UPPER BOUNDS FOR EIGENVALUES OF THE DISCRETE AND CONTINUOUS LAPLACE OPERATORS

F. R. K. Chung
University of Pennsylvania, Philadelphia, Pennsylvania 19104

A. Grigor'yan
Imperial College, London, SW7 2B2 UK

S.-T. Yau
Harvard University, Cambridge, Massachusetts 02138

Introduction

In this paper, we are concerned with upper bounds of eigenvalues of Laplace operator on compact Riemannian manifolds and finite graphs. While on the former the Laplace operator is generated by the Riemannian metric, on the latter it reflects combinatorial structure of a graph. Respectively, eigenvalues have many applications in geometry as well as in combinatorics and in other fields of mathematics.

We develop a universal approach to upper bounds on both continuous and discrete structures based upon certain properties of the corresponding heat kernel. This approach is perhaps much more general than its realization here. Basically, we start with the following entries:

1° an underlying space M with a finite measure μ;
2° a well-defined Laplace operator Δ on functions on M so that Δ is a self-adjoint operator in $L^2(M,\mu)$ with a discrete spectrum;
3° if M has a boundary, then the boundary condition should be chosen so that it does not disrupt self-adjointness of Δ and is of dissipative nature;
4° a distance function $\operatorname{dist}(x,y)$ on M so that $|\nabla \operatorname{dist}| \le 1$ for an appropriate notion of gradient.

Let $\lambda_i, i = 0,1,2,\cdots$ denote the i-th eigenvalue of $-\Delta$ so that $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_i \le \cdots$. Then one of our results states that for any pair of disjoint subsets $X, Y \subset M$

$$\lambda_1 \le \frac{4}{\operatorname{dist}(X,Y)^2}\left(\log \frac{2\mu M}{\sqrt{\mu X \mu Y}}\right)^2 \tag{0.1}$$

We want to emphasize that validity of (0.1) does not depend on any a priori assumption on M which would restrict its geometry except compactness. A similar inequality holds in general for the difference $\lambda_i - \lambda_0$ for any $i \ge 1$.

Let us note that the most non-trivial term in (0.1) is a logarithm on the right hand side. In fact, the logarithm comes from a Gaussian exponential term which enters normally heat kernel upper bounds. A similar inequality for λ_i can be proved involving distances among

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k disjoint sets by constructing cutoff functions (close to the indicators of the sets X, Y) and by using them as trial functions in the minimax property of the eigenvalues.

Let M denote now a graph with a combinatorial Laplacian. We will prove the following inequality. For any two vertex subsets X, Y of a graph on n vertices which is not a complete graph, we will relate λ_1 to the least distance between a vertex in X to a vertex in Y as follows:

$$\operatorname{dist}(X,Y) \le \left\lceil \frac{\log \frac{\mu M}{\sqrt{\mu X \mu Y}}}{\log \frac{1}{1-\lambda}} \right\rceil \tag{0.2}$$

where $\lambda = \lambda_1$ if $1-\lambda_1 \ge \lambda_{n-1}-1$ and $\lambda = 1-(\lambda_{n-1}-\lambda_1)/2$ otherwise. Note that, in general $\lambda \ge \lambda_1/2$. Here, for example, μX can denote the sum of degrees of vertices of X and $\operatorname{dist}(X,Y)$ denotes the length of a shortest path joining a vertex in X and a vertex in Y. This is best possible within a constant factor for expander graphs with degree k and $\lambda = O(1/\sqrt{k})$. In general, we have

$$\lambda \le \log \frac{1}{1-\lambda} \le \left\lceil \frac{\log \frac{\mu M}{\sqrt{\mu X \mu Y}}}{\operatorname{dist}(X,Y)} \right\rceil$$

for any two subsets X, Y of the vertex set of the graph. We note the inequality (0.2) generalizes an earlier result in [2] (also see [4]):

$$\operatorname{diam} M \le \left\lceil \frac{\log(n-1)}{\log \frac{1}{1-\lambda}} \right\rceil$$

