homologies applies for $\mathcal{S}$.

In this case, $\mathcal{A}_p$ consists of linear combinations of all p-dimensional simplexes in $\mathcal{S}$ and $\Omega_p = \mathcal{A}_p$ because $\partial e_{i_0...i_p}$ is always allowed if $e_{i_0...i_p}$ is allowed. Hence, the path homology theory of a path complex $\mathcal{S}$ coincides with the simplicial homology theory of $\mathcal{S}$.



2.2 Hasse diagram

Let $\mathcal{S}$ be a simplicial complex with the vertex set V as above. Define the digraph $G_{\mathcal{S}}$ (the *Hasse diagram* of $\mathcal{S}$) as follows: the vertex set of $G_{\mathcal{S}}$ is $\mathcal{S}$, and $\sigma \to \tau$ for two simplices $\sigma, \tau \in \mathcal{S}$ if $\tau \subset \sigma$ and $|\tau| = |\sigma| - 1$ (that is, τ is a face of σ of codim $= 1$).



If $\mathcal{S}$ is realized geometrically as a collection of simplexes in $\mathbb{R}^n$ then $G_{\mathcal{S}}$ can be realized with the set of vertices $B_{\mathcal{S}}$ consisting of barycenters of the simplexes of $\mathcal{S}$ as on the picture.

Theorem 2.1 *We have*

$$H_*^{simpl}(\mathcal{S}) \simeq H_*(G_{\mathcal{S}}).$$

2.3 Triangulation as a closed path

Given a closed oriented n-dimensional manifold M, let T be its triangulation, that is, a partition into n-dimensional simplexes. Denote by $V = \{0, 1, ...\}$ the set of all vertices of the simplexes from T and by E – the set of all edges, so that (V, E) is a graph embedded on M.

Let us introduce make each edge $(i, j) \in E$ into an arrow $i \to j$ if $i < j$ and into $j \to i$ if $i > j$. Then each simplex from T becomes a digraph-simplex. Denote by $\overrightarrow{T}$ the set of all digraph simplexes constructed in this way. That is, $i_0...i_n \in \overrightarrow{T}$ if $i_0...i_n$ is a monotone increasing sequence that determines a simplex from T. Clearly, any such path $i_0...i_p$ is allowed.

For any simplex from T with the vertices $i_0...i_n$ define the quantity $\sigma^{i_0...i_n}$ to be equal to 1 if the orientation of the simplex $i_0...i_n$ matches the orientation of the manifold M, and -1 otherwise. Then consider the following allowed n-path on the digraph $G = (V, E)$:

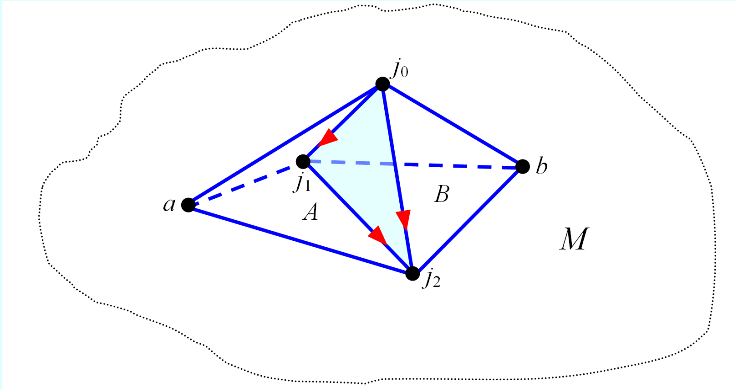
$$\sigma = \sum_{i_0...i_n \in \overrightarrow{T}} \sigma^{i_0...i_n} e_{i_0...i_n}. \tag{2.1}$$

Lemma 2.2 *The path σ is closed, that is, $\partial\sigma = 0$, which, in particular, implies that σ is ∂-invariant.*

Proof. Observe that $\partial\sigma$ is the a linear combination with coefficients ± 1 of the terms $e_{j_0 \dots j_{n-1}}$ where the sequence $j_0, \dots, j_{n-1}$ is monotone increasing and forms an $(n-1)$-dimensional face of one of the n-simplexes from T. In fact, every $(n-1)$-face arises from *two* n-simplexes, say

$A = j_0 \dots j_{k-1} a j_k \dots j_{n-1}$
and
$B = j_0 \dots j_{l-1} b j_l \dots j_{n-1}$

that is, two n-simplexes A, B have a common $(n-1)$-dimensional face $j_0 \dots j_{n-1}$.



We have

$$\partial e_{j_0 \dots j_{k-1} a j_k \dots j_{n-1}} = \dots + (-1)^k e_{j_0 \dots j_{k-1} j_k \dots j_{n-1}} + \dots .$$

Since interchanging the order of two neighboring vertices in an n-simplex changes its orientation, we have

$$\sigma^{j_0 \dots j_{k-1} a j_k \dots j_{n-1}} = (-1)^k \sigma^{a j_0 \dots j_{k-1} j_k \dots j_{n-1}}.$$

Multiplying the above lines, we obtain

$$\partial\left(\sigma^{A}e_{A}\right)=...+\sigma^{aj_0...j_{n-1}}e_{j_0...j_{n-1}}+...\ ,$$

and in the same way

$$\partial\left(\sigma^{B}e_{B}\right)=...+\sigma^{bj_0...j_{n-1}}e_{j_0...j_{n-1}}+...$$

However, the vertices a and b are located on the opposite sides of the face $j_0...j_{n-1}$, which implies that the simplexes $aj_0...j_{n-1}$ and $bj_0...j_{n-1}$ have the opposite orientations relative to that of M. Hence,

$$\sigma^{aj_0...j_{n-1}}+\sigma^{bj_0...j_{n-1}}=0,$$

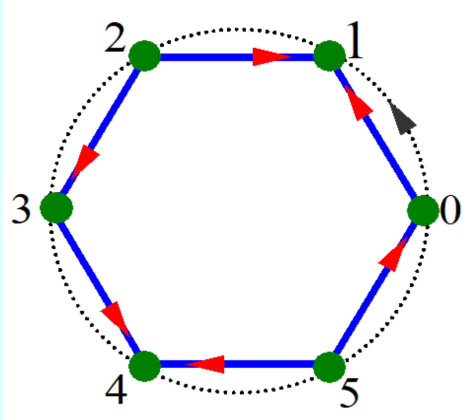
which means that the term $e_{j_0...j_{n-1}}$ cancels out in the sum $\partial\left(\sigma^{A}e_{A}+\sigma^{B}e_{B}\right)$ and, hence, in $\partial\sigma$. This proves that $\partial\sigma=0$. ■

The closed paths σ defined by (2.1) is called a *surface path* on M.

There is a number of examples when a surface path σ happens to be exact, that is, $\sigma=\partial v$ for some $(n+1)$-path v. In this case v is called a *solid path* on M because v represents a "solid" shape whose boundary is given by a surface path. If σ is not exact then σ determines a non-trivial homology class from $H_n(G)$ and, hence, represents a "cavity" in triangulation T.

Example. $M = \mathbb{S}^1$.

A triangulation of $\mathbb{S}^1$ is a polygon,
and the corresponding digraph G is *cyclic*.
On each edge (i, j) of a polygon we choose
an arrow $i \to j$ arbitrary (not necessarily if $i < j$).



We have

$$\sigma = \sum_{i \to j} \sigma^{ij} e_{ij}$$

where $\sigma^{ij} = 1$ if the arrow $i \to j$ goes counterclockwise,
and $\sigma^{ij} = -1$ otherwise.

On the digraph on the picture we have

$$\sigma = e_{01} - e_{21} + e_{23} + e_{34} - e_{54} + e_{50}$$

Proposition 2.3 (*a*) *If a polygon G is neither triangle nor square $\Omega_p = \{0\}$ for $p \geq 2$, $H_1 = \langle \sigma \rangle$ and $H_p = \{0\}$ for all $p \geq 2$.*

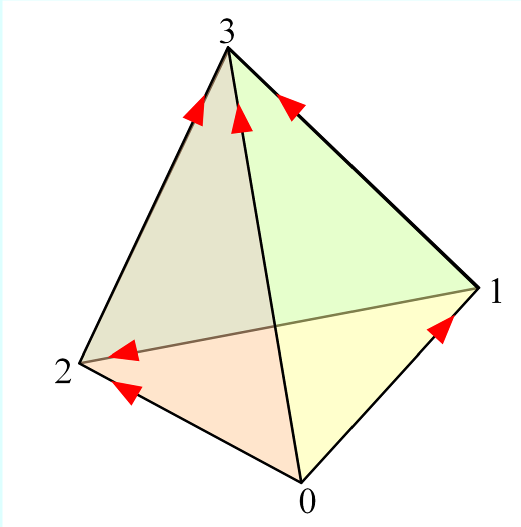
(*b*) *If G is either triangle or square then $\Omega_p = \{0\}$ for $p \geq 3$ and $H_p = \{0\}$ for all $p \geq 1$.*

Example. Let $M = \mathbb{S}^n$ and let triangulation of $\mathbb{S}^n$ be given by an $(n+1)$-simplex. Then G is a $(n+1)$-simplex digraph.

On this picture $n = 2$,

$$\sigma = e_{123} - e_{023} + e_{013} - e_{012} = \partial e_{0123}$$

so that e_{0123} is a solid path representing a tetrahedron.



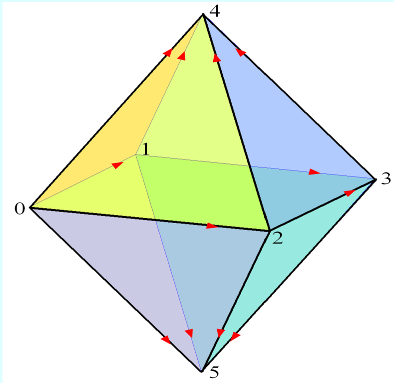
In general we also have

$$\sigma = \partial e_{0\ldots n+1}$$

so that $e_{0\ldots n+1}$ is a solid path representing a $(n+1)$-simplex.

Example. $M = \mathbb{S}^2$, octahedron.

Here is a triangulation of $\mathbb{S}^2$ by an *octahedron* with two ways of numbering.



A

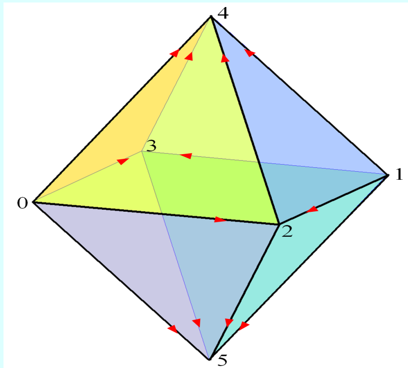
Case A: $\quad H_2 = \{0\}$

$$\sigma = e_{024} - e_{025} - e_{014} + e_{015} - e_{234} + e_{235} + e_{134} - e_{135}$$
$$= \partial \left(e_{0134} - e_{0234} + e_{0135} - e_{0235}\right)$$

Hence,

$$v = e_{0134} - e_{0234} + e_{0135} - e_{0235}$$

is a solid path, and the octahedron represents a solid shape.



B

Case B: $\quad H_2 = \langle \sigma \rangle$

$$\sigma = e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135}$$

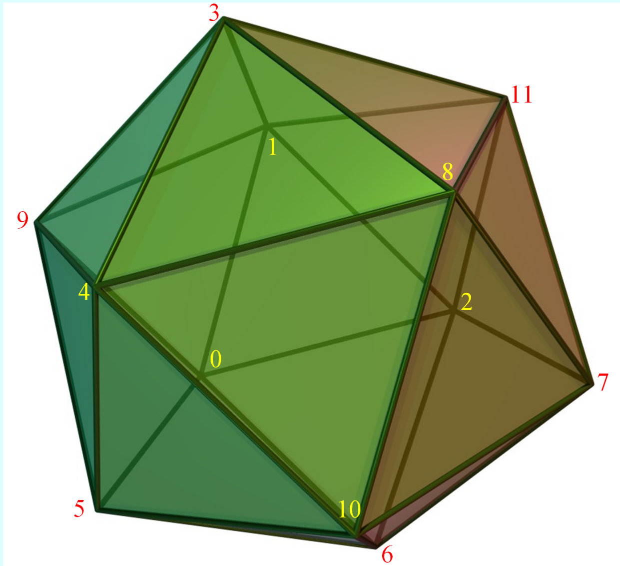
and the octahedron represents a cavity.

Example. $M = \mathbb{S}^2$, icosahedron.

Consider an *icosahedron* as a triangulation of $\mathbb{S}^2$ (here $i \to j$ if $i < j$). We have $|V| = 12$, $|E| = 30$, $H_1 = \{0\}$, and $H_2 = \langle \sigma \rangle$ where

$$\sigma = -e_{0\,1\,9} + e_{0\,1\,2} - e_{1\,2\,11} + e_{0\,2\,6} + e_{0\,5\,9} \\ -e_{0\,5\,6} + e_{5\,6\,10} - e_{1\,3\,9} + e_{1\,3\,11} - e_{2\,6\,7} \\ +e_{6\,7\,10} - e_{2\,7\,11} - e_{3\,4\,9} + e_{3\,4\,8} - e_{4\,8\,10} \\ +e_{3\,8\,11} - e_{4\,5\,9} + e_{4\,5\,10} + e_{7\,8\,10} - e_{7\,8\,11}.$$

Hence, the icosahedron represents a cavity.



Conjecture 2.4 *For icosahedron* $\dim H_2(G) = 1$ *for any numbering of the vertices.*

Conjecture 2.5 *For a general triangulation of* $\mathbb{S}^n$, *the homology group* $H_n(G)$ *is either trivial or is generated by* σ. *All other homology groups* $H_p(G)$ *are trivial.*

2.4 Computational challenge

An interesting paper:

frontiers in Computational Neuroscience

ORIGINAL RESEARCH
published: 12 June 2017
doi: 10.3389/fncom.2017.00048

Check for updates

Cliques of Neurons Bound into Cavities Provide a Missing Link between Structure and Function

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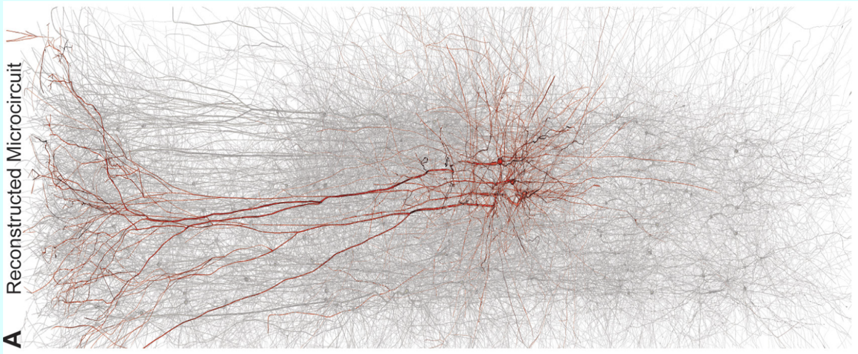
They reconstruct a microcircuit from a rat brain as a graph (neurons and connections between them). The size of the graph is $|V| \sim 31,000$ and $|E| \sim 8,000,000$.



Then they detect cliques in this graph, form out of the cliques a simplicial complex, and compute its Betti numbers over $\mathbb{F}_2$. They were able to compute Betti number β_5 and to show that $\beta_5 > 0$.

Problem 2.6 *Create computational tools capable of computing low dimensional Betti numbers for path homologies of digraphs of similar size.*

At present our program can compute β_1 on a digraph with $|V| \sim 7000$ and $|E| \sim 100,000$, and β_2 on a digraph with $|V| \sim 4000$ and $|E| \sim 25000$.

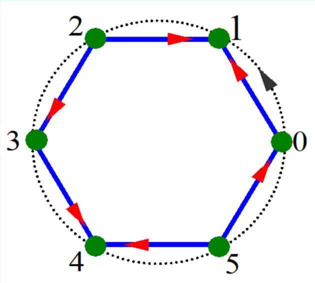
3 Homological dimension

In this section $\mathbb{K} = \mathbb{F}_2$. Define the *homological dimension* of a digraph G by

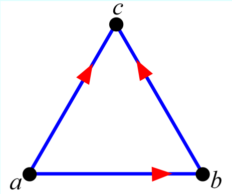
$$\boxed{\dim_h G = \sup\{k : |H_k(G)| > 0\}}.$$

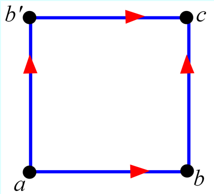
3.1 Some examples

Let G be a polygon (a cyclic digraph). If G is neither triangle nor square then $|H_1| = 1$ and $|H_p| = 0$ for $p \geq 2$ so that $\dim_h G = 1$.

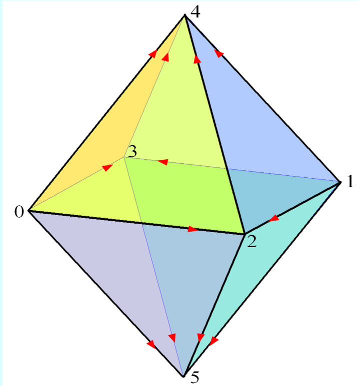


If G is either triangle or square then $|H_p| = 0$ for $p \geq 1$ and, hence $\dim_h G = 0$.





Let G be the octahedron as here:
Then



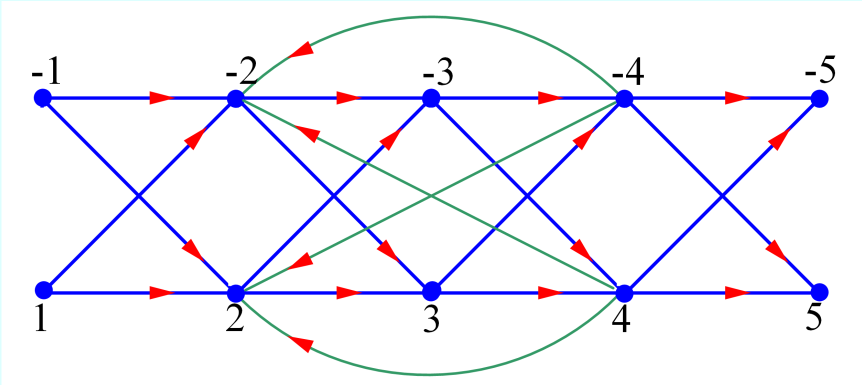
$$|H_2| = 1, \quad |H_p| = 0 \;\text{ for } p \geq 3$$

so that $\dim_h G = 2$.

There are finite digraphs with

$$\dim_h G = \infty$$

as the one on this picture:



This example was constructed by Gabor Lippner and Paul Horn in 2012.

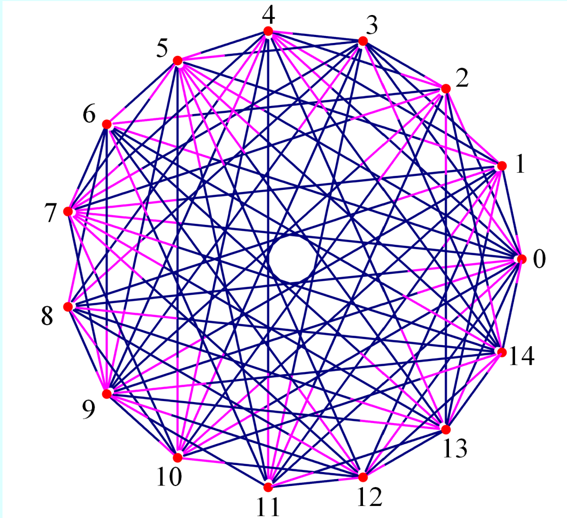
3.2 Random digraphs

We are interested in the homological dimension of a randomly generated digraph G. Fix a finite set of vertices $\{1, ..., V\}$ and two numbers $p, q > 0$ with $p+q \leq 1$. The set of arrows in G is defined as follows: for any two vertices $a < b$, there is either an arrow $a \to b$ with probability p or an arrow $b \to a$ with probability q, or no arrow with probability $1-p-q$. The so constructed probability measure on digraphs will be denoted by $\mathbb{P} = \mathbb{P}_{p,q,V}$.

Here is randomly generated digraph with $p = q = 0.37$, $V = 15$ and $E = 86$.

For this digraph $\dim_h G = 6$.
$\beta_k(G) = \{1, 0, 0, 0, 0, 0, 1\}$.

$H_6(G) = \langle v \rangle$ where v is a sum of 1560 terms:
$v = e_{0\,2\,6\,8\,0\,5\,1} + e_{0\,2\,6\,8\,0\,5\,6} + e_{0\,2\,6\,8\,0\,10\,7} + e_{0\,2\,6\,8\,0\,10\,14}$
$+ e_{0\,2\,6\,8\,0\,12\,1} + e_{0\,2\,6\,8\,0\,12\,14} + e_{0\,2\,6\,8\,9\,0\,6} + e_{0\,2\,6\,8\,9\,0\,7}$
$+ e_{0\,2\,6\,8\,9\,2\,6} + e_{0\,2\,6\,8\,9\,2\,14} + e_{0\,2\,6\,8\,9\,10\,7} + e_{0\,2\,6\,8\,9\,10\,14}$
$+ e_{0\,2\,6\,8\,13\,2\,6} + e_{0\,2\,6\,8\,13\,2\,14} + e_{0\,2\,6\,8\,13\,5\,1} + e_{0\,2\,6\,8\,13\,5\,6}$
$+ \ldots$



Set $r = p + q$. The number E of arrows is random, and it is easy to compute

$$\mathbb{E}\left(E\right) = \frac{r}{2} V \left(V - 1\right) \quad \text{and} \qquad \mathrm{Var}\left(E\right) = \frac{1}{2} r \left(1 - r\right) V \left(V - 1\right). \tag{3.1}$$

Define the *degree of digraph* as the average outcoming degree of the vertices:

$$\boxed{D = \deg G := \frac{E}{V}}.$$

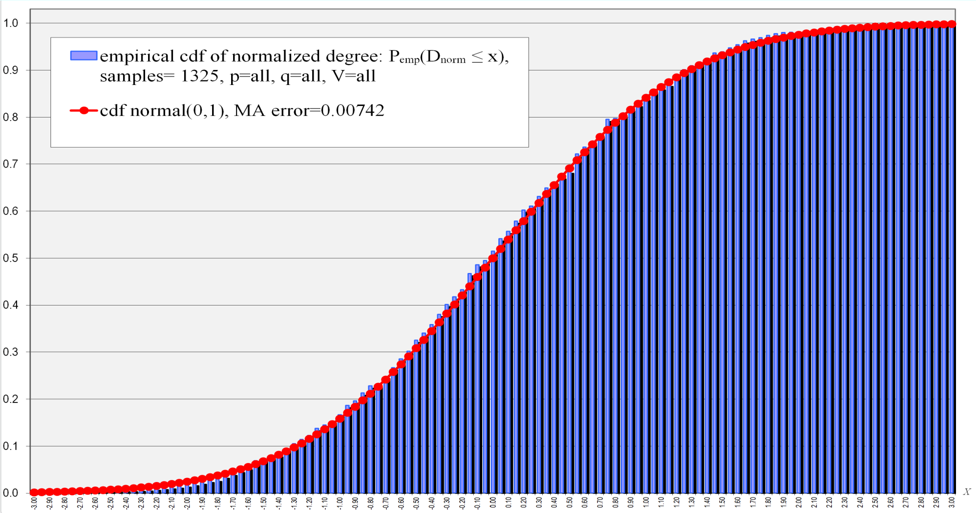
For example, for the above digraph $D = 86/15 \approx 5.7$.

For random digraphs it follows from (3.1) that

$$\mathbb{E}\left(D\right) = \frac{r}{2} \left(V - 1\right) \quad \text{and} \quad \mathrm{Var}\left(D\right) = \frac{1}{2} r \left(1 - r\right) \frac{V - 1}{V}.$$

Moreover, applying the central limit theorem to the sum of indicators of arrows we obtain

$$D_{norm} := \frac{D - \frac{r}{2}\left(V - 1\right)}{\sqrt{\frac{1}{2} r \left(1 - r\right) \frac{V-1}{V}}} \xrightarrow{\mathcal{D}} \mathrm{Normal}\left(0, 1\right) \quad \text{as } V \to \infty.$$



Proposition 3.1 *If $p + q > 0$ then*

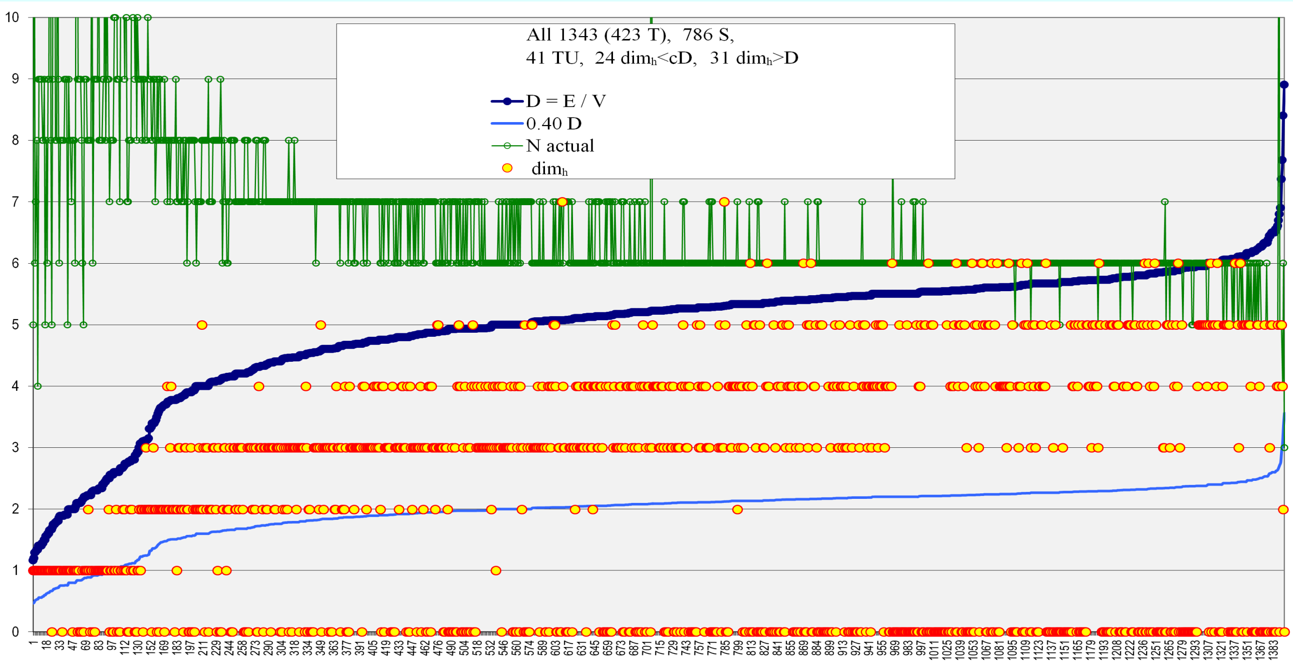
$$\lim_{V \to \infty} \mathbb{P}_{p,q,V}\left(G \text{ } is \text{ } connected\right) = 1$$

that is,

$$\mathbb{P}_{p,q,V}\left(\beta_0\left(G\right) = 1\right) \to 1 \quad as \quad V \to \infty.$$

3.3 Homological dimension and degree

It turns out that $\dim_h G$ for random digraphs is closely related to the degree $D = E/V$. In over 1000 of samples of randomly generated digraphs, we have observed the following dichotomy: $\boxed{\text{with high probability either } \dim_h G = 0 \text{ or } \dim_h G \asymp D.}$



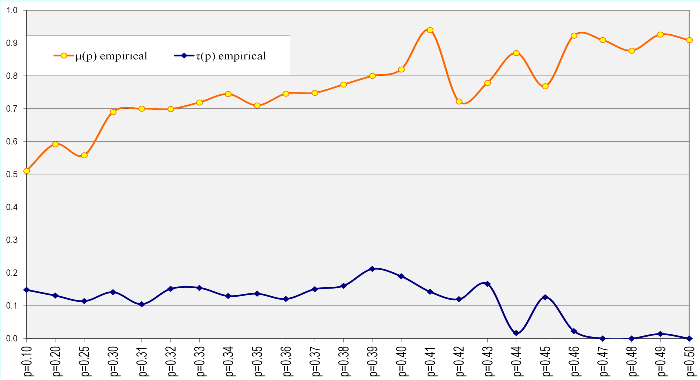
Consider the random variables $Q = \frac{\dim_h G}{D}$ and $Q_+ = (Q \mid Q > 0)$. Everywhere assume that $p = q \in (0, 1/2)$.

Conjecture 3.2 *There exists positive limits*

$$\mu(p) = \lim_{V\to\infty} \mathbb{E}_{p,p,V}(Q_+) \quad \textit{and} \quad \tau^2(p) = \lim_{V\to\infty} \mathrm{Var}_{p,p,V}(Q_+) = \lim_{V\to\infty} \mathbb{E}_{p,p,V}\left(Q_+^2\right) - \mu(p)^2.$$

Besides, we have $\mu(p) > 3\tau(p)$.

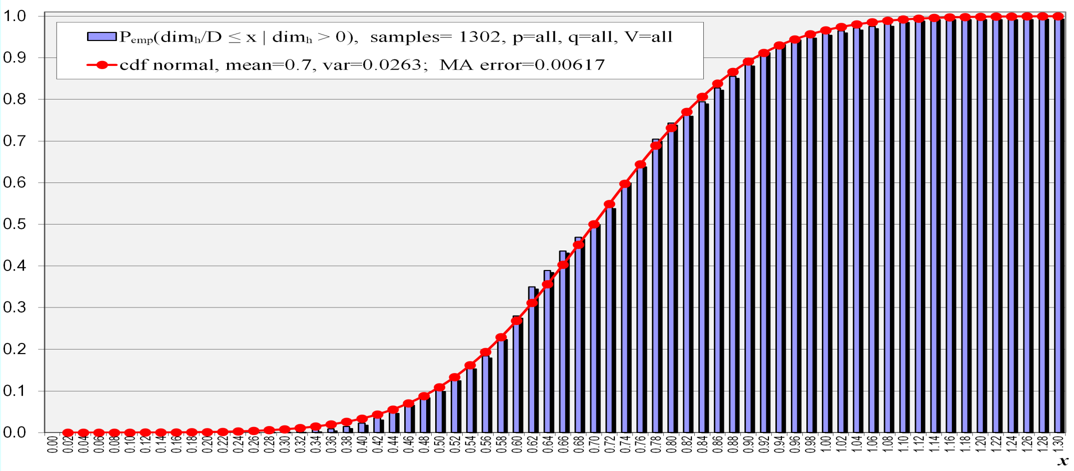
Here are empirical functions $\mu(p)$ and $\tau(p)$ computed using the averages of Q_+ and Q_+^2 among all available samples.



Conjecture 3.3 *We have* $Q_+ \xrightarrow{\mathcal{D}} \frac{1}{Z}\mathrm{Normal}_+(\mu, \tau^2)$ *as* $V \to \infty$, *where* $\mu = \mu(p)$ *and* $\tau = \tau(p)$. *That is, for any* $x \geq 0$,

$$\lim_{V \to \infty} \mathbb{P}_{p,p,V}(Q_+ \leq x) = \frac{1}{Z} \int_0^x \frac{1}{\sqrt{2\pi}\tau} \exp\left(-\frac{(y-\mu)^2}{2\tau^2}\right) dy,$$

where Z *is a normalizing factor.*



As one sees on this diagram, $\mathbb{P}(0.4 \leq Q_+ \leq 1) \approx 0.9$ that is,

$$\mathbb{P}(0.4D \leq \dim_h(G) \leq D \mid \dim_h(G) > 0) \approx 0.9.$$

3.4 Homologically trivial and spherical digraphs

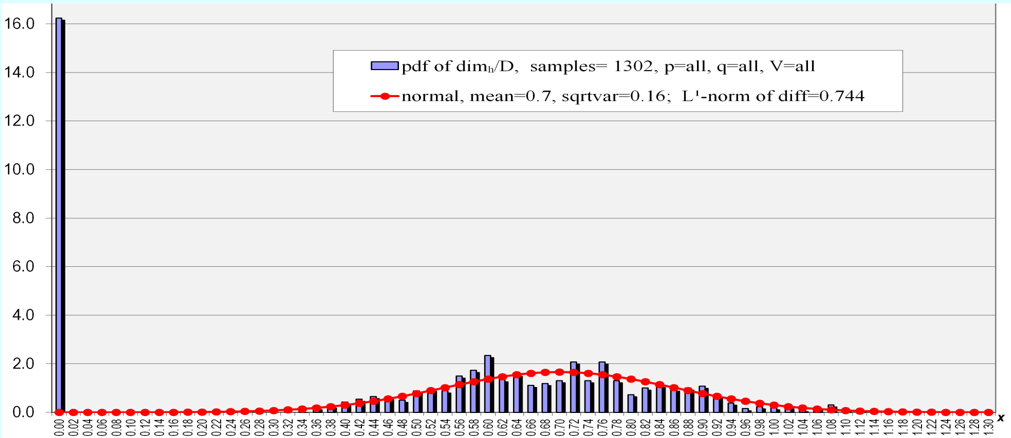
Let call a digraph G *homologically trivial* if $\dim_h G = 0$, that is, $\beta_k(G) = 0$ for all $k \geq 1$.

Conjecture 3.4 *The following limit exists and is positive:*

$$T(p) = \lim_{V\to\infty} \mathbb{P}_{p,p,V}(G \text{ is homologically trivial}).$$

Consequently,

$$\frac{\dim_h G}{D} = Q \xrightarrow{\mathcal{D}} T(p)\,\delta_0 + \frac{1}{Z(1-T(p))}\mathrm{Normal}_+(\mu,\tau^2) \quad \text{as } V \to \infty.$$

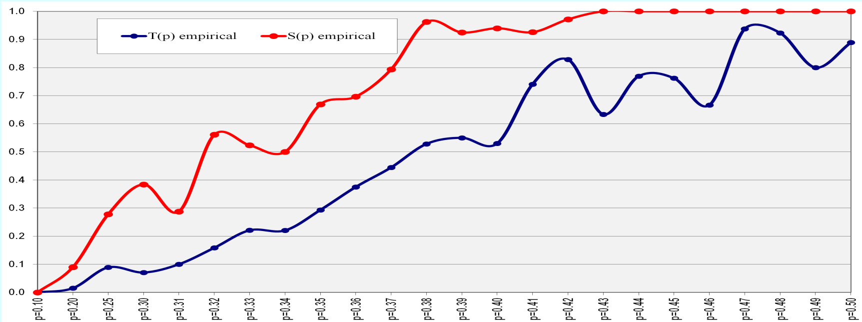


Let us call a digraph G *homologically spherical* of dimension n if $\beta_0(G) = \beta_n(G) = 1$ and all other Betti numbers vanish. In this case $\dim_h G = n$. Any homologically trivial digraph is also spherical of dimension 0.

Conjecture 3.5 *The following limit exists and is positive:*

$$S(p) = \lim_{V\to\infty} \mathbb{P}_{p,p,V}(G \text{ is homologically spherical}).$$

Of course, $S(p) \geq T(p)$ .Here are empirical functions $S(p)$ and $T(p)$ computed as fractions of all homologically spherical resp. trivial digraphs among all available samples.



We see that, for $p \approx 0.5$, a random digraph is homologically spherical with probability nearly 100%, and is homologically trivial with probability $\approx 90\%$.

3.5 Computational limitations

For computation of homology groups and Betti numbers of digraphs we use the aforementioned program of Chao Chen. It computes successively $H_k(G)$ and $\beta_k(G)$ for $k = 1, 2, ...$ until the memory of computer allows. Denote by N_a the largest rank of actually computable Betti number for a digraph G. For randomly generated digraphs with $p = q$ we have found the following empirical formula for N_a:

$$N_e = a \ln\left(1 + \frac{b}{V}\right) / \ln D, \tag{3.2}$$

where $D = E/V$ and a, b are constants to be found experimentally depending on the computer. For a 16GB $i7$ laptop we have $a = 3$ and $b = 400$. If $D > 3$ then usually $|N_a - N_e| \leq 1$ (*show computations*).

Since

$$E \leq \frac{1}{2} V (V - 1),$$

it follows that $D \leq \frac{1}{2}(V - 1)$ and $V > 2D$. Therefore,

$$N_e \leq a \ln\left(1 + \frac{b}{2D}\right) / \ln D.$$

We expect that $\dim_h < D$ with high probability. In order to verify this numerically, we should be able to compute β_k for all $k \leq D$, and for that we need to have $N_e \geq D$ that is,

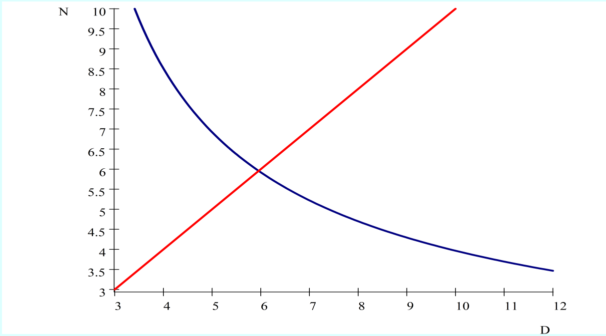
$$a \ln \left(1 + \frac{b}{2D}\right) / \ln D \geq D. \tag{3.3}$$

With these data, the condition (3.3) implies that $D \leq 6$.

Here the graph of the function

$a \ln \left(1 + \frac{b}{2D}\right) / \ln D$

is shown in blue and
the diagonal is shown in red:



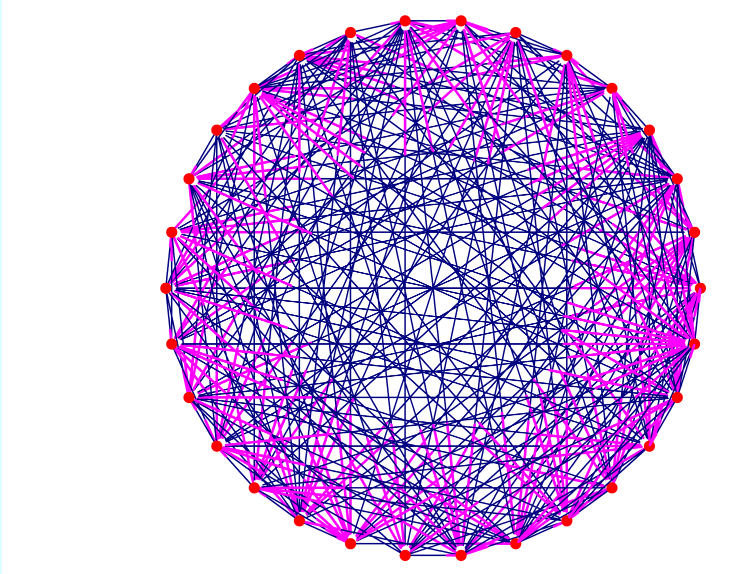
Hence, if for a randomly generated digraph $D > 6$ then computation of $\dim_h G$ becomes unreliable.

Here is a randomly generated digraph with

$V = 30, \quad E = 267, \quad D = 8.9$
$p = q = 0.3$

By (3.2) we have $N_e = 4$,
while $N_a = 3$ and the actually computed Betti numbers are $1, 0, 0, 0$.

Since $D = 8.9 \gg N_a$, no reliable conclusion about the value of $\dim_h G$ can be made.



For such digraphs we need either to use a more powerful computer or to improve the algorithm of the program.

Problem 3.6 *Compute for this digraph* β_k *for all* $k \leq 9$.

4 Combinatorial curvature of digraphs

4.1 Motivation

Let Γ be a finite planar graph. There is the following old notion of a *combinatorial curvature* K_x at any vertex x of Γ:

$$K_x = 1 - \frac{\deg(x)}{2} + \sum_{f \ni x} \frac{1}{\deg(f)}, \tag{4.1}$$

where the sum is taken over all faces f containing x and $\deg(f)$ denotes the number of vertices of f. For example, if all faces are triangles then we obtain

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3}, \tag{4.2}$$

where $\deg_\Delta(x)$ is the number of triangles having x as a vertex.

In general, denoting by E, V and F the number of vertices, edges and faces of Γ and observing that

$$\sum_x \deg(x) = 2E \quad \text{and} \quad \sum_x \sum_{f \ni x} \frac{1}{\deg(f)} = \sum_f \sum_{x \in f} \frac{1}{\deg(f)} = F,$$

we obtain

$$\sum_x K_x = V - E + F = \chi.$$

We try to realize this idea on digraph: to "distribute" the Euler characteristic over all vertices and, hence, to obtain an analog of Gauss curvature that satisfies Gauss-Bonnet.

4.2 Curvature operator

Let $G = (V, E)$ be a finite digraph and $\mathbb{K} = \mathbb{R}$. We would like to generalize (4.1) to arbitrary digraphs, so that the faces in (4.1) should be replaced by the elements of a basis in Ω_p. However, the result should be independent of the choice of a basis.

Fix $p \geq 0$. Any function $f : V \to \mathbb{R}$ on the vertices induces an linear operator

$$T_f : \mathcal{R}_p \to \mathcal{R}_p$$

by

$$T_f e_{i_0 \ldots i_p} = \left(f\left(i_0\right) + \ldots + f\left(i_p\right)\right) e_{i_0 \ldots i_p}.$$

For example, for a constant function $f = \mathbf{1}$ on V, we have $T_{\mathbf{1}} e_{i_0 \ldots i_p} = (p+1)\, e_{i_0 \ldots i_p}$ and, hence,

$$T_{\mathbf{1}} \omega = (p+1)\, \omega \quad \text{for any } \omega \in \mathcal{R}_p. \tag{4.3}$$

If $f = \mathbf{1}_x$ where $x \in V$, then

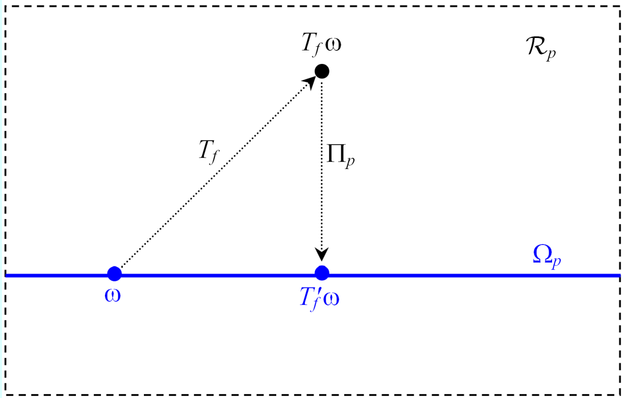
$$T_{\mathbf{1}_x} e_{i_0 \dots i_p} = m e_{i_0 \dots i_p}, \text{ where } m \text{ is the number of occurrences of } x \text{ in } i_0, \dots, i_p. \tag{4.4}$$

Fix in $\mathcal{R}_p$ an inner product $(\cdot, \cdot)$. For example, this can be a *natural inner product* when all regular elementary paths $e_{i_0 \dots i_p}$ form an orthonormal basis in $\mathcal{R}_p$.

