Analysis of elliptic differential equations

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Chapter 0

Introduction

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0.1 Elliptic operators in divergence and non-divergence form

In this course we are concerned with partial differential equations in $\mathbb{R}^n$ of the form $Lu = f$ where f is a given function, u is an unknown function, and L is a second order elliptic differential operator of one of the two forms:

1. $Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij}(x) \partial_j u)$ (a divergence form operator)

2. $Lu = \sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij} u$ (a non-divergence form operator).

In the both cases, the matrix (a_{ij}) depends on x, is symmetric, that is, $a_{ij} = a_{ji}$, and *uniformly elliptic*. The latter means that there is a constant λ such that, for all x from the domain of (a_{ij}) and for all $\xi \in \mathbb{R}^n$,

$$\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2 \tag{0.1}$$

where $|\xi| = \sqrt{\xi_1^2 + ... + \xi_n^2}$. In other words, all the eigenvalues of the matrix $(a_{ij}(x))$ (that are real because the matrix is symmetric) are located in the interval $\left[\lambda^{-1}, \lambda\right]$. The constant λ is called the ellipticity constant of (a_{ij}) or of L.

Of course, the Laplace operator

$$\Delta = \sum_{i=1}^{n} \partial_{ii} u$$

is both divergence and non-divergence form operator with the matrix $(a_{ij}) = \mathrm{id}$. It is uniformly elliptic with $\lambda = 1$.

If (a_{ij}) is a constant matrix, that is, independent of x, and (a_{ij}) is symmetric and positive definite, then the divergence and non-divergence form operators coincide and are uniformly elliptic. Indeed, we have for all $\xi \in \mathbb{R}^n$

$$\lambda_{\min} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \lambda_{\max} |\xi|^2 ,$$

where $\lambda_{\min}$ is the minimal eigenvalue of (a_{ij}) and $\lambda_{\max}$ – the maximal eigenvalues. Hence, (0.1) holds with $\lambda = \max\left(\lambda_{\max}, \lambda_{\min}^{-1}\right)$.

Note that the divergence form operator can be represented in the form

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x)\, \partial_j u\right) = \sum_{i,j=1}^{n} a_{ij}(x)\, \partial_{ij} u + \left(\partial_i a_{ij}\right) \partial_{ju},$$

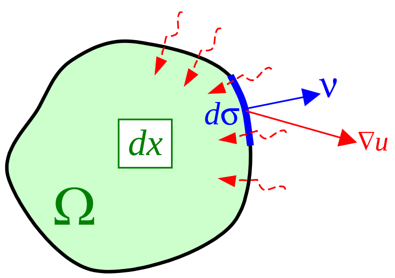
that is the sum of the non-divergence form operator and lower order terms. However, this works only for differentiable coefficients a_{ij}. In fact, the most interesting applications in mathematics requires operators with discontinuous coefficients a_{ij}. Of course, in this case the divergence form operator cannot be understood in the the sense of classical derivatives, and we will define the meaning of Lu in the weak sense.

0.2 Origin of divergence form operators

One of the origins of divergence form operators is *heat diffusion*. Let $u(x,t)$ denote the temperature in some medium at a point $x \in \mathbb{R}^3$ at time t. Fix a region $\Omega \subset \mathbb{R}^3$. By the Fourier law of thermoconductance, the amount dQ of the heat energy that has flown into Ω through a piece $d\sigma$ of its boundary $\partial\Omega$ between the time moments t and $t + dt$ is equal to

$$dQ = \sum_{i,j=1}^{3} a_{ij}(x)\, \nu_i \partial_j u \; d\sigma dt$$

where ν is the outer unit normal vector field to $\partial\Omega$ at a point $x \in d\sigma$ and $a_{ij}(x)$ is the tensor of the thermal conductance of the material of the body (the dependence of a_{ij} of x means that the conductance may be different at different points, and the dependence on the indices i, j reflects the fact that the conductance may be different in different directions).



The expression

$$\sum_{i,j=1}^{3} a_{ij}(x)\, \nu_i \partial_j u$$

can be regarded as an inner product of the vectors ν and ∇u with the coefficients $a_{ij}(x)$ (symmetry and positive definiteness of this matrix are needed for that). Hence, the total energy Q that has flown into Ω through its entire boundary between time moments t and $t+h$ is

$$Q = \int_t^{t+h} \int_{\partial\Omega} \sum_{i,j=1}^{3} a_{ij}(x)\, \nu_i \partial_j u \; d\sigma dt,$$

On the other hand, the amount of heat energy dQ' acquired by a piece dx of Ω from time t to time $t+h$ is equal to

$$dQ' = (u(x, t+h) - u(x,t))\, c\rho dx$$

where ρ is the density of the material of the body and c is its heat capacity (both c and ρ are functions of x). Indeed, the volume element dx has the mass ρdx, and increase of its temperature by one degree requires $c\rho dx$ of heat energy. Hence, increase of the temperature from $u(x,t)$ to $u(x, t+dt)$ requires $(u(x,t+h) - u(x,t))\, c\rho dx$ of heat energy. The total amount Q' of energy absorbed by the entire body Ω from time t to time $t+h$ is equal to

$$Q' = \int_\Omega (u(x, t+h) - u(x,t))\, c\rho dx.$$

By the law of conservation of energy, in absence of heat sources we have $Q = Q'$, that is,

$$\int_t^{t+h} \left(\int_{\partial\Omega} \sum_{i,j=1}^{3} a_{ij} \nu_i \partial_j u \; d\sigma \right) dt = \int_\Omega (u(x, t+h) - u(x,t))\, c\rho dx.$$

Dividing by h and passing to the limit as $h \to 0$, we obtain

$$\int_{\partial\Omega} \sum_{i,j=1}^{3} a_{ij} \nu_i \partial_j u \; d\sigma = \int_\Omega (\partial_t u)\, c\rho dx.$$

Applying the divergence theorem to the vector field $\overrightarrow{F}$ with components

$$F_i = \sum_{j=1}^{3} a_{ij} \partial_j u$$

we obtain

$$\int_{\partial\Omega} \sum_{i,j=1}^{3} a_{ij} \nu_i \partial_j u \; d\sigma = \int_{\partial\Omega} \overrightarrow{F} \cdot \nu \; d\sigma = \int_\Omega \operatorname{div} \overrightarrow{F} dx = \int_\Omega \sum_{i=1}^{3} (\partial_i F_i)\, dx = \int_\Omega \sum_{i,j=1}^{3} \partial_i (a_{ij} \partial_j u)\, dx,$$

which implies

$$\int_\Omega c\rho\, \partial_t u\, dx = \int_\Omega Lu\, dx,$$

where

$$L = \sum_{i,j=1}^{3} \partial_i \left(a_{ij} \partial_j u\right)$$

is the divergence form operator. Since this identity holds for any region Ω, it follows that the function u satisfies the following *heat equation*

$$c\rho\, \partial_t u = Lu.$$

In particular, if u is stationary, that is, does not depend on t, then u satisfies $Lu = 0$.

We have seen that in the above derivation the operator L comes out exactly in the divergence form because of an application of the divergence theorem.

0.3 Origin of non-divergence form operators

The operators in non-divergence form originate from different sources, in particular, from *stochastic diffusion* processes. A stochastic diffusion process in $\mathbb{R}^n$ is mathematical model of Brownian motion in inhomogeneous media. It is described by the family $\{\mathbb{P}_x\}_{x\in\mathbb{R}^n}$ of probability measures, where $\mathbb{P}_x$ is the probability measure on the set Ω_x of all continuous paths $\omega : [0, \infty) \to \mathbb{R}^n$ such that is $\omega(0) = x$.