Special cases of (0.2) for regular graphs were investigated by Kahale in [7] . For subsets X_i, $i = 0, 1, \cdots, k$, the distance among X_i is defined to be the least distance $\operatorname{dist}(X_i, X_j)$ for $i \ne j$. We will generalize (0.2) by relating λ_k to the distance among $k+1$ subsets of the vertex set. Namely,

$$\min_{i \ne j} \operatorname{dist}(X_i, X_j) \le \max_{i \ne j} \left\lceil \frac{\log \sqrt{\frac{\mu \bar{X}_i \, \mu \bar{X}_j}{\mu X_i \, \mu X_j}}}{\log \frac{1}{1-\lambda'_k}} \right\rceil$$

where $\lambda'_k = \lambda_k$ if $1-\lambda_k \ge \lambda_{n-1}-1$ and $\lambda'_k = 1-(\lambda_{n-1}-\lambda_k)/2$, otherwise.

In the next Section we prove our main results for the continuous case of manifolds. The discrete cases for graphs will be considered in Section 2 with similar but different and simpler proofs. A more general setting will be considered in [10] .

Eigenvalues on manifolds

Let M be a smooth connected compact Riemannian manifold and Δ be a Laplace operator associated with the Riemannian metric i.e. in coordinates $x_1, x_2, \ldots x_n$

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x_j} \right)$$

where g^{ij} are contra-variant components of the metric tensor and $g = \det \|g_{ij}\|$ and u is a smooth function on M.

We admit that the manifold M has a boundary ∂M. If this is the case, we introduce a boundary condition

$$\alpha u + \beta \frac{\partial u}{\partial \nu} = 0 \tag{1.1}$$

where $\alpha(x), \beta(x)$ are non-negative smooth functions on M such that $\alpha(x) + \beta(x) > 0$ for all $x \in \partial M$.

For example, both Dirichlet and Neumann boundary conditions suit these assumptions. We note that

1° the Laplace operator with the boundary condition (1.1) is self-adjoint and has a discrete spectrum in $L^2(M, \mu)$, where μ is the Riemannian measure;

2° the condition (1.1) implies

$$u \frac{\partial u}{\partial \nu} \leq 0$$

where ν is the outer normal field on ∂M.

Let us introduce also a distance function $\operatorname{dist}(x, y)$ on $M \times M$ which may be equal to the geodesic distance, but in general we shall not assume so. Other than being a distance function the function, $\operatorname{dist}(x, y)$ must be Lipschitz and, moreover, for all $x, y \in M$

$$|\nabla \operatorname{dist}(x, y)| \leq 1.$$

Let us denote by Φ_i the eigenfunction corresponding to the i-th eigenvalue λ_i and normalized in $L^2 M, \mu$ so that $\{\Phi_i\}$ is an orthonormal frame in $L^2(M, \mu)$.

Theorem 1.1 *Suppose that we have chosen $k+1$ disjoint subsets $X_1, X_2, ... X_{k+1}$ of M such that the distance between any pair of them is at least $D > 0$. Then for any $k > 1$*

$$\lambda_k - \lambda_0 \leq \frac{1}{D^2} \max_{i \neq j} \left(\log \frac{4}{\int_{X_i} \Phi_0^2 \int_{X_j} \Phi_0^2} \right)^2 \tag{1.2}$$

We remark that, for example, if either the manifold has no boundary or the Neumann boundary condition has been chosen the first eigenvalue is 0 and the first eigenfunction is the constant

$$\Phi_0 = \frac{1}{\sqrt{\mu M}}$$

An immediate consequence of Theorem 1.1 is that for any $k > 1$

$$\lambda_k \leq \frac{4}{D^2} \max_{i \neq j} \left(\log \frac{2 \mu M}{\sqrt{\mu X_i \mu X_j}} \right)^2$$

Proof of Theorem 1.1: The proof is based upon two fundamental facts about the heat kernel $p(x, y, t)$ being by definition the unique fundamental solution to the heat equation

$$\frac{\partial u}{\partial t} u(x, t) - \Delta u(x, t) = 0$$

with the boundary condition (1.1) if the boundary ∂M is non-empty. The first fact is the eigenfunction expansion

$$p(x,y,t)=\sum_{i=0}^{\infty}e^{-\lambda_i t}\Phi_i(x)\Phi_i(y) \tag{1.3}$$

and the second is the following universal estimate:

$$\int_X\int_Y p(x,y,t)f(x)g(y)\mu(dx)\mu(dy)\le\left(\int_X f^2\int_Y g^2\right)^{\frac{1}{2}}\exp\left(-\frac{D^2}{4t}-\lambda_0 t\right) \tag{1.4}$$

which is true for any functions $f,g\in L^2(M,\mu)$ and for any two disjoint Borel sets $X,Y\subset M$ where $D=\operatorname{dist}(X,Y)$.