Let $\Pi_p : \mathcal{R}_p \to \Omega_p$ be the orthogonal projection onto Ω_p.

Considering T_f as an operator from Ω_p to $\mathcal{R}_p$, we obtain the following operator in Ω_p:

$$T'_f := \Pi_p \circ T_f : \Omega_p \to \Omega_p$$



Definition. Define the *incidence* of f and Ω_p by

$$[f, \Omega_p] := \operatorname{trace} T'_f.$$

Definition. For any $\omega = \sum \omega^{i_0 \dots i_p} e_{i_0 \dots i_p} \in \Omega_p$ define the *incidence* of f and ω by

$$[f, \omega] := (T_f \omega, \omega)$$

Lemma 4.1 *For any orthogonal basis $\{\omega_k\}$ in Ω_p we have*

$$[f, \Omega_p] = \sum_k \frac{[f, \omega_k]}{\|\omega_k\|^2}. \tag{4.5}$$

Proof. It suffices to prove (4.5) for orthonormal basis when $\|\omega_k\| = 1$ for all k. By the definition of the trace

$$\operatorname{trace} T_f' = \sum_k \left(T_f' \omega_k, \omega_k\right).$$

For any $\omega \in \Omega_p$ we have

$$\left(T_f' \omega, \omega\right) = (\Pi_p T_f \omega, \omega) = (T_f \omega, \Pi_p \omega) = (T_f \omega, \omega) = [f, \omega],$$

whence (4.5) follows. ■

Definition. For any $N \in \mathbb{N}$ define the *curvature operator* $K^{(N)} : \mathbb{R}^V \to \mathbb{R}$ of order N by

$$\boxed{K^{(N)} f = \sum_{p=0}^{N} \frac{(-1)^p}{p+1} [f, \Omega_p]}.$$

If $\Omega_p = \{0\}$ for all $p > N$, then write $K_f^{(N)} = K_f$.

For $f = \mathbf{1}_x$ where $x \in V$, we write

$$[x, \Omega_p] := [\mathbf{1}_x, \Omega_p] \quad \text{and} \quad [x, \omega] := [\mathbf{1}_x, \omega],$$

If $\{\omega_k\}$ is an orthogonal basis of Ω_p, then by (4.5)

$$\boxed{[x, \Omega_p] = \sum_k \frac{[x, \omega_k]}{\|\omega_k\|^2}}.$$

If the inner product is natural so that $\{e_{i_0 \ldots i_p}\}$ is orthonormal then by (4.4)

$$\boxed{\left[x, e_{i_0 \ldots i_p}\right] = m, \text{ where } m \text{ is the number of occurrences of } x \text{ in } i_0, ..., i_p}.$$

For example,

$$[a, e_{abca}] = 2, \qquad [b, e_{abca}] = 1, \quad [d, e_{abca}] = 0.$$

In this case, for $\omega = \sum \omega^{i_0 \dots i_p} e_{i_0 \dots i_p}$ we have

$$\boxed{[x, \omega] = \sum_{i_0 \dots i_p \in V} \left(\omega^{i_0 \dots i_p}\right)^2 \left[x, e_{i_0 \dots i_p}\right]}.$$

Definition. For any $N \in \mathbb{N}$ define the *curvature of order* N at a vertex x by

$$\boxed{K_x^{(N)} := K^{(N)} \mathbf{1}_x = \sum_{p=0}^{N} \tfrac{(-1)^p}{p+1} [x, \Omega_p]}.$$

Proposition 4.2 (Gauss-Bonnet) *For any choice of the inner product in $\mathcal{R}_p$ and for any N we have*

$$\sum_{x \in V} K_x^{(N)} =: \boxed{K_{total}^{(N)} = \chi^{(N)}} := \sum_{p=0}^{N} (-1)^p \dim \Omega_p.$$

Proof. Since $\sum_{x \in V} \mathbf{1}_x = \mathbf{1}$, we obtain that

$$K_{total}^{(N)} = \sum_{x \in V} K_x^{(N)} = \sum_{x \in V} K^{(N)} \mathbf{1}_x = K^{(N)} \mathbf{1} = \sum_{p=0}^{N} (-1)^p \frac{[\mathbf{1}, \Omega_p]}{p+1}.$$

On the other hand, by (4.3)

$$[\mathbf{1},\omega]=(T_{\mathbf{1}}\omega,\omega)=(p+1)\left\|\omega\right\|^{2}.$$

If $\{\omega_k\}$ is an orthogonal basis in Ω_p then by (4.5)

$$[\mathbf{1},\Omega_p]=\sum_k\frac{[\mathbf{1},\omega_k]}{\left\|\omega_k\right\|^2}=(p+1)\dim\Omega_p,$$

which implies

$$K_{total}^{(N)}=\sum_{p=0}^{N}(-1)^p\dim\Omega_p=\chi^{(N)}.$$

■

Remark. If $\Omega_p=\{0\}$ for all $p>N$ then

$$\chi:=\sum_{p=0}^{N}(-1)^p\dim\Omega_p=\sum_{p=0}^{N}(-1)^p\dim H_p.$$

Remark. It can happen that $\Omega_p\neq\{0\}$ for all p. One example is given on p.40. Here is a much simpler example: $G=\{a\rightleftarrows b\}$. For this digraph we have

$$\Omega_0=\langle e_a,e_b\rangle,\quad \Omega_1=\langle e_{ab},e_{ba}\rangle,\quad \Omega_3=\langle e_{aba},e_{bab}\rangle,\quad \Omega_4=\{e_{abab},e_{baba}\},\quad \text{etc},$$

so that $|\Omega_p| = 2$ for all $p \geq 0$. Indeed, $e_{aba} \in \mathcal{A}_2$ and

$$\partial e_{aba} = e_{ba} - e_{aa} + e_{ab} = e_{ba} + e_{ab} \in \mathcal{A}_1$$

so that $e_{aba} \in \Omega_2$. Similarly, $e_{abab} \in \mathcal{A}_3$ and

$$\partial e_{abab} = e_{bab} - e_{aab} + e_{abb} - e_{aba} = e_{bab} - e_{aba} \in \mathcal{A}_2$$

so that $e_{abab} \in \Omega_3$, etc.

Problem 4.3 *How to decide whether the sequence $\{\Omega_p(G)\}$ vanishes for all large p?*

Alternatively, one can always truncate the chain complex to make it finite by setting by definition $\Omega_{N+1} = \{0\}$ for some N :

$$0 \ \leftarrow \ \Omega_0 \ \stackrel{\partial}{\leftarrow} \ \Omega_1 \ \stackrel{\partial}{\leftarrow} \ \ldots \ \stackrel{\partial}{\leftarrow} \ \Omega_{N-1} \ \stackrel{\partial}{\leftarrow} \ \Omega_N \ \leftarrow 0$$

and work with homology groups of this complex. This corresponds to the following modification of the notion of allowed paths: all paths of length $> N$ are declared non-allowed.

4.3 Examples of computation of curvature

Let us fix in $\mathcal{R}_p$ the natural inner product. Using the orthonormal basis $\{e_i\}$ in Ω_0 we obtain

$$[x, \Omega_0] = \sum_i [x, e_i] = 1$$

and, using the orthonormal basis $\{e_{ij}\}$ with $i \to j$ in Ω_1, we obtain

$$[x, \Omega_1] = \sum_{i \to j} [x, e_{ij}] = \deg(x) .$$

Therefore,

$$K_x^{(1)} = 1 - \frac{\deg(x)}{2}$$

and, for any $N \geq 1$,

$$K_x^{(N)} = 1 - \frac{\deg(x)}{2} + \sum_{p=2}^{N} \frac{(-1)^p}{p+1} [x, \Omega_p] . \tag{4.6}$$

By Proposition 1.2, in the absence of double arrows the space Ω_2 has always a basis of triangles and squares (but this basis is not necessarily orthogonal).

For a triangle $e_{abc} \in \Omega_2$ we have

$$[x, e_{abc}] = \begin{cases} 1, & x \in \{a, b, c\} \\ 0, & \text{otherwise} \end{cases}$$

and for a square $e_{abc} - e_{ab'c} \in \Omega_2$

$$[x, e_{abc} - e_{ab'c}] = \begin{cases} 2, & x \in \{a, c\} \\ 1, & x \in \{b, b'\} \\ 0, & \text{otherwise} \end{cases}$$

In particular, if G has no square then Ω_2 has a basis $\{\omega_k\}$ that consists of all triangles in G. This basis is orthonormal and

$$[x, \Omega_2] = \sum_k [x, \omega_k] = \deg_\Delta(x) := \#\text{triangles containing } x.$$

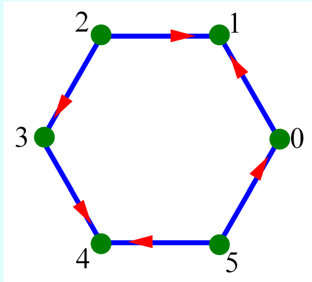
It follows that

$$K_x^{(2)} = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3},$$

which matches (4.2).

Example. Let G be a line digraph, for example, $\cdots \bullet \to \bullet \leftarrow \bullet \to \bullet \ldots$. Then by (4.6) $K_x = \frac{1}{2}$ for the endpoints, and $K_x = 0$ for the interior points.

Example. Let G be a cyclic digraph (polygon) different from triangle or square:



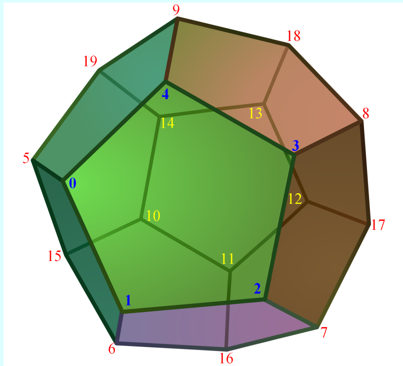
Then we have $\Omega_p = \{0\}$ for $p > 1$.
Hence by (4.6), for any vertex x,

$$K_x = 1 - \frac{\deg(x)}{2} = 0.$$

and $K_{total} = 0$.
For comparison,

$$\chi = |\Omega_0| - |\Omega_1| = 6 - 6 = 0.$$

Example. Consider a dodecahedron (with any orientation of edges):

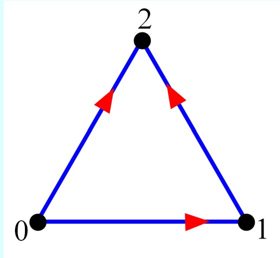


We have $|\Omega_0| = 20$, $|\Omega_1| = 30$, $|\Omega_2| = 0$,
and $|H_1| = 11$, $|H_p| = 0$ for $p > 1$.
Then, for any vertex x,

$$K_x = 1 - \frac{\deg(x)}{2} = -\frac{1}{2}$$

and $K_{total} = -10$.
For comparison,
$\chi = 1 - 11 = 20 - 30 = -10$.

Example. Let G be a triangle. We have $\Omega_2 = \langle e_{012} \rangle$ and $\Omega_p = \{0\}$ for $p > 2$.

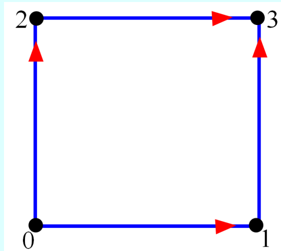


Hence, for each vertex x,

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3} = \frac{1}{3}.$$

and $K_{total} = 1$.
For comparison, $\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| = 3 - 3 + 1 = 1$.

Example. Let G be a square. Then $\Omega_2 = \langle e_{013} - e_{023} \rangle$ and $\Omega_p = \{0\}$ for $p > 2$.



Since $\|e_{013} - e_{023}\|^2 = 2$, we obtain

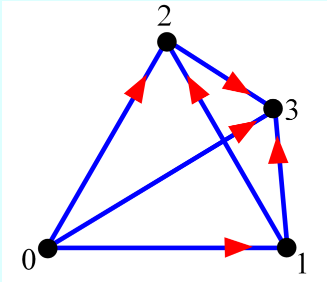
$$[0, \Omega_2] = \tfrac{1}{2}[0, e_{013} - e_{023}] = 1, \qquad [3, \Omega_2] = 1$$

$$[1, \Omega_2] = \tfrac{1}{2}[1, e_{013} - e_{023}] = \tfrac{1}{2}, \qquad [2, \Omega_2] = \tfrac{1}{2}$$

It follows that

$$K_3 = K_0 = 1 - \frac{\deg(0)}{2} + \frac{1}{3} = \frac{1}{3}, \qquad K_2 = K_1 = 1 - \frac{\deg(1)}{2} + \frac{1}{6} = \frac{1}{6}, \quad K_{total} = 1 = \chi.$$

Example. Let G be a 3-simplex



We have

$$\Omega_2 = \langle e_{012}, e_{013}, e_{023}, e_{123} \rangle$$

and

$$\Omega_3 = \langle e_{0123} \rangle,$$

while $\Omega_p = 0$ for $p > 3$. It follows that, for any vertex x,

$$[x, \Omega_2] = \deg_\Delta (x) = 3 \quad \text{and} \quad [x, \Omega_3] = 1$$

whence

$$K_x = 1 - \frac{\deg (x)}{2} + \frac{[x, \Omega_2]}{3} - \frac{[x, \Omega_3]}{4} = \frac{1}{4}, \qquad K_{total} = 1 = \chi.$$

Example. Let G be an *n-simplex*, that is, a digraph with a set of vertices $\{0,1,...,n\}$ and edges $i \to j$ whenever $i < j$. Then, for any $p = 0,1,...,n$

$$\Omega_p = \mathcal{A}_p = \langle e_{i_0...i_p} : i_0 < i_1 < ... < i_p \rangle$$

so that $\dim \Omega_p = \binom{n+1}{p+1}$. It follows that, for any vertex x,

$$[x, \Omega_p] = \# \left\{ e_{i_0...i_p} \text{ such that } x \in \{i_0, ..., i_p\} \right\} = \binom{n}{p},$$

and

$$K_x = \sum_{p=0}^{n} (-1)^p \frac{\binom{n}{p}}{p+1}.$$

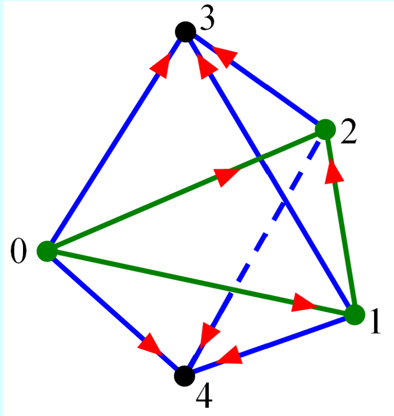
Change $j = p + 1$ gives

$$(n+1) K_x = \sum_{j=1}^{n+1} (-1)^{j-1} \frac{(n+1)\binom{n}{j-1}}{j} = \sum_{j=1}^{n+1} (-1)^{j-1} \binom{n+1}{j} = 1,$$

whence

$$K_x = \frac{1}{n+1} \quad \text{and} \quad K_{total} = 1.$$

Example. Let G be a bipyramid:



We have $|\Omega_0| = 5, \quad |\Omega_1| = 9,$

$\Omega_2 = \langle e_{013}, e_{123}, e_{023}, e_{014}, e_{124}, e_{024}, e_{012} \rangle$

$\Omega_3 = \langle e_{0123}, e_{0124} \rangle$

and $|\Omega_p| = 0$ for $p \geq 4$.

Hence,
$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 5 - 9 + 7 - 2 = 1.$

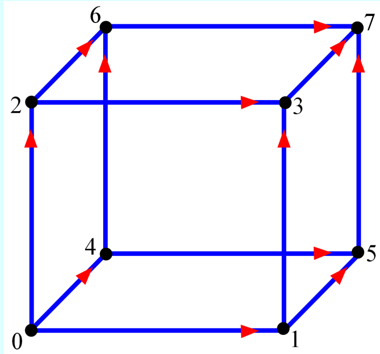
Let us compute the curvature:

x	$[x, \Omega_2]$	$[x, \Omega_3]$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
$3, 4$	3	1	$1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}$	$= \frac{1}{4}$
$0, 1, 2$	5	2	$1 - \frac{4}{2} + \frac{5}{3} - \frac{2}{4}$	$= \frac{1}{6}$

Hence, $K_{total} = \frac{2}{4} + \frac{3}{6} = 1.$

Example. Let G be a 3-cube. We have

$$\Omega_2 = \langle e_{013} - e_{023},\ e_{015} - e_{045},\ e_{026} - e_{046},\ e_{137} - e_{157},\ e_{237} - e_{267},\ e_{457} - e_{467}\rangle$$



(note that this above basis in Ω_2 is orthogonal)

$$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}\rangle$$

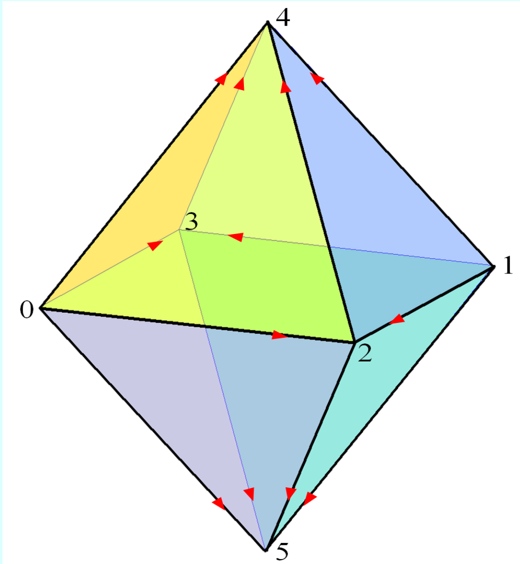
$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 8 - 12 + 6 - 1 = 1$$

Let us compute the curvature:

x	$\frac{[x,\Omega_2]}{}$	$\frac{[x,\Omega_3]}{}$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
$0, 7$	$\frac{6}{2} = 3$	$\frac{6}{6} = 1$	$1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}$	$= \frac{1}{4}$
$1, 2, 3, 4, 5, 6$	$\frac{4}{2} = 2$	$\frac{2}{6} = \frac{1}{3}$	$1 - \frac{3}{2} + \frac{2}{3} - \frac{1}{12} = \frac{1}{12}$	$= \frac{1}{12}$

Consequently, $K_{total} = \frac{2}{4} + \frac{6}{12} = 1 = \chi$.

Example. Consider on octahedron:



We have

$$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \rangle,$$

and $\Omega_p = \{0\}$ for all $p \geq 3$

For any vertex x we obtain

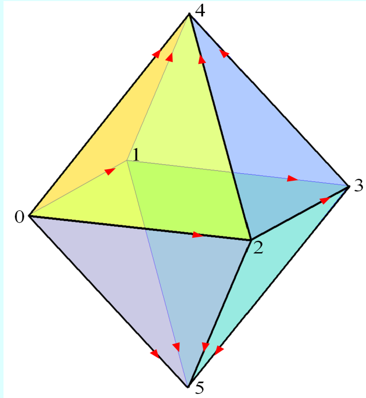
$$[x, \Omega_2] = \deg_\Delta(x) = 4$$

whence

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3} = 1 - \tfrac{4}{2} + \tfrac{4}{3} = \tfrac{1}{3}$$

In particular, $K_{total} = \frac{6}{3} = 2 = \chi$.

Example. Consider on octahedron with a different orientation:



We have the following orthogonal bases:

$$\Omega_2 = \langle e_{024}, e_{025}, e_{014}, e_{015}, e_{234}, e_{235}, e_{134}, e_{135}, e_{013} - e_{023} \rangle$$

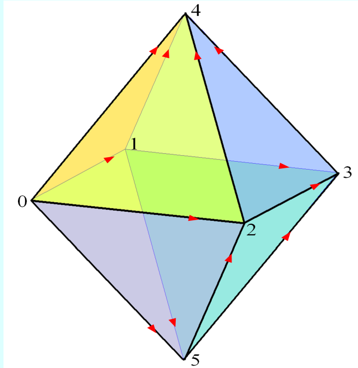
$$\Omega_3 = \langle e_{0234} - e_{0134},\ e_{0235} - e_{0135} \rangle$$

$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 6 - 12 + 9 - 2 = 1$$

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
0	$4 + \frac{2}{2} = 1$	$\frac{4}{2} = 2$	$1 - \frac{4}{2} + \frac{5}{3} - \frac{2}{4}$	$= \frac{1}{6}$
1	$4 + \frac{1}{2} = \frac{9}{2}$	$\frac{2}{2} = 1$	$1 - \frac{4}{2} + \frac{9/2}{3} - \frac{1}{4}$	$= \frac{1}{4}$
2	$4 + \frac{1}{2} = \frac{9}{2}$	$\frac{2}{2} = 1$	$1 - \frac{4}{2} + \frac{9/2}{3} - \frac{1}{4}$	$= \frac{1}{4}$
3	$4 + \frac{2}{2} = 5$	$\frac{4}{2} = 2$	$1 - \frac{4}{2} + \frac{5}{3} - \frac{2}{4}$	$= \frac{1}{6}$
4	4	$\frac{2}{2} = 1$	$1 - \frac{4}{2} + \frac{4}{3} - \frac{1}{4}$	$= \frac{1}{12}$
5	4	$\frac{2}{2} = 1$	$1 - \frac{4}{2} + \frac{4}{3} - \frac{1}{4}$	$= \frac{1}{12}$

$$K_{total} = \tfrac{2}{6} + \tfrac{2}{4} + \tfrac{2}{12} = 1 = \chi.$$

Example. Here is yet another octahedron. We have to orthogonalize the bases:



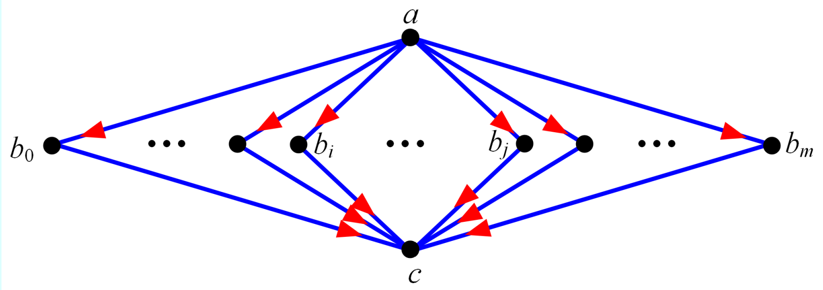
$$
\begin{aligned}
\Omega_2 &= \langle e_{014},\ e_{015},\ e_{024},\ e_{052},\ e_{134},\ e_{153},\ e_{234},\ e_{523},\\
&\qquad e_{013}-e_{023},\ e_{013}-e_{053},\ e_{524}-e_{534}\rangle\\
&= \langle e_{014},\ e_{015},\ e_{024},\ e_{052},\ e_{134},\ e_{153},\ e_{234},\ e_{523},\\
&\qquad e_{013}-e_{023},\ e_{013}+e_{023}-2e_{053},\ e_{524}-e_{534}\rangle\\
\Omega_3 &= \langle e_{0153},\ e_{0523},\ e_{5234},\ e_{0134}-e_{0234},\ e_{0534}-e_{0134}-e_{0524}\rangle\\
&= \langle e_{0153},\ e_{0523},\ e_{5234},\ e_{0134}-e_{0234},\ e_{0134}+e_{0234}-2e_{0534}+2e_{0524}\rangle\\
\Omega_4 &= \langle e_{05234}\rangle,\ \ \Omega_p=\{0\} \text{ for } p\geq 5.
\end{aligned}
$$

$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| + |\Omega_4| \ = 6-12+11-5+1 = 1.$$

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$[x,\Omega_4]$	$1-\frac{\deg(x)}{2}+\frac{[x,\Omega_2]}{3}-\frac{[x,\Omega_3]}{4}+\frac{[x,\Omega_4]}{5}$	$=K_x$
0	$4+\frac{2}{2}+\frac{6}{6}=6$	$2+\frac{2}{2}+\frac{10}{10}=4$	1	$1-\frac{4}{2}+\frac{6}{3}-\frac{4}{4}+\frac{1}{5}$	$=\frac{1}{5}$
1	$4+\frac{1}{2}+\frac{1}{6}=\frac{14}{3}$	$1+\frac{1}{2}+\frac{1}{10}=\frac{8}{5}$	0	$1-\frac{4}{2}+\frac{14/3}{3}-\frac{8/5}{4}$	$=\frac{7}{45}$
2	$4+\frac{1}{2}+\frac{1}{6}+\frac{1}{2}=\frac{31}{6}$	$2+\frac{1}{2}+\frac{5}{10}=3$	1	$1-\frac{4}{2}+\frac{31/6}{3}-\frac{3}{4}+\frac{1}{5}$	$=\frac{31}{180}$
3	$4+\frac{2}{2}+\frac{6}{6}+\frac{1}{2}=\frac{13}{2}$	$3+\frac{2}{2}+\frac{6}{10}=\frac{23}{5}$	1	$1-\frac{4}{2}+\frac{13/2}{3}-\frac{23/5}{4}+\frac{1}{5}=\frac{13}{60}$	$=\frac{13}{60}$
4	$4+\frac{2}{2}=5$	$1+\frac{2}{2}+\frac{10}{10}=3$	1	$1-\frac{4}{2}+\frac{5}{3}-\frac{3}{4}+\frac{1}{5}$	$=\frac{7}{60}$
5	$4+\frac{4}{6}+\frac{2}{2}=\frac{17}{3}$	$3+\frac{8}{10}=\frac{19}{5}$	1	$1-\frac{4}{2}+\frac{17/3}{3}-\frac{19/5}{4}+\frac{1}{5}$	$=\frac{5}{36}$

$$K_{total} = \frac{1}{5}+\frac{7}{45}+\frac{31}{180}+\frac{13}{60}+\frac{7}{60}+\frac{5}{36} = 1 = \chi$$

Example. Consider the following spider-like digraph G:



The space Ω_2 consists of squares $e_{ab_ic} - e_{ab_jc}$ and their linear combinations, while $\Omega_p = \{0\}$ for all $p > 2$. It is easy to see that

$$\Omega_2 = \langle e_{ab_0c} - e_{ab_jc} \rangle_{j=1}^{m} \tag{4.7}$$

so that $|\Omega_2| = m$ and $K_{total} = \chi = |\Omega_0| - |\Omega_1| + |\Omega_2| = (m+3) - 2(m+1) + m = 1$.

Orthogonalization of (4.7) gives the following orthogonal basis in Ω_2:

$$\begin{aligned} \omega_1 &= e_{ab_0c} - e_{ab_1c} \\ \omega_2 &= e_{ab_0c} + e_{ab_1c} - 2e_{ab_2c} \\ &\ldots \end{aligned}$$

$$\omega_i = e_{ab_0c} + ... + e_{ab_{i-1}c} - ie_{ab_ic}$$

$$...$$

$$\omega_m = e_{ab_0c} + ... + e_{ab_{m-1}c} - me_{ab_mc}$$

We have $[a, \omega_i] = [c, \omega_i] = \|\omega_i\|^2 = i(i+1)$ while

$$[b_j, \omega_i] = \begin{cases} 0, & j > i \\ 1, & j < i \\ i^2, & j = i \end{cases}$$

which implies

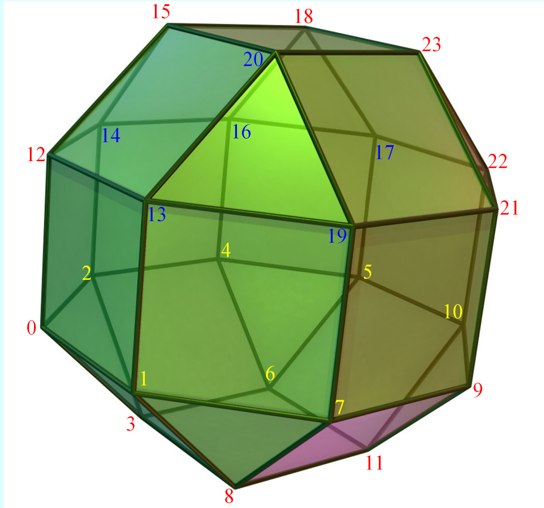
$$K_c = K_a = 1 - \frac{\deg(a)}{2} + \frac{1}{3}\sum_{i=1}^{m} \frac{[a, \omega_i]}{\|\omega_i\|^2} = 1 - \frac{m+1}{2} + \frac{m}{3} = \frac{5}{6} - \frac{m}{6}$$

and

$$K_{b_j} = 1 - \frac{\deg(b_j)}{2} + \frac{1}{3}\sum_{i=1}^{m} \frac{[b_j, \omega_i]}{i(i+1)} = \frac{1}{3}\frac{j^2}{j(j+1)} + \frac{1}{3}\sum_{i=j+1}^{m} \frac{1}{i(i+1)} = \frac{1}{3}\left(1 - \frac{1}{m+1}\right).$$

Example. Consider a rhombicuboctahedron:

It has 24 vertices, 48 edges and 26 faces, among them 8 triangular and 18 rectangular.



Let us make it into a digraph G by choosing direction $i \to j$ on an edge (i,j) if $i < j$. Then each face becomes a triangle or square.

For this digraph $|H_2| = 1$ and $H_p = \{0\}$ for $p = 1$ and $p > 2$.

Spaces Ω_p with $p \geq 3$ are trivial, while $|\Omega_2| = 26$. Space Ω_2 is generated by 8 triangles and 18 squares:

$$\begin{aligned}\Omega_2 \;=\; & \langle e_{023},\ e_{178},\ e_{456},\ e_{9\,10\,11},\ e_{12\,14\,15},\ e_{13\,19\,20},\ e_{16\,17\,18},\ e_{21\,22\,23},\\ & e_{018}-e_{038},\ e_{0\,1\,13}-e_{0\,12\,13},\ e_{0\,2\,14}-e_{0\,12\,14},\ e_{1\,7\,19}-e_{1\,13\,19},\ e_{236}-e_{246},\\ & e_{2\,4\,16}-e_{2\,14\,16},\ e_{3\,6\,11}-e_{3\,8\,11},\ e_{4\,5\,17}-e_{4\,16\,17},\ e_{5\,10\,11}-e_{5\,6\,11},\ e_{5\,10\,22}-e_{5\,17\,22},\\ & e_{7\,8\,11}-e_{7\,9\,11},\ e_{7\,9\,21}-e_{7\,19\,21},\ e_{9\,10\,22}-e_{9\,21\,22},\ e_{12\,13\,20}-e_{12\,15\,20},\\ & e_{14\,15\,18}-e_{14\,16\,18},\ e_{15\,18\,23}-e_{15\,20\,23},\ e_{17\,22\,23}-e_{17\,18\,23},\ e_{19\,20\,23}-e_{19\,21\,23}\rangle,\end{aligned}$$

while the generator of H_2 is a signed sum of all these 2-paths.

This basis in Ω_2 is orthogonal. Hence, we compute the curvature:

$x=$	0,11,23	1,3,4,6,8,9,12,13,15,16,18,20,21	2,5,7,14,17,19,22	10
$[x,\Omega_2]=$	$1+\frac{6}{2}=4$	$1+\frac{4}{2}=3$	$1+\frac{5}{2}=\frac{7}{2}$	$1+\frac{3}{2}=\frac{5}{2}$
$1-\frac{\deg(x)}{2}+\frac{[x,\Omega_2]}{3}=$	$1-\frac{4}{2}+\frac{4}{3}$	$1-\frac{4}{2}+\frac{3}{3}$	$1-\frac{4}{2}+\frac{7/2}{3}$	$1-\frac{4}{2}+\frac{5/2}{3}$
K_x	$=\frac{1}{3}$	$=0$	$=\frac{1}{6}$	$=-\frac{1}{6}$

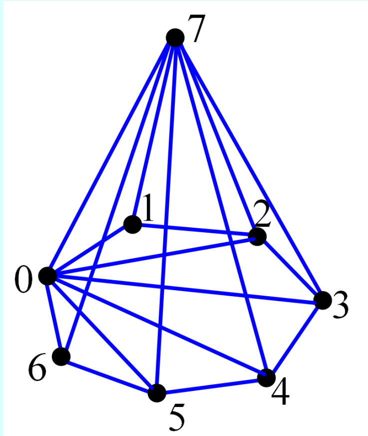
It follows that

$$K_{total}=\frac{3}{3}+\frac{7}{6}-\frac{1}{6}=2.$$

For comparison

$$\begin{aligned}\chi &= |\Omega_0|-|\Omega_1|+|\Omega_2|=24-48+26=2\\ &= |H_0|-|H_1|+|H_2|\,.\end{aligned}$$

Example. Consider the following pyramid:



Let us make it into a digraph G by choosing direction $i \to j$ on an edge (i,j) if $i<j$.
We have $|\Omega_0| = 8, \quad |\Omega_1| = 18,$

$$\Omega_2 = \langle e_{017}, e_{027}, e_{037}, e_{047}, e_{057}, e_{067} \\ e_{012}, e_{023}, e_{034}, e_{045}, e_{056}, \; e_{127}, e_{237}, e_{347}, e_{457}, e_{567} \rangle$$

$$\Omega_3 = \langle e_{0127}, e_{0237}, e_{0347}, e_{0457}, e_{0567} \rangle$$

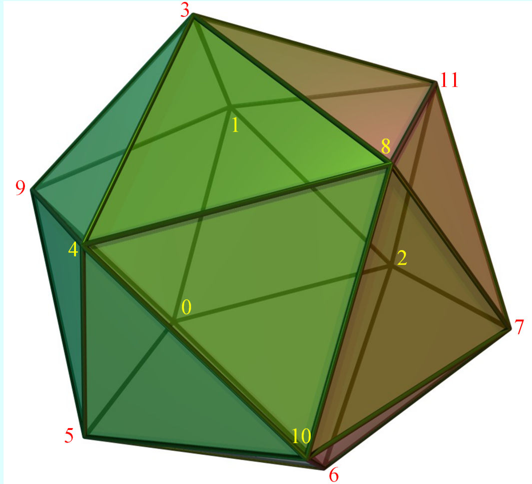
$$\Omega_p = \{0\} \text{ for } p \geq 4.$$

Let us compute the curvature:

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
$0,7$	11	5	$1 - \frac{7}{2} + \frac{11}{3} - \frac{5}{4}$	$= -\frac{1}{12}$
$1,6$	3	1	$1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}$	$= \frac{1}{4}$
$2,3,4,5$	5	2	$1 - \frac{4}{2} + \frac{5}{3} - \frac{2}{4}$	$= \frac{1}{6}$

It follows that $K_{total} = -\frac{2}{12} + \frac{2}{4} + \frac{4}{6} = 1$. For comparison $\chi = 8 - 18 + 16 - 5 = 1$.

Example. Let us compute the curvature of icosahedron (cf. p. 36).



Here we choose direction $i \to j$ if $i < j$. We have

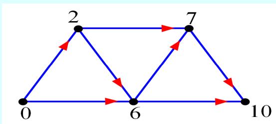
$|H_1| = 0,\ |H_2| = 1,\ |H_p| = 0$ for $p > 2$
$|\Omega_0| = 12,\quad |\Omega_1| = 30,\quad |\Omega_2| = 25,\quad |\Omega_3| = 6,$
$|\Omega_4| = 1$ and $\Omega_p = \{0\}$ for $p \geq 5$.

Hence,
$$\chi = |H_0| - |H_1| + |H_2| = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| + |\Omega_4| = 2.$$

We have

$$\Omega_2 = \langle e_{0\,1\,9}, e_{0\,1\,2}, e_{1\,2\,11}, e_{0\,2\,6}, e_{0\,5\,9}, e_{0\,5\,6}, e_{5\,6\,10}, e_{1\,3\,9}, e_{1\,3\,11}, e_{2\,6\,7},$$
$$e_{6\,7\,10}, e_{2\,7\,11}, e_{3\,4\,9}, e_{3\,4\,8}, e_{4\,8\,10}, e_{3\,8\,11}, e_{4\,5\,9}, e_{4\,5\,10}, e_{7\,8\,10}, e_{7\,8\,11},$$
$$e_{0\,1\,11} - e_{0\,2\,11},\ e_{0\,5\,10} - e_{0\,6\,10},\ e_{2\,6\,10} - e_{2\,7\,10},\ e_{3\,4\,10} - e_{3\,8\,10},\ e_{0\,2\,7} - e_{0\,6\,7}\rangle$$
$$\Omega_3 = \langle e_{0\,1\,2\,11},\ e_{0\,5\,6\,10},\ e_{3\,4\,8\,10},\ e_{0\,2\,6\,7},\ e_{2\,6\,7\,10},\ -e_{0\,6\,7\,10} + e_{0\,2\,7\,10} - e_{0\,2\,6\,10}\rangle$$

$$\Omega_4 = \langle e_{0\,2\,6\,7\,10}\rangle$$



a "snake like" path $e_{i_0 \ldots i_p}$ with $i_k \to i_{k+1}$ and $i_k \to i_{k+2}$ is ∂-invariant

Computation of the curvature:

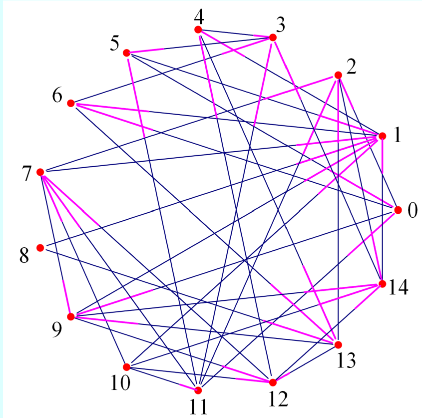
$x=$	0	1	2	3, 11
$[x,\Omega_2]=$	$6+\frac{4}{2}=8$	$5+\frac{1}{2}=\frac{11}{2}$	$5+\frac{4}{2}=7$	$5+\frac{2}{2}=6$
$[x,\Omega_3]=$	$3+\frac{3}{3}=4$	1	$3+\frac{2}{3}=\frac{11}{3}$	1
$[x,\Omega_4]=$	1	0	1	0
$\sum_{p=0}^{4}(-1)^p\frac{[x,\Omega_p]}{p+1}$	$1-\frac{5}{2}+\frac{8}{3}-\frac{4}{4}+\frac{1}{5}$	$1-\frac{5}{2}+\frac{11/2}{3}-\frac{1}{4}$	$1-\frac{5}{2}+\frac{7}{3}-\frac{11/3}{4}+\frac{1}{5}$	$1-\frac{5}{2}+\frac{6}{3}-\frac{1}{4}$
K_x	$=\frac{11}{30}$	$=\frac{1}{12}$	$=\frac{7}{60}$	$=\frac{1}{4}$

4, 5, 8	6	7	9	10
$5+\frac{1}{2}=\frac{11}{2}$	$5+\frac{3}{2}=\frac{13}{2}$	$5+\frac{3}{2}=\frac{13}{2}$	5	$5+\frac{6}{2}=8$
1	$3+\frac{2}{3}=\frac{11}{3}$	$2+\frac{2}{3}=\frac{8}{3}$	0	$3+\frac{3}{3}=4$
0	1	1	0	1
$1-\frac{5}{2}+\frac{11/2}{3}-\frac{1}{4}$	$1-\frac{5}{2}+\frac{13/2}{3}-\frac{11/3}{4}+\frac{1}{5}$	$1-\frac{5}{2}+\frac{13/2}{3}-\frac{8/3}{4}+\frac{1}{5}$	$1-\frac{5}{2}+\frac{5}{3}$	$1-\frac{5}{2}+\frac{8}{3}-\frac{4}{4}+\frac{1}{5}$
$=\frac{1}{12}$	$=-\frac{1}{20}$	$=\frac{1}{5}$	$=\frac{1}{6}$	$=\frac{11}{30}$

Note that $K_6 = -\frac{1}{20} < 0$.

The total curvature: $K_{total} = \frac{11}{30}\cdot 2 + \frac{1}{12}\cdot 4 + \frac{7}{60} + \frac{1}{4}\cdot 2 - \frac{1}{20} + \frac{1}{5} + \frac{1}{6} = 2.$

Example. Consider a randomly generated digraph:



We have $V = 15, \quad E = 39$
$|H_1| = 2, \quad |H_2| = 1, \quad H_p = \{0\}$ for $p \geq 3$
$|\Omega_2| = 28, \quad |\Omega_3| = 4, \quad \Omega_p = \{0\}$ for $p \geq 4$.