Define for any $t \geq 0$ a random variable $X(t)$ on Ω_x by $X(t)(\omega) = \omega(t)$. The *random path* $t \mapsto X(t)$ can be viewed as a stochastic movement of a microscopic particle. The diffusion process is described by its infinitesimal means

$$\mathbb{E}_x \left(X_i(t + dt) - X_i(t)\right) = b_i dt + o(dt) \quad \text{as } dt \to 0,$$

its infinitesimal covariances

$$\mathbb{E}_x \left(\left(X_i(t + dt) - X_i(t)\right)\left(X_j(t + dt) - X_j(t)\right)\right) = a_{ij} dt + o(dt) \quad \text{as } dt \to 0,$$

where b_i and a_{ij} are some functions that in general depend in x and t, but we assume for simplicity that they depend only on x.

By construction, the matrix (a_{ij}) is symmetric and positive definite, as any covariance matrix. The functions a_{ij} and b_i determine the non-divergence form operator with lower order terms:

$$Lu = \sum_{i,j=1}^{n} a_{ij} \partial_{ij} u + \sum_{i=1}^{n} b_i \partial_i u,$$

that has the following relation to the process: for any bounded continuous function f on $\mathbb{R}^n$, the function

$$u(x, t) = \mathbb{E}_x \left(f(X(t))\right)$$

satisfies the heat equation

$$\partial_t u = Lu$$

with the above operator L. This equation is called the *Kolmogorov backward equation.* This operator L is called the *generator* of the diffusion process because it contains all the information about this stochastic process.

Chapter 1

Weak Dirichlet problem for divergence form operators

In this Chapter we deal with the divergence form operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \left(x \right) \partial_j u \right)$$

defined in an open set $\Omega \subset \mathbb{R}^n$. We always assume that the coefficients $a_{ij}\left(x\right)$ are measurable functions of x (not necessarily continuous), the matrix (a_{ij}) is symmetric, that is, $a_{ij} = a_{ji}$, and positive definite at any $x \in \Omega$. Then the operator L is called elliptic. Should the condition (0.1) be satisfied then L is called uniformly elliptic.

Since the coefficients a_{ij} may be not differentiable, we have to specify exactly how the equation $Lu = f$ is understood.

1.1 Distributions

Let Ω be an open subset of $\mathbb{R}^n$. Denote by $\mathcal{D}\left(\Omega\right)$ the linear topological space that as a set coincides with $C_0^\infty\left(\Omega\right)$, the linear structure in $\mathcal{D}\left(\Omega\right)$ is defined with respect to addition of functions and multiplication by scalars from $\mathbb{R}$, and the topology in $\mathcal{D}\left(\Omega\right)$ is defined by means of the following convergence: a sequence $\{\varphi_k\}$ of functions from $\mathcal{D}\left(\Omega\right)$ converges to $\varphi \in \mathcal{D}\left(\Omega\right)$ in the space $\mathcal{D}\left(\Omega\right)$ if the following two conditions are satisfied:

1. $\varphi_k \rightrightarrows \varphi$ in Ω and $D^\alpha \varphi_k \rightrightarrows D^\alpha \varphi$ for any multiindex α of any order;
2. there is a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_k \subset K$ for all k.

It is possible to show that this convergence is indeed topological, that is, given by a certain topology.

Any linear topological space $\mathcal{V}$ has a dual space $\mathcal{V}'$ that consists of continuous linear functionals on $\mathcal{V}$.

Definition. Any linear continuos functional $f : \mathcal{D}\left(\Omega\right) \to \mathbb{R}$ is called a *distribution* in Ω (or generalized functions). The set of all distributions in Ω is denoted by $\mathcal{D}'\left(\Omega\right)$. If $f \in \mathcal{D}'\left(\Omega\right)$ then the value of f on a *test function* $\varphi \in \mathcal{D}\left(\Omega\right)$ is denoted by (f, φ).

Any locally integrable function $f : \Omega \to \mathbb{R}$ can be regarded as a distribution as follows: it acts on any test function $\varphi \in \mathcal{D}(\Omega)$ by the rule

$$(f, \varphi) := \int_\Omega f \varphi \, dx. \tag{1.1}$$

Note that two locally integrable functions f, g correspond to the same distribution if and only if $f = g$ almost everywhere, that is, if the set

$$\{x \in \Omega : f(x) \neq g(x)\}$$

has measure zero. We write shortly in this case

$$f = g \text{ a.e.} \tag{1.2}$$

Clearly, the relation (1.2) is an equivalence relation, that gives rise to equivalence classes of locally integrable functions. The set of all equivalence classes of locally integrable functions is denoted[1] by $L^1_{loc}(\Omega)$. The identity (1.1) establishes the injective mapping $L^1_{loc}(\Omega) \to \mathcal{D}'(\Omega)$ so that $L^1_{loc}(\Omega)$ can be regarded as a subspace of $\mathcal{D}'(\Omega)$.

There are distributions that are not represented by any L^1_{loc} function, that is, the difference $\mathcal{D}'(\Omega) \setminus L^1_{loc}(\Omega)$ is not empty. For example, define the *delta-function* δ_{x_0} for any $x_0 \in \Omega$ as follows:

$$(\delta_{x_0}, \varphi) = \varphi(x_0).$$

Although historically δ_{x_0} is called delta-*function*, it is a distribution that does not correspond to any function.

Definition. Let $f \in \mathcal{D}'(\Omega)$. Fix a multiindex α. A *distributional partial derivative* $D^\alpha f$ is a distribution that acts on test functions $\varphi \in \mathcal{D}(\Omega)$ as follows:

$$(D^\alpha f, \varphi) = (-1)^{|\alpha|} (f, D^\alpha \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega), \tag{1.3}$$

where $D^\alpha \varphi$ is the classical (usual) derivative of φ.

Note that the right hand side of (1.3) makes sense because $D^\alpha \varphi \in \mathcal{D}(\Omega)$. Moreover, the right hand side of (1.3) is obviously a linear continuous functions in $\varphi \in \mathcal{D}(\Omega)$, which means that $D^\alpha f$ exists *always* as a distribution.

If there is a function $g \in L^1_{loc}(\Omega)$ such that

$$(g, \varphi) = (-1)^{|\alpha|} (f, D^\alpha \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega),$$

then g is called a weak D^α derivative of f. In this case the distributional derivative $D^\alpha f$ is represented by the function g.

If $f \in C^k(\Omega)$ then its classical derivative $D^\alpha f$ with $|\alpha| \leq k$ coincides with the weak and, hence, distributional derivative.

[1] Sometimes $L^1_{loc}(\Omega)$ is loosely used to denote the set of all locally integrable functions in Ω. However, in a strict sense, the elements of $L^1_{loc}(\Omega)$ are not functions but equivalence classes of functions.

1.2 Sobolev spaces

As before, let Ω be an open subset of $\mathbb{R}^n$. Fix $p \in [1, \infty)$. A Lebesgue measurable function $f : \Omega \to \mathbb{R}$ is called *p-integrable* if

$$\int_\Omega |f|^p \, dx < \infty.$$

Two measurable functions in Ω (in particular, p-integrable functions) are called equivalent if

$$f = g \text{ a.s.}$$

This is an equivalence relation, and the set of all equivalence classes of p-integrable functions in Ω is denoted by $L^p(\Omega)$. It follows from the Hölder inequality, that $L^p(\Omega) \subset L^1_{loc}(\Omega)$. In particular, all the elements of $L^p(\Omega)$ can be regarded as distributions.

The set $L^p(\Omega)$ is a linear space over $\mathbb{R}$. Moreover, it is a Banach space (=complete normed space) with respect to the norm

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p \, dx \right)^{1/p}.$$

The Banach spaces $L^p(\Omega)$ are called *Lebesgue spaces*.