An inequality of type (1.4) appeared first in the paper by B.Davies [5] and [8] (on page 73). It was improved in [6] by introducing the term $-\lambda_0 t$ in the exponent on the right-hand side of (1.4) . The previous papers treated slightly different situations (for example, without a boundary) and this is why we will show at the end of this Section how to prove (1.4) .

Let us explain first the main idea behind the proof of the Theorem 1.1 in a particular case $k=2$. We start with integrating the eigenvalue expansion (1.3) as follows

$$I(f,g)\equiv\int_X\int_Y p(x,y,t)f(x)g(y)\mu(dx)\mu(dy)=\sum_{i=0}^{\infty}e^{-\lambda_i t}\int_X f\Phi_i\int_Y g\Phi_i \tag{1.5}$$

Let us denote by f_i the Fourier coefficients of the function $f\ 1_X$ with respect to the frame $\{\Phi_i\}$ and by g_i - that of $g\ 1_Y$. Then

$$I(f,g)=e^{-\lambda_0 t}f_0g_0+\sum_{i=1}^{\infty}e^{-\lambda_i t}f_ig_i\ge e^{-\lambda_0 t}f_0g_0-e^{-\lambda_1 t}\|f\ 1_X\|_2\ \|g\ 1_Y\|_2 \tag{1.6}$$

where we used the fact that

$$\left|\sum_{i=1}^{\infty}e^{-\lambda_i t}f_ig_i\right|\le e^{-\lambda_1 t}\left(\sum_{i=1}^{\infty}f_i^2\sum_{i=1}^{\infty}g_i^2\right)^{\frac{1}{2}}\le e^{-\lambda_1 t}\|f\ 1_X\|_2\|g\ 1_Y\|_2.$$

By comparing (1.6) and (1.4) we get

$$\exp(-(\lambda_1-\lambda_0))\|f\ 1_X\|_2\|g\ 1_Y\|_2\ge f_0g_0-\|f\ 1_X\|_2\|g\ 1_Y\|_2\exp\left(-\frac{D^2}{4t}\right). \tag{1.7}$$

Let us choose t so that the second term on the right-hand side (1.7) is equal to one half of the first one (here we take advantage of the Gaussian exponential since it can be made arbitrarily close to 0 by taking t small enough):

$$t=\frac{D^2}{4\log\frac{2\|f\ 1_X\|_2\|g\ 1_Y\|_2}{f_0g_0}}.$$

For this t, we have

$$\exp(-(\lambda_1-\lambda_0))\|f\ 1_X\|_2\|g\ 1_Y\|_2\ge\frac{1}{2}f_0g_0$$

which implies

$$\lambda_1 - \lambda_0 \le \frac{1}{t} \log \frac{2\|f\ 1_X\|_2 \|g\ 1_Y\|_2}{f_0 g_0}$$

and after substituting the value of t, we have

$$\lambda_1 - \lambda_0 \le \frac{4}{D^2} \left(\log \frac{2\|f\ 1_X\|_2 \|g\ 1_Y\|_2}{f_0 g_0} \right)^2$$

Finally, we choose $f = g = \Phi_0$ and taking into account that

$$f_0 = \int_X f\Phi_0 = \int_X \Phi_0^2,$$

and

$$\|f\ 1_X\|_2 = \left(\int_X \Phi_0^2 \right)^{\frac{1}{2}} = \sqrt{f_0}$$

and similar identities hold for g we obtain

$$\lambda_1 - \lambda_0 \le \frac{1}{D^2} \left(\log \frac{4}{\int_X \Phi_0^2 \int_Y \Phi_0^2} \right)^2$$

