Blatt 8. Abgabe bis 8.12.23

Additional problems are marked by *

Everywhere Ω is an open subset of $\mathbb{R}^n$.

39. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n)$.

    (a) Prove that $f * \varphi \in C(\mathbb{R}^n)$.

    (b) Prove that $f * \varphi \in C^\infty(\mathbb{R}^n)$ and, for any multiindex α,

    $$D^\alpha (f * \varphi) = f * D^\alpha \varphi. \tag{34}$$

    (c) Assume that $\operatorname{supp} \varphi \in B_r(0)$ for some $r > 0$. Prove that $\operatorname{supp}(f * \varphi)$ is a subset of the closed r-neighborhood of $\operatorname{supp} f$.

40. Let φ be a mollifier in $\mathbb{R}^n$, that is, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\operatorname{supp} \varphi \subset B_1(0)$, $\varphi \geq 0$, and $\int_{\mathbb{R}^n} \varphi dx = 1$. Let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty]$.

    (a) Prove that $f * \varphi \in L^p(\mathbb{R}^n)$ and

    $$\|f * \varphi\|_{L^p} \leq \|f\|_{L^p}. \tag{35}$$

    Hint: In the case $p \in (1, \infty)$, in order to estimate $\|f * \varphi\|_{L^p}$, use the identity

    $$\int_{\mathbb{R}^n} \varphi F = \int_{\mathbb{R}^n} \varphi^{\frac{1}{q}} (\varphi^{\frac{1}{p}} F),$$

    where q is the Hölder conjugate of p, and then apply the Hölder inequality.

    (b) Prove that if $p < \infty$ then

    $$f * \varphi_l \overset{L^p(\mathbb{R}^n)}{\longrightarrow} f \text{ as } l \to \infty, \tag{36}$$

    where $\varphi_l(x) = l^n \varphi(lx)$.
    Hint: Use the following two facts: (i) the set $C_0(\mathbb{R}^n)$ of continuos compactly supported functions is dense in $L^p(\mathbb{R}^n)$ if $p < \infty$; (ii) if $g \in C_0(\mathbb{R}^n)$ then $g * \varphi_l \rightrightarrows g$ as $l \to \infty$.
    Remark: The case $p = 2$ was also considered in Exercise 4.

41. Let φ be a mollifier in $\mathbb{R}^n$ and let $f \in W^{k,p}(\mathbb{R}^n)$ for some $p \in [1, \infty]$ and $k \in \mathbb{Z}_+$.

    (a) Prove that $f * \varphi \in W^{k,p}(\mathbb{R}^n)$ and

    $$D^\alpha (f * \varphi) = (D^\alpha f) * \varphi. \tag{37}$$

    (b) Prove that if $p < \infty$ then

    $$f * \varphi_l \overset{W^{k,p}(\mathbb{R}^n)}{\longrightarrow} f \text{ as } l \to \infty. \tag{38}$$

    Remark: The case $p = 2$ was also considered in Exercise 4.

42. For any $p \in [1,\infty]$ and $k \in \mathbb{Z}_+$, denote by $W_c^{k,p}(\Omega)$ the set of functions from $W^{k,p}(\Omega)$ that have compact support in Ω, and by $W_0^{k,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$.

(a) Prove that

$$W_c^{k,p}(\Omega) \subset W_c^{k,p}(\mathbb{R}^n),$$

where any function $f \in W_c^{k,p}(\Omega)$ extends to a function on $\mathbb{R}^n$ by setting $f = 0$ outside Ω.

(b) Prove that if $p < \infty$ then

$$W_c^{k,p}(\Omega) \subset W_0^{k,p}(\Omega). \tag{39}$$

Remark. The case $p = 2$ was also considered in Exercise 8.

43. * The purpose of this exercise is to prove that $Lip_{loc}(\Omega) = W_{loc}^{1,\infty}(\Omega)$.

(a) Let K be a compact subset of $\mathbb{R}^n$, and let $\{f_k\}$ be a sequence of functions on K such that (i) functions $\{f_k\}$ are uniformly bounded on K; (ii) functions $\{f_k\}$ are uniformly Lipschitz on K (that is, they have the same Lipschitz constant). Prove that there exists a subsequence that converges uniformly on K to a Lipschitz function.

Hint: Use the Arzela-Ascoli theorem: if a sequence of functions on a compact set K is uniformly bounded and equicontinuous then there exists a subsequence that converges uniformly on K.

(b) Prove that

$$Lip_c(\Omega) = W_c^{1,\infty}(\Omega).$$

Hint: It was proved in lectures that $Lip_c(\Omega) \subset W_c^{1,\infty}(\Omega)$. You need to prove the opposite inclusion

$$W_c^{1,\infty}(\Omega) \subset Lip_c(\Omega),$$

which means that, for any $f \in W_c^{1,\infty}(\Omega)$, there is a function $g \in Lip_c(\Omega)$ such that $f = g$ a.e. (that is, g is a Lipschitz representative of f).
For the proof, assuming that $f \in W_c^{1,\infty}(\Omega)$, extend first f to a function from $W_c^{1,\infty}(\mathbb{R}^n)$ (Exercise 42(a)), then use mollification (Exercise 40(a)) and the claim of (a).

(c) Prove that

$$Lip_{loc}(\Omega) = W_{loc}^{1,\infty}(\Omega).$$