The case $p = 2$ is of special importance because the space $L^2(\Omega)$ has inner product

$$(f, g) = \int_\Omega f g \, dx,$$

whose norm coincides with $\|f\|_2$ as

$$\|f\|_{L^2} = \left(\int_\Omega f^2 dx \right)^{1/2} = \sqrt{(f, f)}.$$

Hence, $L^2(\Omega)$ is a Hilbert space.

Definition. Define the Sobolev space $W^{k,p}$ for arbitrary non-negative integer k and $p \in [1, \infty)$

$$W^{k,p}(\Omega) = \{ f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \le k \}, \tag{1.4}$$

where $D^\alpha f$ is distributional derivative.

In words, $W^{k,p}(\Omega)$ is a subspace of $L^p(\Omega)$ that consists of functions having in $L^p(\Omega)$ all weak partial derivatives of the order $\le k$. In particular, $W^{0,p} = L^p$. It is easy to see that $C_0^\infty(\Omega) \subset W^{k,p}(\Omega)$ for any k and p.

Let us introduce the $W^{k,p}(\Omega)$ the following norm:

$$\|f\|_{W^{k,p}}^p = \sum_{\alpha : |\alpha| \le k} \int_\Omega |D^\alpha f|^p \, dx.$$

It is possible to show that $\|\cdot\|_{W^{k,p}}$ is indeed a norm, and $W^{k,p}$ is a Banach space with this norm. In the case $p = 2$ this norm is given by the inner product:

$$(f, g)_{W^{k,2}} = \sum_{\alpha : |\alpha| \le k} \int_\Omega D^\alpha f \; D^\alpha g \, dx,$$

so that $W^{k,2}$ is a Hilbert space.

Similarly, define the space

$$W_{loc}^{k,p}(\Omega) = \{f \in L_{loc}^p(\Omega) : D^\alpha f \in L_{loc}^p(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \le k\}. \tag{1.5}$$

It is easy to see that $C^\infty(\Omega) \subset W_{loc}^{k,p}(\Omega)$ for any k and p.

1.3 The weak Dirichlet problem

As above, let $(a_{ij}(x))$ be an x-dependant matrix in Ω with the following properties: functions $a_{ij}(x)$ are measurable in x, the matrix is symmetric, that is, $a_{ij} = a_{ji}$, and uniformly elliptic, that is, for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 \tag{1.6}$$

for some constant λ. We are going to define how to understand the divergence form operator

$$Lu = \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u) \tag{1.7}$$

in this case.

Definition. Let $u \in W_{loc}^{1,2}$ and $f \in L_{loc}^2(\Omega)$. We say that the equation $Lu = f$ is satisfied in a *weak sense* or *weakly* if, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi\, dx = -\int_\Omega f\varphi\, dx. \tag{1.8}$$

Note that the integral on the right hand side of (1.8) makes sense because the integration can be reduced to a compact set $\operatorname{supp}\varphi$, where φ is bounded and f is integrable. The left hand side makes sense similarly because $\partial_j u \in L_{loc}^2$ and hence is integrable on $\operatorname{supp}\varphi$, while $\partial_i\varphi$ and a_{ij} are bounded (the latter follows from (1.6)).

Motivation for this definition is as follows. Assume that $a_{ij} \in C^1$ and $u \in C^2$. Then the equation $Lu = f$ can be understood in the classical sense. Multiplying it by $\varphi \in \mathcal{D}(\Omega)$ and integrating in Ω using integration by parts, we obtain

$$\int f\varphi\, dx = \sum_{i,j=1}^n \int \partial_i(a_{ij}\partial_j u)\varphi\, dx = -\sum_{i,j=1}^n \int a_{ij}\partial_j u \partial_i\varphi\, dx,$$

that is the identity (1.8). Hence, the weak meaning of the equation $Lu = f$ is consistent with the classical one.

Define $W_0^{1,2}(\Omega)$ as the subspace of $W^{1,2}(\Omega)$ that is obtained by taking the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$.

Lemma 1.1 *Let $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$. Then $Lu = f$ holds in a weak sense if and only if* (1.8) *holds for all $\varphi \in W_0^{1,2}(\Omega)$.*

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Proof. If (1.8) holds for all $\varphi \in W_0^{1,2}(\Omega)$ then, of course, it holds also for all $\varphi \in C_0^\infty(\Omega)$. Let us prove the converse statement. For any $\varphi \in W_0^{1,2}(\Omega)$ there is a sequence $\{\varphi_k\}$ of functions from $C_0^\infty(\Omega)$ such that $\varphi_k \to \varphi$ in the norm of $W^{1,2}(\Omega)$. Any φ_k satisfies (1.8), and we would like to pass to the limit as $k \to \infty$. For that, it suffices to verify that the both sides of (1.8) are continuous functions of $\varphi \in W^{1,2}(\Omega)$.

Clearly, the functional $\varphi \mapsto \int_\Omega f\varphi dx$ is continuous in $L^2(\Omega)$ and, hence, in $W^{1,2}(\Omega)$. Let us show that the functional

$$\varphi \mapsto A(\varphi) := \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi \, dx$$

is continuous in $W^{1,2}(\Omega)$. It is linear, so that its continuity is equivalent to the boundedness. Hence, it suffices to prove that

$$|A(\varphi)| \le C \|\varphi\|_{W^{1,2}} \tag{1.9}$$

for some constant C and all $\varphi \in W^{1,2}(\Omega)$. Fix $x \in \Omega$ and consider in $\mathbb{R}^n$ the inner product

$$(\xi, \eta)_a := \sum_{i,j=1}^n a_{ij}(x)\xi_i\eta_j.$$

Indeed, it is bilinear, symmetric and positive definite by the ellipticity. By the Cauchy-Schwarz inequality and the uniform ellipticity condition, we obtain

$$|(\xi,\eta)_a| \le \sqrt{(\xi,\xi)_a}\sqrt{(\eta,\eta)_a} \le \lambda|\xi||\eta|$$

It follows that

$$\begin{aligned} |A(\varphi)| &\le \int_\Omega \left| \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi \right| dx \le \int_\Omega \lambda |\nabla u| |\nabla \varphi| \, dx \\ &\le \lambda \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2} \left(\int_\Omega |\nabla \varphi|^2 dx \right)^{1/2}, \end{aligned}$$

whence

$$|A(\varphi)| \le \lambda \|u\|_{W^{1,2}} \|\varphi\|_{W^{1,2}}, \tag{1.10}$$

which proves (1.9). ■

Definition. Given a divergence form operator L in an open set Ω as above, consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

that is understood in the weak sense as follows:

$$\begin{cases} Lu = f & \text{weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{1.11}$$

In other words, the weak meaning of the boundary condition $u = 0$ on $\partial\Omega$ is $u \in W_0^{1,2}(\Omega)$.