Hence, $\chi = |H_0| - |H_1| + |H_2|$
$= |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 0$

$$\begin{aligned}\Omega_2 \;=\; \langle\, & e_{13\,2\,14} - e_{13\,12\,14},\ e_{13\,2\,14} - e_{13\,9\,14},\ e_{0\,2\,14} - e_{0\,9\,14},\ e_{1\,4\,3} - e_{1\,6\,3}, \\ & e_{1\,4\,13} - e_{1\,6\,13},\ e_{5\,0\,6} - e_{5\,1\,6},\ e_{7\,2\,14} - e_{7\,9\,14},\ e_{9\,1\,4} - e_{9\,12\,4}, \\ & e_{10\,1\,4} - e_{10\,12\,4},\ e_{10\,7\,2} - e_{10\,11\,2},\ e_{10\,11\,3} - e_{10\,14\,3},\ e_{11\,0\,9} - e_{11\,7\,9}, \\ & e_{11\,5\,1} - e_{11\,7\,1},\ e_{12\,4\,3} - e_{12\,14\,3},\ e_{12\,7\,1} - e_{12\,14\,1},\ e_{7\,9\,1},\ e_{9\,12\,14},\ e_{9\,14\,1}, \\ & e_{10\,7\,1},\ e_{10\,11\,7},\ e_{10\,12\,7},\ e_{10\,12\,14},\ e_{10\,14\,1}, e_{11\,0\,2},\ e_{11\,3\,5},\ e_{11\,5\,0},\ e_{11\,7\,2},\ e_{13\,9\,12}\rangle\end{aligned}$$

$$\Omega_3 = \langle e_{10\,11\,7\,2},\ e_{13\,9\,12\,14},\ e_{10\,12\,7\,1} - e_{10\,12\,14\,1},\ e_{11\,0\,2\,14} - e_{11\,0\,9\,14} + e_{11\,7\,9\,14} - e_{11\,7\,2\,14}\rangle$$

$$\{K_x\}_{x=0}^{14} = \left\{-\tfrac{7}{24}, -\tfrac{1}{12}, -\tfrac{23}{72}, -\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, -\tfrac{1}{3}, \tfrac{1}{6}, 0, \tfrac{13}{72}, \tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{18}, -\tfrac{11}{12}, \tfrac{13}{24}\right\}.$$

4.4 Digraphs of constant curvature

Recall that a graph is called regular if $\deg(x)$ is constant. We say that a digraph G is *strongly regular* if the function $x \mapsto [x, \Omega_p]$ is constant for any p (in particular, G is regular because $\deg(x) = [x, \Omega_1]$ is constant). In this case the function $x \mapsto K_x$ is constant and we set

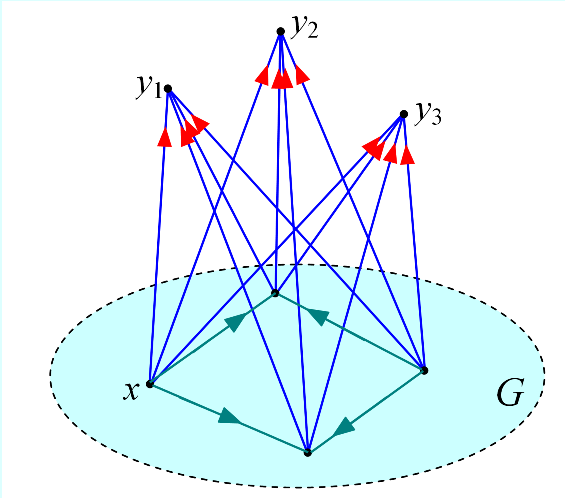
$$K(G) := K_x = \frac{\chi(G)}{|V|}.$$

For any digraph G and any $m \in \mathbb{N}$ let us construct a new digraph by adding to G m new vertices $\{y_1, ..., y_m\}$ and all arrows

$$x \to y_i$$

for all $x \in X$.

This digraph is called *m-suspension* of G and is denoted by $\operatorname{sus}_m G$.



Theorem 4.4 *Let G be a strongly regular digraph, such that for some $k, m \in \mathbb{N}$ and any $p \geq 0$*

$$\dim \Omega_p(G) = \binom{k}{p+1} m^{p+1}. \tag{binom(k,m)}$$

Then $\mathrm{sus}_m G$ *is strongly regular, and for all* $p \geq 0$

$$\dim \Omega_p(\mathrm{sus}_m G) = \binom{k+1}{p+1} m^{p+1}. \tag{binom(k+1,m)}$$

For the digraph G as in Theorem 4.4 we have

$$\chi(G) = \sum_{p\geq 0} (-1)^p \dim \Omega_p = \sum_{p=0}^{k-1} (-1)^p \binom{k}{p+1} m^{p+1} = -\sum_{j=1}^{k} (-1)^j \binom{k}{j} m^j = 1-(1-m)^k.$$

It follows that

$$K(G) = \frac{\chi(G)}{|V|} = \frac{\chi(G)}{\dim \Omega_0} = \frac{1-(1-m)^k}{km}.$$

Of course, the same formula is true for $K(\mathrm{sus}_m G)$ with k replaced by $k+1$:

$$K(\mathrm{sus}_m G) = \frac{1-(1-m)^{k+1}}{(k+1)\, m}$$

Example. We have seen that a triangle (= 2-simplex) is strongly regular and

$$\dim \Omega_0 = 3, \quad \dim \Omega_1 = 3, \quad \dim \Omega_2 = 1, \quad \dim \Omega_p = 0 \ \text{ for } p \geq 3$$

that is, the sequence $\{\dim \Omega_p\}_{p \geq 0}$ is the sequence $\binom{3}{p+1}$ that satisfies (binom$(3,1)$). The 1-suspension of an n-simplex is an $(n+1)$-simplex. Hence, we obtain by induction that the n-simplex is strongly regular and satisfies (binom$(n+1,1)$). In particular,

$$K\,(n\text{-simplex}) = \frac{1}{n+1}.$$

For any $m \in \mathbb{N}$ denote by $D^{(m)}$ a digraph with m vertices and no arrows. Then

$$\dim \Omega_0 \left(D^{(m)}\right) = m = \binom{1}{p+1} m^{p+1} \ \text{ for } p = 0$$

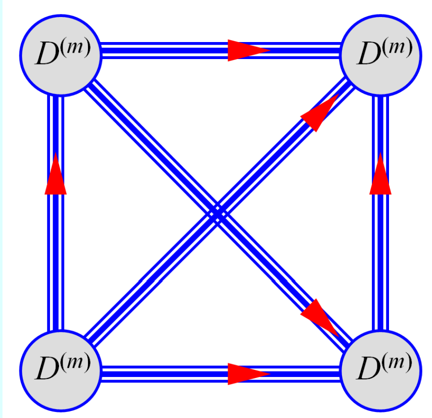
$$\dim \Omega_p \left(D^{(m)}\right) = 0 = \binom{1}{p+1} m^{p+1} \ \text{ for } p \geq 1$$

so that (binom$(1,m)$) is satisfied. Clearly, $D^{(m)}$ is strongly regular.

Define inductively a sequence of digraphs $\left\{D_k^{(m)}\right\}_{k=1}^{\infty}$ by

$D_1^{(m)} = D^{(m)}$,
$D_{k+1}^{(m)} = \mathrm{sus}_m\, D_k^{(m)}$

In fact, $D_k^{(m)}$ is a digraph version of a complete k-partite graph $K_{\underbrace{m,\,m,\,...,\,m}_{k}}$



$D_4^{(m)}$

By induction we obtain that $D_k^{(m)}$ is strongly regular and satisfies (binom(k,m)).

Hence, $D_k^{(m)}$ has a constant curvature

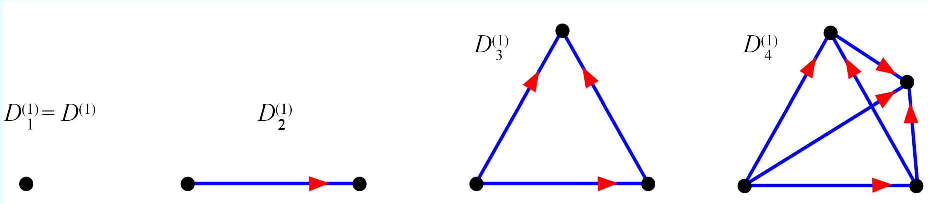
$$K(D_k^{(m)}) = \frac{1-(1-m)^k}{km}. \tag{4.8}$$

One can show that the only non-trivial Betti number of $D_k^{(m)}$ is

$$\beta_{k-1} = (m-1)^k .$$

Example. For $m = 1$ we have by (4.8) $K(D_k^{(1)}) = \frac{1}{k}$.

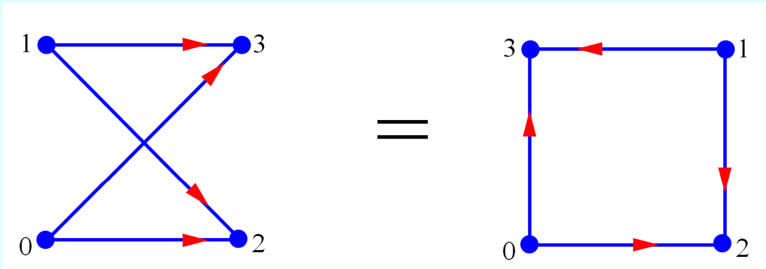
$D_k^{(1)}$ is a $(k-1)$-simplex:



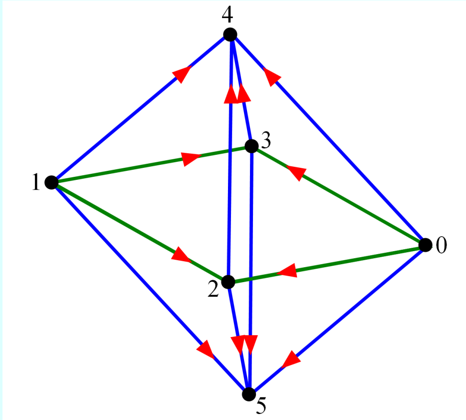
Example. For $m = 2$ we have by (4.8)

$$K(D_k^{(2)}) = \begin{cases} 0, & k \text{ even}, \\ \frac{1}{k}, & k \text{ odd}. \end{cases}$$

For example, $D_2^{(2)}$ is a 4-cycle:
It is an analogue of 1-sphere.
It has constant curvature 0.

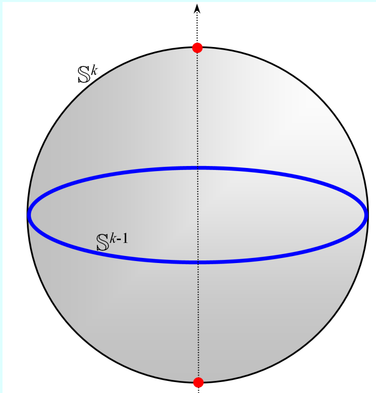


$D_3^{(2)}$ is the octahedron:
It is an analogue of 2-sphere.
It has constant curvature $\frac{1}{3}$.



$D_4^{(2)}$ is an analogue of 3-sphere.
It has constant curvature 0.

$D_{k+1}^{(2)}$ is a digraph analogue of a k-sphere $\mathbb{S}^k$ because $D_{k+1}^{(2)}$ is obtained from $D_k^{(2)}$ by 2-suspension.

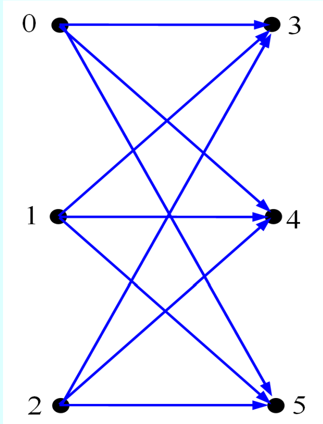


Besides, the only non-trivial Betti number of $D_{k+1}^{(2)}$ is $\beta_k = 1$ like Betti numbers for $\mathbb{S}^k$.

Example. For $m = 3$ we have by (4.8)

$$K(D_k^{(3)}) = \frac{1-(-2)^k}{3k} = \frac{1}{3k}\begin{cases} 1-2^k, & k \text{ even}, \\ 1+2^k, & k \text{ odd}. \end{cases}$$

For example, $D_2^{(3)}$ is a directed version of $K_{3,3}$:



$D_2^{(3)}$

We have

$$K(D_2^{(3)}) = -\frac{1}{2}$$

and

$$K(D_3^{(3)}) = 1.$$

4.5 Some problems

Problem 4.5 *Compare this notion of curvature with other definitions of curvature of graphs.*

Problem 4.6 *Is it true that for icosahedron (see p. 76)* $|\Omega_2| = 25$ *for any numbering of the vertices?*

Problem 4.7 *Devise an efficient algorithm/software for computation of the spaces* Ω_p *for arbitrary digraphs, possibly avoiding null-spaces of large matrices. Such algorithms exist for* Ω_2 *and* Ω_3*.*

Problem 4.8 *Let a digraph* G *be determined by a triangulation of* $\mathbb{S}^2$ *(see Section 2.3). Assume that* $\deg(x) \leq 4$ *for all* $x \in G$*. Is it true that* $K_x \geq 0$ *for all* $x \in G$*?*

For triangulations of $\mathbb{S}^1$ we have always $K_x \geq 0$: these are triangles and squares with $K_x > 0$ and other polygons with $K_x \equiv 0$.

For triangulations of $\mathbb{S}^2$ we have verified above that $K_x \geq 0$ for simplex, bipyramid, octahedron, but with specific orientations of edges (the question remains open when the numbering of vertices is arbitrary). All these digraphs have $\deg(x) \leq 4$. We have seen that $K_x < 0$ can occur for icosahedron with $\deg(x) = 5$ and for a pyramid with $\deg(x) = 7$.

Problem 4.9 *Denote $D = \max_{x \in G} \deg(x)$. Is it true that $|K_x| \leq C_D$ for some constant C_D depending only on D? The same question about $K_x^{(2)}$ and $K_x^{(3)}$.*

Note that K_x can be arbitrarily large, for example, for a strongly regular digraph satisfying (binom(k, m)), we have

$$K_x = \frac{1 - (1 - m)^k}{km}$$

while $\deg(x) = (k - 1)\, m$.

Problem 4.10 *What can be said about the curvature of random digraphs?*

Problem 4.11 *Let $\mathcal{S}$ be a simplicial complex and $G_{\mathcal{S}}$ be its Hasse diagram (see Section 2.2). Is there any relation of $K_x(G_{\mathcal{S}})$ to properties of $\mathcal{S}$? For example, we have*

$$K_{total}(G_S) = \chi(G_{\mathcal{S}}) = \chi_{simp}(\mathcal{S}).$$

Can one give an explicit formula for computing $K_\sigma(G_{\mathcal{S}})$ for any simplex $\sigma \in \mathcal{S}$?

5 Homology and Cartesian product of digraphs

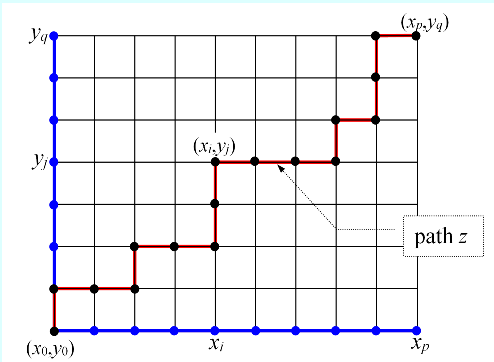
5.1 Cross product of paths

Given two finite sets X, Y, consider their product

$$Z = X \times Y = \{(a, b) : a \in X \text{ and } b \in Y\}.$$

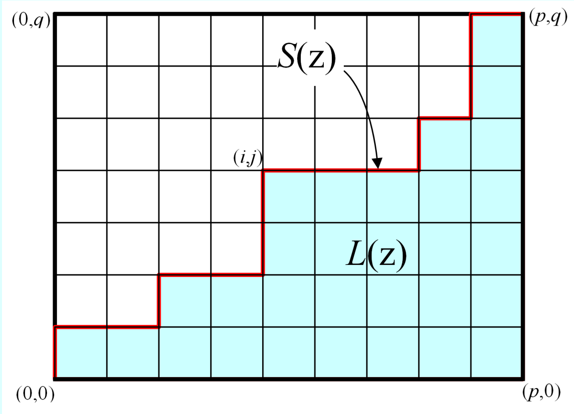
Let $z = z_0 z_1 ... z_r$ be a regular elementary r-path on Z, where $z_k = (a_k, b_k)$ with $a_k \in X$ and $b_k \in Y$. We say that z is *stair-like* if, for any $k = 1, ..., r$, either $a_{k-1} = a_k$ or $b_{k-1} = b_k$ is satisfied. That is, any couple $z_{k-1} z_k$ of consecutive vertices is either vertical (when $a_{k-1} = a_k$) or horizontal (when $b_{k-1} = b_k$).

Given a stair-like path z on Z, define its projection onto X as an elementary path x on X obtained from z by removing Y-components in all the vertices of z and then by collapsing in the resulting sequence any subsequence of repeated vertices to one vertex.
In the same way define projection of z onto Y and denote it by y.



Projections $x = x_0 ... x_p$ and $y = y_0 ... y_q$ are regular elementary paths, and $p + q = r$.

Every vertex (x_i, y_j) of path z can be represented as a point (i, j) of $\mathbb{Z}^2$ so that path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting points $(0, 0)$ and (p, q).



Define the *elevation* $L(z)$ of z as the number of cells in $\mathbb{Z}^2_+$ below the staircase $S(z)$.

For given elementary regular paths x on X and y on Y, denote by $\Sigma_{x,y}$ the set of all stair-like paths z on Z whose projections on X and Y are respectively x and y.

Definition. Define the *cross product* of the paths e_x and e_y as a path $e_x \times e_y$ on Z as follows:

$$e_x \times e_y = \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)} e_z \tag{5.1}$$

and it extend by linearity to all $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ so that $u \times v \in \mathcal{R}_{p+q}(Z)$.

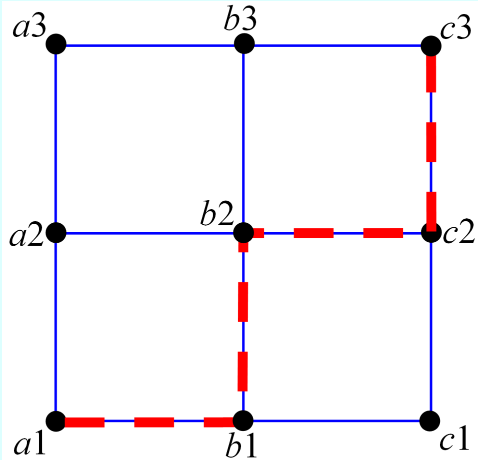
Example. Let us denote the vertices on X by letters a, b, c etc and the vertices on Y by integers $1, 2, 3,$ etc so that the vertices on Z can be denoted as $a1, b2$ etc as the fields on the chessboard. Then we have

$$e_a \times e_{12} = e_{a1\,a2}, \quad e_{ab} \times e_1 = e_{a1\,b1}$$

$$e_{ab} \times e_{12} = e_{a1\,b1\,b2} - e_{a1\,a2\,b2}$$

$$e_{ab} \times e_{123} = e_{a1\,b1\,b2\,b3} - e_{a1\,a2\,b2\,b3} + e_{a1\,a2\,a3\,b3}$$

$$\begin{aligned} e_{abc} \times e_{123} = {} & e_{a1\,b1\,c1\,c2\,c3} - e_{a1\,b1\,b2\,c2\,c3} + e_{a1\,b1\,b2\,b3\,c3} \\ & + e_{a1\,a2\,b2\,c2\,c3} - e_{a1\,a2\,b2\,b3\,c3} + e_{a1\,a2\,a3\,b3\,c3} \end{aligned}$$



Lemma 5.1 *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial(u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v). \tag{5.2}$$

5.2 Cartesian product of digraphs

Denote a digraph and its set of vertices by the same letters to simplify notation. Given two digraphs X and Y, define there Cartesian product as a digraph $Z = X \square Y$ as follows:

- the set of vertices of Z is $X \times Y$, that is, the vertices of Z are the couples (a, b) where $a \in X$ and $b \in Y$;

- the edges in Z are of two types: $(a, b) \rightarrow (a', b)$ where $a \rightarrow a'$ (a *horizontal* edge) and $(a, b) \rightarrow (a, b')$ where $b \rightarrow b'$ (a *vertical* edge):

$$
\begin{array}{cccccc}
b'\bullet & \dots & \overset{(a,b')}{\bullet} & \longrightarrow & \overset{(a',b')}{\bullet} & \dots \\
\uparrow & & \uparrow & & \uparrow & \\
b\bullet & \dots & \overset{(a,b)}{\bullet} & \longrightarrow & \overset{(a',b)}{\bullet} & \dots \\
& & & & & \\
{}^{Y}\diagup_{X} & \dots & \underset{a}{\bullet} & \longrightarrow & \underset{a'}{\bullet} & \dots
\end{array}
$$

It follows that any allowed elementary path in Z is stair-like.

Moreover, any regular elementary path on Z is allowed if and only if it is stair-like and its projections onto X and Y are allowed.

It follows from definition (5.1) of the cross product that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \quad \Rightarrow \quad u \times v \in \mathcal{A}_{p+q}(Z). \tag{5.3}$$

Furthermore, the following is true.

Lemma 5.2 *If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_{p+q}(Z)$.*

Proof. $u \times v$ is allowed by (5.3). Since ∂u and ∂v are allowed, by (5.3) also $\partial u \times v$ and $u \times \partial v$ are allowed. By (5.2), $\partial(u \times v)$ is also allowed. Hence, $u \times v \in \Omega_{p+q}(Z)$. ∎

Theorem 5.3 *Any ∂-invariant path w on $Z = X \Box Y$ admits a representation in the form*

$$w = \sum_{i=1}^{m} u_i \times v_i$$

for some finite m, where u_i and v_i are ∂-invariant paths on X and Y, respectively.

5.3 Künneth formula

Here is the main result of this chapter.

Theorem 5.4 *Let X, Y be two finite digraphs. Then, for any $r \geq 0$,*

$$\Omega_r\left(X \square Y\right) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} \Omega_p\left(X\right) \otimes \Omega_q\left(Y\right), \tag{5.4}$$

where the isomorphism is given by

$$u \otimes v \mapsto u \times v$$

for $u \in \Omega_p\left(X\right)$ and $v \in \Omega_q\left(Y\right)$. Consequently, we have

$$H_r\left(X \square Y\right) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} H_p\left(X\right) \otimes H_q\left(Y\right) \tag{5.5}$$

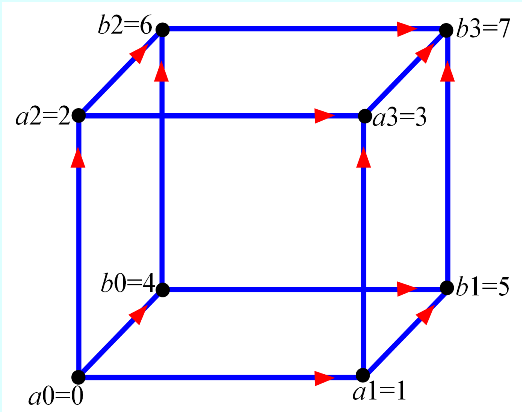
and

$$\beta_r\left(X \square Y\right) = \sum_{\{p,q \geq 0 : p+q=r\}} \beta_p\left(X\right) \beta_q\left(Y\right).$$

Example. Let X be an interval and Y be a square:

$$X = {}^{a}\bullet \longrightarrow \bullet^{b} \quad \text{and } Y = \begin{array}{ccc} {}_2\bullet & \longrightarrow & \bullet_3 \\ \uparrow & & \uparrow \\ {}_0\bullet & \longrightarrow & \bullet_1 \end{array}$$

Then $Z = X \Box Y$ is a cube:



$Z = X \Box Y$

We have:

$\Omega_1(X) = \langle e_{ab} \rangle$
$\Omega_p(X) = 0$ for $p \geq 2$

$\Omega_1(Y) = \langle e_{01}, e_{13}, e_{23}, e_{02} \rangle$
$\Omega_2(Y) = \langle e_{013} - e_{023} \rangle$
$\Omega_q(Y) = 0$ for $q \geq 3$.

By (5.4) we obtain

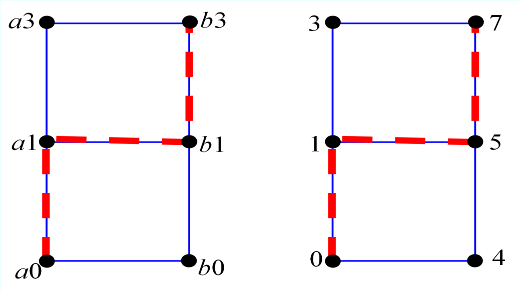
$$\Omega_3(Z) \cong \Omega_1(X) \otimes \Omega_2(Y) = \langle e_{ab} \times (e_{013} - e_{023}) \rangle.$$

Let us compute the cross-products:

$$
\begin{aligned}
e_{ab} \times e_{013} &= e_{a0\,b0\,b1\,b3} - e_{a0\,a1\,b1\,b3} + e_{a0\,a1\,a3\,b3} \\
&= e_{0457} - e_{0157} + e_{0137}
\end{aligned}
$$

and

$$
e_{ab} \times e_{023} = e_{0467} - e_{0267} + e_{0237}
$$



Hence, we obtain

$$
\Omega_3\left(Z\right) = \langle e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}\rangle
$$

that is the ∂-invariant 3-path associated with 3-cube.

Define n-cube as follows:

$$
n\text{- cube} = \underbrace{I\Box I\Box ...\Box I}_{n},
$$

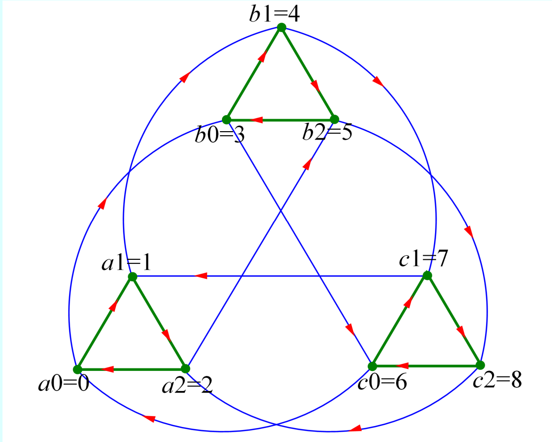
where $I = \{\bullet \to \bullet\}$. Similarly one shows that $\Omega_n\left(n\text{- cube}\right)$ is spanned by a single n-path that is an alternating sum of $n!$ elementary n-paths connecting the vertices 0 and $2^n - 1$. This corresponds to partitioning of a solid n-dim cube into $n!$ simplexes.

By the Künneth formula, $H_p\left(n\text{-cube}\right) = \{0\}$ for all $p \geq 1$.

5.4 An example: 2-torus

Example. Denote by T the following 3-cycle (=1-torus):

$$T = \begin{tikzcd} & b \arrow[dr] & \\ a \arrow[ur] & & c \arrow[ll] \end{tikzcd} = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd}$$



Consider a 2-torus $G = T \square T$ shown here:

Let us compute $\Omega_r(G)$, $H_r(G)$, $K_x(G)$.

We know that

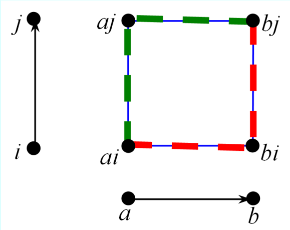
$$\Omega_0(T) = \langle e_0, e_1, e_2 \rangle, \quad \Omega_1(T) = \langle e_{01}, e_{12}, e_{20} \rangle, \quad \Omega_p(T) = \{0\} \text{ for } p \geq 2$$

By (5.4) we obtain $\Omega_r = \{0\}$ for $r \geq 3$ and

$$\Omega_2(G) = \Omega_1(T) \otimes \Omega_1(T)$$
$$= \langle e_{ab} \times e_{01}, e_{ab} \times e_{12}, e_{ab} \times e_{20}, e_{bc} \times e_{01}, e_{bc} \times e_{12}, e_{bc} \times e_{20}, e_{ca} \times e_{01}, e_{ca} \times e_{12}, e_{ca} \times e_{20} \rangle$$

Using

$$e_{ab} \times e_{ij} = e_{ai\,bi\,bj} - e_{ai\,aj\,bj}$$



we obtain that

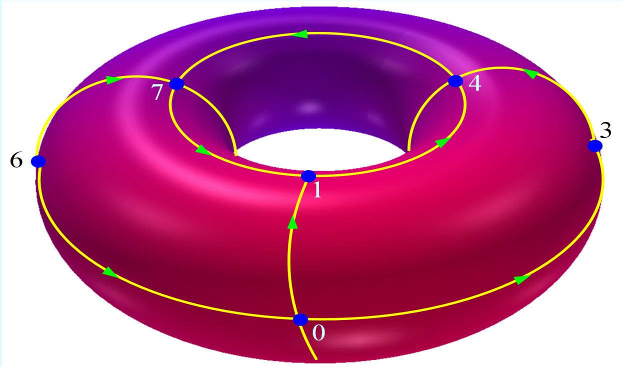
$$\begin{aligned}\Omega_2(G) = \langle & e_{a0\,b0\,b1} - e_{a0\,a1\,b1},\; e_{a1\,b1\,b2} - e_{a1\,a2\,b2},\; e_{a2\,b2\,b0} - e_{a2\,a0\,b0},\\ & e_{b0\,c0\,c1} - e_{b0\,b1\,c1},\; e_{b1\,c1\,c2} - e_{b1\,b2\,c2},\; e_{b2\,c2\,c0} - e_{b2\,b0\,c0},\\ & e_{c0\,a0\,a1} - e_{c0\,c1\,a1},\; e_{c1\,a1\,a2} - e_{c1\,c2\,a2},\; e_{c2\,a2\,a0} - e_{c2\,c0\,a0}\rangle\end{aligned}$$

that is,

$$\begin{aligned}\Omega_2(G) = \langle & e_{034} - e_{014},\; e_{145} - e_{125},\; e_{253} - e_{203},\\ & e_{367} - e_{347},\; e_{478} - e_{458},\; e_{586} - e_{536}\\ & e_{601} - e_{671},\; e_{712} - e_{782},\; e_{820} - e_{860}\rangle.\end{aligned}$$

We see that $\Omega_2(G)$ is generated by 9 squares.

This can be visualized using the following embedding of $G = T \square T$ on a topological torus:



Using $\Omega_2(G)$, let us compute the curvature K_x on G.
The above basis in $\Omega_2(G)$ is orthogonal and $\|\omega\|^2 = 2$ for any element ω of the basis.

Besides, for any vertex x, we have $[x, \omega] = 2$ for two of ω, $[x, \omega] = 1$ for two of ω, and $[x, \omega] = 0$ for the rest of ω. Hence,

$$[x, \Omega_2] = \sum_{\omega} \frac{[x, \omega]}{\|\omega\|^2} = \frac{2 \cdot 2 + 2 \cdot 1}{2} = 3$$

and

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{[x, \Omega_2]}{3} = 1 - \frac{4}{2} + \frac{3}{3} = 0.$$

Let us compute the homology groups of G. We know that

$$H_0(T) = \langle e_0 \rangle, \quad H_1(T) = \langle e_{01} + e_{12} + e_{20} \rangle, \quad H_p(T) = \{0\} \text{ for } p \geq 2.$$

By (5.5) we have

$$H_1(G) = H_0(T) \otimes H_1(T) + H_1(T) \otimes H_0(T) = \langle v_1, v_2 \rangle$$

where $v_1 = e_a \times (e_{01} + e_{12} + e_{20}) = e_{a0\,a1} + e_{a1\,a2} + e_{a2\,a0} = e_{01} + e_{12} + e_{20}$

$$v_2 = (e_{ab} + e_{bc} + e_{ca}) \times e_0 = e_{a0\,b0} + e_{b0\,c0} + e_{c0\,a0} = e_{03} + e_{36} + e_{60}.$$

Again by (5.5)

$$H_2(G) = H_1(T) \otimes H_1(T) = \langle u \rangle,$$

where $u = (e_{ab} + e_{bc} + e_{ca}) \times (e_{01} + e_{12} + e_{20})$, and $H_r(Z) = 0$ for all $r \geq 2$. Hence,

$$\begin{aligned} u &= e_{a0\,b0\,b1} - e_{a0\,a1\,b1} + e_{a1\,b1\,b2} - e_{a1\,a2\,b2} + e_{a2\,b2\,b0} - e_{a2\,a0\,b0} \\ &\quad + e_{b0\,c0\,c1} - e_{b0\,b1\,c1} + e_{b1\,c1\,c2} - e_{b1\,b2\,c2} + e_{b2\,c2\,c0} - e_{b2\,b0\,c0} \\ &\quad + e_{c0\,a0\,a1} - e_{c0\,c1\,a1} + e_{c1\,a1\,a2} - e_{c1\,c2\,a2} + e_{c2\,a2\,a0} - e_{c2\,c0\,a0} \end{aligned}$$

that is $u = (e_{034} - e_{014}) + (e_{145} - e_{125}) + (e_{253} - e_{203}) + (e_{367} - e_{347}) + (e_{478} - e_{458})$

$$+ (e_{586} - e_{536}) + (e_{601} - e_{671}) + (e_{712} - e_{782}) + (e_{820} - e_{860}).$$

5.5 Cartesian product and curvature

Proposition 5.5 *Let X be any digraph with a finite chain sequence $\{\Omega_p\}$ and Y be a cyclic digraph*

$$Y = \{0 \to 1 \to 2 \to ... \to m \to 0\}$$

with $m \geq 2$. Then, with respect to the natural inner product,

$$K_z\left(X \square Y\right) = 0$$

for any $z \in X \square Y$. In particular, $K(T^{\square n}) = 0$ where T is an 1-torus.

Consider an n-cube$= I^{\square n}$ where $I = \{0 \to 1\}$. Then any vertex x of the n-cube is represented by a binary sequence $(x_1, ..., x_n)$. Set $|x| = x_1 + ... + x_n$.

Proposition 5.6 *For any vertex x of the n-cube we have*

$$K_x\left(n\text{- cube}\right) = \frac{1}{(n+1)\binom{n}{|x|}}.$$

Problem 5.7 *How to compute $K\left(X \square Y\right)$ in general?*

5.6 Strong product

Define a *strong product* $X\boxtimes Y$ of digraphs as follows: the set of vertices of $X\boxtimes Y$ is $X\times Y$, while the arrows are defined as follows: $(a,b)\to(a',b)$ where $a\to a'$ (a horizontal edge), $(a,b)\to(a,b')$ where $b\to b'$ (a vertical edge), and $(a,b)\to(a',b')$ where $a\to a'$ and $b\to b'$ (a diagonal edge):

$$\begin{array}{ccc} \overset{(a,b')}{\bullet} & \longrightarrow & \overset{(a',b')}{\bullet} \\ \uparrow & \nearrow & \uparrow \\ \underset{}{\overset{(a,b)}{\bullet}} & \longrightarrow & \overset{(a',b)}{\bullet} \end{array}$$

Conjecture 5.8 *The Künneth formula holds for the strong product:*

$$H_r\left(X\boxtimes Y\right)\cong\bigoplus_{\{p,q\geq 0:p+q=r\}}\left(H_p\left(X\right)\otimes H_q\left(Y\right)\right),$$

where the isomorphism is given by $u\otimes v\mapsto u\times v$.

It suffices to prove an analogue of the theorem of Eilenberg-Zilber: there are chain maps

$$F:\Omega_*\left(X\boxtimes Y\right)\to\Omega_*\left(X\right)\otimes\Omega_*\left(Y\right)$$

and

$$G : \Omega_* (X) \otimes \Omega_* (Y) \to \Omega_* (X \boxslash Y)$$

such that $FG = \mathrm{id}$ and GF is *chain-homotopic* to id.

In fact, one can define G by $G(u \otimes v) = u \times v$, while the main difficulty is in construction of F. In the setting of Theorem 5.4, one uses Theorem 5.3 to show that G is bijective so that one can take $F = G^{-1}$.

6 Path cohomology

As before, V is a finite set and $\mathbb{K}$ is a field. Recall that Λ_p is a $\mathbb{K}$-linear space spanned by all elementary p-paths $e_{i_0 \ldots i_p}$.

6.1 p-forms and exterior derivative

Definition. For any $p \geq 0$ define a *p-form* on V as any linear functional $\omega : \Lambda_p \to \mathbb{K}$. The linear space of all p-forms is denoted by Λ^p. That is, Λ^p is the dual space of Λ_p.

If $\omega \in \Lambda^p$ and $v \in \Lambda_p$ then write $(\omega, v) \equiv \omega(v)$. For any elementary p-path $e_{i_0 \ldots i_p}$ there is a dual *elementary p-form* $e^{i_0 \ldots i_p}$ such that

$$\left(e^{i_0 \ldots i_p}, e_{j_0 \ldots j_p}\right) = \delta^{i_0 \ldots i_p}_{j_0 \ldots j_p}.$$

Any p-form $\omega \in \Lambda^p$ can be represented as a linear combination of elementary p-forms

$$\omega = \sum_{i_0, \ldots, i_p \in V} \omega_{i_0 \ldots i_p} e^{i_0 \ldots i_p},$$

where $\omega_{i_0 \ldots i_p} = \left(\omega, e_{i_0 \ldots i_p}\right) \in \mathbb{K}$.

For any p-path

$$v = \sum_{i_0,\ldots,i_p \in V} v^{i_0\ldots i_p} e_{i_0\ldots i_p} \in \Lambda_p,$$

we have then

$$(\omega, v) = \sum_{i_0,\ldots,i_p \in V} \omega_{i_0\ldots i_p} v^{i_0\ldots i_p}.$$

Definition. For any $p \geq 1$, define *the exterior derivative* $d : \Lambda^{p-1} \to \Lambda^p$ by

$$\boxed{(d\omega)_{i_0\ldots i_p} = \sum_{q=0}^{p} (-1)^q \, \omega_{i_0\ldots\widehat{i_q}\ldots i_p}} \quad \text{for any } \omega \in \Lambda^{p-1}. \tag{6.1}$$

Recall for comparison that

$$\partial e_{i_0\ldots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0\ldots\widehat{i_q}\ldots i_p}. \tag{6.2}$$

For example, for 0-form $\omega = \sum \omega_i e^i$ we have

$$(d\omega)_{ij} = \omega_j - \omega_i,$$

for a 1-form $\omega = \sum \omega_{ij} e^{ij}$ we have

$$(d\omega)_{ijk} = \omega_{jk} - \omega_{ik} + \omega_{ij}.$$

It follows from (6.1) that

$$\boxed{de^{i_0 \ldots i_p} = \sum_{k \in V} \sum_{q=0}^{p+1} (-1)^q \, e^{i_0 \ldots i_{q-1} k i_q \ldots i_p}}. \tag{6.3}$$

For example,

$$de^i = \sum_{k \in V} \left(e^{ki} - e^{ik} \right) \quad \text{and} \quad de^{ij} = \sum_{k \in V} \left(e^{kij} - e^{ikj} + e^{ijk} \right).$$

Proposition 6.1 *(Stokes's theorem) Let $p \geq 1$. For any p-path u and any $(p-1)$-form ω, the following identity holds*

$$(d\omega, u) = (\omega, \partial u).$$

Hence, the operators $d : \Lambda^{p-1} \to \Lambda^p$ and $\partial : \Lambda_p \to \Lambda_{p-1}$ are dual, and $d^2 = 0$.

Proof. It suffices to prove this identity for $u = e_{i_0 \ldots i_p}$. Using (6.1) and (6.2), we obtain

$$(d\omega, u) = (d\omega)_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q \, \omega_{i_0 \ldots \widehat{i_q} \ldots i_p}$$

and

$$(\omega, \partial u) = \left(\omega, \sum_{q=0}^{p} (-1)^q \, e_{i_0 \ldots \widehat{i_q} \ldots i_p}\right) = \sum_{q=0}^{p} (-1)^q \, \omega_{i_0 \ldots \widehat{i_q} \ldots i_p},$$

whence the required identity follows. ∎

Consider the following regular subspace of Λ^p:

$$\mathcal{R}^p = \langle e^{i_0 \ldots i_p} : \ i_0 \ldots i_p \text{ is regular} \rangle.$$

Then the spaces $\mathcal{R}^p$ and $\mathcal{R}_p$ are dual with the same pairing $(\cdot, \cdot)$.

Lemma 6.2 *If $\omega \in \mathcal{R}^p$ then $d\omega \in \mathcal{R}^{p+1}$. Moreover, the operator $d : \mathcal{R}^p \to \mathcal{R}^{p+1}$ and the regular boundary operator $\partial : \mathcal{R}^{p+1} \to \mathcal{R}^p$ are dual.*

Proof. It suffices to prove this for an elementary regular p-form $\omega = e^{i_0 \ldots i_p}$. By (6.3) we have

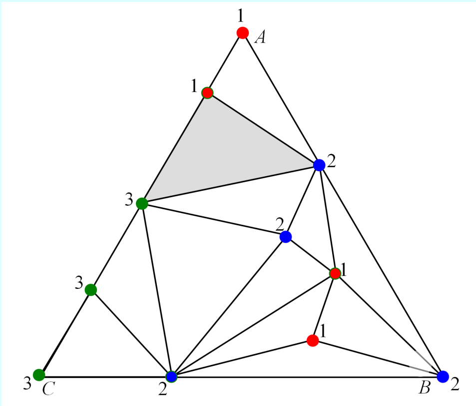
$$d\omega = de^{i_0 \ldots i_p} = \sum_{k \in V} \sum_{q=0}^{p+1} (-1)^q \, e^{i_0 \ldots i_{q-1} k i_q \ldots i_p}.$$

A $(p+1)$-path $e^{i_0 \ldots i_{q-1} k i_q \ldots i_p}$ can be non-regular only if $k = i_q$ or $k = i_{q-1}$. For example, let $k = i_q$. The above sum contains also the term $(-1)^{q+1} \, e^{i_0 \ldots i_{q-1} \underline{i_q k} \ldots i_p}$ that cancels out with $(-1)^q \, e^{i_0 \ldots i_{q-1} \underline{k i_q} \ldots i_p}$ so that $d\omega$ is a sum of regular terms. ∎

6.2 Example: Sperner's lemma

Consider a triangle ABC on the plane $\mathbb{R}^2$ and its triangulation T. The set of vertices of T is colored with three colors $1, 2, 3$ in such a way that the following conditions are satisfied:

- the vertices A, B, C are colored with $1, 2, 3$ respectively;
- each vertex on any edge of ABC is colored with one of the two colors of the endpoints of the edge.



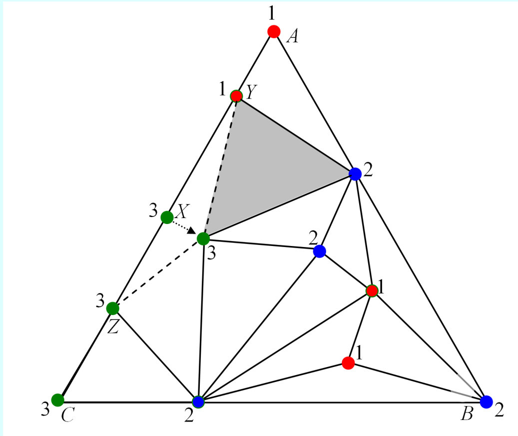
Sperner's coloring

A classical lemma of Sperner says the following: under the above hypotheses, there exists in T a 3-*color triangle*, that is, a triangle, whose vertices are colored with the three different colors. Moreover, the number of 3-color triangles is odd.

We give here a proof using Stokes's formula of Proposition 6.1.

Step 1. Let us modify the triangulation T so that there are no vertices on the edges AB, AC, BC except for A, B, C.