Now we turn to the general case $k > 2$. Let us consider a function $f(x)$ and denote by f_i^j the i-th Fourier coefficient of the function $f\ 1_{X_j}$ i.e.

$$f_i^j = \int_{X_j} f\Phi_i.$$

Let us put also in analogy to the case $k = 2$

$$I_{lm}(f, f) = \int_{X_l} \int_{X_m} p(x, y, t) f(x) f(y) \mu(dx)\mu(dy),$$

then we have the upper bound for $I_{lm}(f, f)$

$$I_{lm}(f, f) \le \|f\ 1_{X_l}\|_2 \|f\ 1_{X_m}\|_2 \exp\left(-\frac{D^2}{4t} - \lambda_0 t \right) \tag{1.8}$$

while we rewrite the lower bound (1.6) in another way:

$$I_{lm}(f, f) \ge e^{-\lambda_0 t} f_0^l f_0^m + \sum_{i=1}^{k-1} e^{-\lambda_i t} f_i^l f_i^m - e^{-\lambda_k t} \|f\ 1_{X_l}\|_2 \|f\ 1_{X_m}\|_2 \tag{1.9}$$

Now we want to kill the middle term on the right-hand side (1.9) by choosing appropriate l and m. To that end, let us consider $k+1$ vectors $f^m = (f_1^m, f_2^m, \cdots, f_{k-1}^m), m = 1, 2, \cdots, k+1$ in R^{k-1} and let us supply this $(k-1)$-dimensional space with a scalar product given by

$$(v, w) = \sum_{i=0}^{k-2} v_i w_i e^{-\lambda_{i+1} t}.$$

We suppose that we have chosen a value of t from the very beginning so t will not vary (although the optimal value of t will be found later). Let us apply the following elementary fact: out of any $k+1$ vectors in $(k-1)$-dimensional Euclidean space there are always two vectors with non-negative scalar product (see the end of this section for the proof). Therefore, we can find different l, m so that $(f^l, f^m) \geq 0$ and due to that choice we are able to cancel the second term on the right-hand side (1.9) .

Comparing (1.8) and (1.9) we get

$$e^{-(\lambda_k-\lambda_0)t}\|f\ 1_{X_l}\|_2\|f\ 1_{X_m}\|_2 \leq f_0^l f_0^m - \|f\ 1_{X_l}\|_2\|f\ 1_{X_m}\|_2 \exp\left(-\frac{D^2}{4t}\right) \tag{1.10}$$

Now, similar to the case $k=2$ we choose t so that the right-hand side is at least $\frac{1}{2}f_0^l f_0^m$. Since t must be independent on l, m we put simply

$$t = \min_{l \neq m} \frac{D^2}{4\log\frac{2\|f\ 1_{X_l}\|_2\|f\ 1_{X_m}\|_2}{f_0^l f_0^m}}$$

and obtain from (1.10)

$$\lambda_k - \lambda_0 \leq \frac{1}{t}\log\frac{2\|f\ 1_{X_l}\|_2\|f\ 1_{X_m}\|_2}{f_0^l f_0^m}$$

whence (1.2) follows by substituting t from above and by taking $f = \Phi_0$.

Now we will prove two auxiliary facts used in the course of the proof of Theorem 1.1.

Lemma 1.1 *For any two Borel sets $X, Y \subset M$ and for any functions $f, g \in L^2(M, \mu)$ we have:*

$$\int_X\int_Y p(x,y,t)f(x)g(y)\mu(dx)\mu(dy) \leq \left(\int_X f^2 \int_Y g^2\right)^{\frac{1}{2}} \exp\left(-\frac{D^2}{4t} - \lambda_1 t\right)$$

where $D = \operatorname{dist}(X, Y)$.

Proof of Lemma 1.1: Let us put

$$u(x,t) = \int_Y p(x,y,t)g(y)\mu(dy)$$

then $u(x,t)$ is a solution to the heat equation $u_t = \Delta u$ with the initial data $u(x,0) = g\ 1_Y$ and with the boundary condition (1.1) . As was shown in [6] , for any Lipschitz function $\xi(x,t)$ such that for all $x \in M, t > 0$

$$\xi_t + \frac{1}{2}|\nabla|\xi \leq 0$$

the integral

$$e^{2\lambda_0 t}\int_M u^2 e^{\xi(x,t)}\mu(dx)$$

is a decreasing function of t. Actually, this was proved for the Dirichlet boundary value problem but the proof used only that

$$u\frac{\partial u}{\partial \nu} \leq 0$$

that is true in our setting.