Theorem 1.2 *Let Ω be a bounded domain. Then the weak Dirichlet problem* (1.11) *with the operator* (1.7) *has exactly one solution for any $f \in L^2(\Omega)$.*

Proof. Consider in $W_0^{1,2}$ the following bilinear symmetric form

$$[u,v]_a := \int_\Omega a_{ij}(x)\,\partial_i u(x)\,\partial_j v(x)\,dx$$

(the integral converges because a_{ij} are bounded and $\partial_i u, \partial_i v \in L^2(\Omega)$). By the uniform ellipticity we have

$$[u,u]_a = \int_\Omega a_{ij}(x)\,\partial_i u(x)\,\partial_j u(x)\,dx \le \lambda \int_\Omega |\nabla u|^2\,dx \le \lambda \|u\|^2_{W^{1,2}},$$

and

$$[u,u]_a \ge \lambda^{-1} \int_\Omega |\nabla u|^2\,dx.$$

On the other hand, by the Friedrichs inequality we have, for any $u \in W_0^{1,2}(\Omega)$ that

$$\int_\Omega |\nabla u|^2\,dx \ge c \int_\Omega u^2 dx,$$

with some positive constant $c = c(\Omega)$. Assuming without loss of generality that $c < 1$, we obtain

$$\int_\Omega |\nabla u|^2\,dx \ge \frac{c}{2}\int_\Omega \left(u^2 + |\nabla u|^2\right)dx = \frac{c}{2}\|u\|^2_{W^{1,2}},$$

whence it follows that

$$\frac{c}{2\lambda}\|u\|^2_{W^{1,2}} \le [u,u]_a \le \lambda \|u\|^2_{W^{1,2}}.$$

In particular, $[u,v]_a$ is positive definite and, hence, is an inner product in $W_0^{1,2}$. Since the norm $[u,u]_a^{1/2}$ is equivalent to $\|u\|_{W^{1,2}}$, we see that $W_0^{1,2}$ with the inner product $[\cdot,\cdot]_a$ is a Hilbert space.

The weak equation $Lu = f$ can be rewritten in the form

$$[u,\varphi]_a = -\int_\Omega f\varphi\,dx \quad \forall \varphi \in W_0^{1,2}.$$

The right hand side $\int_\Omega f\varphi\,dx$ is obviously a bounded linear functional of $\varphi \in W_0^{1,2}$. Therefore, the existence of $u \in W_0^{1,2}$ that solves this equation, follows from the Riesz representation theorem. Indeed, the latter says that in any Hilbert space H with inner product $[\cdot,\cdot]$, the equation

$$[u,\varphi] = l(\varphi) \quad \forall \varphi \in H$$

has a unique solution $u \in H$ provided $l(\varphi)$ is a bounded linear functional. ■

1.4 Weak Dirichlet problem with lower order terms

Here we consider a more general operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_i u \right) + \sum_{i=1}^{n} b_i \left(x \right) \partial_i u \tag{1.12}$$

in an open set $\Omega \subset \mathbb{R}^n$. We assume that the coefficients a_{ij}, b_i are measurable functions, the second order part $\sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_i u \right)$ is uniformly elliptic divergence form operator, and that all functions b_i are bounded, that is, there is a constant b, such that

$$\sum_{i=1}^{m} |b_i| \leq b \text{ a.e. in } \Omega.$$

Assuming that $u \in W^{1,2} \left(\Omega \right)$ and $f \in L^2 \left(\Omega \right)$, the equation $Lu = f$ is understood weakly as follows: for any $\varphi \in W_0^{1,2} \left(\Omega \right)$

$$\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \varphi - \sum_{i=1}^{n} b_i \left(\partial_i u \right) \varphi \right) dx = - \int_\Omega f \varphi. \tag{1.13}$$

1.4.1 Uniqueness

Theorem 1.3 (Uniqueness) *Let Ω be a bounded domain and L be the operator* (1.12). *Then the weak Dirichlet problem*

$$\begin{cases} Lu = f & \textit{weakly in } \Omega \\ u \in W_0^{1,2} \left(\Omega \right) \end{cases}$$

has at most one solution.

For the proof we need some facts about weak derivatives that will be proved later on.

Lemma 1.4 *If $u \in W_0^{1,2} \left(\Omega \right)$ then, for any $\alpha \geq 0$, also $\left(u - \alpha \right)_+ \in W_0^{1,2} \left(\Omega \right)$ and*

$$\nabla \left(u - \alpha \right)_+ = \begin{cases} \nabla u & \text{a.e.} \textit{on the set } \{u > \alpha\} \\ 0 & \text{a.e.} \textit{on the set } \{u \leq \alpha\} \end{cases} \tag{1.14}$$

Lemma 1.5 *If $u \in W_0^{1,2} \left(\Omega \right)$ then, for any $\alpha \in \mathbb{R}$,*

$$\nabla u = 0 \text{ a.e. } \textit{on the set } \{u = \alpha\}.$$

Besides we are going to use the following inequality that also will be proved later (see Corollary 1.9).

Sobolev inequality. *If $n > 2$ then, for any $\varphi \in W_0^{1,2} \left(\Omega \right)$,*

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq c_n \left(\int_\Omega |\varphi|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}},$$

where c_n is a positive constant depending only on n.

If $n = 2$ and Ω is bounded then, for any $q \geq 1$ and for any $\varphi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq c \left(\int_\Omega |\varphi|^{2q} \, dx \right)^{1/q},$$

where c is a positive constant depending on q and Ω.

Proof of Theorem 1.3. We need to prove that if $u \in W_0^{1,2}(\Omega)$ and $Lu = 0$ then $u = 0$ a.e. in Ω. It suffices to prove that $u \leq 0$ a.e. in Ω since $u \geq 0$ a.e. follows by the same argument applied to $-u$.

We use the notion of the *essential supremum* that is defined by

$$\operatorname*{esssup}_\Omega u = \inf \{ k \in \mathbb{R} : u \leq k \text{ a.e.} \}.$$

Then $u \leq 0$ a.e. is equivalent to $\operatorname{esssup} u \leq 0$. Let us assume from the contrary that

$$\alpha_0 := \operatorname*{esssup}_\Omega u > 0$$

and bring this to contradiction (note that $\alpha_0 = \infty$ is allowed). The weak equation $Lu = 0$ implies that, for any $\varphi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^n a_{ij} \partial_j u \partial_i \varphi \, dx = \int_\Omega \sum_{i=1}^n b_i \partial_i u \, \varphi \, dx. \tag{1.15}$$

The right hand side of (1.15) admits a simple estimate

$$\int_\Omega \sum_{i=1}^n b_i \left(\partial_i u \right) \varphi \, dx \leq b \int_\Omega |\nabla u| \, |\varphi| \, dx. \tag{1.16}$$

Now we specify function φ as follows: choose α from the interval

$$0 \leq \alpha < \alpha_0$$

and set

$$\varphi = (u - \alpha)_+ .$$

By Lemma 1.4, $\varphi \in W_0^{1,2}(\Omega)$ so that we can use this φ in (1.15). Consider the set

$$S_\alpha := \{ x \in \Omega : \alpha < u(x) < \alpha_0 \}$$

and let us verify that

$$\nabla \varphi = \begin{cases} \nabla u & \text{a.e. in } S_\alpha, \\ 0 & \text{a.e. in } S_\alpha^c, \end{cases} \tag{1.17}$$

where $S_\alpha^c = \Omega \setminus S_\alpha$. Indeed, $S_\alpha \subset \{u > \alpha\}$, so that the first line in (1.17) follows from that in (1.14). Note that

$$S_\alpha^c = \{ u \leq \alpha \} \cup \{ u \geq \alpha_0 \} .$$

On the set $\{u \leq \alpha\}$ we have by (1.14) $\nabla\varphi = 0$. Since the set $\{u > \alpha_0\}$ has measure 0 by definition of α_0, we see that

$$u = \alpha_0 \text{ a.e. on } \{u \geq \alpha_0\}.$$

By Lemmas 1.4 and 1.5 we conclude that $\nabla\varphi = \nabla u = 0$ a.e. on $\{u \geq \alpha_0\}$, which finishes the proof of (1.17).