Indeed, if X is a vertex on AB then we move X a bit inside the triangle ABC.

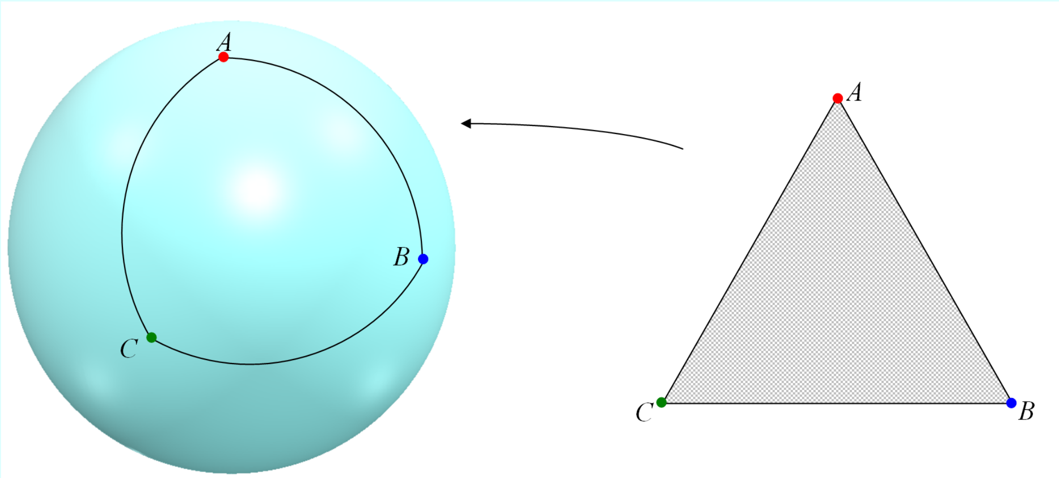


This gives rise to a new triangle that is formed by X and its former neighbors, say Y and Z, on the edge AB (while keeping all old triangles).

However, since all X, Y, Z are colored with two colors, no 3-color triangle emerges after that move.

Repeating this procedure, we remove all the vertices from the interior of edges of ABC.

Step 2. We map the triangle ABC and the triangulation T onto the sphere $\mathbb{S}^2$ and add to the set T the triangle ABC itself from the other side of the sphere.

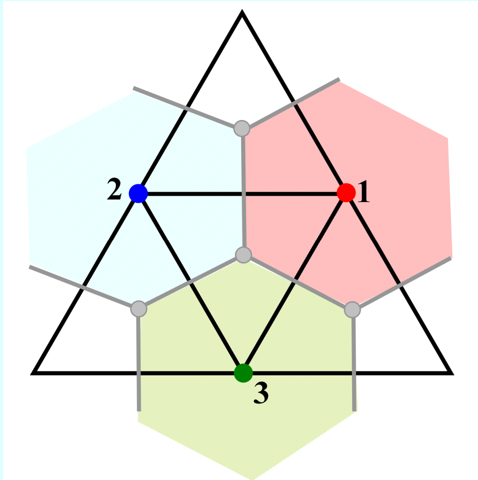


Then we obtain a triangulation of $\mathbb{S}^2$; denote it again by T. Now we need to prove that the number of 3-color triangles in T is *even* (because the newly added triangle ABC is 3-color). *From now on we do not need any restriction on coloring of the vertices of T – it can be arbitrary.*

Step 3. Let us regard T as a graph on $\mathbb{S}^2$ and construct a dual graph G.

Chose at each face (triangle) of T a point and regard these points as the vertices of the dual graph G.

The vertices in G are connected by an edge if the corresponding triangles in T have a common edge.

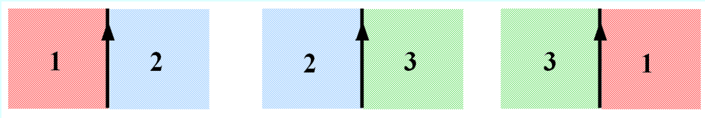


The graphs T (black) and G (grey)

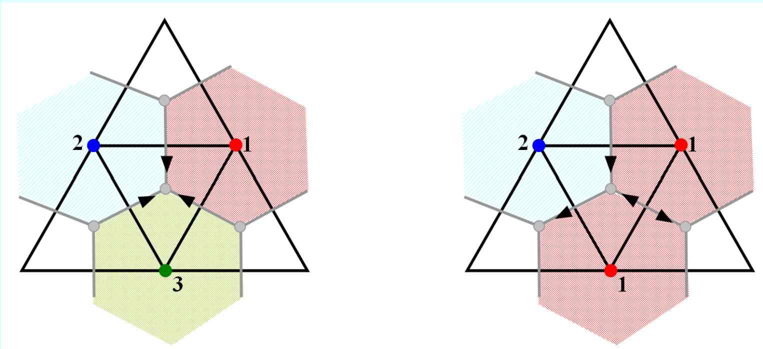
Then the faces of G are in one-to-one correspondence to the vertices of T, and we color each face of G in the same color as the corresponding vertex of T.

Hence, we obtain a planar graph G on $\mathbb{S}^2$ such that each vertex of G has degree 3 and each face is colored with one of the colors $1, 2, 3$. We need to prove that the number of 3-color vertices of G (that is, the vertices, whose adjacent faces have all three colors) is *even*.

Step 4. Let us make G into a *digraph* as follows. Choose the orientation of any edge ξ of G according to the color of the faces from the both sides of ξ as follows:



If the colors are the same from the both sides then ξ becomes a double arrow $\rightleftarrows$. Examples of such orientations are shown here:



Step 5. Consider an 1-path on the digraph G:

$$v=\sum_{i\rightarrow j}e_{ij}.$$

We have for any vertex $a\in V$ of G

$$\begin{aligned}(\partial v)^a &= (\partial v,e^a)=\sum_{i\rightarrow j}(\partial e_{ij},e^a)=\sum_{i\rightarrow j}(e_j-e_i,e^a)=\sum_{i\rightarrow a}1-\sum_{a\rightarrow j}1\\ &= \#\{\text{incoming arrows at } a\}-\#\{\text{outcoming arrows at } a\}.\end{aligned}$$

If a is 3-color, then either all three arrows at a are incoming or all are outcoming so that $(\partial v)^a=+3$ or -3, respectively. If a is not 3-color then $(\partial v)^a=0$.
Denoting by n_1 the total number of 3-color vertices with all incoming arrows and by n_2 the total number of 3-color vertices with outcoming arrows, we obtain that

$$\sum_{a\in V}(\partial v)^a=3\left(n_1-n_2\right).$$

On the other hand, we have by Proposition 6.1

$$\sum_{a\in V}(\partial v)^a=(\partial v,\sum_{a\in V}e^a)=(\partial v,1)=(v,d1)=0.$$

Hence, we conclude that $n_1=n_2$. Consequently, the total number of 3-color vertices is equal to $2n_1$, that is, even, which was to be proved. ■

6.3 d-invariant forms

Let $G = (V, E)$ be a digraph. For any $p \geq 0$, consider the following subspaces of $\mathcal{R}^p$:

$$\mathcal{A}^p = \langle e^{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is allowed} \rangle$$

$$\mathcal{N}^p = \langle e^{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is non-allowed but regular} \rangle$$

so that

$$\mathcal{R}^p = \mathcal{A}^p \oplus \mathcal{N}^p. \tag{6.4}$$

Set

$$\boxed{J^p = \mathcal{N}^p + d\mathcal{N}^{p-1}} \subset \mathcal{R}^p$$

(where $\mathcal{N}^{-1} = \{0\}$) and

$$\boxed{\Omega^p = \mathcal{R}^p \,/\, J^p}.$$

Definition. The elements of Ω^p are called *d-invariant* p-forms.

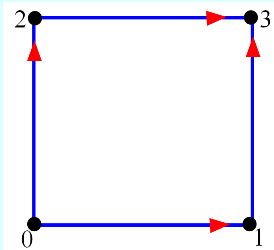
For $\varphi, \psi \in \mathcal{R}^p$ we write $\varphi \simeq \psi$ if $\varphi = \psi \mod J^p$, that is, if φ and ψ represent the same element of Ω^p. In other words, the symbol $\simeq$ means equality in Ω^p.

Using (6.4) it is easy to see that

$$\Omega^p = \mathcal{A}^p \,/ (J^p \cap \mathcal{A}^p) \, ,$$

that is, $\boxed{d\text{-invariant } p\text{-forms are allowed } p\text{-forms considered mod } J^p}$. Since all allowed elementary p-forms $e^{i_0 \dots i_p}$ constitute a basis in $\mathcal{A}^p$, choosing from the sequence $\{e^{i_0 \dots i_p} \bmod J^p\}$ a maximal linearly independent subsequence, we obtain a basis in Ω^p. Note also that $J^0 = \{0\}$ and $J^1 \cap \mathcal{A}^1 = \{0\}$ so that $\Omega^0 = \mathcal{A}^0$ and $\Omega^1 = \mathcal{A}^1$.

Example. Let G be a square. We have $e^{03} \in \mathcal{N}^1$ and



$$de^{03} = \sum_k e^{k03} - \sum_k e^{0k3} + \sum_k e^{03k} = -e^{013} - e^{023} + \varphi,$$

where $\varphi \in \mathcal{N}^2$. It follows that

$$e^{013} + e^{023} = \varphi - de^{03} \in \mathcal{N}^2 + d\mathcal{N}^1 = J^2.$$

Hence, $e^{013} \simeq -e^{023}$ that is, e^{013} and $-e^{023}$ represent the same element of Ω^2.

Lemma 6.3 *If $\omega \in J^p$ then $d\omega \in J^{p+1}$. Hence, d is well defined on spaces $\Omega^p = \mathcal{R}^p / J^p$.*

Proof. For $\omega \in J^p$ then $\omega = \alpha + d\beta$ where $\alpha \in \mathcal{N}^p$ and $\beta \in \mathcal{N}^{p-1}$. It follows that

$$d\omega = d\alpha + d^2\beta = d\alpha \in d\mathcal{N}^p \subset J^{p+1}.$$

■

Lemma 6.4 *Let $v \in \mathcal{R}_p$. Then v is an annihilator of J^p if and only of $v \in \Omega_p$, that is,*

$$(\omega, v) = 0 \;\; \textit{for all}\; \omega \in J^p \;\; \Leftrightarrow \;\; v \in \Omega_p$$

Hence, the pairing (ω, v) is well defined for all $\omega \in \Omega^p$ and $v \in \Omega_p$, and is non-degenerate.

Proof. Let $\omega = \alpha + d\beta$ where $\alpha \in \mathcal{N}^p$ and $\beta \in \mathcal{N}^{p-1}$. Then

$$(\omega, v) = (\alpha + d\beta, v) = (\alpha, v) + (\beta, \partial v).$$

This sum vanishes for all $\alpha \in \mathcal{N}^p$ and $\beta \in \mathcal{N}^{p-1}$ if and only is $(\alpha, v) = 0$ and $(\beta, \partial v) = 0$, which is the case if and only if both v and ∂v are allowed, that is, $v \in \Omega_p$. ■

Consequently, the spaces Ω^p and Ω_p are dual, and the operators d on Ω^* and ∂ on Ω_* are also dual. We obtain the duality of that cochain complex

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n \xrightarrow{d} \Omega^{n+1} \xrightarrow{d} \ldots \tag{6.5}$$

and the chain complex

$$0 \xleftarrow{\partial} \Omega_0 \xleftarrow{\partial} \ldots \xleftarrow{\partial} \Omega_n \xleftarrow{\partial} \Omega_{n+1} \xleftarrow{\partial} \ldots$$

Every allowed p-form $\omega \in \mathcal{A}^p$ determines a d-invariant p-form $\omega \bmod J^p$. The following lemma is useful for determination of the linear independence of a sequence of d-invariant p-forms $\omega_i \bmod J^p$.

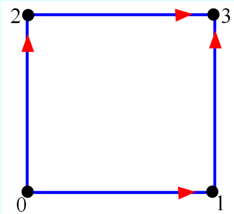
Lemma 6.5 *Let $\{v_j\}_{j=1}^n$ be a basis in Ω_p and $\{\omega_i\}_{i=1}^m$ be a sequence of allowed p-forms. Then the rank of $\{\omega_i \operatorname{mod} J^p\}_{i=1}^m$ in Ω^p is equal to the rank of the $m \times n$ matrix (ω_i, v_j).*

Particular cases: (i) an allowed p-form ω determines a non-zero element $\omega \operatorname{mod} J^p$ of Ω^p if and only if one of the values (ω, v_j) is non-zero;

(ii) if $\{\omega_i\}_{i=1}^n$ is a sequence of allowed p-forms (for example, of some allowed elementary p-forms $e^{i_0 \dots i_p}$) then $\{\omega_i \operatorname{mod} J^p\}_{i=1}^n$ is a basis of Ω^p if and only if the $n \times n$ matrix (ω_i, v_j) is non-singular.

Example. Let G be a square.

We know that $\Omega_2 = \langle e_{013} - e_{023} \rangle$.



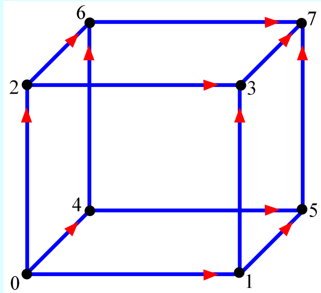
Since $(e^{013}, e_{013} - e_{023}) = 1 \neq 0$,

we obtain by (ii) that $\Omega^2 = \langle e^{013} \rangle$.

We have seen above that $e^{013} \simeq -e^{023}$.

This follows also from (i) because $(e^{013} + e^{023}, e_{013} - e_{023}) = 0$ and, hence, $e^{013} + e^{023} \simeq 0$.

Example. Let G be the 3-cube. We know that



$$\Omega_2 = \langle e_{013} - e_{023}, e_{046} - e_{026}, e_{157} - e_{137}, \\ e_{015} - e_{045}, e_{237} - e_{267}, e_{457} - e_{467} \rangle$$

and

$$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \rangle$$

By Lemma 6.5, we obtain

$$\Omega^2 = \langle e^{013}, e^{046}, e^{157}, e^{015}, e^{237}, e^{457} \rangle$$

because the matrix (ω_i, v_j) is in this case id_6 . Similarly we have

$$\Omega^3 = \langle e^{0237} \rangle = \langle e^{0137} \rangle = \langle e^{0157} \rangle = \left\langle e^{0457} \right\rangle = \left\langle e^{0467} \right\rangle = \left\langle e^{0267} \right\rangle .$$

Observe also, that

$$e^{0157} \simeq -e^{0137}$$

because $e^{0157} + e^{0137}$ annihilates Ω_3:

$$\left(e^{0157} + e^{0137}, e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \right) = 0.$$

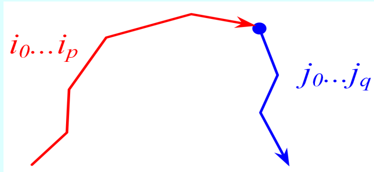
6.4 Concatenation of forms

Definition. For $p, q \geq 0$ and for any two forms $\varphi \in \Lambda^p$ and $\psi \in \Lambda^q$, define their *concatenation* $\varphi\psi \in \Lambda^{p+q}$ by

$$(\varphi\psi)_{i_0 \ldots i_{p+q}} = \varphi_{i_0 \ldots i_p} \psi_{i_p i_{p+1} \ldots i_{p+q}}. \tag{6.6}$$

For elementary forms $e^{i_0 \ldots i_p}$ and $e^{j_0 \ldots j_q}$ we have

$$e^{i_0 \ldots i_p} e^{j_0 \ldots j_q} = \begin{cases} 0, & i_p \neq j_0, \\ e^{i_0 \ldots i_p j_1 \ldots i_q}, & i_p = j_0. \end{cases}$$



Clearly, concatenation is associative.

For example, $e^{12} e^{234} = e^{1234}$ and $e^{12} e^{345} = 0$.

Example. For the 0-form

$$\sigma := 1 = \sum_{i \in V} e^i \in \Lambda^0$$

and any other form $\varphi \in \Lambda^p$ we have $\sigma\varphi = \varphi\sigma = \varphi$ because $(\varphi\sigma)_{i_0 \ldots i_p} = \varphi_{i_0 \ldots i_p} \sigma_{i_p} = \varphi_{i_0 \ldots i_p}$.

Lemma 6.6 *For all $p, q \geq 0$ and $\varphi \in \Lambda^p$, $\psi \in \Lambda^q$, we have*

$$d(\varphi\psi) = (d\varphi)\,\psi + (-1)^p\,\varphi d\psi. \tag{6.7}$$

Clearly, if $\varphi \in \mathcal{R}^p$, $\psi \in \mathcal{R}^q$ then $\varphi\psi \in \mathcal{R}^{p+q}$ and if $\varphi \in \mathcal{A}^p$, $\psi \in \mathcal{A}^q$ then $\varphi\psi \in \mathcal{A}^{p+q}$.

Lemma 6.7 *If $\varphi \in J^p$ or $\psi \in J^q$ then $\varphi\psi \in J^{p+q}$. Consequently, concatenation is well-defined as an operation from $\Omega^p \times \Omega^q$ to $\Omega^{p+q\cdot}$.*

Proof. If $\varphi \in \mathcal{N}^p$ then clearly $\varphi\psi \in \mathcal{N}^{p+q}$. If $\varphi \in J^p$ then $\varphi = \alpha + d\beta$ where $\alpha \in \mathcal{N}^p$ and $\beta \in \mathcal{N}^{p-1}$. We have

$$\varphi\psi = \alpha\psi + (d\beta)\,\psi = \alpha\psi + d(\beta\psi) - (-1)^{p-1}\,\beta d\psi.$$

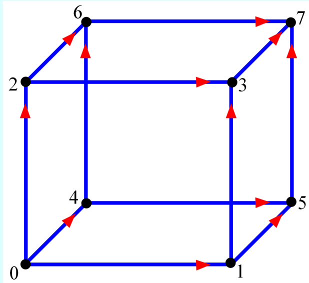
Since $\alpha\psi$, $\beta\psi$ and $\beta d\psi$ are non-allowed, $\varphi\psi \in J^{p+q}$. The case $\psi \in J^q$ is similar.

Elements in Ω^p and Ω^q have representatives $\varphi \in \mathcal{A}^p$ and $\psi \in \mathcal{A}^q$. Then $\varphi\psi \in \mathcal{A}^{p+q}$ and if $\varphi' \simeq \varphi$ and $\psi' \simeq \psi$ are other representatives of the same elements then

$$\varphi'\psi' - \varphi\psi = (\varphi' - \varphi)\,\psi' + \varphi\,(\psi' - \psi) \in J^{p+q}\ ,$$

whence $\varphi\psi \simeq \varphi'\psi'$. ■

Example. Let G be the 3-cube. We have



$$de^{01}=\sum_k\left(e^{k01}-e^{0k1}+e^{01k}\right)\simeq e^{015}+e^{013}$$

$$de^{13}=\sum_k\left(e^{k13}-e^{1k3}+e^{13k}\right)\simeq e^{013}+e^{137}$$

It follows that

$$de^{013}=d\left(e^{01}e^{13}\right)=\left(de^{01}\right)e^{13}-e^{01}de^{13}\simeq\left(e^{015}+e^{013}\right)e^{13}-e^{01}\left(e^{013}+e^{137}\right)=-e^{0137}.$$

Proposition 6.8 *If* $\dim\Omega^n\leq 1$ *then* $\Omega^p=\{0\}$ *for all* $p\geq n+1$.

Proof. Assume first that $\dim\Omega^n=0$ so that $e^{i_0\ldots i_n}\simeq 0$ for all allowed paths $i_0 \ldots i_n$. For any $p>n$ we obtain for any allowed path $i_0 \ldots i_p$ that $e^{i_0\ldots i_p}=e^{i_0\ldots i_n}e^{i_n\ldots i_p}\simeq 0$ whence $\Omega^p=\{0\}$.

Assume now $\dim \Omega^n = 1$. We have for any $p > n$ and any allowed path $i_0 ... i_p$

$$e^{i_0 \dots i_p} = \underbrace{e^{i_0 \dots i_n}}_{n\text{-form}} e^{i_n \dots i_p} = e^{i_0 i_1} \underbrace{e^{i_1 \dots i_{n+1}}}_{n\text{-form}} e^{i_{n+1} \dots i_p}. \tag{6.8}$$

If

$$e^{i_0 \dots i_n} \simeq 0 \quad \text{or} \quad e^{i_1 \dots i_{n+1}} \simeq 0, \tag{6.9}$$

then we obtain $e^{i_0 \dots i_p} \simeq 0$. If (6.9) fails then the both n-forms $e^{i_0 \dots i_n}$ and $e^{i_1 \dots i_{n+1}}$ represent non-zero elements of Ω^n. Since $\dim \Omega^n = 1$, there is $c \in \mathbb{K}$,

$$e^{i_1 \dots i_{n+1}} \simeq c e^{i_0 \dots i_n}.$$

Substituting into (6.8), we obtain

$$e^{i_0 \dots i_p} \simeq c e^{i_0 i_1} e^{i_0 \dots i_n} e^{i_{n+1} \dots i_p}.$$

Since the path $i_0 ... i_p$ is allowed and, hence, regular, we have $i_0 \neq i_1$. It follows that $e^{i_0 i_1} e^{i_0 \dots i_n} = 0$, whence $e^{i_0 \dots i_p} \simeq 0$, which finishes the proof. ■

Proposition 6.9 *If G contains no double arrow and if $\dim \Omega^n \leq 2$ then $\Omega^p = \{0\}$ for all $p \geq n+2$.*

Problem 6.10 *Find practical criteria for $\Omega^p = \{0\}$ for all large p.*

6.5 Cohomology classes

Define the *cohomology groups* of the chain complexes

$$0 \to \Omega^0 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{p-1} \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \ldots$$

by

$$H^p = \ker d|_{\Omega^p} / \operatorname{Im} d|_{\Omega^{p-1}}.$$

A p-form $\varphi \in \Omega^p$ is called *closed* if $d\varphi = 0$, and *exact* if $\varphi = d\psi$ for some $\psi \in \Omega^{p-1}$.

If φ, ψ are two closed p-forms then we write $\varphi \sim \psi$ if φ and ψ represent the same cohomology class, that is, if $\varphi - \psi$ is exact.

Lemma 6.11 *The pairing (φ, v) with $\varphi \in H^p$ and $v \in H_p$ is well defined and is non-degenerate. Hence, the spaces H^p and H_p are dual.*

Proof. Indeed, if $\varphi' \sim \varphi$ and $v' \sim v$ then $\varphi' = \varphi + d\psi$ and $v' = v + \partial u$, and we obtain

$$(\varphi', v') = (\varphi, v) + (d\psi, v) + (\varphi, \partial u) + (d\psi, \partial u).$$

Since $(d\psi, v) = (\psi, \partial v) = 0$ and similarly all other terms vanish, we obtain $(\varphi', v') = (\varphi, v)$. ■

Lemma 6.12 *If $\varphi \in \Omega^p$ and $\psi \in \Omega^q$ are closed forms then $\varphi\psi$ is also closed. If in addition one of the forms φ, ψ is exact then $\varphi\psi$ is also exact. Consequently, concatenation is well defined for $\varphi \in H^p$, $\psi \in H^q$ and results in $\varphi\psi \in H^{p+q}$.*

Proof. If φ and ψ are closed then

$$d(\varphi\psi) = (d\varphi)\,\psi + (-1)^p\,\varphi d\psi = 0$$

so that $\varphi\psi$ is closed. If φ is exact, say $\varphi = d\alpha$ then

$$d(\alpha\psi) = (d\alpha)\,\psi + (-1)^{p+1}\,\alpha d\psi = \varphi\psi$$

so that $\varphi\psi$ is exact. ■

Example. Consider an 1-torus

$$T = \begin{tikzcd} & \overset{1}{\bullet} \arrow[dr] & \\ \overset{0}{\bullet} \arrow[ur] & & \overset{2}{\bullet} \arrow[ll] \end{tikzcd}$$

We have $\Omega^0 = \langle e^0, e^1, e^2 \rangle$ and $\Omega^1 = \langle e^{01}, e^{12}, e^{20} \rangle$ while $\Omega^p = \{0\}$ for $p \geq 2$. Since

$$de^0 \;=\; \sum_k e^{k0} - \sum_k e^{0k} \simeq e^{20} - e^{01},$$

$$
\begin{aligned}
de^1 &= \sum_k e^{k1} - \sum_k e^{1k} \simeq e^{01} - e^{12},\\
de^2 &= \sum_k e^{k2} - \sum_k e^{2k} \simeq e^{12} - e^{20},
\end{aligned}
$$

we see that

$$\ker d|_{\Omega^0} = \langle e^0 + e^1 + e^2 \rangle$$

and

$$\operatorname{Im} d|_{\Omega^0} = \langle e^{20} - e^{01}, e^{01} - e^{12} \rangle. \tag{6.10}$$

In particular,

$$H^0 = \ker d|_{\Omega^0} = \langle e^0 + e^1 + e^2 \rangle.$$

Since $\Omega^2 = \{0\}$, we have

$$\ker d|_{\Omega^1} = \Omega^1 = \langle e^{01}, e^{12}, e^{20} \rangle.$$

Note that $e^{01} \sim e^{20} \sim e^{12}$ because their differences belong to $\operatorname{Im} d|_{\Omega^0}$ by (6.10). It follows that

$$H^1 = \ker d|_{\Omega^1} / \operatorname{Im} d|_{\Omega_0} = \langle e^{01} \rangle.$$

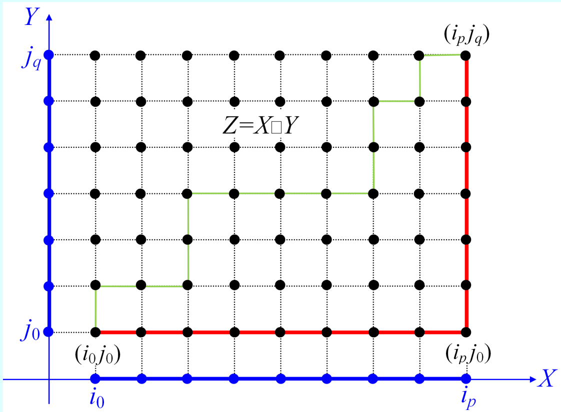
Remark. For a connected digraph G we have always $\beta_0 = 1$ and $H_0 = \langle e_0 \rangle$. We claim that in this case $H^0 = \langle \sigma \rangle$, where $\sigma = 1 = \sum_i e^i \in \Omega^0$. Indeed, we have $(d\sigma)^{ij} = \sigma^j - \sigma^i = 0$ and, hence, $d\sigma = 0$, while $\sigma \not\sim 0$ as $(\sigma, e_0) = 1$.

6.6 Star product and Künneth formula

Definition. Let X and Y be two digraphs. For a p-form φ on X and a q-form ψ on Y, define their *star product* $\varphi \star \psi$ as a $(p+q)$-form on $Z = X \Box Y$ as follows: for elementary forms set

$$e^{i_0 \ldots i_p} \star e^{j_0 \ldots j_q} = e^{(i_0 j_0)(i_1 j_0) \ldots (i_p j_0)(i_p j_1) \ldots (i_p j_q)}$$

where $i \in X$, $j \in Y$ and (ij) is a vertex in $X \Box Y$, and then extend this operation using bilinearity.



Clearly, if φ and ψ are allowed then $\varphi \star \psi$ is also allowed.

In the next statement we use pairing (φ, u) that so far was defined for $\varphi \in \mathcal{R}^p(G)$ and $u \in \mathcal{R}_p(G)$. Let us set $(\varphi, u) = 0$ if $\varphi \in \mathcal{R}^p(G)$ and $u \in \mathcal{R}_{p'}(G)$ with $p' \neq p$.

Lemma 6.13 *For all $\varphi \in \mathcal{R}^p(X)$, $\psi \in \mathcal{R}^q(Y)$ and $u \in \mathcal{R}_{p'}(X)$, $v \in \mathcal{R}_{q'}(Y)$ we have*

$$(\varphi \star \psi, u \times v) = (\varphi, u)(\psi, v). \tag{6.11}$$

Lemma 6.14 *If $\varphi \simeq 0$ or $\psi \simeq 0$ then $\varphi \star \psi \simeq 0$. Consequently, the operation $\varphi \star \psi$ is well defined for all $\varphi \in \Omega^p(X)$, $\psi \in \Omega^q(Y)$, and $\varphi \star \psi \in \Omega^{p+q}(Z)$.*

Proof. If $\varphi \simeq 0$ then $\varphi = \alpha + d\beta$ for $\alpha, \beta \in \mathcal{N}^*(X)$. For all $u \in \Omega_*(X)$ and $v \in \Omega_*(Y)$ we have

$$(\alpha \star \psi, u \times v) = (\alpha, u)(\psi, v) = 0$$

because $\alpha \in \mathcal{N}^*(X)$ and $u \in \mathcal{A}_*(X)$. Similarly,

$$(d\beta \star \psi, u \times v) = (d\beta, u)(\psi, v) = (\beta, \partial u)(\psi, v) = 0$$

because $\beta \in \mathcal{N}^*(X)$ and $\partial u \in \mathcal{A}_*(X)$. Hence,

$$(\varphi \star \psi, u \times v) = 0.$$

By Theorem 5.3, $\Omega_*(Z)$ is spanned by the terms like $u \times v$, which implies that $\varphi \star \psi$ annihilates $\Omega_*(Z)$ and, hence, $\varphi \star \psi \simeq 0$. ∎

Lemma 6.15 *For all* $\varphi \in \Omega^p(X)$, $\psi \in \Omega^q(Y)$, *we have*

$$d(\varphi \star \psi) = d\varphi \star \psi + (-1)^p \varphi \star d\psi. \tag{6.12}$$

Proof. For arbitrary $u \in \Omega_{p'}(X)$ and $v \in \Omega_{q'}(Y)$, we have by the duality of d and ∂ and by the product rule for the cross product:

$$\begin{aligned}
(d(\varphi \star \psi), u \times v) &= (\varphi \star \psi, \partial(u \times v)) \\
&= (\varphi \star \psi, \partial u \times v) + \left(\varphi \star \psi, (-1)^{p'} u \times \partial v\right) \\
&= (\varphi, \partial u)(\psi, v) + (-1)^{p'}(\varphi, u)(\psi, \partial v) \\
&= (d\varphi, u)(\psi, v) + (-1)^p(\varphi, u)(d\psi, v) \\
&= (d\varphi \star \psi, u \times v) + (-1)^p(\varphi \star d\psi, u \times v) \\
&= (d\varphi \star \psi + (-1)^p \varphi \star d\psi, u \times v).
\end{aligned}$$

The proof is concluded by application of Theorem 5.3 as above. ■

It follows from (6.12) that $\varphi \star \psi$ is well defined for cohomology classes $\varphi \in H^p(X)$, $\psi \in H^q(Y)$, and $\varphi \star \psi \in H^{p+q}(Z)$.

Note that for the forms φ and ψ from $\mathcal{R}^*$ the product rule (6.12) is *not* true. In this case the above proof fails at the last step because $\mathcal{R}_*(Z)$ is not spanned by the terms $u \times v$.

Theorem 6.16 (Künneth formula for product in cohomology) *Let $Z = X \Box Y$. We have, for any $r \geq 0$*

$$\Omega^r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} (\Omega^p(X) \otimes \Omega^q(Y)) \tag{6.13}$$

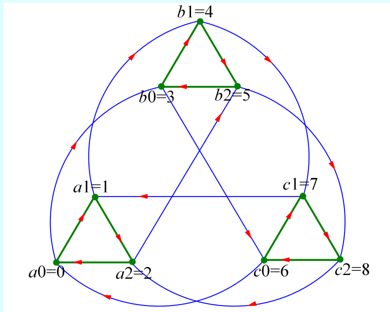
and

$$H^r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} (H^p(X) \otimes H^q(Y)) \tag{6.14}$$

where the isomorphism is given by the map $\varphi \otimes \psi \mapsto \varphi \star \psi$.

Example. Consider the digraph $G = T \Box T$ where T an 1-torus:

$$T = \begin{tikzcd}[column sep=small] & \bullet^{b} \arrow[dr] & \\ {}_{a}\bullet \arrow[ur] & & \bullet^{c} \arrow[ll] \end{tikzcd} = \begin{tikzcd}[column sep=small] & \bullet^{1} \arrow[dr] & \\ {}_{0}\bullet \arrow[ur] & & \bullet^{2} \arrow[ll] \end{tikzcd}$$

$G =$ 

Let us compute $\Omega^p(G)$ and $H^p(G)$.

We know that

$$\Omega^0(T) = \langle e^0, e^1, e^2 \rangle, \quad \Omega^1(T) = \langle e^{01}, e^{12}, e^{20} \rangle \text{ and } \Omega^p = \{0\} \text{ for } p \geq 2$$

and

$$H^0(T) = \langle e^0 + e^1 + e^2 \rangle, \quad H^1(T) = \langle e^{01} \rangle.$$

By the Künneth formula of Theorem 6.16, we obtain

$$\begin{aligned} H^1(G) &= H^0(T) \otimes H^1(T) + H^1(T) \otimes H^0(T) \\ &= \langle \left(e^a + e^b + e^c\right) \star e^{01},\ e^{ab} \star \left(e^0 + e^1 + e^2\right) \rangle \\ &= \langle e^{a0\,a1} + e^{b0\,b1} + e^{c0\,c1},\ e^{a0\,b0} + e^{a1\,b1} + e^{a2\,b2} \rangle \\ &= \langle e^{01} + e^{34} + e^{67},\ e^{03} + e^{14} + e^{25} \rangle \end{aligned}$$

and

$$H^2(G) = H^1(T) \otimes H^1(T) = \langle e^{ab} \star e^{01} \rangle = \langle e^{a0\,b0\,b1} \rangle = \langle e^{034} \rangle$$

Similarly, we have

$$\Omega^2(G) = \Omega^1(T) \otimes \Omega^1(T) = \langle e^{ab}, e^{bc}, e^{ca} \rangle \otimes \langle e^{01}, e^{12}, e^{20} \rangle$$

that is

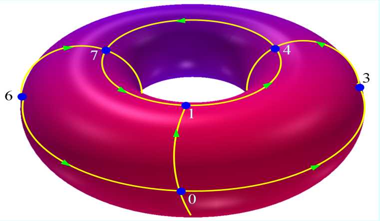
$$\Omega^2(G) = \langle e^{ab} \star e^{01}, e^{ab} \star e^{12}, e^{ab} \star e^{20}, e^{bc} \star e^{01}, e^{bc} \star e^{12}, e^{bc} \star e^{20}, e^{ca} \star e^{01}, e^{ca} \star e^{12}, e^{ca} \star e^{20} \rangle.$$

Next we compute

$$e^{ab} \star e^{01} = e^{a0\,b0\,b1} = e^{034}$$

$$e^{ab} \star e^{12} = e^{a1\,b1\,b2} = e^{145}$$
$$e^{ab} \star e^{20} = e^{a2\,b2\,b0} = e^{253}$$
$$e^{bc} \star e^{01} = e^{b0\,c0\,c1} = e^{367}$$
$$e^{bc} \star e^{12} = e^{b1\,c1\,c2} = e^{478}$$
$$e^{bc} \star e^{20} = e^{b2\,c2\,c0} = e^{586}$$
$$e^{ca} \star e^{01} = e^{c0\,a0\,a1} = e^{601}$$
$$e^{ca} \star e^{12} = e^{c1\,a1\,a2} = e^{712}$$
$$e^{ca} \star e^{20} = e^{c2\,a2\,a0} = e^{820}$$

$$\Rightarrow \qquad \Omega^2(G) = \langle e^{034}, e^{145}, e^{253}, e^{367}, e^{478}, e^{586}, e^{601}, e^{712}, e^{820} \rangle$$



Recall for comparison that

$$\Omega_2(G) = \langle e_{034} - e_{014},\ e_{145} - e_{125},\ e_{253} - e_{203}, e_{367} - e_{347},$$
$$e_{478} - e_{458}, e_{586} - e_{536}, e_{601} - e_{671},\ e_{712} - e_{782},\ e_{820} - e_{860} \rangle$$

7 Intersection forms

7.1 Summary of d-invariant forms and cohomology

Let V be a finite set, $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}$. Space Λ^p of p-forms is generated by elementary p-forms $e^{i_0 \ldots i_p}$, where $i_0 \ldots i_p$ is any sequence of $p+1$ vertices. Any p-form $\omega \in \Lambda^p$ has a form

$$\omega = \sum_{i_0, \ldots, i_p \in V} \omega_{i_0 \ldots i_p} e^{i_0 \ldots i_p} \quad \text{where } \omega_{i_0 \ldots i_p} \in \mathbb{K}.$$

The spaces Λ^p and Λ_p are dual with the pairing

$$\left(e^{i_0 \ldots i_p}, e_{j_0 \ldots j_p}\right) = \delta^{i_0 \ldots i_p}_{j_0 \ldots j_p}.$$

The exterior derivative $d : \Lambda^{p-1} \to \Lambda^p$ is defined by

$$(d\omega)_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q \, \omega_{i_0 \ldots \widehat{i_q} \ldots i_p} \ \text{ for any } \omega \in \Lambda^{p-1}.$$

Concatenation of p-forms and q-forms is defined by

$$e^{i_0 \ldots i_p} e^{j_0 \ldots j_q} = \begin{cases} 0, & i_p \neq j_0, \\ e^{i_0 \ldots i_p j_1 \ldots i_q}, & i_p = j_0. \end{cases}$$

The operator d satisfies the product rule with respect to concatenation:

$$d(\varphi\psi) = (d\varphi)\,\psi + (-1)^p\,\varphi d\psi.$$

Both d and concatenation are well defined on the spaces $\mathcal{R}^p$ of regular p-forms spanned by elementary p-forms $e^{i_0 \ldots i_p}$ with regular paths $i_0 \ldots i_p$.

Given a digraph $G = (V, E)$, consider the following subspaces of $\mathcal{R}^p$:

$$\mathcal{A}^p = \langle e^{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is allowed} \rangle$$

$$\mathcal{N}^p = \langle e^{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is non-allowed but regular} \rangle$$

so that $\mathcal{R}^p = \mathcal{A}^p \oplus \mathcal{N}^p$. Set $J^p = \mathcal{N}^p + d\mathcal{N}^{p-1}$ and define the space of d-invariant p-forms:

$$\Omega^p = \mathcal{R}^p / J^p = \mathcal{A}^p / (J^p \cap \mathcal{A}^p)$$

so that any d-invariant p-form is an allowed p-form considered modulo J^p.

Both d and concatenation are well defined on spaces Ω^*. The cochain complex

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n \xrightarrow{d} \Omega^{n+1} \xrightarrow{d} \ldots \tag{7.1}$$

is dual to the chain complex

$$0 \xleftarrow{\partial} \Omega_0 \xleftarrow{\partial} \ldots \xleftarrow{\partial} \Omega_n \xleftarrow{\partial} \Omega_{n+1} \xleftarrow{\partial} \ldots$$

The cohomology groups

$$H^p = \ker d|_{\Omega^p} / \operatorname{Im} d|_{\Omega^{p-1}}$$

and the homology groups H_p are dual. Concatenation is well defined on cohomology classes: for $\varphi \in H^p$ and $\psi \in H^q$ we have $\varphi\psi \in H^{p+q}$.

7.2 Graded symmetry

Conjecture 7.1 *The concatenation of cohomology classes is graded-symmetric: for all $\varphi \in H^p$ and $\psi \in H^q$*

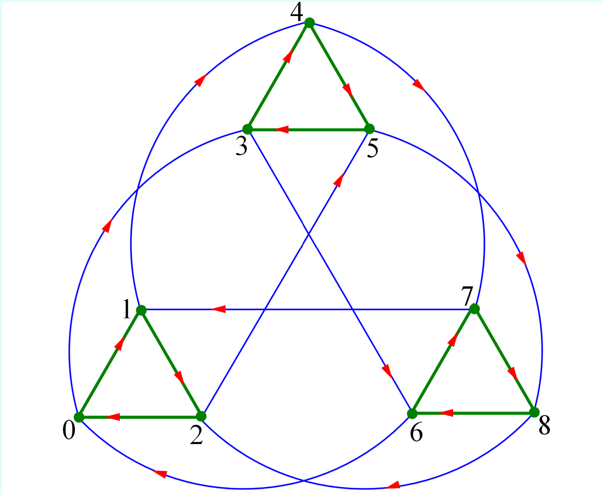
$$\varphi\psi = (-1)^{pq}\,\psi\varphi. \tag{7.2}$$

Note that concatenation is *not* graded-symmetric in $\Omega^* \times \Omega^* \to \Omega^*$. For example, if $a \to b$ then $e^a \in \Omega^0$, $e^{ab} \in \Omega^1$ and

$$e^a e^{ab} = e^{ab} \quad \text{and} \quad e^{ab} e^a = 0.$$

On the other hand, it is easy to verify (7.2) if $p = 0$. For example, if G is connected then $|H^0| = 1$ and, hence, $\varphi = c\sigma$ where $\sigma = \sum_i e^i$, and (7.2) is trivially satisfied.

Example. Let $G = T\Box T$ where $T = \{0 \to 1 \to 2 \to 0\}$ is an 1-torus.



We have seen above that

$$H^1(G) = \langle \varphi_1, \varphi_2 \rangle$$

where

$$\varphi_1 = e^{01} + e^{34} + e^{67}$$

$$\varphi_2 = e^{03} + e^{14} + e^{25}$$

Let us verify that the concatenation is graded symmetric in H^1, that is,

$$\varphi\psi = -\psi\varphi \text{ for all } \varphi, \psi \in H^1.$$

We clearly have

$$\varphi_1\varphi_1 = \varphi_2\varphi_2 = 0$$

while

$$\varphi_1\varphi_2 = \left(e^{01} + e^{34} + e^{67}\right)\left(e^{03} + e^{14} + e^{25}\right) = e^{014}$$

and

$$\varphi_2\varphi_1 = \left(e^{03} + e^{14} + e^{25}\right)\left(e^{01} + e^{34} + e^{67}\right) = e^{034}.$$

It remains to verify that

$$e^{034} \sim -e^{014}\,. \tag{7.3}$$

For that we use that

$$H_2\left(G\right) = \langle u\rangle,$$

where

$$\begin{aligned} u = {} & (e_{034} - e_{014}) + (e_{145} - e_{125}) + (e_{253} - e_{203}) + (e_{367} - e_{347}) + (e_{478} - e_{458}) \\ & + (e_{586} - e_{536}) + (e_{601} - e_{671}) + (e_{712} - e_{782}) + (e_{820} - e_{860}) \end{aligned}$$

(see p. 99). Since

$$\left(e^{034}, u\right) = 1 \quad \text{and} \quad \left(e^{014}, u\right) = -1,$$

we see that (7.3) is satisfied.

