Blatt 10. Abgabe bis 05.01.23

49. Let $u \in C^1(B_R \setminus \{0\})$, where B_R is a ball in $\mathbb{R}^n$. Assume that the function u satisfies in $B_R \setminus \{0\}$ the following inequality:

$$|u(x)| \leq C\,|x|^s\,,$$

for some constants $C > 0$ and

$$s > 1 - n.$$

Prove that if the classical derivative $\partial_i u$ belongs to $L^1_{loc}(B_R)$ then $\partial_i u$ is also the weak derivative of u in B_R.

Hint: You meed to verify that, for any $\varphi \in \mathcal{D}(B_R)$,

$$\int_{B_R} \partial_i u\,\varphi\,dx = -\int_{B_R} u\,\partial_i\varphi\,dx.$$

For that apply the integration-by-parts formula in $B_R \setminus \overline{B}_\varepsilon$, for a small $\varepsilon > 0$, and then pass to the limit as $\varepsilon \to 0$.

50. Consider the function $u(x) = |x|^s$ in a ball B_R in $\mathbb{R}^n$. Prove that if

$$s > k - n/p, \tag{45}$$

where $p \in [1, \infty)$ and $k \geq 0$ is an integer, then $u \in W^{k,p}(B_R)$.

Hint: Prove the following statements:

(i) the classical derivative $D^\alpha u$ of any order $l = |\alpha|$ satisfies in $\mathbb{R}^n \setminus \{0\}$ the inequality

$$|D^\alpha u(x)| \leq C\,|x|^{s-l}\,;$$

(ii) any classical derivative $D^\alpha u$ with $|\alpha| \leq k$ belongs to $L^p(B_R)$;

(iii) any classical derivative $D^\alpha u$ with $|\alpha| \leq k$ is also the weak derivative of u (use Exercise 49).

51. Consider in $\mathbb{R}^n$ a non-divergence form operator

$$Lu = \sum_{i,j=1}^n a_{ij}\partial_{ij}u$$

with the coefficients

$$a_{ij}(x) = \begin{cases} \delta_{ij} + c\frac{x_i x_j}{|x|^2}, & x \neq 0, \\ \delta_{ij}, & x = 0, \end{cases}$$

where c is a positive constant and $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

(a) Prove that L is uniformly elliptic in $\mathbb{R}^n$.

(b) Prove that if
$$1 > s > 2 - \frac{n}{2}$$
and $c = \frac{n-2+s}{1-s}$ then the function
$$u(x) = |x|^s - R^s$$
belongs to $W^{2,2}(B_R)$ and solves the strong Dirichlet problem
$$\begin{cases} Lu = 0 & \text{in } B_R, \\ u \in W_0^{1,2}(B_R). \end{cases}$$

Hint: Use Exercise 50, the computation of $L|x|^s$ from Exercise 5, and Exercise 28.

Remark: This example shows non-uniqueness in the strong Dirichlet problem for non-divergence form operator if the coefficients a_{ij} are discontinuous. If the coefficients a_{ij} are Lipschitz then the existence and uniqueness in the strong Dirichlet problem were proved in lectures.

52. Let Ω be a bounded domain in $\mathbb{R}^n$.

(a) Consider a divergence form uniformly elliptic operator in Ω with measurable coefficients:
$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right).$$
Fix some
$$q \in [2, \infty] \cap (n/2, \infty]. \tag{46}$$
Prove that
$$\text{if } u \in W^{1,2}_{loc}(\Omega) \text{ and } Lu \in L^q_{loc}(\Omega) \text{ then } u \in L^\infty_{loc}(\Omega).$$
Hint: Use Theorem 1.15 that says the following:
$$\text{if } u \in W_0^{1,2}(\Omega) \text{ and } Lu \in L^q(\Omega) \text{ then } u \in L^\infty(\Omega).$$

(b) Let B be the unit ball in $\mathbb{R}^n$ where $n > 4$. For any $q \in [2, n/2)$, give an example of a function u such that
$$u \in W^{1,2}(B) \text{ and } \Delta u \in L^q(B) \text{ but } u \notin L^\infty_{loc}(B).$$
Hint: Use Exercise 51.

Remark. The example of (b) shows that the restriction $q > n/2$ in (a) is essential.