Let us now prove that

$$|\nabla u|\,\varphi = \begin{cases} |\nabla\varphi|\,\varphi & \text{a.e. in } S_\alpha, \\ 0, & \text{a.e. in } S_\alpha^c. \end{cases} \tag{1.18}$$

Indeed, on the first line in (1.18) follows from that of (1.17). On the set $\{u \leq \alpha\}$ we have $\varphi = 0$, while on $\{u \geq \alpha_0\}$ we have as above $\nabla u = 0$, which proves the second line in (1.18). It follows that

$$\int_\Omega |\nabla u|\,\varphi\,dx = \int_{S_\alpha} |\nabla\varphi|\,\varphi dx \leq \left(\int_{S_\alpha} \varphi^2 dx\right)^{1/2} \left(\int_{S_\alpha} |\nabla\varphi|^2\,dx\right)^{1/2}.$$

For the left hand side of (1.15) we have by (1.17) and the uniform ellipticity

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi\,dx = \int_{S_\alpha} \sum_{i,j=1}^n a_{ij}\partial_j \varphi \partial_i \varphi\,dx \geq \lambda^{-1} \int_{S_\alpha} |\nabla\varphi|^2\,dx.$$

Combining the above two calculations with (1.15), we obtain

$$\lambda^{-1} \int_{S_\alpha} |\nabla\varphi|^2\,dx \leq b \left(\int_{S_\alpha} \varphi^2 dx\right)^{1/2} \left(\int_{S_\alpha} |\nabla\varphi|^2\,dx\right)^{1/2}. \tag{1.19}$$

It follows that

$$\int_{S_\alpha} \varphi^2 dx \geq c \int_{S_\alpha} |\nabla\varphi|^2\,dx$$

where $c = (\lambda b)^{-2} > 0$.

Assume $n > 2$. By the Sobolev inequality we have

$$\int_{S_\alpha} |\nabla\varphi|^2\,dx = \int_\Omega |\nabla\varphi|^2\,dx \geq c' \left(\int_\Omega \varphi^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}},$$

where $c' = c'(n) > 0$. On the other hand, by the Hölder inequality,

$$\begin{aligned} \int_{S_\alpha} \varphi^2 dx &= \int_{S_\alpha} 1 \cdot \varphi^2 dx \leq \left(\int_{S_\alpha} dx\right)^{\frac{2}{n}} \left(\int_\Omega \left(\varphi^2\right)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \\ &= |S_\alpha|^{2/n} \left(\int_\Omega \varphi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}, \end{aligned}$$

where $|S_\alpha|$ is the Lebesgue measure of the set S_α. Combining the above inequalities, we obtain

$$|S_\alpha|^{2/n} \left(\int_\Omega \varphi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \geq cc' \left(\int_\Omega \varphi^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}$$

and, hence,

$$|S_\alpha| \geq c'', \tag{1.20}$$

for some positive constant c'' that is independent of α.

In the case $n = 2$ the same argument works where the exponent $\frac{n}{n-2}$ should be
19.04.16 replaced by any $q > 1$.

Now let us bring (1.20) to contradiction. Consider an increasing sequence $\{\alpha_k\}_{k=1}^{\infty}$ that converges to α_0 as $k \to \infty$. Then the sequence of sets S_{α_k} is decreasing and

$$\bigcap_{k=1}^{\infty} S_{\alpha_k} = \{x \in \Omega : \forall k \ \ \alpha_k < u(x) < \alpha_0\} = \emptyset.$$

Hence, by the continuity property of the Lebesgue measure,

$$\lim_{k\to\infty} |S_{\alpha_k}| = \left|\bigcap_{k=1}^{\infty} S_{\alpha_k}\right| = 0,$$

which contradicts (1.20), thus finishing the proof. ■

1.4.2 Some properties of weak derivatives

Here Ω is an open subset of $\mathbb{R}^n$.

Lemma 1.6 (Chain rule in $W_0^{1,2}$) *Let ψ be a C^∞-function on $\mathbb{R}$ such that*

$$\psi(0) = 0 \ \ \text{and} \ \ \sup_{t\in\mathbb{R}} |\psi'(t)| < \infty. \tag{1.21}$$

Then $u \in W_0^{1,2}(\Omega)$ implies $\psi(u) \in W_0^{1,2}(\Omega)$ and

$$\nabla\psi(u) = \psi'(u)\nabla u. \tag{1.22}$$

Proof. If $u \in C_0^\infty$ then obviously $\psi(u)$ is also in C_0^∞ and hence in $W_0^{1,2}$, and the chain rule (1.22) is trivial.

An arbitrary function $u \in W_0^{1,2}$ can be approximated by a sequence $\{u_k\}$ of C_0^∞-functions, which converges to u in $W^{1,2}$-norm, that is,

$$u_k \xrightarrow{L^2} u \ \text{ and } \ \ \nabla u_k \xrightarrow{L^2} \nabla u.$$

By selecting a subsequence, we can assume that also $u_k(x) \to u(x)$ for almost all $x \in \Omega$.

By (1.21) we have $|\psi(u)| \leq C|u|$ where $C = \sup|\psi'|$, whence it follows that $\psi(u) \in L^2$. The boundedness of ψ' implies also that $\psi'(u)\nabla u \in \vec{L}^2$. Let us show that

$$\psi(u_k) \xrightarrow{L^2} \psi(u) \quad \text{and} \quad \nabla\psi(u_k) \xrightarrow{L^2} \psi'(u)\nabla u, \tag{1.23}$$

which will imply that the distributional gradient of $\psi(u)$ is equal to $\psi'(u)\nabla u$. The latter, in turn, yields that $\psi(u)$ is in $W^{1,2}$ and, moreover, in $W_0^{1,2}$.

The convergence $\psi(u_k) \overset{L^2}{\to} \psi(u)$ trivially follows from $u_k \overset{L^2}{\to} u$ and

$$|\psi(u_k) - \psi(u)| \leq C|u_k - u|\ .$$

To prove the second convergence in (1.23) observe that

$$\begin{aligned}\left|\nabla\psi\left(u_k\right)-\psi'\left(u\right)\nabla u\right| &= \left|\psi'\left(u_k\right)\nabla u_k-\psi'\left(u\right)\nabla u\right| \\ &\leq \left|\psi'\left(u_k\right)\left(\nabla u_k-\nabla u\right)\right|+\left|\left(\psi'\left(u_k\right)-\psi'\left(u\right)\right)\nabla u\right|,\end{aligned}$$

whence

$$\left\|\nabla\psi\left(u_k\right)-\psi'\left(u\right)\nabla u\right\|_{L^2}\leq C\left\|\nabla u_k-\nabla u\right\|_{L^2}+\left\|\left(\psi'\left(u_k\right)-\psi'\left(u\right)\right)\nabla u\right\|_{L^2}. \tag{1.24}$$

The first term on the right hand side of (1.24) goes to 0 because $\nabla u_k \xrightarrow{L^2} \nabla u$. By construction, we have also $u_k\left(x\right)\to u\left(x\right)$ a.e. , whence

$$\psi'\left(u_k\right)-\psi'\left(u\right)\longrightarrow 0 \text{ a.e.}$$

Since

$$\left|\psi'\left(u_k\right)-\psi'\left(u\right)\right|^2\left|\nabla u\right|^2\leq 4C^2\left|\nabla u\right|^2$$

and the function $\left|\nabla u\right|^2$ is integrable on Ω, we conclude by the dominated convergence theorem that

$$\int_\Omega\left|\psi'\left(u_k\right)-\psi'\left(u\right)\right|^2\left|\nabla u\right|^2 d\mu\longrightarrow 0,$$

which finishes the proof. ■

Lemma 1.7 *Let $\{\psi_k\left(t\right)\}$ be a sequence of C^∞-smooth functions on $\mathbb{R}$ such that*

$$\psi_k\left(0\right)=0 \quad \textit{and} \quad \sup_k\sup_{t\in\mathbb{R}}\left|\psi_k'\left(t\right)\right|<\infty. \tag{1.25}$$