Let us put

$$\xi(x,t) = \frac{d^2(x)}{2(t+\varepsilon)}$$

where $d(x) \equiv \mathrm{dist}(x,Y)$ and $\varepsilon > 0$. Then

$$e^{2\lambda_0 t}\int_X u^2(x,t)e^{\xi(x,t)}\mu(dx) \leq \int_M u^2(x,0)e^{\xi(x,0)}\mu(dx)$$

or, taking into account that $u(x,0) = g\ 1_Y$, $\xi|_Y = 0$, and $\xi|_X \geq \frac{D^2}{2(t+\varepsilon)}$ and letting $\varepsilon \to 0$ we get

$$\int_X u^2(x,t)\mu(dx) \leq \exp\left(-\frac{D^2}{2t} - 2\lambda_0 t\right)\int_Y g^2(x)\mu(dx).$$

Finally, we obtain upon an application of the Cauchy-Schwarz inequality

$$\begin{aligned}\int_X\int_Y p(x,y,t)f(x)g(y)\mu(dx)\mu(dy) &= \int_X u(x,t)f(x)\mu(dx)\\ &\leq \left(\int_X f^2\right)^{\frac{1}{2}}\left(\int_X u^2(x,t)\right)^{\frac{1}{2}}\\ &\leq \left(\int_X f^2\right)^{\frac{1}{2}}\left(\int_Y g^2\right)^{\frac{1}{2}}\exp\left(-\frac{D^2}{4t} - \lambda_0 t\right)\end{aligned}$$

what was to be proved.

Now let us prove the next geometric lemma.

Lemma 1.2 *Let E be an n-dimensional Euclidean space and $v_1, v_2, \cdots, v_{n+2}$ be $n+2$ arbitrary vectors in E. Then there are two of them, say, v_i, v_j (where $i \neq j$) such that $(v_i, v_j) \geq 0$ where $(\cdot,\cdot)$ denotes the scalar product in E.*

Proof of Lemma 1.2: For $n = 1$ this is obvious. Let us prove the inductive step from $n-1$ to n. Suppose that for each pair of the given vectors their scalar product is negative. Let E' be a hyperplane orthogonal to v_{n+2} and let $v_i{}'$ be a projection of v_i on E', $i = 1, 2, ...n+1$. We claim that $(v_i{}', v_j{}') < 0$ provided $i \neq j$. Indeed, since $(v_i, v_{n+2}) < 0$ (where we assume $i \leq n+1$) all vectors v_i lie on the same half-space with respect to E' which implies that each of them is represented in the form

$$v_i = v_i{}' + a_i e$$

where $a_i > 0$ and e is a unit vector orthogonal to E' and directed to the same half-space as all v_i. Hence, we have

$$0 > (v_i, v_j) = (v_i{}' - a_i e, v_j{}' - a_j e) = (v_i{}', v_j{}') + a_i a_j$$

whence $(v_i{}', v_j{}') < 0$ follows. On the other hand, by the induction hypothesis out of $n+1$ vectors $v_i{}', i = 1, 2, ...n+1$ in the $(n-1)$ -dimensional space E' there are two vectors with non-negative scalar product. This contradiction proves the lemma.

Eigenvalues on graphs

Let G denote a graph on vertex set $V(G)$ and edge set $E(G)$. For a vertex v, the degree of v is denoted by d_v and for a subset X of $V(G)$, we define the volume of X to be

$$\mu X = \sum_{x \in X} d_x$$

The Laplacian of the graph is defined to be

$$\mathcal{L}(x,y) = \begin{cases} 1 & \text{if } x = y; \\ -\frac{1}{\sqrt{d_x d_y}} & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases}$$

Suppose $\mathcal{L}$ has eigenvalues $\lambda_0 = 0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ where $n = |V(G)|$. Then

$$\lambda_1 = \inf_f \frac{\sum\limits_{x \sim y} [f(x) - f(y)]^2}{\sum_x f^2(x) d_x}$$

where f ranges over functions satisfying

$$\sum_x f(x) d_x = 0.$$

Let T be an $n \times n$ function with the (x,x)-entry of value d_x. Then $T^{1/2}1$ is the eigenfunction associated with eigenvalue 0 where 1 denotes the function with all entries 1. It is not difficult to see that $\lambda_1 \le 1$ for any graph which is not a complete graph and $1 < \lambda_{n-1} \le 2$. More discussions on the eigenvalues λ_i can be found in [3] . Let $\bar{X}$ denote the complement of X in $V(G)$.