7.3 Intersection form and signature

Definition. We say that a homology class $u \in H_r$ is *proper* if, for all $\varphi \in H^p$ and $\psi \in H^q(G)$

$$(\varphi\psi, u) = (-1)^{pq}(\psi\varphi, u). \tag{7.4}$$

If Conjecture 7.1 is true then all homology classes are proper.

For any homology class $u \in H_{2p}$ consider the following bilinear form

$$Q_u(\varphi, \psi) = (\varphi\psi, u) \quad \text{where } \varphi, \psi \in H^p,$$

that is called the *intersection form* of u. If u is proper and if p is even then (7.4) implies that Q_u is a symmetric bilinear form in H^p. Hence, the notion of signature of Q_u is well-defined:

$$\sigma(Q_u) = a - b,$$

where a and b are the numbers of positive resp. negative eigenvalues of Q_u.

Definition. Let $u \in H_r$ be proper. Define the signature $\sigma(u)$ of u as follows:
- if r is divisible by 4 then set $\sigma(u) = \sigma(Q_u)$;
- if r is not divisible by 4 then set $\sigma(u) = 0$.

Theorem 7.2 *Assume that the homology classes* $u \in H_*(X)$ *and* $v \in H_*(Y)$ *are proper. Then* $u \times v \in H_*(X \Box Y)$ *is also proper and*

$$\sigma(u \times v) = \sigma(u)\sigma(v). \tag{7.5}$$

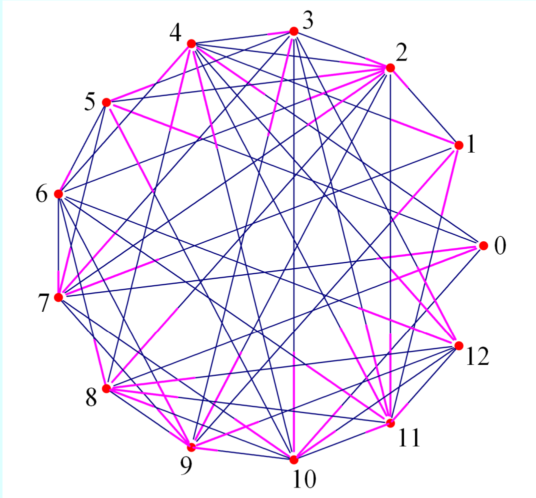
Conjecture 7.3 *There exists a digraph* G *and a proper homology class* $w \in H_4(G)$ *such that* $\sigma(w) \neq 0$.

Note that such a path w cannot be constructed as a product $w = u \times v$ because u and v must have orders < 4 whence $\sigma(u) = \sigma(v) = 0$, and by Theorem 7.2 also $\sigma(w) = 0$.

Here is an approach how one can try to construct $w \in H_4$ with $\sigma(w) \neq 0$. It is known that $\sigma(\mathbb{C}P^2) \neq 0$ and the Betti numbers of $\mathbb{CP}^2$ are $1, 0, 1, 0, 1$. We may try to find digraphs with the same Betti numbers and compute $\sigma(w)$ for a generator $w \in H_4$. Let φ be a generator of H^2. Then the question amounts to verification of the fact that

$$Q_w(\varphi, \varphi) = (\varphi\varphi, w) \neq 0.$$

One of digraphs with Betti $= 1, 0, 1, 0, 1$ is shown here:



Another possibility is as follows. Let S be a simplicial complex that is a triangulation of $\mathbb{C}P^2$ with the same Betti numbers $1, 0, 1, 0, 1$. Let G_S be the Hasse diagram of S, that is, the vertices of G_S are all simplices of S, and for two simplices $s, t \in S$ we have an arrow $s \to t$ in G_S if and only if t is a face of s of the codimension 1.

By Theorem 2.1, we have

$$H_*^{simp}(S) \simeq H_*(G_S),$$

Hence, the Betti numbers of G_S are also $1, 0, 1, 0, 1$.

If Conjecture 7.3 is true then a question arises how to characterize homology classes u with $\sigma(u) \neq 0$. For simplicity we denote by u also its representative path. Note that if $\partial u = 0$ on G then also $\partial u = 0$ on any *larger* digraph $G' \supset G$. Hence, u determines a homology class not only on G, but also on G'. However, it can happen that $u \neq 0$ in $H_*(G)$ while $u = 0$ in $H_*(G')$, that is, u is a boundary on G'.

Conjecture 7.4 *Assume that $u \in H_*(G)$ is proper. Suppose that u is a boundary on a certain larger digraph $G' \supset G$. Then $\sigma(u) = 0$.*

If Conjecture 7.3 is true then G' cannot be arbitrary. Indeed, by adding all possible arrows to G, we obtain a complete digraph G' with $H_*(G') = \{0\}$ so that all cycles in G' are boundaries. Hence, one must put certain restrictions on G'.

Note that in order to determine a symmetric bilinear form Q_u up to isomorphism, it is not enough to know just the signature $\sigma(Q_u)$: one needs also the rank of Q_u (=the number of non-zero eigenvalues) and/or the nullity of Q_u (=the number of zero eigenvalues). If $u \in H_{2p}$ then

$$\operatorname{rank}(Q_u) + \operatorname{nullity}(Q_u) = \dim H^p = \beta_p.$$

Problem 7.5 *How to compute* $\operatorname{rank}(Q_{u\times v})$ *and/or* $\operatorname{nullity}(Q_{u\times v})$?

7.4 An example of computation of intersection form

For any p-path u we write $|u| = p$, and for any p-form φ we write $|\varphi| = p$.

Lemma 7.6 *Let X, Y be two digraphs and $Z = X \Box Y$. Let $u \in \mathcal{R}_*(X)$, $v \in \mathcal{R}_*(Y)$ and $\varphi_1, \varphi_2 \in \mathcal{R}^*(X)$, $\psi_1, \psi_2 \in \mathcal{R}^*(Y)$. Then for pairing on Z we have*

$$((\varphi_1 \star \psi_1)(\varphi_2 \star \psi_2), u \times v) = (-1)^{|\psi_1||\varphi_2|} (\varphi_1\varphi_2, u)(\psi_1\psi_2, v) \tag{7.6}$$

that is,

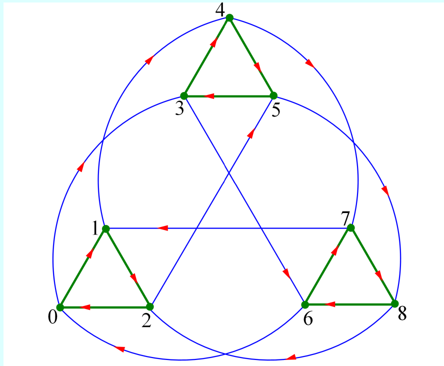
$$Q_{u\times v}(\varphi_1 \star \psi_1, \varphi_2 \star \psi_2) = (-1)^{|\psi_1||\varphi_2|} Q_u(\varphi_1, \varphi_2) Q_v(\psi_1, \psi_2).$$

If $\{\varphi_i\}$ is a basis in $H^*(X)$ and $\{\psi_j\}$ is a basis in $H^*(Y)$ then $\{\varphi_i \star \psi_j\}$ is a basis in $H^*(Z)$ by the Künneth formula of Theorem 6.16. Hence, Lemma 7.6 allows to determine $Q_{u\times v}$ via Q_u and Q_v.

Example. Consider the digraph $G = T^{4\Box}$ where $T = \{0 \to 1 \to 2 \to 0\}$ is 1-torus.

Here is $X = T^{2\Box} = T \Box T$:

while $G = X \Box X$.

$X =$ 

$\beta(T) = \{1, 1\}$
$\beta(X) = \{1, 2, 1\}$
$\beta(G) = \{1, 4, 6, 4, 1\}$

We compute Q_w in $H^2(G)$, where w is a generator of $H_4(G)$. By the Künneth formula we have

$$H^2(G) = H^2(X) \otimes H^0(X) + H^1(X) \otimes H^1(X) + H^0(X) \otimes H^2(X)$$

and

$$H_4(G) = H_2(X) \otimes H_2(X).$$

We have seen above that

$$H^0(X) = \langle \varphi_0 \rangle \qquad \text{where} \quad \varphi_0 = e^0 + ... + e^8$$

$$H^1(X) = \langle \varphi_1, \varphi_2 \rangle \quad \text{where} \quad \varphi_1 = e^{01} + e^{34} + e^{67}, \quad \varphi_2 = e^{03} + e^{14} + e^{25}$$
$$H^2(X) = \langle \omega \rangle \qquad \text{where} \quad \omega = e^{034},$$

and $H_2(X) = \langle u \rangle$ where

$$u = (e_{034} - e_{014}) + (e_{145} - e_{125}) + (e_{253} - e_{203}) + (e_{367} - e_{347}) + (e_{478} - e_{458}) + (e_{586} - e_{536}) + (e_{601} - e_{671}) + (e_{712} - e_{782}) + (e_{820} - e_{860})$$

Hence,

$$H_4(G) = \langle w \rangle \quad \text{where } w = u \times u$$

and

$$H^2(G) = \langle \varphi_0 \star \omega,\ \varphi_1 \star \varphi_1,\ \varphi_1 \star \varphi_2,\ \varphi_2 \star \varphi_1,\ \varphi_2 \star \varphi_2,\ \omega \star \varphi_0, \rangle$$

Computation by means of (7.6) shows that the matrix M_w of Q_w in this basis of $H^2(G)$ is antidiagonal:

$$M_w = \begin{pmatrix} \mathbf{0} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & 1 & & & \\ & -1 & & & & \\ 1 & & & & & \mathbf{0} \end{pmatrix}.$$

This matrix has the eigenvalues 1 and -1, each with multiplicity 3. Hence,

$$\sigma(w) = 0, \quad rank(w) = 6, \quad nullity(w) = 0.$$

The signature can also be computed by Theorem 7.2: $\sigma(w) = \sigma(u)^2 = 0$ because $|u| = 2$.

Let us show how to compute the entries of M_w. For example, the $(3,4)$-entry is

$$\begin{aligned} Q_w(\varphi_1 \star \varphi_2, \varphi_2 \star \varphi_1) &= ((\varphi_1 \star \varphi_2)(\varphi_2 \star \varphi_1), u \times u) \\ &= -(\varphi_1\varphi_2, u)(\varphi_2\varphi_1, u) \\ &= -\left(e^{014}, u\right)\left(e^{034}, u\right) = -(-1) \cdot 1 = 1, \end{aligned}$$

the $(2,5)$-entry is

$$\begin{aligned} Q_w(\varphi_1 \star \varphi_1, \varphi_2 \star \varphi_2) &= ((\varphi_1 \star \varphi_1)(\varphi_2 \star \varphi_2), u \times u) \\ &= -(\varphi_1\varphi_2, u)(\varphi_1\varphi_2, u) \\ &= -\left(e^{014}, u\right)\left(e^{014}, u\right) = -1, \end{aligned}$$

and the $(1,6)$-entry is

$$\begin{aligned} Q_u(\varphi_0 \star \omega, \omega \star \varphi_0) &= ((\varphi_0 \star \omega)(\omega \star \varphi_0), u \times u) \\ &= (\varphi_0\omega, u)(\omega\varphi_0, u) \\ &= \left(e^{034}, u\right)\left(e^{034}, u\right) = 1. \end{aligned}$$

8 Hodge Laplacian

Here $\mathbb{K} = \mathbb{R}$. Let us fix an arbitrary inner product $\langle \cdot, \cdot \rangle$ in each of the spaces $\mathcal{R}_p$ so that we have an inner product also in all Ω_p. In all examples we use the natural inner product.

8.1 Definition and spectral properties of Δ_p

For the operator $\partial : \Omega_p \to \Omega_{p-1}$ consider the adjoint operator $\partial^* : \Omega_{p-1} \to \Omega_p$ so that

$$\langle \partial u, v \rangle = \langle u, \partial^* v \rangle \quad \text{for all } u \in \Omega_p \text{ and } v \in \Omega_{p-1}.$$

Definition. Define the *Hodge-Laplace operator on paths* $\Delta_p : \Omega_p \to \Omega_p$ by

$$\Delta_p u = \partial^* \partial u + \partial \partial^* u. \tag{8.1}$$

Here we use the following operators ∂ and ∂^*: $\quad \Omega_{p-1} \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_p \quad$ and $\quad \Omega_p \underset{\partial}{\overset{\partial^*}{\rightleftarrows}} \Omega_{p+1}$.

Proposition 8.1 *The operator Δ_p is self-adjoint and non-negative definite.*

Proof. We have for all $u, v \in \Omega_p$

$$\langle \Delta_p u, v \rangle = \langle \partial^* \partial u + \partial \partial^* u, v \rangle = \langle \partial u, \partial v \rangle + \langle \partial^* u, \partial^* v \rangle = \langle u, \Delta_p v \rangle$$

so that Δ_p is symmetric, and

$$\langle \Delta_p u, u \rangle = \|\partial u\|^2 + \|\partial^* u\|^2 \geq 0, \tag{8.2}$$

so that $\Delta_p \geq 0$. Hence, the spectrum of Δ_p is real, non-negative and consists of a finite sequence of eigenvalues. ■

Proposition 8.2 *Denote* $D = \max_{i \in V} \deg(i)$. *If* $\langle \cdot, \cdot \rangle$ *is natural then* $\operatorname{spec} \Delta_0 \subset [0, 2D]$.

Proof. By the variational principle, it suffices to prove that for all $u \in \Omega_0$

$$\frac{\langle \Delta_0 u, u \rangle}{\|u\|^2} \leq 2D.$$

Since $\partial u = 0$, we have by (8.2)

$$\langle \Delta_0 u, u \rangle = \|\partial^* u\|^2.$$

Since for any $i \to j$

$$\langle \partial^* u, e_{ij} \rangle = \langle u, \partial e_{ij} \rangle = \langle u, e_j - e_i \rangle = u^j - u^i,$$

it follows that

$$\|\partial^* u\|^2 = \sum_{i \to j} \left(u^j - u^i\right)^2 \leq 2 \sum_{i \to j} \left(u^j\right)^2 + 2 \sum_{i \to j} \left(u^i\right)^2 = 2 \sum_i \deg(i) \left(u^i\right)^2 \leq 2D \|u\|^2, \quad (8.3)$$

whence the claim follows. ■

The bottom eigenvalue of Δ_0 is always 0 because if all $u^k = 1$ then by (8.3) $\partial^* u = 0$ and, hence, $\Delta_0 u = \partial \partial^* u = 0$. If $G = K_{D,D}$ – a complete bipartite graph, then G is D-regular and $2D$ is the top eigenvalue of Δ_0.

For a general p, *the multiplicity of* 0 *as an eigenvalue of* Δ_p *is equal to the Betti number* β_p as we will see below.

Problem 8.3 *Find a reasonable upper bounds for* $\operatorname{spec} \Delta_p$. *The question amounts to obtaining an upper bound for the Rayleigh quotient for non-zero* $u \in \Omega_p$:

$$\frac{\|\partial u\|^2 + \|\partial^* u\|^2}{\|u\|^2} \leq ?$$

Problem 8.4 *Find estimates of the eigenvalues of* Δ_p *in terms of geometric and combinatorial properties of* G.

8.2 Matrix of Δ_p

Let $\{\alpha_i\}$ be an orthonormal basis in Ω_p, $\{\beta_m\}$ be an orthonormal basis in Ω_{p-1} and $\{\gamma_n\}$ be an orthonormal basis in Ω_{p+1} :

$$\begin{array}{ccccc} \Omega_{p-1} & \overset{\partial}{\underset{\partial^*}{\leftrightarrows}} & \Omega_p & \overset{\partial}{\underset{\partial^*}{\leftrightarrows}} & \Omega_{p+1} \\ \{\beta_m\} & & \{\alpha_i\} & & \{\gamma_n\} \end{array}$$

The operator $\partial : \Omega_p \to \Omega_{p-1}$ has in the bases $\{\alpha_i\}$ and $\{\beta_m\}$ the matrix

$$B = (\langle \beta_m, \partial \alpha_i \rangle)_{m,i} \tag{8.4}$$

where m is the row index and i is the column index.

Similarly, the operator $\partial^* : \Omega_p \to \Omega_{p+1}$ has the matrix

$$C = (\langle \gamma_n, \partial^* \alpha_i \rangle)_{n,i} = (\langle \partial \gamma_n, \alpha_i \rangle)_{n,i} \ . \tag{8.5}$$

Since $\Delta_p = \partial^* \partial + (\partial^*)^* \partial^*$, we obtain the matrix of Δ_p in the basis $\{\alpha_i\}$:

$$\boxed{\text{matrix of } \Delta_p = B^T B + C^T C}. \tag{8.6}$$

More explicitly, the (i,j)-entry of the matrix of Δ_p in the basis $\{\alpha_i\}$ is given by

$$\boxed{\langle \Delta_p \alpha_i, \alpha_j \rangle = \sum_m \langle \partial \alpha_i, \beta_m \rangle \langle \partial \alpha_j, \beta_m \rangle + \sum_n \langle \alpha_i, \partial \gamma_n \rangle \langle \alpha_j, \partial \gamma_n \rangle}. \tag{8.7}$$

Example. Recall that $\Omega_{-1} = \{0\}$, $\Omega_0 = \{e_i : i \in V\}$ and $\Omega_1 = \langle e_{kl} : k \to l \rangle$. Assuming that $\langle \cdot, \cdot \rangle$ is the natural inner product, we obtain by (8.7) that the matrix of Δ_0 is

$$\begin{aligned}
\langle \Delta_0 e_i, e_j \rangle &= \sum_{k \to l} \langle e_i, \partial e_{kl} \rangle \langle e_j, \partial e_{kl} \rangle \\
&= \sum_{k \to l} \langle e_i, e_l - e_k \rangle \langle e_j, e_l - e_k \rangle \\
&= \sum_{k \to l} (\delta_{il} - \delta_{ik})(\delta_{jl} - \delta_{jk}) \\
&= \sum_{k \to i} \delta_{ij} + \sum_{i \to l} \delta_{ij} - \mathbf{1}_{\{i \to j\}} - \mathbf{1}_{\{j \to i\}} \\
&= \deg(i)\, \delta_{ij} - \mathbf{1}_{\{i \to j\}} - \mathbf{1}_{\{j \to i\}}.
\end{aligned}$$

If G has no double arrow then the matrix of $\Delta_0 = \operatorname{diag}(\deg(i)) - \mathbf{1}_{\{i \sim j\}}$ where $\mathbf{1}_{\{i \sim j\}}$ is the adjacency matrix of G. Hence, Δ_0 is the usual unnormalized Laplacian (=Kirchhoff operator) on functions on G.

Consequently, $\operatorname{trace} \Delta_0 = \sum_{i \in V} \deg(i) = 2E$.

8.3 Examples of computation of Δ_1

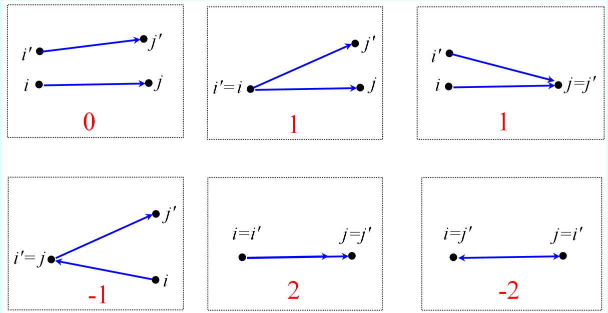
Let us compute Δ_1 for the natural inner product. We use the orthonormal bases $\{e_m\}$ in Ω_0 and $\{e_{ij} : i \to j\}$ in Ω_1. Let $\{\gamma_n\}$ be an orthonormal basis in Ω_2.
The matrix of Δ_1 has dimensions $E \times E$ and, by (8.7), its entries are

$$\langle \Delta_1 e_{ij}, e_{i'j'} \rangle = \sum_m \langle \partial e_{ij}, e_m \rangle \langle \partial e_{i'j'}, e_m \rangle + \sum_n \langle e_{ij}, \partial \gamma_n \rangle \langle e_{i'j'}, \partial \gamma_n \rangle \tag{8.8}$$

for all arrows $i \to j$ and $i' \to j'$. For the first sum in (8.8) we have

$$\begin{aligned} \sum_m \langle \partial e_{ij}, e_m \rangle \langle \partial e_{i'j'}, e_m \rangle &= \sum_m \langle e_j - e_i, e_m \rangle \langle e_{j'} - e_{i'}, e_m \rangle = \sum_m (\delta_{jm} - \delta_{im})(\delta_{j'm} - \delta_{i'm}) \\ &= \delta_{jj'} - \delta_{ij'} - \delta_{ji'} + \delta_{ii'} =: [ij, i'j'] \,. \end{aligned}$$

The values of $[ij, i'j']$ are shown here:

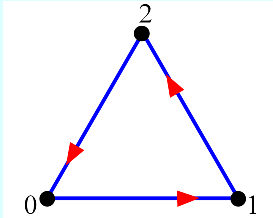


Hence, in the case $p = 1$, we have

$$B^T B = ([ij, i'j']) \,.$$

In particular, diagonal entries of $B^T B$ are 2.

Example. Consider an 1-torus

$T =$ 

In this case $\Omega_1 = \langle e_{01}, e_{12}, e_{20} \rangle$, $\Omega_2 = \{0\}$, $|H_1| = 1$. Hence, we obtain

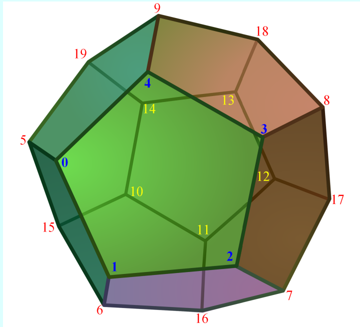
$$\text{the matrix of } \Delta_1 = B^T B = ([ij, i'j'])$$

$$= \begin{pmatrix} & e_{01} & e_{12} & e_{20} \\ e_{01} & [01,01] & [01,12] & [01,20] \\ e_{12} & [12,01] & [12,12] & [12,20] \\ e_{20} & [20,01] & [20,12] & [20,20] \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

The eigenvalues of Δ_1 are $\{0, 3, 3\}$.

Example. Consider a dodecahedron (like on p.4.3):

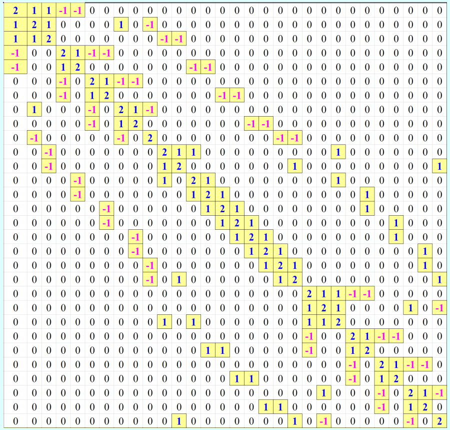


We have $V = 20$, $E = 30$,

$\Omega_2 = \{0\}$ and $|H_1| = 11$.

In particular, $C^T C = 0$.

The matrix of $\Delta_1 = B^T B$ is shown here:



2	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	2	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	2	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	-1	0	2	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-1	0	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	0	0	0	0	0	-1	0	2	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0	2	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
0	0	0	0	-1	0	0	0	0	0	1	0	2	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	-1	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	1	0	0	0	0	1	0	-1
0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	1	2	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	2	1	-1	-1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	-1	0	0	1	2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	2	1	-1	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	0	1	2	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	-1	0	2	1	-1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	0	1	2	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	-1	0	0	0	0	0	-1	0	2

The eigenvalues of Δ_1 are:

$$0_{11},\ 2_5,\ 3_4,\ 5_4,\ \left(3 \pm \sqrt{5}\right)_3,$$

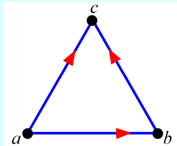
where the subscripts show multiplicity.

For a general digraph G with $\Omega_2 \neq \{0\}$, let us compute the entry $\langle e_{ij}, \partial\gamma_n \rangle$ of the matrix C assuming that $\gamma_n = \gamma$ is a triangle or square (note that although Ω_2 has always a basis of triangles and squares, the squares in this basis do not have to be orthogonal).
If $\gamma = e_{abc}$ is a triangle then we have

$$\langle e_{ij}, \partial\gamma \rangle = \langle e_{ij}, e_{ab} + e_{bc} - e_{ac} \rangle = [ij, \gamma] \,,$$

where

$$\boxed{[ij, \gamma] := \begin{cases} 1, & \text{if } ij \in \{ab, bc\} \\ -1 & \text{if } ij = ac \\ 0, & \text{otherwise.} \end{cases}}$$

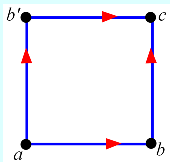


If $\gamma = \frac{e_{abc} - e_{ab'c}}{\sqrt{2}}$ is a (normalized) square then

$$\langle e_{ij}, \partial\gamma \rangle = \frac{1}{\sqrt{2}} \langle e_{ij},\, e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \rangle = \frac{1}{\sqrt{2}} [ij, \gamma] \,,$$

where

$$\boxed{[ij, \gamma] = \begin{cases} 1, & \text{if } ij \in \{ab\,, bc\} \\ -1 & \text{if } ij \in \{ab', b'c\} \\ 0, & \text{otherwise.} \end{cases}}$$



Example. Let G be a triangle $\{0 \to 1 \to 2, 0 \to 2\}$. Then $\Omega_1 = \langle e_{01}, e_{12}, e_{02} \rangle$ and

$$B^T B = ([ij, i'j']) = \begin{array}{c|ccc} & e_{01} & e_{12} & e_{02} \\ \hline e_{01} & [01, 01] & [01, 12] & [01, 20] \\ e_{12} & [12, 01] & [12, 12] & [12, 20] \\ e_{02} & [02, 01] & [02, 12] & [02, 02] \end{array} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The basis $\{\gamma_n\}$ of Ω_2 consists of a single triangle $\gamma = e_{012}$ so that

$$C = \begin{array}{c|ccc} & e_{01} & e_{12} & e_{02} \\ \hline e_{012} & [01, \gamma] & [12, \gamma] & [02, \gamma] \end{array} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$$

$$C^T C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\text{matrix of } \Delta_1 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Example. Let G be a square $\{0 \to 1 \to 3, 0 \to 2 \to 3\}$. Then $\Omega_1 = \langle e_{01}, e_{02}, e_{13}, e_{23} \rangle$ and

$$B^T B = ([ij, i'j']) = \begin{array}{c} \\ e_{01} \\ e_{02} \\ e_{13} \\ e_{23} \end{array} \begin{array}{c} \begin{array}{cccc} e_{01} & e_{02} & e_{13} & e_{23} \end{array} \\ \begin{pmatrix} [01,01] & [01,02] & [01,13] & [01,23] \\ [02,01] & [02,02] & [02,13] & [02,23] \\ [12,01] & [13,02] & [13,13] & [13,23] \\ [23,01] & [23,02] & [23,13] & [23,23] \end{pmatrix} \end{array} = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

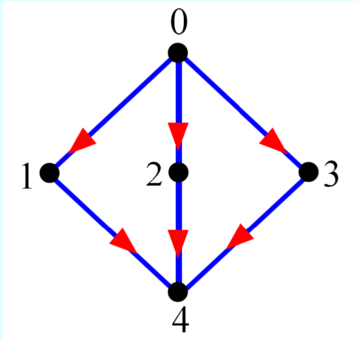
The basis $\{\gamma_n\}$ of Ω_2 consists of a single square $\gamma = \frac{1}{\sqrt{2}} (e_{013} - e_{023})$ so that

$$C = \frac{1}{\sqrt{2}} \begin{array}{c} \\ \gamma \end{array} \begin{array}{c} \begin{array}{cccc} e_{01} & e_{02} & e_{13} & e_{23} \end{array} \\ \begin{pmatrix} [01,\gamma] & [02,\gamma] & [13,\gamma] & [23,\gamma] \end{pmatrix} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}$$

$$C^T C = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$$\text{matrix of } \Delta_1 = B^T B + C^T C = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{pmatrix}, \quad \text{the eigenvalues are } \{2_3, 4\}.$$

Example. Consider a following digraph:



Here $|\Omega_1| = E = 6, \quad |\Omega_2| = 2$ and

$\Omega_2 = \langle e_{014} - e_{024}, e_{014} - e_{034} \rangle$

However, this basis is not orthogonal.

Orthogonalization gives an orthonormal basis in Ω_2:

$$\begin{aligned} \gamma_1 &= \tfrac{1}{\sqrt{2}} \left(e_{014} - e_{024} \right), \\ \gamma_2 &= \tfrac{1}{\sqrt{6}} \left(e_{014} + e_{024} - 2e_{034} \right). \end{aligned}$$

Since

$$\begin{aligned} \partial\gamma_1 &= \tfrac{1}{\sqrt{2}} \left(e_{01} + e_{14} - e_{02} - e_{24} \right), \\ \partial\gamma_2 &= \tfrac{1}{\sqrt{6}} \left(e_{01} + e_{04} + e_{02} + e_{24} - 2e_{03} - 2e_{34} \right), \end{aligned}$$

we compute the matrix C:

$$C = (\langle e_{ij}, \partial\gamma_n\rangle) = \begin{array}{c} \\ \partial\gamma_1 \\ \partial\gamma_2 \end{array} \begin{array}{c} \begin{matrix} \quad e_{01} & \quad e_{14} & \quad e_{02} & \quad e_{24} & \quad e_{03} & \quad e_{34} \end{matrix} \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \end{array}$$

and

$$C^T C = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

We compute also B:

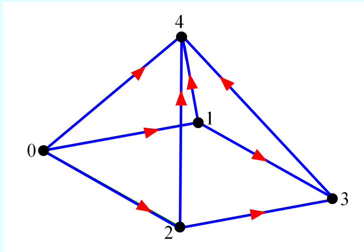
$$B^T B = ([e_{ij}, e_{i'j'}]) = \begin{pmatrix} 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \end{pmatrix}$$

whence

$$\text{matrix of } \Delta_1 = B^T B + C^T C = \begin{pmatrix} \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} \end{pmatrix}.$$

The spectrum of Δ_1 is $\{2_4, 3, 5\}$.

Example. Consider the following pyramid:



Here $|\Omega_0| = 5, \ |\Omega_1| = 8, \ |\Omega_2| = 5,$

and

$\Omega_2 = \langle e_{014}, e_{024}, e_{134}, e_{234}, e_{013} - e_{023} \rangle$.

We have

$$B^T B = ([ij, i'j']) = \begin{array}{cc} & \begin{array}{cccccccc} e_{01} & e_{02} & e_{13} & e_{23} & e_{04} & e_{14} & e_{24} & e_{34} \end{array} \\ \begin{array}{c} e_{01} \\ e_{02} \\ e_{13} \\ e_{23} \\ e_{04} \\ e_{14} \\ e_{24} \\ e_{34} \end{array} & \begin{pmatrix} 2 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & -1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 2 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 2 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

$$C = \begin{array}{cc} & \begin{array}{cccccccc} e_{01} & e_{02} & e_{13} & e_{23} & e_{04} & e_{14} & e_{24} & e_{34} \end{array} \\ \begin{array}{c} e_{014} \\ e_{024} \\ e_{134} \\ e_{234} \\ \frac{1}{\sqrt{2}}(e_{013} - e_{023}) \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$C^TC = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -1 & 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & -1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 & 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$$\text{matrix of } \Delta_1 = B^TB + C^TC = \begin{pmatrix} \frac{7}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{7}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{7}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 \end{pmatrix}$$

The eigenvalues of Δ_1 are $\{3_5,\ 5_3\}$.

Example. Let G be an $(n-1)$-simplex, that is, the vertices are $\{0, 1, ..., n-1\}$ and

$$i \to j \Leftrightarrow i < j.$$

Let us show that

$$A := matrix\ of\ \Delta_1 = \mathrm{diag}\,(n)\,.$$

Let ij and $i'j'$ be two arrows. Then $(ij, i'j')$-entry of A is

$$A_{ij,i'j'} = \left(B^T B\right)_{ij,i'j'} + \left(C^T C\right)_{ij,i'j'} = [ij, i'j'] + \sum_n [ij, \gamma_n]\,[i'j', \gamma_n]\,, \tag{8.9}$$

where $\{\gamma_n\}$ is an orthonormal basis of Ω_2 that in this case consists of all triangles in G.

If $ij = i'j'$ then $[ij, i'j'] = 2$. Since the arrow ij belongs to $(n-2)$ triangles γ_n, we obtain

$$A_{ij,ij} = 2 + (n-2) = n$$

that is, all the diagonal entries of Δ_1 are equal to n. It remains to show that if $ij \neq i'j'$ then

$$A_{ij,i'j'} = 0. \tag{8.10}$$

If ij and $i'j'$ have no common vertex then they cannot belong to the same triangle γ_n and, hence, all the terms in (8.9) vanish.

Let $i = i'$ while $j \neq j'$:

$$\begin{tikzcd} & j \\ \begin{smallmatrix} i \\ i' \end{smallmatrix} \bullet \arrow[ur] \arrow[r] & \bullet\, j' \end{tikzcd}$$

Then $[ij, i'j'] = 1$ while $[ij, \gamma_n]\,[i'j', \gamma_n]$ does not vanish only of γ_n is the triangle formed by i, j, j'. In this case the arrows ij and $i'j'$ have opposite orientations with respect to γ_n, whence $[ij, \gamma_n]\,[i'j', \gamma_n] = -1$ and (8.10).

Let $i = j'$ while $j \neq i'$:

$$\begin{tikzcd} & j \\ \begin{smallmatrix} i \\ j' \end{smallmatrix} \bullet \arrow[ur] & \bullet\, i' \arrow[l] \end{tikzcd}$$

Then $[ij, i'j'] = -1$ while $[ij, \gamma_n]\,[i'j', \gamma_n]$ does not vanish only if γ_n is the triangle $i'ij$, and in this case the arrows ij and $i'j'$ have the same orientation with respect to γ_n, whence $[ij, \gamma_n]\,[i'j', \gamma_n] = 1$ and again (8.10).

The cases $j = i'$ and $j = j'$ are similar.

Problem 8.5 *Describe all digraphs where Δ_1 has only one eigenvalue.*

Problem 8.6 *Devise a program for computing the matrix and spectrum of Δ_1 for large digraphs.*

8.4 Trace of Δ_1

Recall that

$$\operatorname{trace} \Delta_0 = \sum_{i \in V} \deg(i) = 2E.$$

There is a similar result for the trace of Δ_1.

Theorem 8.7 *Let T be the number of triangles in Ω_2, S be the number of linearly independent squares in Ω_2, and D be the number of double arrows $a \rightleftarrows b$. Then*

$$\operatorname{trace} \Delta_1 = 2E + 3T + 2S + 4D. \tag{8.11}$$

By a square here we mean an allowed 2-path $e_{abc} - e_{ab'c}$ such that $a \neq c$ and $a \not\to c$.

For example, for the pyramid on p.157 we have $E = 8$, $T = 4$, $S = 1$ and $D = 0$, whence

$$\operatorname{trace} \Delta_1 = 2 \cdot 8 + 3 \cdot 4 + 2 \cdot 1 = 30,$$

which matches the sum of the eigenvalues as well as the sum of the diagonal values of the matrix of Δ_1 in this example.

Proof. Let $\{\gamma_n\}$ be an orthogonal basis in Ω_2. Let us first prove that

$$\operatorname{trace}\Delta_1 = 2E + \sum_n \frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2}. \tag{8.12}$$

By (8.6), $\operatorname{trace}\Delta_1 = \operatorname{trace}B^T B + \operatorname{trace}C^T C$. As we have seen above (see p.149), all the diagonal entries of $B^T B$ are equal to 2 so that

$$\operatorname{trace}B^T B = 2E.$$

Let us compute $\operatorname{trace}C^T C$. Without loss of generality assume that the basis $\{\gamma_n\}$ is orthonormal basis. Let $\{\alpha_i\}$ be the sequence of all arrows. Since $\{\alpha_i\}$ is an orthonormal basis in Ω_1, we have by (8.5)

$$C = (\langle\partial\gamma_n, \alpha_i\rangle)_{n,i}$$

and, hence,

$$\left(C^T C\right)_{ij} = \sum_n \langle\partial\gamma_n, \alpha_i\rangle \langle\partial\gamma_n, \alpha_j\rangle .$$

It follows that

$$\operatorname{trace}C^T C = \sum_i \sum_n \langle\partial\gamma_n, \alpha_i\rangle^2 = \sum_n \sum_i \langle\partial\gamma_n, \alpha_i\rangle^2 = \sum_n \|\partial\gamma_n\|^2,$$

whence (8.12) follows.

As we know, Ω_2 has a basis $\{\gamma_n\}$ that consists of triangles, squares and double arrows. The only non-orthogonal pairs in this basis can be pairs of squares containing the same elementary 2-path, like $e_{abc} - e_{ab'c}$ and $e_{abc} - e_{ab''c}$. Assume first that the entire basis $\{\gamma_n\}$ is orthogonal.

A double arrow $a \rightleftarrows b$ gives two elements of the basis $\{\gamma_n\}$: e_{aba} and e_{bab}. If $\gamma_n = e_{aba}$ then

$$\|\gamma_n\|^2 = 1, \quad \partial\gamma_n = e_{ba} + e_{ab}, \quad \|\partial\gamma_n\|^2 = 2$$

and

$$\frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2} = 2.$$

The same is true for $\gamma_n = e_{bab}$ so that each double arrow contributes 4 to the sum

$$\sum_n \frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2}. \tag{8.13}$$

If γ_n is a triangle e_{abc} then

$$\|\gamma_n\|^2 = 1, \qquad \partial\gamma_n = e_{bc} - e_{ac} + e_{ab}, \quad \|\partial\gamma_n\|^2 = 3,$$

whence

$$\frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2} = 3,$$

so that each triangle contributes 3 to the sum (8.13).

If γ_n is a square $e_{abc} - e_{ab'c}$ then

$$\|\gamma_n\|^2 = 2, \quad \partial\gamma_n = e_{ab} + e_{bc} - e_{ab'} - e_{b'c}, \quad \|\partial\gamma_n\|^2 = 4,$$

so that

$$\frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2} = 2,$$

so that each square contributes 2 to the sum (8.13). Hence, we obtain that the sum (8.13) is equal to $3T + 2S + 4D$, which proves (8.11) in this case.

In the general case, assume that there is an allowed 2-path e_{abc} that forms m squares:

$$e_{abc} - e_{ab_1c}, \; e_{abc} - e_{ab_2c}, \; \ldots, \; e_{abc} - e_{ab_mc} \; .$$

They are linearly independent but not orthogonal. Orthogonalization gives a sequence

$$\omega_1 = e_{abc} - e_{ab_1c}$$

$$
\begin{aligned}
\omega_2 &= e_{abc} + e_{ab_1c} - 2e_{ab_2c} \\
&\ldots \\
\omega_k &= e_{abc} + \ldots + e_{ab_{k-1}c} - ke_{ab_kc} \\
&\ldots \\
\omega_m &= e_{abc} + \ldots + e_{ab_{n-1}c} - me_{ab_mc}.
\end{aligned}
$$

(see Example on p.71). We have

$$
\begin{aligned}
\partial\omega_k &= (e_{ab} + e_{bc}) + \ldots + \left(e_{ab_{k-1}} + e_{b_{k-1}c}\right) - k\left(e_{ab_k} + e_{b_kc}\right) \\
\|\partial\omega_k\|^2 &= 2k + 2k^2, \quad \|\omega_k\|^2 = k + k^2,
\end{aligned}
$$

whence

$$
\frac{\|\partial\omega_k\|^2}{\|\omega_k\|^2} = 2.
$$

Hence, each ω_k contributes 2 to the sum (8.13), which completes the proof. ■

Since the sum of all eigenvalues is trace Δ_1 and the eigenvalue 0 has the multiplicity β_1, we obtain that the average value of positive eigenvalues is

$$
\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1}.
$$

8.5 An estimate of $\lambda_{\max}(\Delta_1)$

Denote by $\lambda_{\max}(A)$ the maximal eigenvalue of a symmetric operator A. Recall that, by Proposition 8.2,

$$\lambda_{\max}(\Delta_0) \leq 2\max_i \deg(i).$$

For any arrow $i \to j$ in G denote by $\deg_\Delta(ij)$ the number of triangles containing the arrow $i \to j$, and by $\deg_\square(ij)$ the number of squares containing $i \to j$.