Assume that, for some functions $\psi\left(t\right)$ and $\varphi\left(t\right)$ on $\mathbb{R}$,

$$\psi_k\left(t\right)\to\psi\left(t\right) \quad \textit{and} \quad \psi_k'\left(t\right)\to\varphi\left(t\right) \quad \textit{for all } t\in\mathbb{R}. \tag{1.26}$$

Then, for any $u\in W_0^{1,2}\left(\Omega\right)$, the function $\psi\left(u\right)$ is also in $W_0^{1,2}\left(\Omega\right)$ and

$$\nabla\psi\left(u\right)=\varphi\left(u\right)\nabla u.$$

Proof. The function $\psi\left(u\right)$ is the pointwise limit of measurable functions $\psi_k\left(u\right)$ and, hence, is measurable; by the same argument, $\varphi\left(u\right)$ is also measurable. By (1.25), there is a constant C such that

$$\left|\psi_k\left(t\right)\right|\leq C\left|t\right|, \tag{1.27}$$

for all k and $t\in\mathbb{R}$, and the same holds for function ψ. Therefore, $\left|\psi\left(u\right)\right|\leq C\left|u\right|$, which implies $\psi\left(u\right)\in L^2\left(\Omega\right)$. By (1.25), we have also $\left|\varphi\left(t\right)\right|\leq C$, whence $\varphi\left(u\right)\nabla u\in\vec{L}^2$.

Since each function ψ_k is smooth and satisfies (1.21), Lemma 1.6 yields that

$$\psi_k\left(u\right)\in W_0^{1,2}\left(\Omega\right) \text{ and } \nabla\psi_k\left(u\right)=\psi_k'\left(u\right)\nabla u.$$

Let us show that

$$\psi_k\left(u\right)\xrightarrow{L^2}\psi\left(u\right) \quad \text{and} \quad \nabla\psi_k\left(u\right)\xrightarrow{L^2}\varphi\left(u\right)\nabla u, \tag{1.28}$$

which will settle the claim. The dominated convergence theorem implies that

$$\int_\Omega |\psi_k(u) - \psi(u)|^2 \, d\mu \longrightarrow 0,$$

because the integrand functions tend pointwise to 0 as $k \to \infty$ and, by (1.27),

$$|\psi_k(u) - \psi(u)|^2 \le 4C^2 u^2,$$

whereas u^2 is integrable on Ω. Similarly, we have

$$\int_\Omega |\nabla \psi_k(u) - \varphi(u) \nabla u|^2 \, d\mu = \int_\Omega |\psi_k'(u) - \varphi(u)|^2 |\nabla u|^2 \, d\mu \longrightarrow 0,$$

because the sequence of functions $|\psi_k'(u) - \varphi(u)|^2 |\nabla u|^2$ tends pointwise to 0 as $k \to \infty$ and is uniformly bounded by the integrable function $4C^2 |\nabla u|^2$. ■

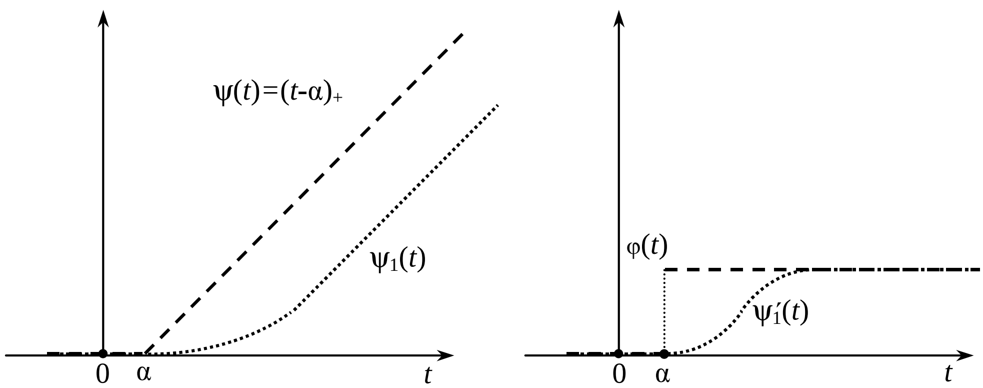
Proof of Lemma 1.4. Consider the functions

$$\psi(t) = (t-\alpha)_+ \quad \text{and} \quad \varphi(t) = \begin{cases} 1, & t > \alpha, \\ 0, & t \le \alpha, \end{cases}$$

that can be approximated as in (1.26) as follows. Fix any smooth function $\eta(t)$ on $\mathbb{R}$ such that

$$\eta(t) = \begin{cases} t-1, & t \ge 2, \\ 0, & t \le 0 \end{cases}$$

Such function $\eta(t)$ can be obtained by twice integrating a suitable function from $C_0^\infty(0,2)$.



Define ψ_k for any $k \in \mathbb{N}$ by

$$\psi_k(t) = \frac{1}{k} \eta(k(t-\alpha)).$$

If $t \le \alpha$ then $\psi_k(t) = 0$. If $t > \alpha$ then, for large enough k, we have $k(t-\alpha) > 2$ whence

$$\psi_k(t) = \frac{1}{k}(k(t-\alpha) - 1) = t - \alpha - \frac{1}{k} \to t - \alpha \ \text{ as } k \to \infty.$$

Hence, $\psi_k(t) \to \psi(t)$ for all $t \in \mathbb{R}$.

Similarly, if $t \leq \alpha$ then $\psi_k'(t) = 0$, and, for $t > \alpha$,

$$\psi_k'(t) = \eta'(k(t-\alpha)) \to 1 \text{ as } k \to \infty.$$

Hence, $\psi_k'(t) \to \varphi(t)$ for all $t \in \mathbb{R}$.

By Lemma 1.7, we conclude that $(u-\alpha)_+ \in W_0^{1,2}$ and

$$\nabla (u-\alpha)_+ = \varphi(u) \nabla u = \begin{cases} \nabla u, & u > \alpha, \\ 0, & u \leq \alpha, \end{cases}$$

which finishes the proof. ■

Proof of Lemma 1.5. By Lemma 1.4 with $\alpha = 0$, we have $u_+ \in W_0^{1,2}$ and

$$\nabla u_+ = \begin{cases} \nabla u, & u > 0, \\ 0, & u \leq 0. \end{cases} \tag{1.29}$$

Applying this to function $(-u)$, we obtain that $u_- \in W_0^{1,2}$ and

$$\nabla u_- = \begin{cases} 0, & u \geq 0, \\ -\nabla u, & u < 0. \end{cases} \tag{1.30}$$

Consequently, since $\nabla u_+ = \nabla u_- = 0$ on the set $\{u = 0\}$, we obtain

$$\nabla u = 0 \text{ a.e. on } \{u = 0\}. \tag{1.31}$$

In particular, (1.31) implies the following: if u, v are two functions from $W_0^{1,2}(\Omega)$ and S is a subset of Ω then

$$u = v \text{ a.e. on } S \quad \Rightarrow \nabla u = \nabla v \text{ a.e. on } S.$$

Let us now prove that, for any $\alpha \in \mathbb{R}$,

$$\nabla u = 0 \text{ a.e. on } \{u = \alpha\}. \tag{1.32}$$

If the constant function $v \equiv \alpha$ were in $W_0^{1,2}$ then by

$$u = v \text{ on } \{u = \alpha\}$$

we could obtain

$$\nabla u = \nabla v = 0 \text{ a.e.on } \{u = \alpha\}$$

thus proving (1.32). However, the constant function is not in $W_0^{1,2}$ and we argue as follows. Choose a compact set $K \subset \Omega$ and a function $v \in C_0^\infty(\Omega)$ such that $v = \alpha$ in a neighborhood of K. Then

$$u = v \text{ on } K \cap \{u = \alpha\}$$

which implies that

$$\nabla u = \nabla v = 0 \text{ a.e. on } K \cap \{u = \alpha\}.$$

Covering Ω by a countable family of compact sets K, we obtain (1.32). ■

1.4.3 Sobolev inequality

Theorem 1.8 *Assume $1 \leq p < n$. Then there is a constant $C = C(p,n)$ such that, for all $u \in W_0^{1,p}(\mathbb{R}^n)$,*