Theorem 2.1 *Suppose G is not a complete graph. For $X, Y \subset V(G)$, we have*

$$dist(X,Y) \le \left\lceil \frac{\log \sqrt{\frac{\mu \bar{X} \mu \bar{Y}}{\mu X \, \mu Y}}}{\log \frac{1}{1-\lambda}} \right\rceil \tag{2.1}$$

where $\lambda = \lambda_1$ if $1 - \lambda_1 \ge \lambda_{n-1} - 1$ and $\lambda = 1 - (\lambda_{n-1} - \lambda_1)/2$ otherwise.

Proof: For $X \subset V(G)$, we define

$$f_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

If we can show that for some integer t and a polynomial $p_t(z)$ of degree t,

$$\langle T^{1/2} f_Y, \, p_t(\mathcal{L})(T^{1/2} f_X) \rangle > 0$$

then there is a path of length at most t joining a vertex in X to a vertex in Y. Therefore we have $dist(X,Y) \le t$.

Let a_i denote the Fourier coefficient of $T^{1/2} f_X$, i.e.,

$$T^{1/2}f_X = \sum_{i=0}^{n-1} a_i\varphi_i$$

where φ_i's are eigenfunctions of $\mathcal{L}$. In particular, we have

$$\begin{aligned} a_0 &= \frac{\langle T^{1/2}f_X, T^{1/2}1\rangle}{\langle T^{1/2}1, T^{1/2}1\rangle} \\ &= \frac{\mu X}{\mu V}T^{1/2}1 \end{aligned}$$

Similarly, we write

$$T^{1/2}f_Y = \sum_{i=0}^{n-1} b_i\varphi_i$$

We choose

$$p_t(z) = \begin{cases} (1-z)^t & \text{if } 1-\lambda_1 \geq \lambda_{n-1}-1\ ; \\ (\frac{(\lambda_1+\lambda_{n-1})}{2} - z)^t & \text{otherwise} \end{cases}$$

Since G is not a complete graph, $\lambda_1 \neq \lambda_{n-1}$ and

$$|p_t(\lambda_i)| \leq (1-\lambda)^t$$

for all $i = 1, \cdots, n-1$. Therefore we have

$$\begin{aligned} \langle T^{1/2}f_Y, p_t(\mathcal{L})T^{1/2}f_X\rangle &= a_0b_0 + \sum_{i>0} p_t(\lambda_i)a_ib_i \\ &> a_0b_0 - (1-\lambda)^t\sqrt{\sum_{i>0} a_i^2 \sum_{i>0} b_i^2} \\ &= \frac{\mu X\,\mu Y}{\mu V} - (1-\lambda)^t\frac{\sqrt{\mu X\,\mu\bar{X}\,\mu Y\,\mu\bar{Y}}}{\mu V} \end{aligned}$$

Note that

$$\begin{aligned} \sum_{i>0} a_i^2 &= \|T^{1/2}f_X\|^2 - \frac{(\mu X)^2}{\mu V} \\ &= \frac{\mu X\,\mu\bar{X}}{\mu V} \end{aligned}$$

If we choose

$$t \geq \frac{\log\sqrt{\frac{\mu\bar{X}\,\mu\bar{Y}}{\mu X\,\mu Y}}}{\log\frac{1}{1-\lambda}}$$

we have

$$\langle T^{1/2}f_Y, p_t(\mathcal{L})T^{1/2}f_X\rangle > 0$$

Therefore we have

$$\mathrm{dist}(X,Y) \leq t.$$

To improve the inequality in some cases, we use the same approach as in [4] by considering the Chebychev polynomials of the first kind.

