Blatt 9. Abgabe bis 15.12.23

Additional problems are marked by *

44. The purpose of this question is to investigate the validity of the identity

$$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n). \tag{40}$$

(a) Prove (40) for all $p \in [1, \infty)$ and $k \geq 1$.
Hint: Use Exercise 42(b) and show that any function $f \in W^{k,p}(\mathbb{R}^n)$ can be approximated by a sequence of functions from $W_c^{k,p}(\mathbb{R}^n)$.

(b) Prove that (40) does not hold if $p = \infty$ and $k = 1$, that is, $W_0^{1,\infty}(\mathbb{R}^n) \subsetneqq W^{1,\infty}(\mathbb{R}^n)$.
Hint. Show that the function $u \equiv 1$ does not belong to $W_0^{1,\infty}(\mathbb{R}^n)$.

45. Let Ω be a bounded domain in $\mathbb{R}^n$ and

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u)$$

be a uniformly elliptic operator in Ω with measurable coefficients. Consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{weakly in } \Omega \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{41}$$

Prove that if $f \in L^2(\Omega)$ then

$$\|\nabla u\|_{L^2} \leq C |\Omega|^{\frac{1}{n}} \|f\|_{L^2}, \tag{42}$$

where $C = C(n, \lambda)$.
Hint. Use the same approach as in Exercise 22, but instead of the Friedrichs inequality use the Faber-Krahn inequality.

46. Consider in a bounded domain $\Omega \subset \mathbb{R}^n$ a uniformly elliptic divergence form operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) + \sum_{j=1}^{n} b_j \partial_j u + cu,$$

where the coefficients a_{ij}, b_j and c are bounded measurable functions and

$$c(x) \leq 0 \quad \text{a.e. in } \Omega.$$

Prove that the Dirichlet problem

$$\begin{cases} Lu = f & \text{weakly in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{43}$$

has at most one solution.

Hint: Use the following fact from the proof of Theorem 1.3: if $u \in W_0^{1,2}(\Omega)$ satisfies the inequality

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi \, dx \le b \int_\Omega |\nabla u| \, |\varphi| \, dx$$

for some constant b and for a function $\varphi = (u - \alpha)_+$ with any $\alpha > 0$, then $u \le 0$.

47. *(Chain rule for L)* Consider in Ω a uniformly elliptic divergence form operator

$$Lu = \sum \partial_i \left(a_{ij} \partial_j u \right)$$

with measurable coefficients.

(a) Let J be a closed interval and ψ be a C^∞-function on J such that

$$\sup_J |\psi'| < \infty \quad \text{and} \quad \sup_J |\psi''| < \infty.$$

Consider a function $u : \Omega \to J$ so that the composition $\psi(u)$ is well-defined. Prove that if

$$u \in W^{1,2}_{loc}(\Omega) \ \text{ and } Lu \in L^2_{loc}(\Omega)$$

then

$$L\psi(u) \in L^1_{loc}(\Omega)$$

and

$$L\psi(u) = \psi'(u) \, Lu + \psi''(u) \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i u. \tag{44}$$

Hint: Use Exercises 14 and 17.

(b) Assume that $u \in W^{1,2}_{loc}(\Omega)$ and $\operatorname{essinf}_\Omega u > 0$. Prove that if $Lu = 0$ in Ω then

$$L \ln \frac{1}{u} \ge 0 \text{ in } \ \Omega.$$

48. * For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and any distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, define the convolution $f * \varphi$ as a function on $\mathbb{R}^n$ as follows:

$$f * \varphi(x) = (f, \varphi(x - \cdot)),$$

where $\varphi(x - \cdot)$ denotes the test function $y \mapsto \varphi(x - y)$.

(a) Prove that $f * \varphi \in C(\mathbb{R}^n)$.

(b) Prove that $f * \varphi \in C^\infty(\mathbb{R}^n)$ and, for any multiindex α,

$$D^\alpha (f * \varphi) = f * D^\alpha \varphi = D^\alpha f * \varphi.$$