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{pn}{n-p}}\,dx\right)^{\frac{n-p}{n}} \leq C \int_{\mathbb{R}^n} |\nabla u|^p\,dx. \tag{1.33}$$

In the proof we will use the following extended Hölder inequality for non-negative functions on $\mathbb{R}$:

$$\int_{\mathbb{R}} \prod_{i=1}^{m} f_i^{\frac{1}{m}}\,dt \leq \prod_{i=1}^{m}\left(\int_{\mathbb{R}} f_i dt\right)^{1/m}. \tag{1.34}$$

Indeed, for $m = 1$ this is trivial, and in the case $m = 2$ this is a Cauchy-Schwarz inequality. For a general m, let us make the inductive step from $m-1$ to m as follows:

$$\begin{aligned}
\int_{\mathbb{R}} f_1^{\frac{1}{m}}...f_m^{\frac{1}{m}}\,dt &\leq \left(\int_{\mathbb{R}}\left(f_1^{\frac{1}{m}}...f_{m-1}^{\frac{1}{m}}\right)^{\frac{m}{m-1}}dt\right)^{\frac{m-1}{m}}\left(\int_{\mathbb{R}}\left(f_m^{\frac{1}{m}}\right)^m dt\right)^{\frac{1}{m}} \\
&= \left(\int_{\mathbb{R}} f_1^{\frac{1}{m-1}}...f_{m-1}^{\frac{1}{m-1}}dt\right)^{\frac{m-1}{m}}\left(\int_{\mathbb{R}} f_m dt\right)^{\frac{1}{m}} \\
&\leq \left(\left(\int_{\mathbb{R}} f_1 dt\right)^{\frac{1}{m-1}}...\left(\int_{\mathbb{R}} f_{m-1}dt\right)^{\frac{1}{m-1}}\right)^{\frac{m-1}{m}}\left(\int_{\mathbb{R}} f_m dt\right)^{\frac{1}{m}} \\
&= \left(\int_{\mathbb{R}} f_1 dt\right)^{\frac{1}{m}}...\left(\int_{\mathbb{R}} f_m dt\right)^{\frac{1}{m}}
\end{aligned}$$

which is equivalent to (1.34). 21.04.16

Proof of Theorem 1.8. *Step 0.* Let us first show that it suffices to prove (1.33) for $u \in C_0^\infty(\mathbb{R}^n)$. Indeed, if (1.33) is known to be true for $u \in C_0^\infty(\mathbb{R}^n)$ then choose for any $u \in W_0^{1,p}(\mathbb{R}^n)$ a sequence $\{u_k\}$ from $C_0^\infty(\mathbb{R}^n)$ such that $u_k \to u$ in the norm of $W^{1,p}$. It follows that

$$\int_{\mathbb{R}^n} |\nabla u_k|^p\,dx \to \int_{\mathbb{R}^n} |\nabla u|^p\,dx \text{ as } k \to \infty.$$

In particular, for $\forall \varepsilon > 0$ and for all large enough k

$$\int_{\mathbb{R}^n} |\nabla u_k|^p\,dx \leq \int_{\mathbb{R}^n} |\nabla u|^p\,dx + \varepsilon.$$

Since (1.33) holds for each function u_k, we have

$$\left(\int_{\mathbb{R}^n} |u_k|^{\frac{pn}{n-p}}\,dx\right)^{\frac{n-p}{n}} \leq C\int_{\mathbb{R}^n} |\nabla u_k|^p\,dx \leq C\left(\int_{\mathbb{R}^n} |\nabla u|^p\,dx + \varepsilon\right).$$

Since $u_k \to u$ in L^p, choosing a subsequence we can assume that $u_k \to u$ a.e.. Hence, by Fatou's lemma, we conclude that

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{pn}{n-p}}\,dx\right)^{\frac{n-p}{n}} \leq C\left(\int_{\mathbb{R}^n} |\nabla u|^p\,dx + \varepsilon\right).$$

Since $\varepsilon > 0$ is arbitrary, we obtain that (1.33) holds for arbitrary $u \in W_0^{1,p}(\mathbb{R}^n)$.

Step 1. Let us prove (1.33) in the case $p = 1$ for any $u \in C_0^1(\mathbb{R}^n)$. For $p = 1$ (1.33) becomes

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le C \int_{\mathbb{R}^n} |\nabla u| \, dx, \tag{1.35}$$

assuming that $n > 1$. Since u has a compact support, we have, for any index $i = 1, ..., n$,

$$u(x) = \int_{-\infty}^{x} \partial_i u(x_1, .., x_{i-1}, y_i, x_{i+1}, ..., x_n) \, dy_i,$$

which implies

$$|u(x)| \le \int_{-\infty}^{\infty} |\nabla u| (x_1, .., x_{i-1}, y_i, x_{i+1}, ..., x_n) \, dy_i. \tag{1.36}$$

Consider function $F = |\nabla u|$ and let us use the following notation: for any sequence $i_1, ..., i_k$ of distinct indices, set

$$F_{i_1 \ldots i_k} = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} F(x) \, dx_{i_1} dx_{i_2} \ldots dx_{i_k}.$$

We consider $F_{i_1 \ldots i_k}$ as a function of x that does not depend on $x_{i_1}, ..., x_{i_k}$ but depends on all other components x_j.

Inequality (1.36) can be written in a short form

$$|u(x)| \le F_i(x).$$

Multiplying these inequalities for $i = 1, ..., n$ and taking to the power $\frac{1}{n-1}$, we obtain

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} F_i^{\frac{1}{n-1}}.$$

Let us integrate this inequality in x_1. Since F_1 does not depend on x_1, we obtain, using (1.34) with $m = n - 1$, that

$$\begin{aligned}
\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} dx_1 &\le F_1^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(\prod_{i=2}^{n} F_i^{\frac{1}{n-1}}\right) dx_1 \\
&\le F_1^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} F_i dx_1\right)^{\frac{1}{n-1}} \\
&= F_1^{\frac{1}{n-1}} \prod_{i=2}^{n} F_{1i}^{\frac{1}{n-1}}.
\end{aligned}$$

Now let us integrate the last inequality in x_2, noticing that F_{12} does not depend on x_2 and using (1.34):

$$\begin{aligned}
\int_{\mathbb{R}^2} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 &\le F_{12}^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(F_1^{\frac{1}{n-1}} \prod_{i=3}^{n} F_{1i}^{\frac{1}{n-1}}\right) dx_2 \\
&\le F_{12}^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} F_1 dx_2\right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{\mathbb{R}} F_{1i} dx_2\right)^{\frac{1}{n-1}} \\
&= F_{12}^{\frac{2}{n-1}} \prod_{i=3}^{n} F_{12i}^{\frac{1}{n-1}}.
\end{aligned}$$

Integrating the last inequality in x_3, noticing that F_{123} does not depend on x_3 and using (1.34), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 dx_3 &\leq F_{123}^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(F_{12}^{\frac{1}{n-1}} F_{12}^{\frac{1}{n-1}} \prod_{i=4}^{n} F_{12i}^{\frac{1}{n-1}} \right) dx_3 \\ &\leq F_{123}^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} F_{12} dx_3 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} F_{12} dx_3 \right)^{\frac{1}{n-1}} \\ &\quad \times \prod_{i=4}^{n} \left(\int_{\mathbb{R}} F_{12i} dx_3 \right)^{\frac{1}{n-1}} \\ &= F_{123}^{\frac{3}{n-1}} \prod_{i=4}^{n} F_{123i}^{\frac{1}{n-1}}.\end{aligned}$$

Continuing further by induction, we obtain, for any $1 \leq k \leq n$, that

$$\int_{\mathbb{R}^k} |u(x)|^{\frac{n}{n-1}} dx_1 ... dx_k \leq F_{1...k}^{\frac{k}{n-1}} \prod_{i=k+1}^{n} F_{1...ki}^{\frac{1}{n-1}}.$$

In particular, for $k = n$ we obtain

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq F_{12...n}^{\frac{n}{n-1}} = \left(\int_{\mathbb{R}^n} |\nabla u| \, dx \right)^{\frac{n}{n-1}},$$

which proves the Sobolev inequality (1.35) in the case $p = 1$. Note that in this case $C = 1$.