$$\begin{aligned} T_0(z) &= 1, \\ T_1(z) &= z, \\ T_{t+1}(z) &= 2zT_t(z) - T_{t-1}(z), \quad \text{for integer } t > 1. \end{aligned}$$

Or, equivalently,

$$T_t(z) = \cosh(t \cosh^{-1}(z)).$$

In the place of $p_t(\mathcal{L})$, we will use $S_t(\mathcal{L})$ where

$$S_t(x) = \frac{T_t(\frac{\lambda_1+\lambda_{n-1}-2x}{\lambda_{n-1}-\lambda_1})}{T_t(\frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1})}$$

Then we have

$$\max_{x\in[\lambda_1,\lambda_{n-1}]} \geq S_t(\lambda_1) \geq \frac{1}{T_t(\frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1})}$$

Suppose we take

$$t \geq \frac{\cosh^{-1}\sqrt{\frac{\mu\,\bar{X}\,\mu\bar{Y}}{\mu X\,\mu Y}}}{\cosh^{-1}\frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}}$$

Then we have

$$\langle T^{1/2}f_Y, S_t(\mathcal{L})T^{1/2}f_X\rangle > 0$$

Theorem 2.2 *Suppose G is not a complete graph. For $X, Y \subset V(G)$, we have*

$$\mathrm{dist}(X,Y) \leq \left\lceil \frac{\cosh^{-1}\sqrt{\frac{\mu\bar{X}\mu\bar{Y}}{\mu X\,\mu Y}}}{\cosh^{-1}\frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil$$

We can easily derive isoperimetric inequalities by using (2.1) . These isoperimetric inequalities are generalizations of the inqualities concerning vertex or edge "expansion" in Tanner [9] and in Alon and Miller [1] for regular graphs.

For a subset $X \subset V$, we define the s-neighborhood of X by

$$N_s(X) = \{y : \mathrm{dist}(x,y) \leq s, \text{ for some}\, x \in X\}$$

Suppose we choose $Y = V - N_s(X)$ in (2.1) . Theorem 2.1 implies the following result which gives a lower bound for the expansion of the neighborhood.

$$\mu N_t(X) \geq \frac{\mu X}{\frac{\mu X}{\mu V} + (1-\frac{\mu X}{\mu V})(1-\lambda)^{2(t-1)}}$$

Theorem 2.1 can be generalized for the case of relating the distance of $k+1$ disjoint subsets of vertices and the eigenvalues λ_k. The line of proof is quite similar to that in Section 2.

Theorem 2.3 *Suppose G is not a complete graph. For $X_i \subset V(G), i = 0, 1, \cdots, k$, we have*

$$\min_{i \neq j} \operatorname{dist}(X_i, X_j) \leq \max_{i \neq j} \left\lceil \frac{\log \sqrt{\frac{\mu \bar{X}_i \, \mu \bar{X}_j}{\mu X_i \, \mu X_j}}}{\log \frac{1}{1 - \lambda'_k}} \right\rceil$$

where $\lambda'_k = \lambda_k$ if $1 - \lambda_k \geq \lambda_{n-1} - 1$ and $\lambda'_k = 1 - (\lambda_{n-1} - \lambda_k)/2$ otherwise.

Proof:

There exist two distinct subsets in X_i's, denoted by X and Y, satisfying

$$\begin{aligned} \langle T^{1/2} f_Y, p_t(\mathcal{L}) T^{1/2} f_X \rangle &\geq a_0 b_0 + \sum_{i=1}^{k-1} p_t(\lambda_i) a_i b_i + \sum_{i \geq k} p_t(\lambda_i) a_i b_i \\ &> \frac{\mu X \, \mu Y}{\mu V} - (1 - \lambda'_k)^t \frac{\sqrt{\mu X \, \mu \bar{X} \, \mu Y \, \mu \bar{Y}}}{\mu V} \end{aligned}$$

by using Lemma 1.2 and by associating to each X a vector $(p_t(\lambda_1)^{1/2} a_1, \cdots, p_t(\lambda_{k-1})^{1/2} a_{k-1})$ where $f_X = \sum a_i \varphi_i$.

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