Theorem 8.8 *Assume that there is an orthogonal basis $\{\gamma_n\}$ in Ω_2 that consists of triangles and squares. Then*

$$\lambda_{\max}(\Delta_1) \leq 2\max_i \deg(i) + 3\max_{i\to j}\deg_\Delta(ij) + 2\max_{i\to j}\deg_\square(ij). \tag{8.14}$$

Proof. Recall that

$$\lambda_{\max}(\Delta_1) = \sup_{u\in\Omega_1\setminus\{0\}}\left(\frac{\|\partial u\|^2}{\|u\|^2} + \frac{\|\partial^* u\|^2}{\|u\|^2}\right).$$

Since the operators $\partial : \Omega_1 \to \Omega_0$ and $\partial^* : \Omega_0 \to \Omega_1$ are dual, the have the same norm. The norm of the latter was estimated in the proof of Proposition 8.2 (cf. (8.3)), whence we obtain the same estimate for the norm of the former, that is, for any non-zero $u \in \Omega_1$,

$$\frac{\|\partial u\|^2}{\|u\|^2} \le 2 \max_{i \in V} \deg(i).$$

Let us prove that

$$\frac{\|\partial^* u\|^2}{\|u\|^2} \le 3 \max_{i \to j} \deg_{\Delta}(ij) + 2 \max_{i \to j} \deg_{\square}(ij). \tag{8.15}$$

Let $u = \sum_{i \to j} u^{ij} e_{ij}$ and, hence,

$$\|u\|^2 = \sum_{i \to j} \left(u^{ij}\right)^2$$

Using the basis $\{\gamma_n\}$ in Ω_2, we obtain

$$\partial^* u = \sum_n \frac{\langle \partial^* u, \gamma_n \rangle^2}{\|\gamma_n\|^2} = \sum_n \frac{\langle u, \partial \gamma_n \rangle^2}{\|\gamma_n\|^2}.$$

If γ_n is a triangle e_{abc} then $\|\gamma_n\| = 1$,

$$\langle u, \partial \gamma_n \rangle = \langle u, e_{ab} - e_{ac} + e_{ab} \rangle = u^{ab} - u^{ac} + u^{ab},$$

$$\langle u, \partial\gamma_n\rangle^2 \leq 3\left((u^{ab})^2 + (u^{ac})^2 + (u^{ab})^2\right).$$

Summing up over all triangles γ_n and using that any arrow $i \to j$ occurs in $\deg_\Delta(ij)$ triangles, we obtain

$$\sum_{n:\gamma_n \text{ is triangle}} \frac{\langle u, \partial\gamma_n\rangle^2}{\|\gamma_n\|^2} \leq 3\sum_{i\to j}\left(u^{ij}\right)^2 \deg_\Delta(ij) \leq 3\|u\|^2 \max_{i\to j}\deg_\Delta(ij). \tag{8.16}$$

Let now γ_n be a square $e_{abc} - e_{ab'c}$ (such that $a \not\to c$). Then $\|\gamma_n\|^2 = 2$,

$$\langle u, \partial\gamma_n\rangle = \langle u, e_{ab} + e_{bc} - e_{ab'} + e_{b/c}\rangle = u^{ab} + u^{bc} - u^{ab'} - u^{b'c},$$

$$\langle u, \partial\gamma_n\rangle^2 \leq 4\left((u^{ab})^2 + (u^{bc})^2 + (u^{ab'})^2 + (u^{b'c})^2\right).$$

Summing up over all squares γ_n and using that any arrow $i \to j$ occurs in $\deg_\square(ij)$ squares, we obtain

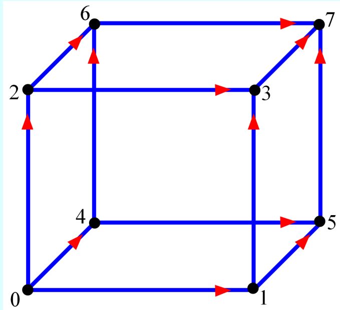
$$\sum_{n:\gamma_n \text{ is square}} \frac{\langle u, \partial\gamma_n\rangle^2}{\|\gamma_n\|^2} \leq 2\sum_{i\to j}\left(u^{ij}\right)^2 \deg_\square(ij) \leq 2\|u\|^2 \max_{i\to j}\deg_\square(ij). \tag{8.17}$$

Adding up (8.16) and (8.17), we obtain (8.15). ∎

Problem 8.9 *How sharp is the upper bound of $\lambda_{\max}(\Delta_1)$ in (8.14)? Is it attained on some digraphs? Extend (8.14) to the general case.*

8.6 Examples of computation of $\operatorname{spec} \Delta_1$

Example. Consider a 3-cube:



Here $V = 8, \quad E = 12, \quad |\Omega_2| = 6,$
$H_p = \{0\}$ for $p \geq 1$.

Space Ω_2 is generated by 6 squares.

Using $S = 6, \ \ T = 0$ we obtain

$$\operatorname{trace} \Delta_1 = 2E + 2S = 2 \cdot 12 + 2 \cdot 6 = 36.$$

Since $\beta_1 = 0$, we obtain

$$\lambda_{average} = \frac{1}{E - \beta_1} \operatorname{trace} \Delta_1 = 3.$$

In fact, the eigenvalues of Δ_1 are

$$\{2_6, 3_2, 4_3, 6\},$$

where the subscript denotes the multiplicity.

Example. Let G be the n-cube, that is, $G = \underbrace{I \Box I \Box ... \Box I}_{n \text{ times}}$ where $I = \{0 \to 1\}$.

Then

$$V = 2^n, \quad E = n2^{n-1}, \quad S = |\Omega_2| = 2^{n-3} n (n-1)$$

and $T = 0$. Hence,

$$\operatorname{trace} \Delta_1 = 2E + 2S = 2^{n-2} n (n+3)$$

and

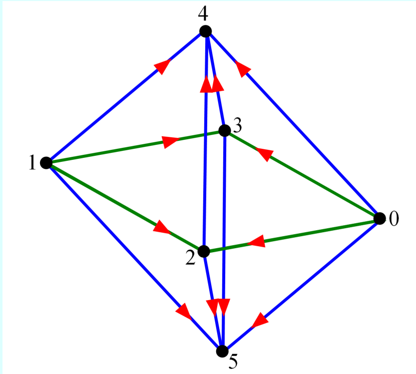
$$\lambda_{average} = \frac{1}{E - \beta_1} \operatorname{trace} \Delta_1 = \frac{2^{n-2} n (n+3)}{n 2^{n-1}} = \frac{n+3}{2}.$$

For example, for a 4-cube we obtain $\operatorname{trace} \Delta_1 = 2^2 \cdot 4 \cdot 7 = 112$. The full spectrum of Δ_1 on a 4-cube is $\{2_{10}, 3_8, 4_9, 6_4, 8\}$.

For a 5-cube we obtain $\operatorname{trace} \Delta_1 = 2^3 \cdot 5 \cdot 8 = 320$. The full spectrum of Δ_1 on a 5-cube is $\{2_{15}, 3_{20}, 4_{25}, 5_4, 6_{10}, 8_5, 10\}$.

Problem 8.10 *Determine the full spectrum of Δ_1 on the n-cube. In particular, prove that $\lambda_{\max} = 2n$ and $\lambda_{\min} = 2_{\frac{n(n+1)}{2}}$. It seems that* $\operatorname{spec} \Delta_1$ *consists of all even integers from 2 to $2n$ and of all odd integers from 3 to n.*
A difficulty is that the method of separation of variables does not work for Δ_1 on Cartesian products.

Example. Consider an octahedron:



We have $V = 6, \quad E = 12, \quad |\Omega_2| = 8.$

The space Ω_2 is generated by 8 triangles:

$\Omega_2 = \langle e_{024}\,, e_{025}\,, e_{034}\,, e_{035}\,, e_{124}\,, e_{125}\,, e_{134}\,, e_{135} \rangle$

Hence, $T = 8, \;\; S = 0$ and we obtain

$$\operatorname{trace} \Delta_1 = 2E + 3T = 2 \cdot 12 + 3 \cdot 8 = 48.$$

Since $\beta_1 = 0$, we obtain

$$\lambda_{average} = \frac{1}{E - \beta_1} \operatorname{trace} \Delta_1 = \frac{48}{12} = 4.$$

The eigenvalues of Δ_1 are

$$\{2_3, 4_6, 6_3\}\,.$$

Consider a family of digraphs: $X_0 = \{0, 1\}$ and

$$X_{n+1} = \operatorname{sus}_2 X_n.$$

For example, X_2 is the above octahedron and X_1 is its middle section (a diamond). The digraph X_n can be regarded as an analogue of *n-sphere*.

Proposition 8.11 *We have for $n \geq 1$*

$$\operatorname{spec} \Delta_1 (X_n) = \left\{ 2(n-1)_{\frac{n(n+1)}{2}}, \ 2n_{n(n+1)}, \ 2(n+1)_{\frac{n(n+1)}{2}} \right\}. \tag{8.18}$$

For example,

$$\operatorname{spec} \Delta_1 (X_1) = \{0, 2_2, 4\}$$

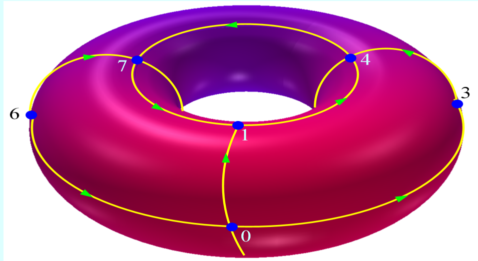
and

$$\operatorname{spec} \Delta_1 (X_2) = \{2_3, 4_6, \ 6_3\}$$

as we have seen above. For $n = 3$ we have

$$\operatorname{spec} \Delta_1 (X_3) = \{4_6, \ 6_{12}, \ 8_6\}.$$

Example. Consider 2-torus $G = T\Box T$ where $T = \{0 \to 1 \to 2 \to 0\}$.



Here $V = 9$, $E = 18$, $|\Omega_2| = 9$, $|H_1| = 2$.
Space Ω_2 is generated by 9 squares, whence

$$\operatorname{trace}\Delta_1 = 2 \cdot 18 + 2 \cdot 9 = 54.$$

In fact, the full spectrum of Δ_1 on 2-torus is

$$\{0_2,\ 1.5_4,\ 3_8,\ 6_4\}.$$

For a 3-torus $G = T^{\Box 3}$ we have $E = 81$, $S = |\Omega_2| = 81$, $|H_1| = 3$.
Hence, $\operatorname{trace}\Delta_1 = 2 \cdot 81 + 2 \cdot 81 = 324$. The full spectrum of Δ_1 on 3-torus is

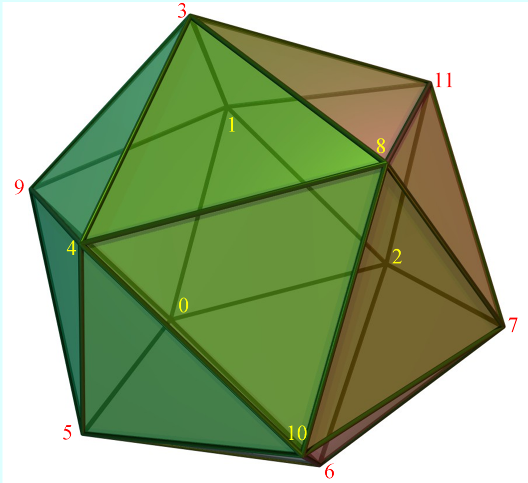
$$\{0_3,\ 1.5_{12},\ 3_{30},\ 4.5_{16},\ 6_{12},\ 9_8\}.$$

For n-torus $G = T^{\Box n}$ we have $E = n3^n$, $S = |\Omega_2| = \frac{n(n-1)}{2}3^n$, $|H_1| = n$, whence

$$\operatorname{trace}\Delta_1 = 2E + 2S = n\,(n+1)\,3^n \quad \text{and} \quad \lambda_{average} = (n+1)\,\frac{3^n}{3^n - 1}.$$

Problem 8.12 *Compute the full spectrum of Δ_1 for n-torus. In particular, prove that $\lambda_{\max} = (3n)_{2^n}$. In fact, $\lambda_{\min} = 0_n$ which is a consequence of $\beta_1 = n$.*

Example. Consider the icosahedron:



Here $V = 12$, $\quad E = 30$, $\quad |\Omega_2| = 25$

Space Ω_2 is generated by 20 triangles and 5 squares (see p.76).

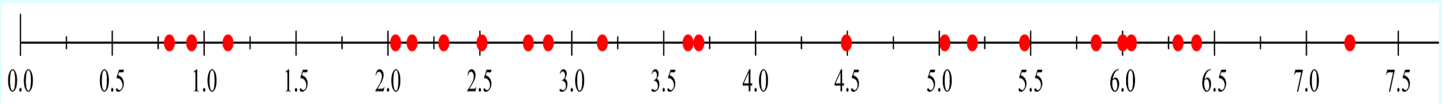
Hence, $T = 20$, $\quad S = 5$ and

$$\operatorname{trace} \Delta_1 = 2 \cdot 30 + 3 \cdot 20 + 2 \cdot 5 = 130.$$

Since $\beta_1 = 0$, we have

$$\lambda_{average} = \frac{1}{E - \beta_1} \operatorname{trace} \Delta_1 = \frac{130}{30} = 4.333...$$

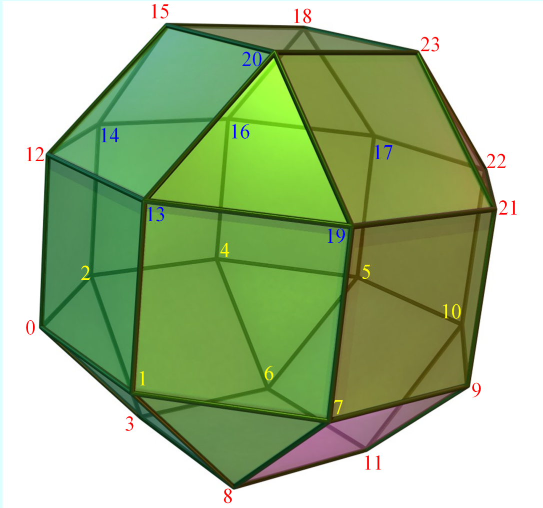
In fact, $\lambda_{\min} = 0.810...$ and $\lambda_{\max} = \left(5 + \sqrt{5}\right)_3$. Other multiple eigenvalues are 6_5 and $\left(5 - \sqrt{5}\right)_3$. The full spectrum of Δ_1 is shown here:



For icosahedron
the matrix of $\Delta_1 =$

4.5	-0.5	1.0	0.0	1.0	0.0	-1.0	0.0	-0.5	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-0.5	5.0	1.0	1.0	-0.5	0.0	0.0	0.0	-0.5	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0
1.0	1.0	4.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0
0.0	1.0	0.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	-0.5	-0.5	1.0	5.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	4.0	1.0	1.0	0.0	-1.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-1.0	0.0	0.0	0.0	0.0	1.0	4.0	0.0	0.0	0.0	0.0	0.0	-1.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	1.0	0.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	1.0	4.5	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0
0.0	-0.5	0.0	0.0	-0.5	-1.0	0.0	0.0	0.0	5.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	-1.0	-0.5	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	-0.5	4.5	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	-0.5	0.0	-0.5	0.0	0.0	0.0
-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	1.0	4.5	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0
0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	4.5	-0.5	0.0	1.0	-1.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0
0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	-0.5	4.5	1.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	-0.5	0.0
0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	1.0	0.0	1.0	4.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	4.0	1.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	-1.0
0.0	0.0	0.0	1.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	1.0	4.5	0.0	0.0	0.0	0.0	1.0	0.0	1.0	0.0	-0.5	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	4.0	1.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.5	0.0	-0.5	0.0	1.0	0.0	1.0	0.0
0.0	-0.5	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	4.5	0.0	-1.0	0.0	-1.0	0.0	0.0
0.0	0.0	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	-0.5	0.0	5.0	0.0	-0.5	0.0	1.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	4.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	-0.5	0.0	4.5	1.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	1.0	4.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	1.0	0.0	1.0	0.0	0.0	0.0	4.5	1.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0

Example. Consider a rhombicuboctahedron (see also p.73):



Here $V = 24$, $E = 48$, $|\Omega_2| = 26$.

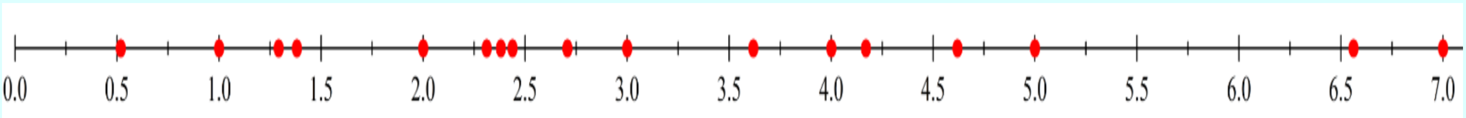
Space Ω_2 is generated by 8 triangles and 18 squares so that $T = 8$ and $S = 18$. Hence,

$$\operatorname{trace} \Delta_1 = 2 \cdot 48 + 3 \cdot 8 + 2 \cdot 18 = 156.$$

Since $\beta_1 = 0$ we have

$$\lambda_{average} = \frac{1}{E - \beta_1} \operatorname{trace} \Delta_1 = \frac{156}{48} = 3.25.$$

We have also $\lambda_{\max} = 7_2$ and $\lambda_{\min} = 0.518...$ There are many multiple eigenvalues: 5_6, 4_4, 3_3, 2_3, 1_3 etc. The spectrum of Δ_1 is here:



For rhombicuboctahedron the matrix of $\Delta_1 =$

$$
\begin{array}{*{48}{r}}
3.0 & 1.0 & 0.5 & -1.0 & -0.5 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
1.0 & 3.5 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.5 & 0.0 & 3.5 & 0.0 & -0.5 & 0.0 & 0.0 & -1.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
-1.0 & 0.0 & 0.0 & 3.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.5 & 0.0 & -0.5 & 0.0 & 3.5 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 3.5 & 0.5 & -0.5 & -1.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.5 & 3.0 & -0.5 & 0.0 & -1.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -1.0 & 0.0 & 0.0 & -0.5 & -0.5 & 3.0 & 0.5 & 0.0 & 0.5 & 0.0 & 1.0 & -0.5 & 0.0 & 0.0 & -0.5 & 0.0 \\
-0.5 & 0.0 & -0.5 & 0.0 & 0.5 & -1.0 & 0.0 & 0.5 & 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 1.0 & 0.0 & -0.5 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 3.5 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & -0.5 & 0.5 & 0.0 & 0.0 & 3.5 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 3.0 & 0.5 & -0.5 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.5 & 3.5 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & -0.5 & 0.0 & -1.0 & -0.5 & -0.5 & 3.0 & 0.0 & 0.0 & 0.5 & 0.0 & 1.0 & 0.5 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 3.5 & 0.5 & -0.5 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 3.0 & -0.5 & -1.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & -0.5 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & -0.5 & -0.5 & 3.0 & 0.0 & 0.5 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 3.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & -0.5 & -0.5 & 0.5 & 0.0 & 3.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
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\end{array}
$$

8.7 Harmonic paths

A path $u \in \Omega_p$ is called *harmonic* if $\Delta_p u = 0$.

Lemma 8.13 *A path $u \in \Omega_p$ is harmonic if and only if $\partial u = 0$ and $\partial^* u = 0$.*

Proof. Indeed, If $\partial u = 0$ and $\partial^* u = 0$ then by (8.1) we have $\Delta_p u = 0$. Conversely, if $\Delta_p u = 0$ then we obtain by (8.2) that

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \langle \Delta_p u, u \rangle = 0,$$

whence $\|\partial u\| = \|\partial^* u\| = 0$. ∎

Denote by $\mathcal{H}_p$ the set of all harmonic paths in Ω_p so that $\mathcal{H}_p$ is a subspace of Ω_p.

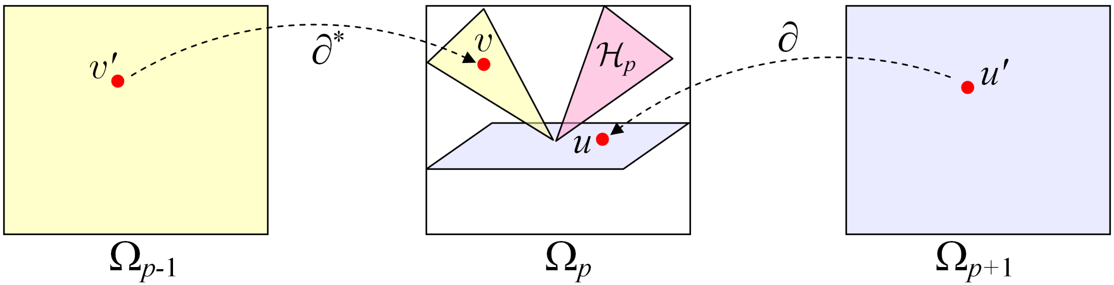
Theorem 8.14 (Hodge decomposition) *The space Ω_p is an orthogonal sum:*

$$\Omega_p = \partial \Omega_{p+1} \bigoplus \partial^* \Omega_{p-1} \bigoplus \mathcal{H}_p. \tag{8.19}$$

Proof. If $u \in \partial \Omega_{p+1}$ and $v \in \partial^* \Omega_{p-1}$ then $u = \partial u'$ and $v = \partial^* v'$, and we have

$$\langle u, v \rangle = \langle \partial u', \partial^* v' \rangle = \left\langle \partial^2 u', v' \right\rangle = 0$$

so that the subspaces $\partial\Omega_{p+1}$ and $\partial^*\Omega_{p-1}$ are orthogonal.



Denote by K the orthogonal complement of $\partial\Omega_{p+1}\bigoplus\partial^*\Omega_{p-1}$ in Ω_p. Then we have

$$w \in K \Leftrightarrow \langle w, u\rangle = 0 \ \ \forall u \in \partial\Omega_{p+1} \quad \text{and} \quad \langle w, v\rangle = 0 \ \forall v \in \partial^*\Omega_{p-1}$$

that is,

$$\begin{aligned}
w \in K \quad &\Leftrightarrow \quad \langle w, \partial u'\rangle = 0 \ \ \forall u' \in \Omega_{p+1} \quad \text{and} \quad \langle w, \partial^* v'\rangle = 0 \ \ \forall v' \in \Omega_{p-1} \\
&\Leftrightarrow \quad \langle \partial^* w, u'\rangle = 0 \ \ \forall u' \in \Omega_{p+1} \quad \text{and} \quad \langle \partial w, v'\rangle = 0 \ \ \forall v' \in \Omega_{p-1} \\
&\Leftrightarrow \quad \partial^* w = 0 \ \ \text{and} \ \ \partial w = 0 \\
&\Leftrightarrow \quad w \in \mathcal{H}_p.
\end{aligned}$$

Hence, $K = \mathcal{H}_p$ which finishes the proof. ■

Corollary 8.15 *There is a natural linear isomorphism*

$$H_p \cong \mathcal{H}_p. \tag{8.20}$$

In particular, $\dim \mathcal{H}_p = \beta_p$*, that is, the multiplicity of* 0 *as an eigenvalue of* Δ_p *is equal to the Betti number* β_p*.*

Proof. Observe that $Z_p := \ker \partial|_{\Omega_p}$ is the orthogonal complement of $\partial^*\Omega_{p-1}$ in Ω_p because, for any $u \in \Omega_p$,

$$u \in Z_p \Leftrightarrow \partial u = 0 \Leftrightarrow \langle \partial u, v \rangle = 0 \ \ \forall v \in \Omega_{p-1} \Leftrightarrow \langle u, \partial^* v \rangle = 0 \ \ \forall v \in \Omega_{p-1} \Leftrightarrow u \perp \partial^*\Omega_{p-1}.$$

Since by (8.19)

$$\Omega_p = \partial\Omega_{p+1} \bigoplus \mathcal{H}_p \bigoplus \partial^*\Omega_{p-1}$$

we obtain

$$Z_p = (\partial^*\Omega_{p-1})^{\perp} = \partial\Omega_{p+1} \bigoplus \mathcal{H}_p \tag{8.21}$$

whence $\mathcal{H}_p \cong Z_p/\partial\Omega_{p+1} = H_p$. ■

Remark. It follows from this argument that $\mathcal{H}_p$ is an orthogonal complement of B_p in Z_p and that any homology class $\omega \in H_p$ has a unique a harmonic representative $u \in \mathcal{H}_p$. In addition, u minimizes the norm $\|\cdot\|$ among all representatives of ω.

9 A fixed point theorem

9.1 Lefschetz number and a fixed point theorem

Everywhere here $\mathbb{K} = \mathbb{R}$ (or $\mathbb{Q}$). Let $f_n : \Omega_n \to \Omega_n$ be a sequence of linear mappings that commutes with ∂, that is,

$$\partial \circ f_{n+1} = f_n \circ \partial \tag{9.1}$$

for any $n \geq 0$. In other words, the following diagram is commutative:

$$\begin{array}{ccccccc}
\cdots \longleftarrow & \Omega_{n-1} & \overset{\partial}{\longleftarrow} & \Omega_n & \overset{\partial}{\longleftarrow} & \Omega_{n+1} & \cdots \longleftarrow \\
 & \downarrow^{f_{n-1}} & & \downarrow^{f_n} & & \downarrow^{f_{n+1}} & \\
\cdots \longleftarrow & \Omega_{n-1} & \overset{\partial}{\longleftarrow} & \Omega_n & \overset{\partial}{\longleftarrow} & \Omega_{n+1} & \cdots \longleftarrow
\end{array} \tag{9.2}$$

Denote

$$Z_n = \ker \partial|_{\Omega_n}, \quad B_n = \operatorname{Im} \partial|_{\Omega_{n+1}}$$

so that

$$H_n = Z_n / B_n.$$

It follows from (9.1) that f_n acts in Z_n, B_n and H_n.

Definition. Denote shortly by f the sequence $\{f_n\}$ of the mappings as above. For any non-negative integer N, define the *Lefschetz number* of f of order N by

$$L^{(N)}(f) = \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{\Omega_n}. \tag{9.3}$$

For example, if each $f_n = \mathrm{id}$ then $L^{(N)}(f) = \sum_{n=0}^{N} (-1)^n \dim \Omega_n = \chi^{(N)}$.

Lemma 9.1 *The following identity holds:*

$$L^{(N)}(f) = \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{H_n} + (-1)^N \operatorname{trace} f_N|_{B_N} \ . \tag{9.4}$$

Proof. Using the following identity (that will be proved later on)

$$\boxed{\operatorname{trace} f_n|_{H_n} = \operatorname{trace} f_n|_{\Omega_n} - \operatorname{trace} f_{n-1}|_{B_{n-1}} - \operatorname{trace} f_n|_{B_n}} \tag{9.5}$$

we obtain

$$
\begin{aligned}
&\sum_{n=0}^{N}(-1)^n \operatorname{trace} f_n|_{H_n} \\
&= \sum_{n=0}^{N}(-1)^n \operatorname{trace} f_n|_{\Omega_n} - \sum_{n=1}^{N}(-1)^n \operatorname{trace} f_{n-1}|_{B_{n-1}} - \sum_{n=0}^{N}(-1)^n \operatorname{trace} f_n|_{B_n} \\
&= \sum_{n=0}^{N}(-1)^n \operatorname{trace} f_n|_{\Omega_n} + \sum_{k=0}^{N-1}(-1)^k \operatorname{trace} f_k|_{B_k} - \sum_{n=0}^{N}(-1)^n \operatorname{trace} f_n|_{B_n} \\
&= \sum_{n=0}^{N}(-1)^n \operatorname{trace} f_n|_{\Omega_n} - (-1)^N \operatorname{trace} f_N|_{B_N} \\
&= L^{(N)}(f) - (-1)^N \operatorname{trace} f_N|_{B_N},
\end{aligned}
$$

whence (9.3) follows. ∎

Let now $f: G \to G$ be a digraph map, that is, $\boxed{i \to j \Rightarrow f(i) \to f(j) \text{ or } f(i) = f(j)}$.

Extend f to a mapping $\Lambda_n \to \Lambda_n$ as follows: first set

$$f(e_{i_0 \dots i_n}) = e_{f(i_0)\dots f(i_n)},$$

and then extend f by linearity to all of Λ_n. If $e_{i_0\dots i_n}$ is non-regular then $f(e_{i_0\dots i_n})$ is also non-regular. Hence, f maps the space $\mathcal{R}_n$ of regular paths into itself.

Next, f maps the space $\mathcal{A}_n$ of allowed paths into itself: if $e_{i_0...i_n}$ is allowed then $i_k \to i_{k+1}$, which implies that either $f(i_k) \to f(i_{k+1})$ for all k and, hence, $f(e_{i_0...i_n})$ is also allowed, or $f(i_k) = f(i_{k+1})$ for some k so that $f(e_{i_0...i_n})$ is non-regular and, hence, $f(e_{i_0...i_n}) = 0$.

Clearly, f commutes with ∂, which implies that f maps also Ω_n into itself. Hence, we obtain the diagram (9.2) where all $f_n = f$. In particular, $L^{(N)}(f)$ is defined.

Theorem 9.2 *Let $f : G \to G$ be a digraph map. If, for some $N \geq 0$, we have $L^{(N)}(f) \neq 0$ then f has a fixed point, that is, a vertex a of G such that $f(a) = a$.*

Definition. Let a, b be two vertices of G. A p-path $v = \sum_{i_0,...,i_p \in V} v^{i_0...i_p} e_{i_0...i_p}$ is called an (a,b)-*cluster* if, for any p-path $i_0...i_p$ with $v^{i_0...i_p} \neq 0$, we have $i_0 = a$ and $i_p = b$. A path v is called a *cluster* if it is a (a,b)-cluster for some a, b.

For example, $e_{abc} - e_{ab'c}$ is an (a,c)-cluster whereas $e_{abc} + e_{acb}$ is not a cluster.

Lemma 9.3 *In each Ω_n there is an orthogonal basis (with respect to the natural inner product) that consists of clusters.*

Proof of Theorem 9.2. Assume that f has no fixed point. We will prove that

$$\operatorname{trace} f|_{\Omega_n} = 0 \quad \text{for any } n \geq 0, \tag{9.6}$$

which gives by (9.3) that $L^{(N)}(f)=0$ thus contradicting the hypothesis that $L^{(N)}(f)\neq 0$.

By Lemma 9.3, there is an orthogonal basis $u_1,...,u_m$ in Ω_n, where all u_k are clusters. Denote by (c_{ij}) the matrix of operator $f:\Omega_n\to\Omega_n$ in this basis, that is,

$$f(u_j)=\sum_{i=1}^{m}c_{ij}u_i,\ \text{whence}\quad c_{ij}=\frac{(f(u_j),u_i)}{\|u_i\|^2}.$$

Consequently, we have

$$\operatorname{trace} f|_{\Omega_n}=\sum_{k=1}^{m}c_{kk}=\sum_{k=1}^{m}\frac{(f(u_k),u_k)}{\|u_k\|^2}.$$

It remains to show that $f(u_k)\perp u_k$, which will imply (9.6). Indeed, let u_k be an (a,b)-cluster, that is, u_k is a linear combination of elementary n-paths of the form

$$e_{ai_1...i_{n-1}b}, \tag{9.7}$$

where a,b are fixed while $i_1,...,i_{n-1}$ are variable. Then $f(u_k)$ is a linear combination of the n-paths

$$e_{f(a)f(j_1)...f(j_{n-1})f(b)}, \tag{9.8}$$

where $j_1,...,j_{n-1}$ are variable. Since $a\neq f(a)$, we see that the paths (9.7) and (9.8) are orthogonal, which implies that $f(u_k)$ and u_k are orthogonal, too, which was to be proved.

■

9.2 A fixed point theorem in terms of homology

Definition. Define the *path dimension* of a digraph G by $\dim_p G = \sup\{n : |\Omega_n| > 0\}$.

Assume that $\dim_p G < \infty$. Then for any $N > \dim_p G$ we have by (9.4)

$$L^{(N)}(f) = \sum_{n=0}^{N} (-1)^n \operatorname{trace} f|_{\Omega_n} = \sum_{n=0}^{N} (-1)^n \operatorname{trace} f|_{H_n}. \tag{9.9}$$

Recall the definition of the homological dimension: $\dim_h G = \sup\{n : |H_n| > 0\}$.

Theorem 9.4 *Let G be a connected digraph. Let $\dim_p G < \infty$ and $\dim_h G = 0$. Then any digraph map $f : G \to G$ has a fixed point.*

Proof. The condition $\dim_h G = 0$ means that $H_n = \{0\}$ for all $n \geq 1$, and the connectedness means that $|H_0| = 1$. The space H_0 is spanned by a single homology class $[e_a]$ where a is one of the vertices. Then $f(e_a) = e_{f(a)} \sim e_a$ so that $f([e_a]) = [e_a]$. It follows that $\operatorname{trace} f|_{H_0} = 1$ while $\operatorname{trace} f|_{H_n} = 0$ for all $n \geq 1$. ...... By (9.9) we obtain $L^{(N)}(f) = 1 \neq 0$, and by Theorem 9.2 we conclude that f has a fixed point. ■

The condition that a mapping $f : G \to G$ is a digraph map can be reformulated as follows. Define a *directed distance* between vertices a, b of G by

$$\overrightarrow{d}(a,b) = \inf\{n : \exists \text{ a path } \underbrace{a \to i_1 \to \ldots \to i_{n-1} \to b}_{n \text{ arrows}}\}.$$

Then f is a digraph map if and only if

$$\overrightarrow{d}(f(a), f(b)) \leq \overrightarrow{d}(a,b) \quad \text{for all } a, b \in V.$$

Let us relax this condition.

Problem 9.5 *Devise a fixed point theorem for maps $f : G \to G$ with*

$$\overrightarrow{d}(f(a), f(b)) \leq C\overrightarrow{d}(a,b) \quad \textit{for all } a, b \in V,$$

where $C > 1$ is a constant.

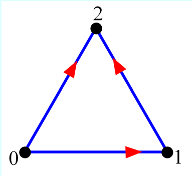
Alternatively, one can strengthen conditions on f, assuming that f is a *digraph isomorphism*, which is equivalent to

$$\overrightarrow{d}(f(a), f(b)) = \overrightarrow{d}(a,b) \quad \text{for all } a, b \in V.$$

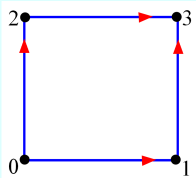
Problem 9.6 *Devise a fixed point theorem for a digraph isomorphism $f : G \to G$.*

9.3 Examples

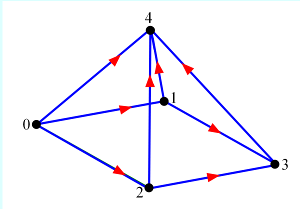
Example. Here are some examples of digraphs satisfying the hypotheses of Theorem 9.4.



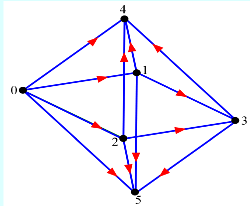
triangle



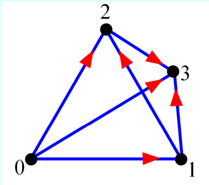
square



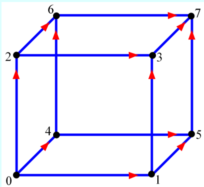
pyramid



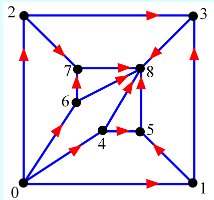
octahedron based on square



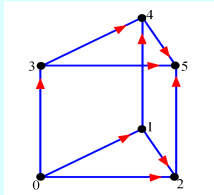
3-*simplex*



3-*cube*



broken cube



prizm

In all these examples the vertices admit a *monotone* numbering: arrows go in direction of increase of numbers. In this case all allowed paths have bounded length and, hence, $\dim_p G < \infty$.

The triviality of H_* (that is, $\dim_h G = 0$) for each of these digraphs was mentioned in the previous sections.

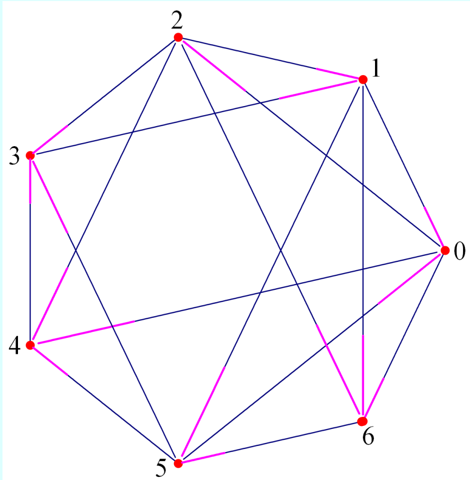
Example. Consider a digraph G with 7 vertices and 16 arrows.

There are arbitrarily long allowed paths because there are loops:

$0 \to 2 \to 1 \to 0$, $5 \to 0 \to 6 \to 5$ etc.

Nevertheless, $\dim_p G < 6$,

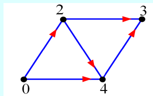
and all homology groups are trivial.



Hence, G satisfies the hypotheses of Theorem 9.4 and we conclude that any digraph map $f : G \to G$ has a fixed point.

Let us show why $\Omega_6 = \{0\}$, which will imply by Proposition 6.8 that $\Omega_p = \{0\}$ $\forall p \geq 6$. For that we first obtain by computation

$$\Omega_3 = \langle e_{0243}\,,\ e_{2165}\,,\ e_{1504} \rangle\,.$$



Hence, out of all allowed elementary 3-forms $e^{i_0 i_1 i_2 i_3}$, only the following are non-zero as elements of Ω^3:

$$e^{0243},\ e^{2165},\ e^{1504} \tag{9.10}$$

(in fact, (9.10) is a basis in Ω^3). It is easy to observe that for any pair of 3-forms φ, ψ from (9.10) the concatenation $\varphi\psi$ vanishes. It follows that

$$\varphi\psi \simeq 0 \ \text{ for all allowed elementary 3-forms } \varphi, \psi, \tag{9.11}$$

because if one of φ, ψ is not from the list (9.10), say, φ, then $\varphi \simeq 0$ whence $\varphi\psi \simeq 0$ by Lemma 6.7.

Any allowed 6-form $e^{i_0 \dots i_6}$ is a concatenation of two allowed 3-forms

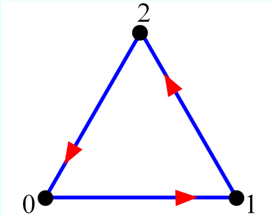
$$e^{i_0 \dots i_6} = e^{i_0 i_1 i_2 i_3} e^{i_3 i_4 i_5 i_6},$$

whence by (9.11) $e^{i_0 \dots i_6} \simeq 0$ and, hence, $\Omega^6 = \{0\}$.

Example. Assume that G contains a double arrow $\{a \rightleftarrows b\}$. Then $\dim_p G = \infty$ since each Ω_p contains p-paths $e_{ababab\dots}$ and $e_{bababa\dots}$. Define a map $f : G \to G$ by $f(a) = b$ and $f(x) = a$ for $x \neq a$. Clearly, f is a digraph map without fixed points. Hence, the hypotheses $\dim_p G < \infty$ is essential for Theorem 9.4.

Example. Here are some examples of digraphs that admit digraph maps f without fixed points. All they have $\dim_p G < \infty$ but $\dim_h G > 0$.

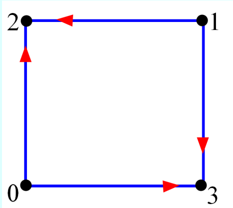
1-*torus*



$|H_1| = 1$

f =rotation

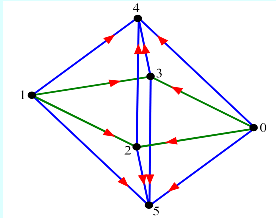
diamond



$|H_1| = 1$

f =central symmetry

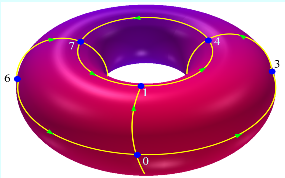
octahedron based on diamond



$|H_2| = 1$

f =central symmetry

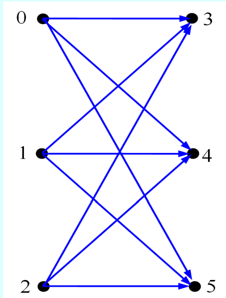
2-torus:
$|H_1| = 2$
$|H_2| = 1$



f = rotation

$\overrightarrow{K}_{3,3}$:
$|H_1| = 2$

$f : 0 \mapsto 1 \mapsto 2 \mapsto 0,$
$3 \mapsto 4 \mapsto 5 \mapsto 3$



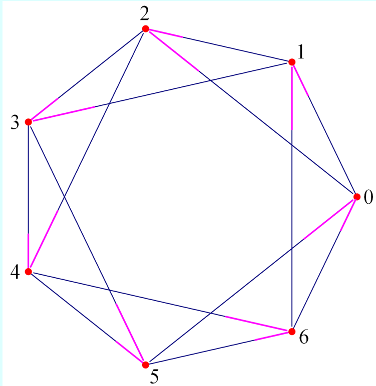
Problem 9.7 *Suppose that* $H_1(G)$ *contains a non-trivial class* $e_{01} + e_{12} + e_{20}$ *(like for 1-torus). Is it true that there exists a digraph map* $f : G \to G$ *without a fixed point?*

Example. Consider the following digraph G with 7 vertices and 14 arrows:

The arrows on G are as follows:

$$i \to i+1 \text{ and } i \to i+2$$

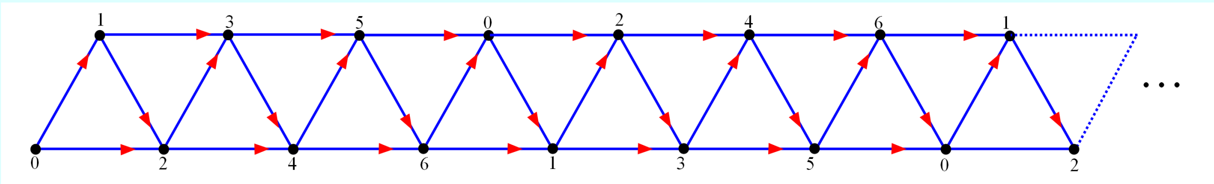
where addition is considered $\bmod 7$.



For this digraph $|\Omega_p| = 14$ for **all** $p \geq 1$ so that $\dim_p G = \infty$, while $\dim_h G = 0$.

The digraph G does not satisfy the hypotheses of Theorem 9.4. In fact, the digraph map $f(i) = i+1$ has no fixed point.

Let us explain why $|\Omega_p| = 14$. This digraph can also be shown as a *periodic snake*:



where the vertices with the same numbers are merged (like a Möbius band).

Each elementary p-path

$$e_{i(i+1)(i+2)\dots(i+p)} \tag{9.12}$$

is snake-like and, hence, is ∂-invariant. Let us refer to any path (9.12) as a p-snake. Hence, we obtain in Ω_p already 7 linearly independent p-snakes. Another group of 7 linearly independent p-paths in Ω_p is given by the boundaries of $(p+1)$-snakes:

$$\partial e_{i(i+1)(i+2)\dots(i+p)(i+p+1)},$$

which makes $\dim \Omega_p = 14$. Since $\partial^2 = 0$, while the boundaries of p-snakes (9.12) are linearly independent for $p \geq 2$, we obtain that $\dim \ker \partial|_{\Omega_p} = 7$. By the rank-nullity theorem $\dim \operatorname{Im} \partial|_{\Omega_{p+1}} = 14 - 7 = 7$, whence $H_p = \{0\}$ for all $p \geq 2$.

For the case $p = 1$ we have

$$H_1 = \langle e_{01} + e_{12} + e_{23} + e_{34} + e_{45} + e_{56} + e_{60} \rangle .$$

It is curious that this digraph is strongly regular and its curvature is $K_x^{(N)} = (-1)^N$.

