# Blatt 11. Abgabe bis 12.01.24

Additional problems are marked by *

Everywhere $\Omega$ is a domain in $\mathbb{R}^n$.

53. Let

$$Lu = \sum_{i,j=1}^{n} \partial_i \left( a_{ij} \partial_j u \right) \tag{47}$$

be a uniformly elliptic operator in $\Omega$ with measurable coefficients. Let $u$ be a non-negative supersolution of $L$ in $\Omega$. Prove that, for any $p < 0$ and for any ball $B_{3R} \subset \Omega$,

$$\operatorname*{essinf}_{B_R} u \geq c \left( \fint_{B_R} u^p dx \right)^{1/p}, \tag{48}$$

where $c = c(n, \lambda, p) > 0$.

*Hint*: In the case $p = -1$ this estimate was proved in lectures (Corollary 3.5). Use the same approach for an arbitrary $p < 0$.

54. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $L$ be the operator (47). For any function $g \in W^{1,2}(\Omega)$, consider the following Dirichlet problem:

$$\begin{cases} Lu = 0 \text{ weakly in } \Omega \\ u - g \in W_0^{1,2}(\Omega) \end{cases} \tag{49}$$

(where the function $g$ plays a role of the boundary condition). Prove that

$$\|u\|_{W^{1,2}(\Omega)} \leq C \|g\|_{W^{1,2}(\Omega)},$$

where $C$ depends on $n, \lambda$ and $\Omega$.

*Hint*: Use substitution $v = u - g$ and then estimate $\|v\|_{W^{1,2}}$ by means of Exercise 24.

*Remark*: The problem (49) has a unique solution by Exercise 7.

55. Let $L$ be an operator (47).

(a) Prove that

$$u \in W_{loc}^{1,2}(\Omega) \text{ and } Lu \in L_{loc}^{\infty}(\Omega) \Rightarrow u \in L_{loc}^{\infty}(\Omega).$$

*Hint*: For any precompact open set $U$ s.t. $\overline{U} \subset \Omega$, solve the Dirichlet problem

$$\begin{cases} Lv = f \text{ weakly in } U, \\ v \in W_0^{1,2}(U), \end{cases} \tag{50}$$

where $f = Lu$. Then use Theorem 1.14 and Corollary 3.3 from lectures.

(b) Prove that

$$u \in W_{loc}^{1,2}(\Omega),\ Lu \in W_{loc}^{1,2}(\Omega) \text{ and } L(Lu) \in L_{loc}^{\infty}(\Omega) \Rightarrow u \in L_{loc}^{\infty}(\Omega).$$

56. * Let $L$ be an operator (47) and let $u \in W^{1,2}(\Omega)$ be a subsolution of $L$ in $\Omega$.

 (a) Let $f \in C^2(\mathbb{R})$ be a function on $\mathbb{R}$ such that, for some $A > 0$,

$$0 \le f'(t) \le A \text{ and } 0 \le f''(t) \le A \quad \text{for all } t \in \mathbb{R}. \tag{51}$$

 Prove that $v = f(u)$ is also a subsolution of $L$ in $\Omega$.

 *Hint*: In order to prove the inequality $Lf(u) \ge 0$ with a test function $\varphi$, use the inequality $Lu \ge 0$ with a test function $\psi = f'(u)\varphi$.

 (b) Prove that the function $v = u_+$ is a subsolution of $L$.

 *Hint*: Approximate the function $f(t) = t_+$ by a sequence of $C^2$ functions $\{f_k\}$ satisfying (51).

57. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) \tag{52}$$

be a uniformly elliptic operator in $\Omega$ with measurable coefficients. Define the Green operator

$$G : L^2(\Omega) \to L^2(\Omega)$$

of $L$ in $\Omega$ as follows: for any $f \in L^2(\Omega)$, set $Gf := u$, where $u$ is the unique solution of the following weak Dirichlet problem:

$$\begin{cases} Lu = -f & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

 (a) Prove that $G$ is a bounded linear operator from $L^2(\Omega)$ to $L^2(\Omega)$.

 (b) Prove that $G$ is a self-adjoint operator in $L^2(\Omega)$.

*Hint:* Use the same approach as in Exercise 30.

58. * Under the hypotheses of Exercise 57, prove the following properties of the Green operator $G$.

 (a) $G$ is a positive definite operator in $L^2(\Omega)$.

 (b) $G$ is a compact operator in $L^2(\Omega)$.

*Hint:* Use the same approach as in Exercise 31, as well as Exercise 57.

59. * Let $\Omega$ be bounded and consider the following eigenvalue problem for the operator (52):

$$\begin{cases} Lv + \gamma v = 0 \text{ in } \Omega \\ v \in W_0^{1,2}(\Omega) \setminus \{0\}, \end{cases} \tag{53}$$

where $\gamma$ is a spectral parameter. If a pair $v, \gamma$ satisfies (53), then $\gamma$ is called an eigenvalue of $L$ in $\Omega$ and $v$ is called an eigenfunction.

(a) Prove that function $v$ is an eigenfunction of the Green operator $G$ with the eigenvalue $\alpha$ then $\alpha > 0$ and $v$ is an eigenfunction of $L$ in $\Omega$ with the eigenvalue $\gamma = \frac{1}{\alpha}$.

(b) Prove that there exists an orthonormal basis $\{v_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ that consists of eigenfunctions of $L$ in $\Omega$; moreover, the corresponding eigenvalues $\gamma_k$ are positive, and the sequence $\{\gamma_k\}$ is monotone increasing and diverges to $+\infty$ as $k \to \infty$.

*Hint*: Apply the Hilbert-Schmidt theorem to the Green operator $G$ from Exercises 57 and 58.