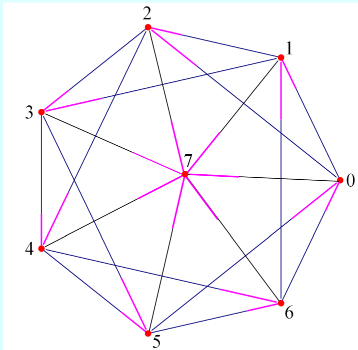
Problem 9.8 *Describe classes of strongly regular digraphs with* $\dim_p G = \infty$ *having a non-trivial periodic sequence* $\left\{K^{(N)}\right\}_{N=1}^{\infty}$.

Problem 9.9 *Devise a fixed point theorem that would work with digraphs containing double arrows. For that we need to impose additional restriction on* $f : G \to G$, *for example, let us assume that* f *is a digraph isomorphism, that is,* $i \to j \Rightarrow f(i) \to f(j)$.

Problem 9.10 *Assume that* G *is connected,* $\dim_h G = 0$ *and that* G ***has no double arrow****. Prove or disprove the claim that any digraph map* $f : G \to G$ *has a fixed point. Of course, the main interest here lies in the case when* $\dim_p G = \infty$.

Example. Here is a candidate for a positive example with $\dim_p G = \infty$.

This is the above snake with an additional vertex 7 such that $i \to 7$ for all $i \in \{0, ..., 6\}$.

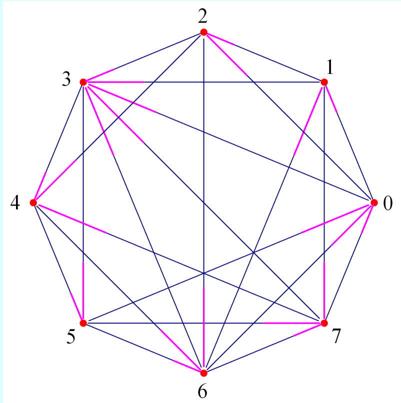


For this digraph we have

$\dim_h G = 0$ and $\dim_p G = \infty$.

Problem: prove that any digraph map $f : G \to G$ for this digraph has a fixed point.

Example. Here is a candidate for a counterexample.



For this digraph again

$\dim_h G = 0$ and $\dim_p G = \infty$,

where the latter is the case because G contains a periodic snake

$$e_{\underline{0123456}0123456\ldots}$$

Problem: construct for this digraph a digraph map f without fixed points (or prove a fixed point theorem for this digraph). Simple rotations $f(i) = i + a \bmod 8$ are not digraph maps here. For example, for $f(i) = i + 4$ the arrow $0 \to 3$ goes to $4 \not\to 7$, for $f(i) = i + 5$ the arrow $5 \to 0$ goes to $2 \not\to 5$.

Problem 9.11 *Create efficient computations tools for computing the spaces* Ω_p *or at least for computing* $\dim \Omega_p$.

Problem 9.12 *Devise convenient sufficient conditions for* $\dim_p G < \infty$.

Example. For all $1 \le k \le n$, the *Johnson digraph* $\overrightarrow{J}(n,k)$ is defined as follows. The vertices of $\overrightarrow{J}(n,k)$ are all k-element subsets of $S_n = \{1, 2, ..., n\}$. To define the arrows, for any subset $a \subset S_n$ denote $\operatorname{sum}(a) = \sum_{i \in a} i$. Then, for two k-element subsets $a, b \subset S_n$,

$$\boxed{a \to b \text{ in } \overrightarrow{J}(n,k) \Leftrightarrow \ a \cap b \text{ contains exactly } k-1 \text{ elements and } \operatorname{sum}(a) > \operatorname{sum}(b)}.$$

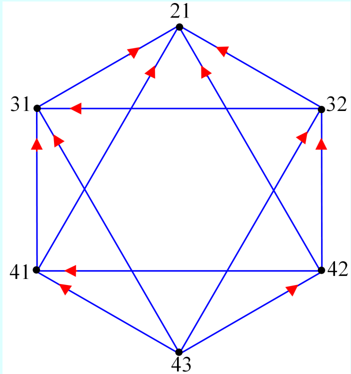
For example, here is $\overrightarrow{J}(4,2)$:

The vertices of $\overrightarrow{J}(4,2)$ are the pairs

$$43,\ 42,\ 41,\ 32,\ 31,\ 21,$$

and there are 12 arrows.

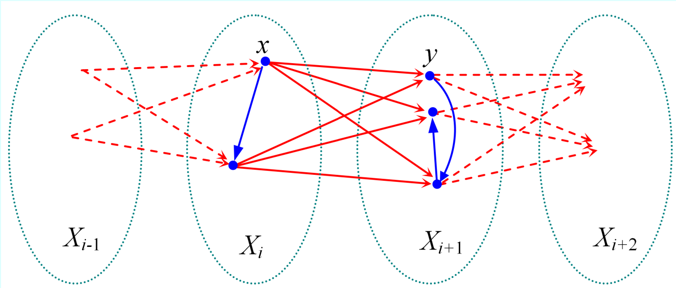
In fact, this is yet another octahedron.



Theorem 9.13 *All digraphs* $\overrightarrow{J}(n,k)$ *are homologically trivial.*

The length of allowed paths in $\overrightarrow{J}(n,k)$ is bounded because $\operatorname{sum}(a)$ decreases along arrows. Hence, $\dim_p \overrightarrow{J}(n,k) < \infty$. Consequently, $\overrightarrow{J}(n,k)$ satisfies the hypotheses of Theorem 9.4 and, hence, any digraph map f in $\overrightarrow{J}(n,k)$ has a fixed point.

Example. Given n digraphs $X_1, ..., X_n$, define *their monotone linear join* $X_1X_2...X_n$ as follows: take first a disjoint union $\bigsqcup_{i=1}^{n} X_i$ and then add arrows from any vertex x of X_i to any vertex y of X_{i+1}.



Theorem 9.14 *Assume that the following two conditions are satisfied:*

(i) $\forall i \quad \dim_p X_i < \infty$

(ii) $\exists i$ *such that X_i is connected and* $\dim_h X_i = 0$.

Then any digraph map f in $X_1...X_n$ has a fixed point.

The proof uses an analogue of Künneth formula for $X = X_1...X_n$ that insures that X is homologically trivial (see Theorem 10.3 below). Then we can apply Theorem 9.4.

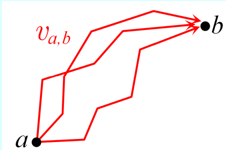
9.4 A cluster basis in Ω_p

We prove below Lemma 9.3. Recall that a p-path $v = \sum v^{i_0...i_p} e_{i_0...i_p}$ is called an (a,b)-*cluster* if, for any p-path $i_0...i_p$ with $v^{i_0...i_p} \neq 0$, we have $i_0 = a$ and $i_p = b$. A p-path v is called a *cluster* if it is a (a,b)-cluster for some a, b.

Lemma 9.15 *Any ∂-invariant p-path is a sum of ∂-invariant clusters.*

Proof. Let $v \in \Omega_p$. For any points $a, b \in V$, denote by $v_{a,b}$ the sum of all terms $v^{i_0...i_p} e_{i_0...i_p}$ with $i_0 = a$ and $i_p = b$.

Then $v_{a,b}$ is a cluster and $v = \sum_{a,b \in V} v_{a,b}$, that is, v is a sum of clusters. Let us prove that each non-zero cluster $v_{a,b}$ is ∂-invariant.



Since v is allowed, also all non-zero terms $v^{i_0...i_p} e_{i_0...i_p}$ are allowed, whence $v_{a,b}$ is also allowed. Let us prove that $\partial v_{a.b}$ is allowed, which will yield the ∂-invariance of $v_{a.b}$. The path $v_{a,b}$ is a linear combination of allowed paths of the form $e_{ai_1...i_{p-1}b}$. We have

$$\partial e_{ai_1...i_{p-1}b} = e_{i_1...i_{p-1}b} + (-1)^p e_{ai_1...i_{p-1}} + \sum_{k=1}^{p-1} (-1)^k e_{ai_1..\widehat{i_k}...i_{p-1}b}.$$

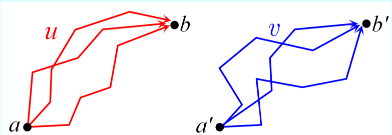
The terms $e_{i_1\ldots i_{p-1}b}$ and $e_{ai_1\ldots i_{p-1}}$ are clearly allowed, while among the terms $e_{ai_1..\widehat{i_k}\ldots i_{p-1}b}$ there may be non-allowed. In the full expansion of

$$\partial v = \sum_{a,b\in V} \partial v_{a,b}$$

all non-allowed terms must cancel out. Since all the terms $e_{ai_1..\widehat{i_k}\ldots i_{p-1}b}$ form a (a,b)-cluster, they cannot cancel with terms containing different values of a or b. Therefore, they have to cancel already within $\partial v_{a,b}$, which implies that $\partial v_{a,b}$ is allowed. ■

Proof of Lemma 9.3. Let us prove that Ω_p has an orthogonal basis that consists of clusters. Let $\mathcal{C}$ be the set of all ∂-invariant clusters in Ω_p. By Lemma 9.15, Ω_p is spanned by $\mathcal{C}$. Choosing in $\mathcal{C}$ a maximal linearly independent subset, we obtain a basis $\mathcal{B}$ in Ω_p that consists of clusters. Let us show how to make an orthogonal basis of clusters. Let u,v be two elements from $\mathcal{B}$, and

let u be a (a,b)-cluster and v be an (a',b')-cluster. If $(a,b)\neq(a',b')$ then we have clearly $u\perp v$.



If $\mathcal{B}$ has more than one (a,b)-cluster, then among all (a,b)-clusters in $\mathcal{B}$, we run a Gram-Schmidt orthogonalization process and obtain an orthogonal set of (a,b)-clusters in $\mathcal{B}$. Note that during this process all newly arising elements are again (a,b)-clusters. Doing that for all pairs (a,b), we obtain an orthogonal basis in Ω_p that consists of clusters. ■

9.5 Rank-nullity formulas for trace

The purpose of this section is to prove the identity (9.5) – see Lemma 9.18 below. Recall that we have a commutative diagram

$$\begin{array}{ccccc}
\Omega_{n-1} & \xleftarrow{\partial} & \Omega_n & \xleftarrow{\partial} & \Omega_{n+1} \\
\downarrow^{f_{n-1}} & & \downarrow^{f_n} & & \downarrow^{f_{n+1}} \\
\Omega_{n-1} & \xleftarrow{\partial} & \Omega_n & \xleftarrow{\partial} & \Omega_{n+1}
\end{array}$$

and $Z_n = \ker \partial|_{\Omega_n}$, $B_n = \operatorname{Im} \partial|_{\Omega_{n+1}}$, $H_n = Z_n / B_n$.

Lemma 9.16 *We have*

$$\operatorname{trace} f_n|_{H_n} = \operatorname{trace} f_n|_{Z_n} - \operatorname{trace} f_n|_{B_n} \quad . \tag{9.13}$$

Proof. Let $u_1, ..., u_l$ be a basis in B_n. Choose in Z_n elements $v_1, ..., v_k$ so that the sequence $u_1, ..., u_l, v_1, ..., v_k$ is a basis in Z_n. Then

$$f_n(u_i) = \sum_{j=1}^{l} a_{ij} u_j$$

and

$$f_n(v_i) = \sum_{j=1}^{k} b_{ij} v_j + \text{terms with } u_j.$$

For the homology classes we have

$$f_n([v_i]) = \sum_{j=1}^{k} b_{ij} [v_j].$$

It follows that

$$\operatorname{trace} f_n|_{Z_n} = \sum_{i=1}^{l} a_{ii} + \sum_{i=1}^{k} b_{ii} = \operatorname{trace} f_k|_{B_n} + \operatorname{trace} f_n|_{H_n},$$

which is equivalent to (9.13). ■

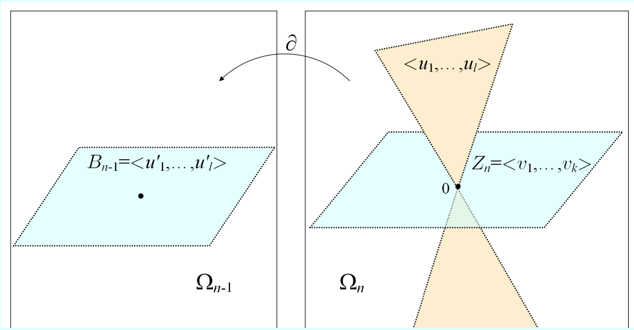
Lemma 9.17 *We have the identity*

$$\operatorname{trace} f_n|_{Z_n} + \operatorname{trace} f_{n-1}|_{B_{n-1}} = \operatorname{trace} f_n|_{\Omega_n}$$

For example, if f_n and f_{n-1} are the identity operators then this becomes the rank-nullity theorem for the operator ∂:

$$\dim Z_n + \dim B_{n-1} = \dim \Omega_n. \tag{9.14}$$

Proof. Let $v_1, ... v_k$ be a basis in Z_n and $u'_1, ..., u'_l$ be a basis in B_{n-1}. Choose any vector $u_i \in \partial^{-1}(u'_i)$, that is, $\partial u_i = u'_i$. Let us show that the sequence $v_1, ..., v_k, u_1, ..., u_l$ is linearly independent in Ω_n.



Indeed, if there is a vanishing linear combination

$$\sum_{i=1}^{l} \alpha_i u_i + \sum_{j=1}^{k} \beta_j v_j = 0,$$

then it follows that

$$0 = \partial \sum_{i=1}^{l} \alpha_i u_i + \partial \sum_{j=1}^{k} \beta_j v_j = \sum_{i=1}^{l} \alpha_i u_i' + 0,$$

whence it follows that all $\alpha_i = 0$. Consequently, $\sum_{j=1}^{k} \beta_j v_j = 0$ and, hence, also all $\beta_j = 0$.

Since by (9.14) $k + l = \dim \Omega_n$, it follows that the sequence $v_1, ..., v_k, u_1, ..., u_l$ is a basis in Ω_n.

Hence, for some coefficients a_{ij} and b_{ij},

$$f_n(u_i) = \sum_{j=1}^{l} a_{ij} u_j + \text{terms with } v_j \tag{9.15}$$

and

$$f_n(v_i) = \sum_{j=1}^{k} b_{ij} v_j.$$

The latter expansion contains no u_j because $f_n(Z_n) \subset Z_n$. Hence,

$$\operatorname{trace} f_n|_{\Omega_n} = \sum_{i=1}^{l} a_{ii} + \sum_{i=1}^{k} b_{ii}.$$

On the other hand, we have

$$\operatorname{trace} f_n|_{Z_n} = \sum_{i=1}^{k} b_{ii}.$$

It remains to prove that

$$\operatorname{trace} f_{n-1}|_{B_{n-1}} = \sum_{i=1}^{l} a_{ii}.$$

Since f_{n-1} maps B_{n-1} into itself, there are coefficients a'_{ij} such that

$$f_{n-1}\left(u'_i\right) = \sum_{j=1}^{l} a'_{ij} u'_j. \tag{9.16}$$

It follows from (9.15) that

$$\partial f_n\left(u_i\right) = \sum_{j=1}^{l} a_{ij} \partial u_j + 0 = \sum_{j=1}^{l} a_{ij} u'_j. \tag{9.17}$$

On the other hand, using (9.1) and (9.16), we obtain that

$$\partial f_n\left(u_i\right) = f_{n-1}\left(\partial u_i\right) = f_{n-1}\left(u'_i\right) = \sum_{j=1}^{l} a'_{ij} u'_j.$$

Comparison with (9.17) shows that $a'_{ij} = a_{ij}$ and, hence,

$$\operatorname{trace} f_{n-1}|_{B_{n-1}} = \sum_{i=1}^{l} a'_{ii} = \sum_{i=1}^{l} a_{ii},$$

which finishes the proof. ■

Finally, we can prove (9.5).

Lemma 9.18 *The following identity holds*

$$\operatorname{trace} f_n|_{H_n} = \operatorname{trace} f_n|_{\Omega_n} - \operatorname{trace} f_{n-1}|_{B_{n-1}} - \operatorname{trace} f_n|_{B_n} \tag{9.18}$$

Proof. By Lemma 9.16 we have

$$\operatorname{trace} f_n|_{H_n} = \operatorname{trace} f_n|_{Z_n} - \operatorname{trace} f_n|_{B_n} \ ,$$

and by Lemma 9.17

$$\operatorname{trace} f_n|_{Z_n} = \operatorname{trace} f_n|_{\Omega_n} - \operatorname{trace} f_{n-1}|_{B_{n-1}},$$

which yields (9.18). ■

10 Reduced homology and join of digraphs

10.1 Augmented chain complex

In this section we use the augmented chain complex

$$\mathbb{K} \stackrel{\partial}{\leftarrow} \Omega_0 \stackrel{\partial}{\leftarrow} \Omega_1 \stackrel{\partial}{\leftarrow} \ldots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \ldots \tag{10.1}$$

where the leftmost ∂ in (10.1) is define by

$$\partial e_i = e = \text{the unity of } \mathbb{K}.$$

The homology groups of (10.1) are called the *reduced* homology groups of G and are denoted by $\widetilde{H}_p(G)$. We have

$$\widetilde{H}_p(G) = H_p(G) \text{ for } p \geq 1 \text{ and } \widetilde{H}_0(G) = H_0(G)/\mathbb{K}.$$

Define the reduced Betti numbers: $\widetilde{\beta}_p(G) = \dim \widetilde{H}_p(G)$. We have

$$\widetilde{\beta}_p(G) = \beta_p(G) \text{ for } p \geq 1 \text{ and } \widetilde{\beta}_0(G) = \beta_0(G) - 1.$$

For a disjoint union $X \sqcup Y$ of two digraphs we have

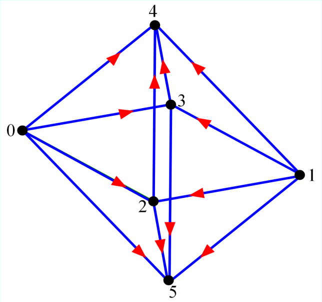
$$\widetilde{\beta}_r(X \sqcup Y) = \widetilde{\beta}_r(X) + \widetilde{\beta}_r(Y) + \mathbf{1}_{\{r=0\}}.$$

10.2 A join of two digraphs

Given two digraphs X, Y, define their *join* $X * Y$ as follows: take first a disjoint union $X \sqcup Y$ and add arrows from any vertex of X to any vertex of Y.

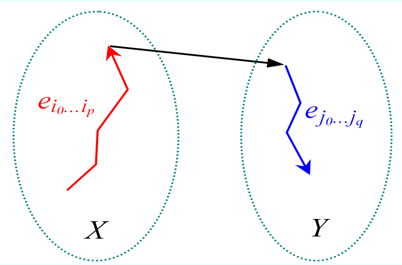
For example,

$$\{0,1\} * \{2,3\} = \begin{array}{ccc} 3 & \leftarrow & 1 \\ \uparrow & & \downarrow \\ 0 & \rightarrow & 2 \end{array} \quad \text{and} \quad \begin{array}{ccc} 3 & \leftarrow & 1 \\ \uparrow & & \downarrow \\ 0 & \rightarrow & 2 \end{array} * \{4,5\} =$$



The *join* uv of p-path u on X and a q-path v on Y is a $(p+q+1)$-path on $X * Y$ that is defined as follows: for elementary paths set

$$e_{i_0 \ldots i_p} e_{j_0 \ldots j_q} = e_{i_0 \ldots i_p j_0 \ldots j_q}$$



and then it extend by linearly to all paths.

If u and v are allowed on X resp. Y then uv is allowed on $Z = X * Y$.

Lemma 10.1 *The join of paths satisfies the product rule for all $p, q \geq -1$:*

$$\partial(uv) = (\partial u)\, v + (-1)^{p+1}\, u \partial v.$$

If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then ∂u and ∂v are allowed, which implies that $\partial(uv)$ is also allowed, that is, $uv \in \Omega_{p+q+1}(Z)$. The product rule implies also that the join uv is well defined for homology classes $u \in \widetilde{H}_p(X)$ and $v \in \widetilde{H}_q(Y)$ so that $uv \in \widetilde{H}_{p+q+1}(Z)$.

Theorem 10.2 (Künneth formula) *We have the following isomorphism: for any $r \geq -1$,*

$$\Omega_r(X * Y) \cong \bigoplus_{\{p,q \geq -1: p+q=r-1\}} \left(\Omega_p(X) \otimes \Omega_q(Y)\right) \tag{10.2}$$

that is given by the map $u \otimes v \mapsto uv$ with $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$, and, for any $r \geq 0$,

$$\widetilde{H}_r(X * Y) \cong \bigoplus_{\{p,q \geq 0: p+q=r-1\}} \widetilde{H}_p(X) \otimes \widetilde{H}_q(Y) \tag{10.3}$$

$$\widetilde{\beta}_r(X * Y) \cong \sum_{\{p,q \geq 0: p+q=r-1\}} \widetilde{\beta}_p(X)\, \widetilde{\beta}_q(Y). \tag{10.4}$$

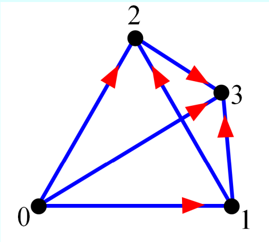
The identity (10.2) means that any paths in $\Omega_r(Z)$ can be obtained as linear combination of joins uv where $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ with $p+q+1=r$, and (10.3) means the same for homology classes.

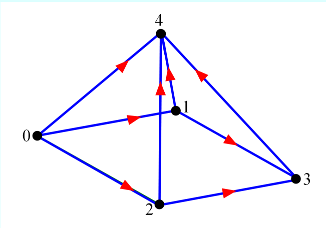
Example. Let Y consist of a single vertex.

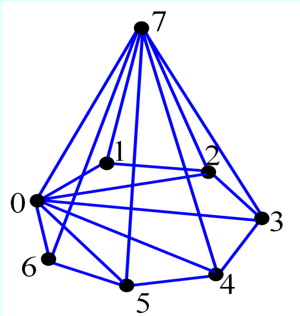
In this case the join $X * Y$ is called a *cone* over X.

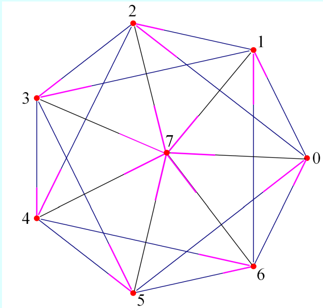
Since all homology groups $\widetilde{H}_*(Y)$ are trivial, the cone $X*Y$ is also homologically trivial.

For example, the following digraphs are cones and, hence, they are homologically trivial.





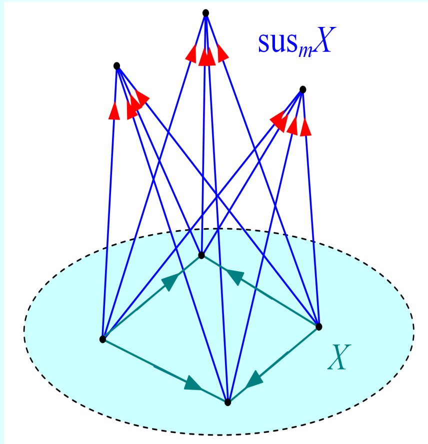




Example. Let Y consist of m vertices without arrows.

Then $X * Y$ coincides with the m-suspension $\operatorname{sus}_m X$.

Here is an example of $\operatorname{sus}_3 X$:



Since $\widetilde{\beta}_0(Y) = m - 1$ and $\widetilde{\beta}_p(Y) = 0$ for $p \geq 1$, we obtain that

$$\widetilde{\beta}_r(\operatorname{sus}_m X) = (m-1)\,\widetilde{\beta}_{r-1}(X)\,.$$

For example, on this picture $X = \operatorname{sus}_2\{\cdot,\cdot\}$, whence $\widetilde{\beta}_1(X) = 1$ and $\widetilde{\beta}_p(X) = 0$ for $p \neq 1$.

For $G = \operatorname{sus}_3 X$: $\quad \widetilde{\beta}_2(G) = 2$ and $\widetilde{\beta}_r(G) = 0$ for $r \neq 2$.

The operation $*$ of digraphs is associative. For a sequence $X_1, ..., X_l$ of l digraphs we obtain by induction from (10.2), (10.3) and (10.4) that

$$\Omega_r(X_1 * X_2 * ... * X_l) \cong \bigoplus_{\{p_i \geq -1:\ p_1+p_2+...+p_l = r-l+1\}} \Omega_{p_1}(X_1) \otimes ... \otimes \Omega_{p_l}(X_l) \quad (10.5)$$

$$\widetilde{H}_r\left(X_1 * X_2 * \dots * X_l\right) \cong \bigoplus_{\{p_i \geq 0:\ p_1+p_2+\dots+p_l=r-l+1\}} \widetilde{H}_{p_1}\left(X_1\right) \otimes \dots \otimes \widetilde{H}_{p_l}\left(X_l\right) \quad (10.6)$$

$$\widetilde{\beta}_r\left(X_1 * X_2 * \dots * X_l\right) = \sum_{\{p_i \geq 0:\ p_1+p_2+\dots+p_l=r-l+1\}} \widetilde{\beta}_{p_1}\left(X_1\right) \dots \widetilde{\beta}_{p_l}\left(X_l\right). \quad (10.7)$$

Example. Consider an octahedron $Z = X_1 * X_2 * X_3$ where $X_1 = \{0, 1\}, X_2 = \{2, 3\}$, $X_3 = \{4, 5\}$ (see p. 208). Then

$$\begin{aligned}
\Omega_2(Z) &= \bigoplus_{\{p_i \geq -1:\ p_1+p_2+p_3=2-3+1\}} \Omega_{p_1}\left(X_1\right) \otimes \Omega_{p_2}\left(X_2\right) \otimes \Omega_{p_3}\left(X_3\right) \\
&= \Omega_0\left(X_1\right) \otimes \Omega_0\left(X_2\right) \otimes \Omega_0\left(X_3\right) \\
&= \langle e_0, e_1 \rangle \otimes \langle e_2, e_3 \rangle \otimes \langle e_4, e_5 \rangle \\
&= \langle e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135} \rangle
\end{aligned}$$

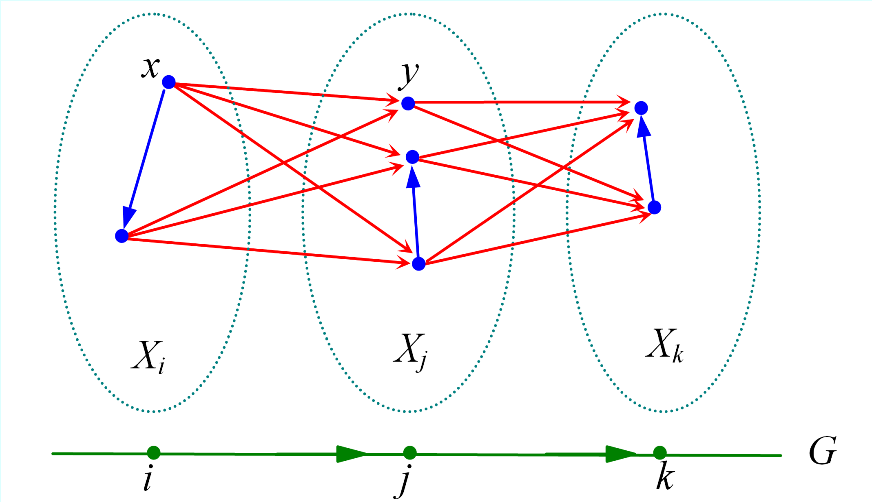
and

$$\begin{aligned}
H_2(Z) = \widetilde{H}_2(Z) &= \bigoplus\nolimits_{\{p_i \geq 0:\ p_1+p_2+p_3=2-3+1\}} \widetilde{H}_{p_1}\left(X_1\right) \otimes \widetilde{H}_{p_2}\left(X_2\right) \otimes \widetilde{H}_{p_3}\left(X_3\right) \\
&= \widetilde{H}_0\left(X_1\right) \otimes \widetilde{H}_0\left(X_2\right) \otimes \widetilde{H}_0\left(X_3\right) \\
&= \langle e_0 - e_1 \rangle \otimes \langle e_2 - e_3 \rangle \otimes \langle e_4 - e_5 \rangle \\
&= \langle e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135} \rangle.
\end{aligned}$$

10.3 A generalized join of digraphs

Given a digraph G of l vertices $\{1, 2, ..., l\}$ and a sequence $X_1, ..., X_l$ of l digraphs, define their *generalized join* $(X_1...X_l)_G = X_G$ as follows: X_G is obtained from the disjoint union $\bigsqcup_i X_i$ of digraphs X_i by keeping all the arrows in each X_i and by adding arrows $x \to y$ whenever $x \in X_i$, $y \in X_j$ and $i \to j$ in G.

Digraph X_G is also referred to as a *G-join* of $X_1, ..., X_l$, and G is called the *base* of X_G.



The main problem to be discussed here is

how to compute the homology groups and Betti numbers of X_G.

Denote by K_l a complete digraph with vertices $\{1,...,l\}$ and arrows

$$i \to j \Leftrightarrow i < j$$

that is, K_l is an $(l-1)$-simplex. For example, $K_2 = \{1 \to 2\}$ and $K_3 = \{1 \to 2 \to 3, 1 \to 3\}$ is a triangle.

The digraph X_{K_l} is called a *complete* join of $X_1, ..., X_l$. It is easy to see that

$$X_{K_l} = X_1 * X_2 * ... * X_l$$

It follows from (10.7) that, for any $r \geq 0$,

$$\widetilde{\beta}_r \left(X_{K_l}\right) = \sum_{\{p_i \geq 0:\ p_1+p_2+...+p_l=r-l+1\}} \widetilde{\beta}_{p_1}\left(X_1\right) ... \widetilde{\beta}_{p_l}\left(X_l\right). \tag{10.8}$$

10.4 A monotone linear join

Denote by I_l a *monotone linear digraph* with the vertices $\{1, ..., l\}$ and arrows $i \to i+1$:

$$I_l = \{1 \to 2 \to ... \to l\}. \tag{10.9}$$

If $G = I_l$ then we use the following simplified notation:

$$(X_1 X_2 ... X_l)_{I_l} = X_1 X_2 ... X_l$$

and refer to this digraph as a *monotone linear join* of $X_1, ..., X_l$.

Clearly, $X_1 X_2 ... X_n$ can be constructed as follows: take first a disjoint union $\bigsqcup_{i=1}^{l} X_i$ and then add arrows from any vertex of X_i to any vertex of X_{i+1} (see p. 213).

In the case $l = 2$ we obviously have $X_1 X_2 = X_1 * X_2$ but in general $X_1 X_2 ... X_l$ is a subgraph of $X_1 * X_2 * ... * X_l$. For example, we have

$$\{0\}\,\{1,2\}\,\{3\} = \begin{array}{ccc} 1 & \to & 3 \\ \uparrow & & \uparrow \\ 0 & \to & 2 \end{array} \quad \text{while} \quad \{0\} * \{1,2\} * \{3\} = \begin{array}{ccc} 1 & \to & 3 \\ \uparrow & \nearrow & \uparrow \\ 0 & \to & 2 \end{array} \tag{10.10}$$

Theorem 10.3 *We have*

$$\widetilde{H}_r(X_1X_2...X_l) \cong \bigoplus_{\{p_i\geq 0:\ p_1+p_2+...+p_l=r-l+1\}} \widetilde{H}_{p_1}(X_1)\otimes ...\otimes \widetilde{H}_{p_l}(X_l) \tag{10.11}$$

and

$$\widetilde{\beta}_r(X_1X_2...X_l) = \sum_{\{p_i\geq 0:\ p_1+p_2+...+p_l=r-l+1\}} \widetilde{\beta}_{p_1}(X_1)...\widetilde{\beta}_{p_l}(X_l). \tag{10.12}$$

By (10.6) and (10.11), $X_1X_2...X_l$ and $X_1 * X_2 * ... * X_l$ are homologically equivalent.

Example. Let the base G be a square:

We have $G = \{1\}\{2,3\}\{4\}$ which implies that

$$X_G = X_1(X_2 \sqcup X_3)X_4.$$

Hence, by Theorem 10.3,

$$\boxed{G = \begin{array}{ccc} 2 & \longrightarrow & 4 \\ \uparrow & & \uparrow \\ 1 & \longrightarrow & 3 \end{array}}$$

$$\begin{aligned}
\widetilde{\beta}_r(X_G) &= \sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1)\widetilde{\beta}_{p_2}(X_2\sqcup X_3)\widetilde{\beta}_{p_3}(X_4) \\
&= \sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1)\left(\widetilde{\beta}_{p_2}(X_2)+\widetilde{\beta}_{p_2}(X_3)+\mathbf{1}_{\{p_2=0\}}\right)\widetilde{\beta}_{p_3}(X_4) \\
&= \widetilde{\beta}_r(X_1X_2X_4)+\widetilde{\beta}_r(X_1X_3X_4)+\widetilde{\beta}_{r-1}(X_1X_4).
\end{aligned} \tag{10.13}$$

For a general base G, if $i_1 ... i_k$ is an arbitrary sequence of vertices in G then denote

$$X_{i_1 ... i_k} = X_{i_1} X_{i_2} ... X_{i_k}.$$

Note that by (10.12)

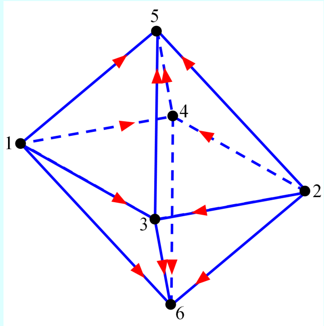
$$\widetilde{\beta}_r \left(X_{i_1 ... i_k} \right) = \sum_{\substack{p_1 + ... + p_k = r - (k-1) \\ p_1, ..., p_k \geq 0}} \widetilde{\beta}_{p_1} \left(X_{i_1} \right) ... \widetilde{\beta}_{p_k} \left(X_{i_k} \right),$$

and we consider the numbers $\widetilde{\beta}_r \left(X_{i_1 ... i_k} \right)$ as known.

Using this notation, we can rewrite (10.13) as follows: if G is a square then

$$\widetilde{\beta}_r \left(X_G \right) = \widetilde{\beta}_r \left(X_{124} \right) + \widetilde{\beta}_r \left(X_{134} \right) + \widetilde{\beta}_{r-1} \left(X_{14} \right).$$

Example. Let G be an octahedron:



We have $G=\{1,2\}*\{3,4\}*\{5,6\}$ whence

$$X_G=(X_1\sqcup X_2)*(X_3\sqcup X_4)*(X_5\sqcup X_6)$$

By (10.8) we obtain

$$\widetilde{\beta}_r\left(X_G\right)=\sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}}\widetilde{\beta}_{p_1}(X_1\sqcup X_2)\widetilde{\beta}_{p_2}(X_3\sqcup X_4)\widetilde{\beta}_{p_3}(X_5\sqcup X_6)$$

$$=\sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}}(\widetilde{\beta}_{p_1}(X_1)+\widetilde{\beta}_{p_1}(X_2)+\mathbf{1}_{\{p_1=0\}})(\widetilde{\beta}_{p_2}(X_3)+\widetilde{\beta}_{p_2}(X_4)+\mathbf{1}_{\{p_2=0\}})$$

$$\times(\widetilde{\beta}_{p_3}(X_5)\sqcup\widetilde{\beta}_{p_3}(X_6)+\mathbf{1}_{\{p_3=0\}})$$

$$=\widetilde{\beta}_r(X_{135})+\widetilde{\beta}_r(X_{145})+\widetilde{\beta}_r(X_{235})+\widetilde{\beta}_r(X_{245})+\widetilde{\beta}_r(X_{136})+\widetilde{\beta}_r(X_{146})+\widetilde{\beta}_r(X_{236})+\widetilde{\beta}_r(X_{246})$$

$$+\widetilde{\beta}_{r-1}(X_{13})+\widetilde{\beta}_{r-1}(X_{23})+\widetilde{\beta}_{r-1}(X_{14})+\widetilde{\beta}_{r-1}(X_{24})+\widetilde{\beta}_{r-1}(X_{15})+\widetilde{\beta}_{r-1}(X_{25})$$

$$+\widetilde{\beta}_{r-1}(X_{35})+\widetilde{\beta}_{r-1}(X_{45})+\widetilde{\beta}_{r-1}(X_{16})+\widetilde{\beta}_{r-1}(X_{26})+\widetilde{\beta}_{r-1}(X_{36})+\widetilde{\beta}_{r-1}(X_{46})$$

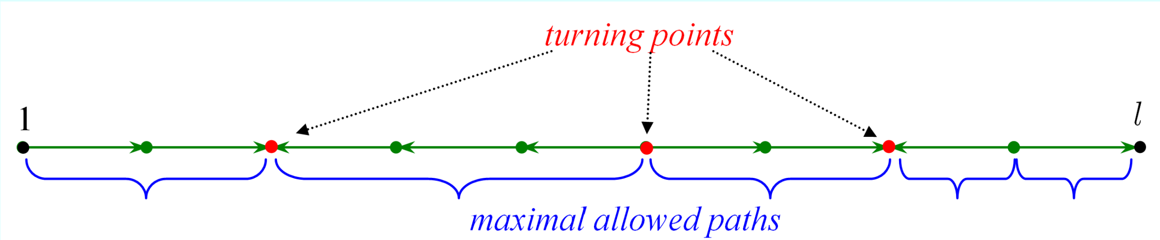
$$+\widetilde{\beta}_{r-2}(X_1)+\widetilde{\beta}_{r-2}(X_2)+\widetilde{\beta}_{r-2}(X_3)+\widetilde{\beta}_{r-2}(X_4)+\widetilde{\beta}_{r-2}(X_5)+\widetilde{\beta}_{r-2}(X_6)+\mathbf{1}_{\{r=2\}}.$$

10.5 An arbitrary linear join

Let now G be a *linear digraph* but not necessarily monotone. That is, the vertex set of G is $\{1, ..., l\}$ and, for any pair $(i, i+1)$ of consecutive numbers there is exactly one arrow: either $i \rightarrow i+1$ or $i \leftarrow i+1$.

Definition. We say that a vertex v of G is a *turning point* if v has either two incoming arrows or two outcoming arrows. Denote by $\mathcal{T}$ the set of all turning points.

An allowed path in G is called *maximal* if it is not a proper subset (as a set of vertices) of another allowed path. Denote by $\mathcal{A}_{\max}$ the family of all maximal allowed paths in G.



Clearly, the end vertices of a maximal path are either turning points or the vertices $1, l$.

Theorem 10.4 *If G is an arbitrary linear digraph then*

$$\widetilde{\beta}_r\left(X_G\right) = \sum_{u\in\mathcal{A}_{\max}} \widetilde{\beta}_r(X_u) + \sum_{v\in\mathcal{T}} \widetilde{\beta}_{r-1}\left(X_v\right).$$

In other words, $\widetilde{\beta}_r\left(X_G\right)$ is the sum of all $\widetilde{\beta}_r$ of the linear joins of X_i along all maximal allowed paths in G plus the sum of $\widetilde{\beta}_{r-1}$ of all X_v sitting at the turning points v.

Example. Consider the base

$$L = \{1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow 5\}.$$

Then $\mathcal{T} = \{2,\ 4\}$,while maximal paths of L are

$$\mathcal{A}_{\max} = \{1 \rightarrow 2, \quad 4 \rightarrow 3 \rightarrow 2, \quad 4 \rightarrow 5\}.$$

Hence, by Theorem 10.4,

$$\widetilde{\beta}_r\left(X_G\right) = \widetilde{\beta}_r\left(X_{12}\right) + \widetilde{\beta}_r\left(X_{432}\right) + \widetilde{\beta}_r\left(X_{45}\right) + \widetilde{\beta}_{r-1}\left(X_2\right) + \widetilde{\beta}_{r-1}\left(X_4\right).$$

Example. Consider the following base:

$$G = \begin{array}{ccccccccc} & & 2 & & & & 5 & & \\ & \nearrow & & \nwarrow & & \swarrow & & \searrow & \\ 1 & & & & 3 & & & & 6 \\ & \searrow & & \swarrow & & \nwarrow & & \nearrow & \\ & & 4 & & & & 7 & & \end{array}$$

It is easy to see that G itself is the following linear join:

$$G = (\{1\}\,\{2,4\}\,\{3\}\,\{5,7\}\,\{6\})_L$$

where $L = \{\alpha \to \beta \leftarrow \gamma \leftarrow \delta \to \varepsilon\}$. Here the turning points of L are $\mathcal{T} = \{\beta,\ \delta\}$, while maximal paths of L are

$$\mathcal{A}_{\max} = \{\alpha \to \beta, \quad \delta \to \gamma \to \beta, \quad \delta \to \varepsilon\}.$$

For L-join we have as above

$$\widetilde{\beta}_r\,(Y_L) = \widetilde{\beta}_r\,(Y_{\alpha\beta}) + \widetilde{\beta}_r\,(Y_{\delta\gamma\beta}) + \widetilde{\beta}_r\,(Y_{\delta\varepsilon}) + \widetilde{\beta}_{r-1}\,(Y_\beta) + \widetilde{\beta}_{r-1}\,(Y_\delta)\,.$$

Setting $Y_\alpha = X_1$, $Y_\beta = X_2 \sqcup X_3$, $Y_\gamma = X_3$, $Y_\delta = X_5 \sqcup X_7$ and $Y_\varepsilon = X_6$ we obtain

$$\widetilde{\beta}_r\,(X_G) = \widetilde{\beta}_r\,((X_1\,(X_2 \sqcup X_3)\,X_3\,(X_5 \sqcup X_7)\,X_6)_L)$$

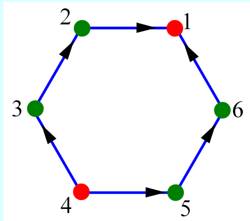
$$
\begin{aligned}
&= \widetilde{\beta}_r(X_1(X_2 \sqcup X_4)) + \widetilde{\beta}_r((X_5 \sqcup X_7)X_3(X_2 \sqcup X_4)) + \widetilde{\beta}_r((X_5 \sqcup X_7)X_6) \\
&\qquad\qquad + \widetilde{\beta}_{r-1}(X_2 \sqcup X_4) + \widetilde{\beta}_{r-1}(X_5 \sqcup X_7) \\
&= \widetilde{\beta}_r\left(X_{12}\right) + \widetilde{\beta}_r\left(X_{14}\right) + \widetilde{\beta}_{r-1}\left(X_1\right) \\
&+ \widetilde{\beta}_r(X_{532}) + \widetilde{\beta}_r(X_{534}) + \widetilde{\beta}_r(X_{732}) + \widetilde{\beta}_r(X_{734}) \\
&\qquad\qquad + \widetilde{\beta}_{r-1}(X_{32}) + \widetilde{\beta}_{r-1}(X_{34}) + \widetilde{\beta}_{r-1}(X_{53}) + \widetilde{\beta}_{r-1}(X_{73}) + \widetilde{\beta}_{r-2}(X_3) \\
&+ \widetilde{\beta}_r\left(X_{56}\right) + \widetilde{\beta}_r\left(X_{76}\right) + \widetilde{\beta}_{r-1}\left(X_6\right) \\
&+ \widetilde{\beta}_{r-1}\left(X_2\right) + \widetilde{\beta}_{r-1}\left(X_4\right) + \mathbf{1}_{\{r=1\}} + \widetilde{\beta}_{r-1}\left(X_5\right) + \widetilde{\beta}_{r-1}\left(X_7\right) + \mathbf{1}_{\{r=1\}}.
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\beta}_r(X_G) \quad = \quad & \widetilde{\beta}_r(X_{534}) + \widetilde{\beta}_r(X_{532}) + \widetilde{\beta}_r(X_{734}) + \widetilde{\beta}_r(X_{732}) \\
& + \widetilde{\beta}_r(X_{12}) + \widetilde{\beta}_r(X_{14}) + \widetilde{\beta}_r(X_{56}) + \widetilde{\beta}_r(X_{76}) \\
& + \widetilde{\beta}_{r-1}(X_{73}) + \widetilde{\beta}_{r-1}(X_{53}) + \widetilde{\beta}_{r-1}(X_{32}) + \widetilde{\beta}_{r-1}(X_{34}) \\
& + \widetilde{\beta}_{r-1}(X_1) + \widetilde{\beta}_{r-1}(X_2) + \widetilde{\beta}_{r-1}(X_4) + \widetilde{\beta}_{r-1}(X_5) + \widetilde{\beta}_{r-1}(X_6) + \widetilde{\beta}_{r-1}(X_7) \\
& + \widetilde{\beta}_{r-2}(X_3) + \mathbf{2}_{\{r=1\}}.
\end{aligned}
$$

10.6 A cyclic join

A digraph G is called *cyclic* if it is connected and each vertex has the undirected degree 2. Let G be a cyclic digraph with the set of vertices $V = \{1, 2, ..., l\}$. We assume that the vertices are ordered so that every vertex $i \in V$ is connected by arrows to $i-1$ and $i+1$ (where l is identified with 0). In the same way as above we define the set $\mathcal{A}_{\max}$ and $\mathcal{T}$.

For example, consider the following hexagon:
Here $\mathcal{T} = \{1, 4\}$ and
$\mathcal{A}_{\max} = \{4 \to 3 \to 2 \to 1,\ 4 \to 5 \to 6 \to 1\}$



Theorem 10.5 *Let G be a cyclic digraph that is neither triangle nor square nor double arrow. Then*

$$\widetilde{\beta}_r(X_G) = \sum_{u \in \mathcal{A}_{\max}} \widetilde{\beta}_r(X_u) + \sum_{v \in \mathcal{T}} \widetilde{\beta}_{r-1}(X_v) + \widetilde{\beta}_r(G). \tag{10.14}$$

Note that in this case $\widetilde{\beta}_r(G) = \mathbf{1}_{\{r=1\}}$. If G is a triangle or square or double arrow then (10.14) is wrong, which is shown in Examples below.

Example. If G is the above hexagon then we obtain

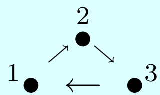
$$\widetilde{\beta}_r\left(X_G\right) = \widetilde{\beta}_r\left(X_{4321}\right) + \widetilde{\beta}_r\left(X_{4561}\right) + \widetilde{\beta}_{r-1}\left(X_1\right) + \widetilde{\beta}_{r-1}\left(X_4\right) + \mathbf{1}_{\{r=1\}}.$$

Example. Consider the following 4-cyclic base:

$$G = \begin{array}{ccc} 2 & \rightarrow & 3 \\ \uparrow & & \downarrow \\ 1 & \rightarrow & 4 \end{array}$$

Since $\mathcal{T} = \{1,4\}$ and $\mathcal{A}_{\max} = \{1 \to 2 \to 3 \to 4, 1 \to 4\}$, we obtain

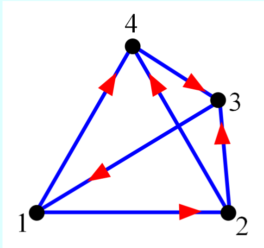
$$\widetilde{\beta}_r(X_G) = \widetilde{\beta}_r(X_{1234}) + \widetilde{\beta}_r(X_{14}) + \widetilde{\beta}_{r-1}\left(X_1\right) + \widetilde{\beta}_{r-1}\left(X_4\right) + \mathbf{1}_{\{r=1\}}. \tag{10.15}$$

Example. Consider the following 3-cyclic base: $G =$ 

.

Then $\mathcal{A}_{\max}$ and $\mathcal{T}$ are empty, and we obtain $\widetilde{\beta}_r\left(X_G\right) = \mathbf{1}_{\{r=1\}} = \widetilde{\beta}_r(G)$.

Example. Consider the following tetrahedron as a base G:



We have $\quad G = C * \{4\} \quad$ where

$$C = \{1 \to 2 \to 3 \to 1\}$$

It follows that

$$X_G = X_C * X_4$$

and

$$\widetilde{\beta}_r (X_G) = \sum_{p+q=r-1} \widetilde{\beta}_p (X_C)\, \widetilde{\beta}_q (X_4) = \sum_{p+q=r-1} \mathbf{1}_{\{p=1\}} \widetilde{\beta}_q (X_4) = \widetilde{\beta}_{r-2} (X_4) \, .$$

Hence, $\widetilde{\beta}_r (X_G) = \widetilde{\beta}_{r-2} (X_4)$.

Example. Let G be a triangle: $G = \begin{tikzcd}[column sep=small] & \overset{2}{\bullet} \arrow[dr] & \\ \overset{1}{\bullet} \arrow[ur] \arrow[rr] & & \overset{3}{\bullet} \end{tikzcd}$. Then $X_G = X_1 * X_2 * X_3$ and we know that

$$\widetilde{\beta}_r (X_G) = \widetilde{\beta}_r (X_{123}) \, .$$

However, the right hand side of (10.14) is in this case

$$\widetilde{\beta}_r (X_{123}) + \widetilde{\beta}_{r-1} (X_1) + \widetilde{\beta}_{r-1} (X_3) \neq \widetilde{\beta}_r (X_G) \, .$$

Example. Let G be a square:

$$G = \begin{array}{ccc} 2 & \longrightarrow & 4 \\ \uparrow & & \uparrow \\ 1 & \longrightarrow & 3 \end{array}$$

Then we that by (10.13)

$$\widetilde{\beta}_r\left(X_G\right) = \widetilde{\beta}_r\left(X_{124}\right) + \widetilde{\beta}_r\left(X_{134}\right) + \widetilde{\beta}_{r-1}\left(X_{14}\right),$$

while the right hand side of (10.14) is in this case

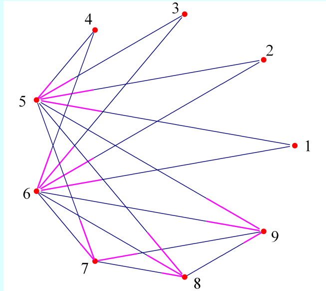
$$\widetilde{\beta}_r\left(X_{124}\right) + \widetilde{\beta}_r\left(X_{134}\right) + \widetilde{\beta}_{r-1}\left(X_1\right) + \widetilde{\beta}_{r-1}\left(X_4\right).$$

Example. Let G be a double arrow: $G = \{1 \rightleftarrows 2\}$. Then

$$X_G = X_1 * X_2 * X_1$$

whence $\widetilde{\beta}_r\left(X_G\right) = \widetilde{\beta}_r\left(X_{121}\right)$. However, in this case $\mathcal{A}_{\max}$ and $\mathcal{T}$ are empty, so that the right hand side of (10.14) is $\widetilde{\beta}_r\left(G\right) = 0$.

Example. Let G be as here:



We have
$G = \{1, 2, 3, 4\}\,\{5, 6\}\,\{7 \to 8 \to 9 \to 7\}$

so that
$X_G = (X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4)\,(X_5 \sqcup X_6)\,X_{\{7\to 8\to 9\to 7\}}$

It follows that

$$\widetilde{\beta}_r\,(X_G) = \sum_{p+q+s=r-2} \left(\widetilde{\beta}_p\,(X_1) + \widetilde{\beta}_p\,(X_2) + \widetilde{\beta}_p\,(X_3) + \widetilde{\beta}_p\,(X_4) + \mathbf{3}_{\{p=0\}}\right) \times \left(\widetilde{\beta}_q\,(X_5) + \widetilde{\beta}_q\,(X_6) + \mathbf{1}_{\{q=0\}}\right) \mathbf{1}_{\{s=1\}}$$

which yields after computation

$$\begin{aligned}\widetilde{\beta}_r(X_G) = {} & \widetilde{\beta}_{r-2}(X_{15}) + \widetilde{\beta}_{r-2}(X_{16}) + \widetilde{\beta}_{r-2}(X_{25}) + \widetilde{\beta}_{r-2}(X_{26}) \\ & + \widetilde{\beta}_{r-2}(X_{35}) + \widetilde{\beta}_{r-2}(X_{36}) + \widetilde{\beta}_{r-2}(X_{45}) + \widetilde{\beta}_{r-2}(X_{46}) \\ & + \widetilde{\beta}_{r-3}(X_1) + \widetilde{\beta}_{r-3}(X_2) + \widetilde{\beta}_{r-3}(X_3) + \widetilde{\beta}_{r-3}(X_4) + 3\widetilde{\beta}_{r-3}(X_5) + 3\widetilde{\beta}_{r-3}(X_6) + \mathbf{3}_{\{r=3\}}.\end{aligned}$$

10.7 Homology of a generalized join

Theorem 10.6 *There exists a finite sequence of paths $\{u_k\}$ in G and a sequence $\{s_k\}$ of non-negative integers such that, for any sequence $\{X_i\}$ of digraphs and any $r \geq 0$,*

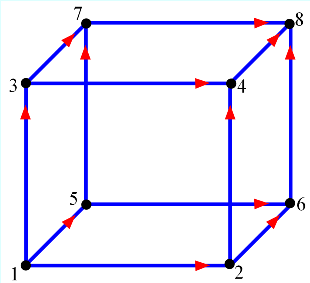
$$\widetilde{\beta}_r(X_G) = \sum_k \widetilde{\beta}_{r-s_k}(X_{u_k}) + \widetilde{\beta}_r(G). \tag{10.16}$$

Besides, the sequence $\{u_k\}$ contains all maximal allowed paths, and $u_k \in \mathcal{A}_{\max} \Leftrightarrow s_k = 0$.

Example. Let the base G be a cube.

Use description of paths u_k from the proof of Theorem 10.6, we obtain

$$\begin{aligned}\widetilde{\beta}_r(X_G) = {} & \widetilde{\beta}_r(X_{1248}) + \widetilde{\beta}_r(X_{1268}) + \widetilde{\beta}_r(X_{1348}) \\ & + \widetilde{\beta}_r(X_{1378}) + \widetilde{\beta}_r(X_{1568}) + \widetilde{\beta}_r(X_{1578}) \\ & + \widetilde{\beta}_{r-1}(X_{178}) + \widetilde{\beta}_{r-1}(X_{168}) + \widetilde{\beta}_{r-1}(X_{148}) \\ & + \widetilde{\beta}_{r-1}(X_{128}) + \widetilde{\beta}_{r-1}(X_{138}) + \widetilde{\beta}_{r-1}(X_{158}) \\ & + \widetilde{\beta}_{r-2}(X_{18})\end{aligned}$$



10.8 Mayer-Vietoris exact sequence

A digraph Y is called a subgraph of a digraph X if both sets of vertices and arrows of Y are subsets of those sets of X. If Y_1 and Y_2 are two subsets of X then $Y_1 \cup Y_2$ is their union, that is, a subset of X whose sets of vertices and arrows are unions of those of Y_1 and Y_2. In the same way one defines the intersection $Y_1 \cap Y_2$.

A subgraph Y of X is called induced if for any two vertices a, b of Y, if there is an arrow $a \to b$ in X then there is also an arrow $a \to b$ in Y. Clearly, the intersection of induced subgraphs is also an induce subgraph.

Assume that a digraph X can be represented as a union of two induced subgraphs Y_1 and Y_2, that is, $X = Y_1 \cup Y_2$. In particular, every arrow of X lies in Y_1 or Y_2. Denote $Z = Y_1 \cap Y_2$.

Any p-path $u \in \mathcal{R}_p(X)$ has a form

$$u = \sum_{i_0 \ldots i_p} u^{i_0 \ldots i_p} e_{i_0 \ldots i_p}$$

with the coefficients $u^{i_0 \ldots i_p} \in \mathbb{K}$. We say that $e_{i_0 \ldots i_p}$ (or $u^{i_0 \ldots i_p} e_{i_0 \ldots i_p}$) is an *elementary term* of u if $u^{i_0 \ldots i_p} \neq 0$.

Theorem 10.7 (Mayer-Vietoris exact sequence) *Assume that, for any $p \geq 2$,*

$$\forall x \in \Omega_p(X) \text{ we have } x = y_1 + y_2 \text{ for some } y_1 \in \Omega_p(Y_1) \text{ and } y_2 \in \Omega_p(Y_2). \tag{10.17}$$

Then we have a long exact sequence of homology groups:

$$\cdots \to \widetilde{H}_n(Z) \to \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(Z) \to \widetilde{H}_{n-1}(Y_1) \oplus \widetilde{H}_{n-1}(Y_2) \to \cdots.$$

Corollary 10.8 *Assume that the hypotheses of Theorem 10.7 are satisfied.*

(a) If, for some n, $\widetilde{H}_n(Z) = \{0\}$ and $\widetilde{H}_{n-1}(Z) = 0$, then

$$\widetilde{H}_n(X) \cong \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2). \tag{10.18}$$

(b) If, for some n, the homology groups $\widetilde{H}_n(Y_1)$, $\widetilde{H}_n(Y_2)$, $\widetilde{H}_{n-1}(Y_1)$, $\widetilde{H}_{n-1}(Y_2)$ are trivial then

$$\widetilde{H}_n(X) \cong \widetilde{H}_{n-1}(Z).$$

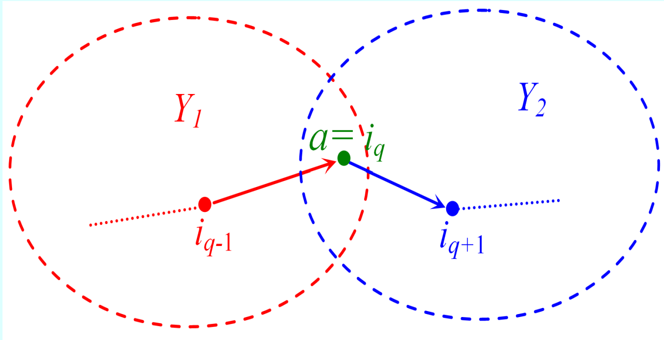
Example. Assume that Z consists of a single vertex a. Let us verify that the hypothesis (10.17) is satisfied. For any $x \in \Omega_p(X)$ with $p \geq 2$, consider an elementary term $ce_{i_0 \ldots i_p}$

of x and show that $e_{i_0...i_p}$ lies in Y_1 or in Y_2. Assume that this is not the case, that is, one of the vertices $i_1, ..., i_{p-1}$ is a, say $a = i_q$, while i_{q-1} and i_{q+1} belong to different Y_1, Y_2.

The path $\partial e_{i_0...i_p}$ contains the term

$$e_{i_0...i_{q-1}i_{q+1}..i_p}$$

that is not allowed because $i_{q-1} \not\to i_{q+1}$. This term must be cancelled in ∂x using other elementary terms of x. However if another elementary term $e_{j_0...j_p}$ x contains $e_{i_0...i_{q-1}i_{q+1}...i_p}$ in its boundary, then

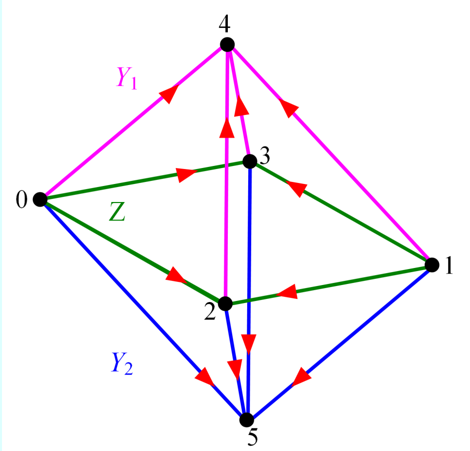


$$i_0...i_{q-1}i_{q+1}...i_p = j_0...j_{q-1}j_{q+1}...j_p$$

which implies $j_q = a$ because this is the only choice of j_q to make $j_0...j_p$ allowed. Hence, $e_{i_0...i_p} = e_{j_0...j_p}$ and the above cancellation is not possible. Finally, denoting by y_k (where $k = 1, 2$) the sum of all elementary terms of x that are contained in Y_k we obtain $y_k \in \Omega_p (Y_k)$ and $y = y_1 + y_2$, which proves (10.17).

Since $\widetilde{H}_* (Z) = \{0\}$, Corollary 10.8($a$) applies in this case and yields (10.18) for all n.

Example. Let $X = Y_1 \cup Y_2$ be an octahedron as here:



$\Omega_2(X)$ is spanned by 8 triangles:

$$e_{024},\ e_{034},\ e_{025},\ e_{035},\ e_{124},\ e_{134},\ e_{125},\ e_{135},$$

each of them lying in Y_1 or Y_2, while $\Omega_p = \{0\}$ for all $p \geq 3$,

Hence, the hypothesis of Theorem 10.7 is satisfied.

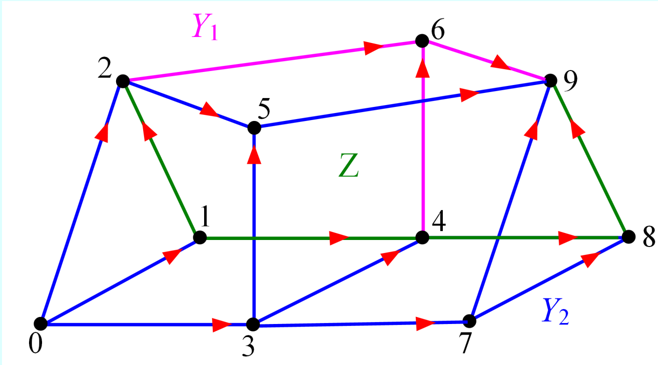
All $\widetilde{H}_*(Y_1)$ and $\widetilde{H}_*(Y_2)$ are trivial, the only nontrivial group $\widetilde{H}_p(Z)$ is

$$H_1(Z) = \{e_{02} - e_{12} + e_{13} - e_{03}\}.$$

By Corollary 10.8(b) we conclude that $H_2(X) \cong H_1(Z)$.

Indeed, we have seen above that $H_2(X)$ is one-dimensional.

Example. Consider the following digraph $X = Y_1 \cup Y_2$:



Y_1 contains the vertices $\{1, 2, 4, 6, 8, 9\}$,

Y_2 contains all the vertices except for 6,

Z contains the vertices $\{1, 2, 4, 8\}$.

All Y_1, Y_2, Z are homologically trivial while $\dim H_2(X) = 1$.

In fact, we have

$$H_2(X) = \langle e_{012} - (e_{014} - e_{034}) + (e_{025} - e_{035}) - (e_{126} - e_{146}) - (e_{259} - e_{269}) - (e_{348} - e_{378}) + (e_{359} - e_{379}) - (e_{469} - e_{489}) + e_{789} \rangle .$$

Therefore, (10.18) fails for $n = 2$. The hypothesis of Theorem 10.7 fails either: the square $x = e_{259} - e_{269}$ is ∂-invariant on X but it does not satisfy (10.17) because e_{269} is not ∂-invariant on Y_1 and e_{259} is not ∂-invariant on Y_2.

11 Homotopy and related notions

11.1 Homotopy equivalent digraphs

For vertices a, b of a digraph, write $a \overrightarrow{=} b$ if either $a \to b$ or $a = b$. Let X and Y be two digraphs.

Definition. A *mapping* $f : X \to Y$ called a *digraph map (or morphism)* if

$$a \to b \text{ on } X \quad \Rightarrow \quad f(a) \overrightarrow{=} f(b) \text{ on } Y.$$

Any digraph map $f : X \to Y$ induces a linear map

$$f_*\colon \mathcal{A}_p(X) \to \mathcal{A}_p(Y), \qquad f_*\left(e_{i_0 \ldots i_p}\right) = e_{f(i_0)\ldots f(i_p)}.$$

It is easy to check that $f_*\partial = \partial f_*$, which implies that f_* provides a morphism of chain complexes $f_* : \Omega_p(X) \to \Omega_p(Y)$ and, consequently, a homomorphism of homology groups $f_* : H_p(X) \to H_p(Y)$.

Definition. For any $n \geq 1$ define a *line digraph* I_n as any digraph with $n+1$ vertices $\{0, 1, \ldots, n\}$ and such that, for any $i = 0, ..., n-1$ holds either $i \to (i+1)$ or $(i+1) \to i$, and there is no other arrow.

Definition. Let X, Y be two digraphs. Two digraph maps $f, g\colon X \to Y$ are called *homotopic* if there exists a line digraph I_n and a digraph map $\Phi\colon X\Box I_n \to Y$ such that

$$\Phi|_{X\times\{0\}} = f \quad \text{and} \quad \Phi|_{X\times\{n\}} = g.$$

In this case we write $f \simeq g$. The map Φ is called a *homotopy* between f and g.

Definition. Two digraphs X and Y are called *homotopy equivalent* if there exist digraph maps

$$f : X \to Y, \quad g : Y \to X \tag{11.1}$$

such that

$$f \circ g \simeq \mathrm{id}_Y, \qquad g \circ f \simeq \mathrm{id}_X\,. \tag{11.2}$$

In this case we write $X \simeq Y$.

Theorem 11.1 *(i) Let $f, g\colon X \to Y$ be two digraph maps. If $f \simeq g$ then they induce the identical maps of homology groups:*

$$f_* : H_p(X) \to H_p(Y) \quad \textit{and} \quad g_* : H_p(X) \to H_p(Y)\,.$$

(ii) If the digraphs X and Y are homotopy equivalent, then $H_*(X) \cong H_*(Y)$.

In particular, if a digraph X is contractible, that is, if $X \simeq \{*\}$, then all the homology groups of X are trivial except for H_0.

We say that a digraph Y is a *subgraph* of X if the set of vertices of Y is a subset of that of X and the arrows of Y are all those arrows of X whose adjacent vertices belong to Y.

Definition. Let X be a digraph and Y be its subgraph. A *retraction* of X onto Y is a digraph map $r : X \to Y$ such that $r|_Y = \mathrm{id}_Y$.

Theorem 11.2 *Let $r : X \to Y$ be a retraction of a digraph X onto a subgraph Y. Assume that*

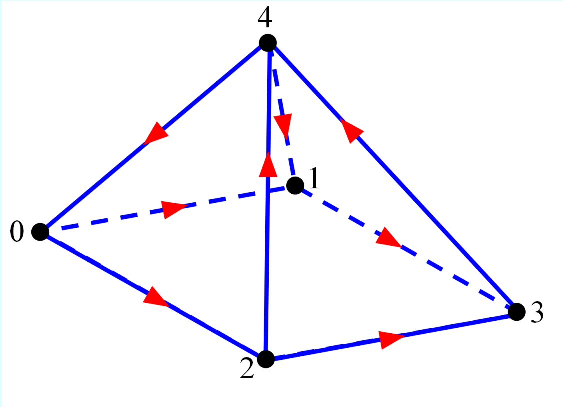
$$\text{either } x \overrightarrow{=} r(x) \quad \text{for all } x \in X \quad \text{or} \quad r(x) \overrightarrow{=} x \quad \text{for all } x \in X. \tag{11.3}$$

Then $X \simeq Y$ and, consequently, $H_*(X) \cong H_*(Y)$.

A retraction that satisfies (11.3) is called a *deformation retraction*.

Example. Let us show that n-cube is contractible. Indeed, a natural projection of n-cube onto $(n-1)$-cube is a deformation retraction. Hence, by induction we obtain n-cube $\simeq \{*\}$.

Example. Consider the digraph X as here.



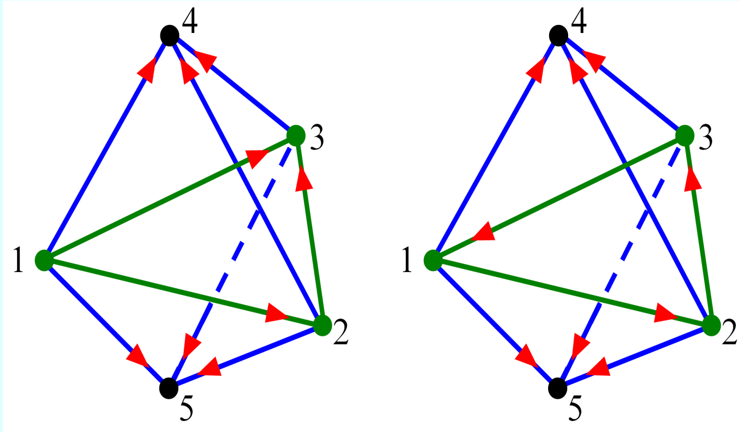
Let Y be its subgraph with the vertex set $\{1,3,4\}$. Consider a retraction $r : X \to Y$ given by $r(0) = 1$, $r(2) = 3$. It is easy to see that r is a deformation retraction, whence $X \simeq Y$. Then we obtain

$$H_1(X) \cong H_1(Y) = \langle e_{13} + e_{34} + e_{41} \rangle \cong \mathbb{K}$$

and $H_p(X) = \{0\}$ for $p \geq 2$.

Example. Consider the following two digraphs.

The digraph at the left panel is contractible as there is a sequence of two deformation retractions reducing it to $\{*\}$:



$$r_1(4) = r_1(5) = 3$$
$$r_2(1) = r_2(2) = 3$$

The digraph at the right panel differs only by one arrow $3 \to 1$, but it is not contractible because $H_2 \neq \{0\}$

In fact, for this digraph

$$H_2 = \langle e_{124} + e_{234} + e_{314} - e_{125} - e_{235} - e_{315} \rangle.$$

11.2 C-homotopy of loops

For any digraph G and a vertex $*$ of G, denote by G^* a *based digraph.*

Definition. A *loop* on G^* is a digraph map $\varphi : I_n \to G$ such that $\varphi(0) = \varphi(n) = *$.

Here I_n is any line digraph with any $n \geq 0$.

Definition. Consider in G^* two loops $\varphi\colon I_n \to G$ and $\psi\colon I_m \to G$. An *one-step direct C-homotopy* from φ to ψ is a digraph map $h : I_n \to I_m$ such that

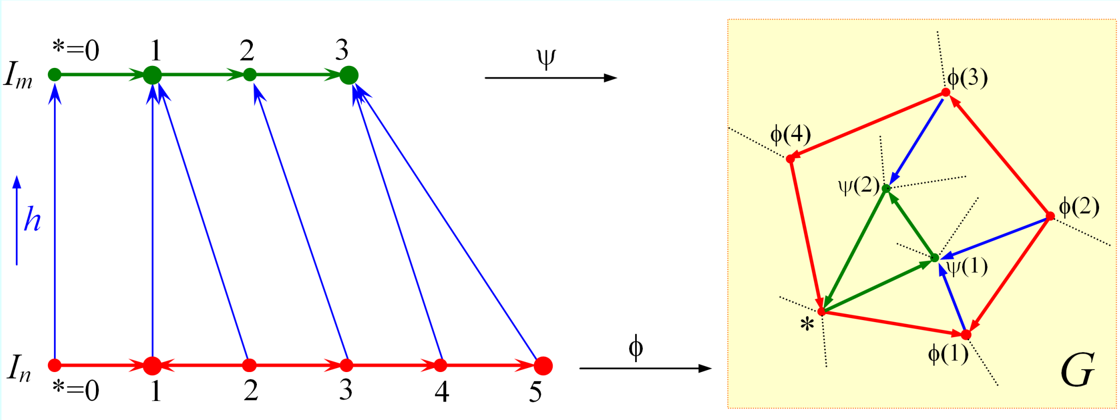
(a) $h(0) = 0$, $h(n) = m$ and $h(i) \leq h(j)$ whenever $i \leq j$;

(b) $\varphi(i) \overset{\rightarrow}{=} \psi(h(i))$ for all $i \in I_n$.

If in (b) holds $\varphi(i) \overset{\leftarrow}{=} \psi(h(i))$ for all $i \in I_n$ then h is called an one-step *inverse* C-homotopy.

We denote an one-step direct C-homotopy with $\varphi \overset{C}{\rightarrow} \psi$ and the one-step inverse C-homotopy with $\varphi \overset{C}{\leftarrow} \psi$.

Example. On the next diagram we have $\varphi \xrightarrow{C} \psi$.



Condition (b) means that φ and ψ provide a digraph map from the digraph on the left panel to G.

Definition. We call two loops φ, ψ *C-homotopic* and write $\varphi \overset{C}{\simeq} \psi$ if there exists a finite sequence $\{\varphi_k\}_{k=0}^m$ of loops in G^* such that $\varphi_0 = \varphi$, $\varphi_m = \psi$ and, for any $k = 0, ..., m-1$, holds $\varphi_k \xrightarrow{C} \varphi_{k+1}$ or $\varphi_k \xleftarrow{C} \varphi_{k+1}$.

Obviously, C-homotopy is an equivalence relation. A loop φ is called contractible if $\varphi \overset{C}{\simeq} e$ where $e : I_0 \to G$ is a trivial loop.

The following theorem gives an efficient way of verifying if two loops are C-homotopic.

Any loop $\varphi\colon I_n \to G$ defines a sequence $\theta_\varphi = \{\varphi(i)\}_{i=0}^{n}$ of vertices of G. We consider θ_φ as a word over the alphabet V.

Theorem 11.3 *Two loops $\varphi : I_n \to G$ and $\psi : I_m \to G$ are C-homotopic if and only if θ_ψ can be obtained from θ_φ by a finite sequence of the following word transformations (or inverses to them):*

(i) $...abc... \mapsto ...ac...$ where a, b, c is a triangle

$$\begin{tikzcd} & b \arrow[dr] & \\ a \arrow[ur] \arrow[rr] & & c \end{tikzcd}$$

in G or any permutation of a triangle.

(ii) $...abc... \mapsto ...adc...$ where a, b, c, d is a square

$$\begin{tikzcd} d \arrow[r] & c \\ a \arrow[u] \arrow[r] & b \arrow[u] \end{tikzcd}$$

in G or any cyclic permutation of a square or an inverse cyclic permutation of a square.

(iii) $...abcd... \mapsto ...ad...$ where a, b, c, d is as in (ii).

(iv) $...aba... \to ...a...$ if $a \to b$ or $b \to a$.

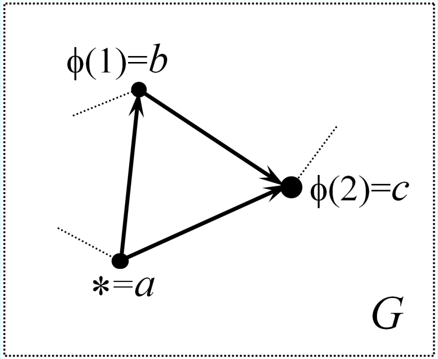
(v) $...aa... \mapsto ...a...$

Examples

1. Consider a triangular loop
$\varphi : (0 \to 1 \to 2 \leftarrow 3) \to G$

It is contractible because

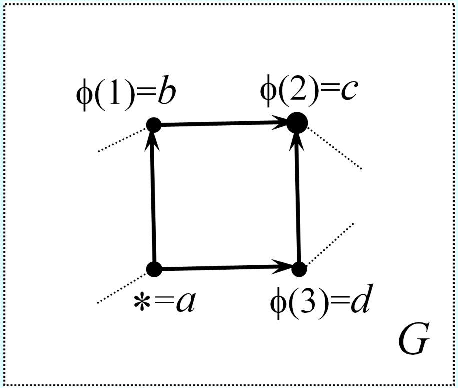
$\theta_\varphi = abca \overset{(i)}{\sim} aca \overset{(iv)}{\sim} a.$



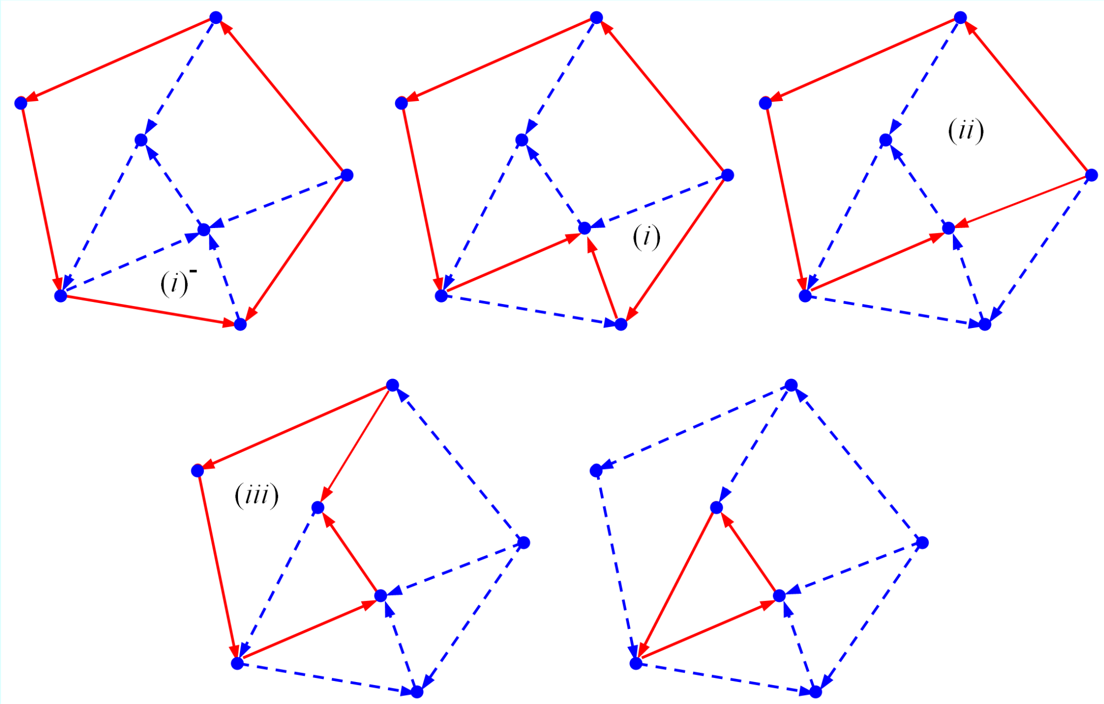
2. Consider a square loop
$\varphi : (0 \to 1 \to 2 \leftarrow 3 \leftarrow 4) \to G$

It is contractible because

$\theta_\varphi = abcda \overset{(iii)}{\sim} ada \overset{(iv)}{\sim} a.$



3. Consider the loops $\varphi : I_5 \to G$ and $\psi : I_3 \to G$ as on p.240. It is shown here how to transform θ_φ to θ_ψ by means of Theorem 11.3: using successively transformations $(i)^-$, (i), (ii) and (iii).



11.3 Fundamental group π_1

The C-homotopy equivalence class of a loop $\varphi : I_n \to G$ will be denoted by $[\varphi]$. For any two loops $\varphi : I_n \to G$ and $\psi : I_m \to G$ define their concatenation $\varphi \vee \psi : I_{n+m} \to G$ by

$$\varphi \vee \psi(i) = \begin{cases} \varphi(i), & 0 \leq i \leq n \\ \psi(i-n), & n \leq i \leq n+m. \end{cases}$$

Then the product $[\varphi] \cdot [\psi] := [\varphi \vee \psi]$ of equivalence classes is then well-defined.

Theorem 11.4 (a) *The set of all equivalence classes* $[\varphi]$ *with the above product is a group with the neutral element* $[e]$. *It is denoted by* $\pi_1(G^*)$.

(b) *Any based digraph map* $f : X^* \to Y^*$ *induces a group homomorphism*

$$\pi_1(f) : \pi_1(X^*) \to \pi_1(Y^*), \quad (\pi_1(f))\,[\phi] = [f \circ \phi].$$

(c) *If* $f, g : X^* \to Y^*$ *are two digraph maps then* $f \simeq g$ *implies* $\pi_1(f) = \pi_1(g)$.

(d) *If* X, Y *are connected and* $X \simeq Y$ *then* $\pi_1(X^*) \cong \pi_1(Y^*)$.

Theorem 11.5 *For any based connected digraph G^* we have an isomorphism*

$$\pi_1(G^*)\,/[\pi_1(G^*), \pi_1(G^*)] \cong H_1(G, \mathbb{Z}),$$

where $[\pi_1(G^*), \pi_1(G^*)]$ is a commutator subgroup.

11.4 An application to graph coloring

An an illustration of the theory of digraph homotopy, we give here a new proof of the classical lemma of Sperner, using the notion the fundamental group of digraphs.

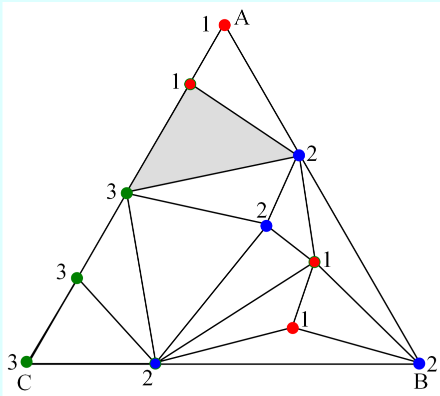
Consider a triangle ABC on the plane $\mathbb{R}^2$ and its triangulation T. Assume that the set of vertices of T is colored in three colors $1, 2, 3$ so that:

• the vertex A in colored in 1, B – in 2, C – in 3;

• each vertex on the side AB is colored in 1 or 2, on the side AC – in 1 or 3, on the side BC – in 2 or 3.

Lemma of Sperner.

Under the above hypotheses,
there exists in T a 3-color triangle,
that is, a triangle, whose vertices
are colored with three different colors.



Let us first modify the triangulation T so that there are no vertices on the sides AB, AC, BC except for A, B, C. If $X \in AB$ then move X a bit inside of ABC. A new triangle XYZ arises, where Y, Z are former neighbors of X on AB. However, since X, Y, Z are colored in two colors, no 3-color triangle emerges after that move. By induction, we remove all the vertices from all sides of ABC.

Consider the triangulation T as a graph and make it into a digraph G as follows. If a, b are two vertices on T and $a \sim b$ then choose direction between a, b using the colors of a, b and the following rule:

$$1 \rightarrow 2, \quad 2 \rightarrow 3, \ 3 \rightarrow 1$$
$$1 \leftrightarrows 1, \quad 2 \leftrightarrows 2, \ 3 \leftrightarrows 3$$

Denote by S the following colored digraph
$$\begin{tikzcd}[column sep=small] & \overset{1}{\bullet} \arrow[dr] & \\ \overset{3}{\bullet} \arrow[ur] & & \overset{2}{\bullet} \arrow[ll] \end{tikzcd}$$
and define a mapping $f : G \rightarrow S$ to preserve colors of vertices. Then f is a digraph map by the choice of arrows in G.

Consider a 3-loop φ on G^* (with $* = A$) with the word

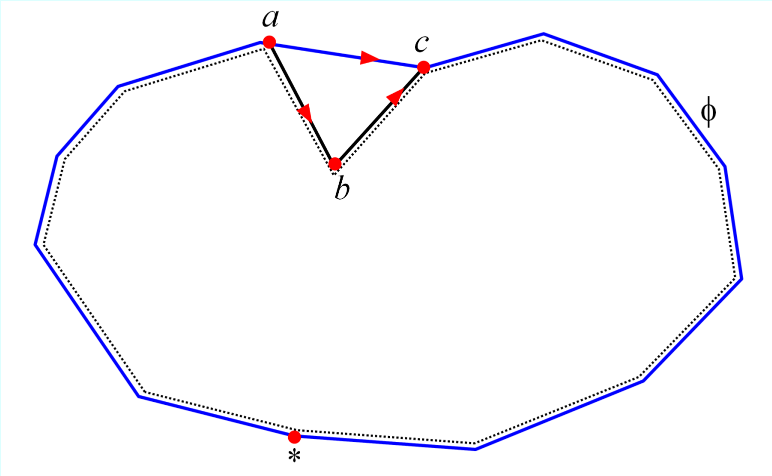
$$\theta_\varphi = ABCA.$$

For the loop $f \circ \varphi$ on S we have $\theta_{f \circ \varphi} = 1231$. This loop is not contractible because none of the transformations of Theorem 11.3 can be applied to the word 1231. By Theorem 11.4(b), the loop φ is also not contractible and, hence, $\pi_1 (G^*) \neq \{0\}$.

Assume now that there is no 3-color triangle in T. Then each triangle from T looks in G like

$$\begin{tikzcd}[column sep=small] & \bullet & \\ \bullet \arrow[ur] \arrow[rr, shift left] & & \bullet \arrow[ul] \arrow[ll, shift left] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd}[column sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[rr, shift left] & & \bullet \arrow[ll, shift left] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd}[column sep=small] & \bullet \arrow[dr, shift left] & \\ \bullet \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, shift left] & & \bullet \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd} \quad .$$

In particular, each of them contains a triangle in the sense of Theorem 11.3. Using the partition of G into the triangles and transformations (ii) and (iv) of Theorem 11.3, we contract any loop on G to the empty word, which contradicts to $\pi_1(G) \neq \{0\}$.



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