Step 2. Let us prove now (1.35) in the case $p > 1$, also for any $u \in C_0^1(\mathbb{R}^n)$. We claim that, for any $\alpha > 1$, the function $|u|^\alpha$ belongs to $C_0^1(\mathbb{R}^n)$ and

$$\partial_i |u|^\alpha = \alpha |u|^{\alpha-1} \operatorname{sgn} u \, \partial_i u. \tag{1.37}$$

Indeed, the the identity (1.37) is easily verified in each of the open sets $\{u > 0\}$, $\{u < 0\}$, $\{u = 0\}^o$. Since the right hand side of (1.37) is continuous in the closure of each of these open sets and vanishes at their boundaries, we see that the right hand side is continuous in $\mathbb{R}^n$, which implies that the identity (1.37) holds in the whole $\mathbb{R}^n$. Consequently, $|u|^\alpha \in C_0^1(\mathbb{R}^n)$.

Applying (1.35) to the function $|u|^\alpha$ and using

$$\nabla |u|^\alpha = \alpha |u|^{\alpha-1} \operatorname{sgn} u \nabla u,$$

we obtain

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\alpha n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla |u|^\alpha| \, dx = \alpha \int_{\mathbb{R}^n} |u|^{\alpha-1} |\nabla u| \, dx.$$

By the Hölder inequality, we have

$$\int_{\mathbb{R}^n} |u|^{\alpha-1} |\nabla u| \, dx \leq \left(\int_{\mathbb{R}^n} |u|^{(\alpha-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$

Choose α so that

$$(\alpha-1)\frac{p}{p-1}=\frac{pn}{n-p}=:q$$

that is,

$$\alpha=1+\frac{n}{n-p}(p-1)=\frac{n-1}{n-p}p.$$

Then also

$$\frac{\alpha n}{n-1}=\frac{pn}{n-p}=q,$$

and we obtain

$$\left(\int_{\mathbb{R}^n}|u|^q\,dx\right)^{\frac{n-1}{n}}\leq\alpha\left(\int_{\mathbb{R}^n}|u|^q\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

It follows that

$$\left(\int_{\mathbb{R}^n}|u|^q\,dx\right)^{\frac{n-1}{n}-\frac{p-1}{p}}\leq\alpha\left(\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right)^{\frac{1}{p}},$$

$$\left(\int_{\mathbb{R}^n}|u|^q\,dx\right)^{\frac{n-p}{np}}\leq\alpha\left(\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right)^{\frac{1}{p}},$$

which is equivalent to (1.33) with

$$C=\alpha^p=\left(\frac{n-1}{n-p}p\right)^p.$$

■

Now let us prove the Sobolev inequality in the form that was used in the proof of Theorem 1.3.

Corollary 1.9 *Let Ω be an open subset of $\mathbb{R}^n$. If $n>2$ then, for any $u\in W_0^{1,2}(\Omega)$,*

$$\int_\Omega|\nabla u|^2\,dx\geq c\left(\int_\Omega|u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\tag{1.38}$$

where $c=c(n)>0$. If $n=2$ and Ω is bounded then, for any $q\geq1$ and any $u\in W_0^{1,2}(\Omega)$

$$\int_\Omega|\nabla u|^2\,dx\geq c\left(\int_\Omega|u|^{2q}\,dx\right)^{1/q},\tag{1.39}$$

where $c=c_0|\Omega|^{-1/q}$ and $c_0=c_0(q)>0$.

Proof. Since $C_0^\infty(\Omega)\subset C_0^\infty(\mathbb{R}^n)$, it follows that $W_0^{1,2}(\Omega)\subset W_0^{1,2}(\mathbb{R}^n)$ (more precisely, any function from $W_0^{1,2}(\Omega)$ that is extended by 0 outside Ω, belongs to $W_0^{1,2}(\mathbb{R}^n)$). Therefore, (1.38) is a particular case of (1.33) with $p=2$.

Assume $n=2$. Then by (1.33) we have, for any $1\leq p<2$,

$$\left(\int_\Omega|u|^{\frac{2p}{2-p}}\,dx\right)^{\frac{2-p}{2}}\leq C\int_\Omega|\nabla u|^p\,dx.$$

Using the Hölder inequality, we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^p\,dx &= \int_\Omega 1\cdot|\nabla u|^p\,dx \le \left(\int_\Omega dx\right)^{1-\frac{p}{2}}\left(\int_\Omega |\nabla u|^{p\frac{2}{p}}\,dx\right)^{\frac{p}{2}} \\ &= |\Omega|^{1-p/2}\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p}{2}}.\end{aligned}$$

Hence, we obtain

$$\left(\int_\Omega |u|^{\frac{2p}{2-p}}\,dx\right)^{\frac{2-p}{2}} \le C\,|\Omega|^{1-p/2}\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p}{2}}$$

or

$$\left(\int_\Omega |u|^{\frac{2p}{2-p}}\,dx\right)^{\frac{2-p}{p}} \le C\,|\Omega|^{\frac{2}{p}-1}\int_\Omega |\nabla u|^2\,dx.$$

It remains to set $q=\frac{p}{2-p}$ and observe that q can be any number from $[1,\infty)$ as p is any number from $[1,2)$. Then $\frac{2}{p}-1=\frac{1}{q}$, and we obtain

$$\int_\Omega |\nabla u|^2\,dx \ge C^{-1}\,|\Omega|^{-1/q}\left(\int_\Omega |u|^{2q}\,dx\right)^{1/q},$$

which was to be proved. ■

1.4.4 Theorem of Lax-Milgram

Theorem 1.10 *Let $B(u,v)$ be a bilinear form in a Hilbert space H. Assume that*

1. *B is bounded, that is, $|B(u,v)|\le C\,\|u\|\,\|v\|$ for all $u,v\in H$ and some constant C.*
2. *B is coercive, that is, $B(u,u)\ge c\,\|u\|^2$ for all $u\in H$, where c is a positive constant.*

Then, for any bounded linear functional l on H, the equation

$$B(u,v)=l(v)\quad\forall v\in H \tag{1.40}$$

has a unique solution $u\in H$. Moreover, for this solution we have

$$\|u\|\le\frac{1}{c}\,\|l\|. \tag{1.41}$$

If the bilinear form $B(u,v)$ is symmetric then this theorem coincides with the Riesz representation theorem. The strength of Theorem 1.10 is that it allows non-symmetric B. 26.04.16

Proof. For any $u\in H$, the function $v\mapsto B(u,v)$ is a bounded linear functional on H. Hence, by the Riesz representation theorem, the equation

$$(w,v)=B(u,v)\quad\forall v\in H$$

has a unique solution $w \in H$. Since w depends on u, we obtain a mapping $A : H \to H$, defined by $Au = w$. In other words, A is defined by the identity

$$(Au, v) = B(u, v) \quad \forall v \in H. \tag{1.42}$$

Operator A is called the *generator* of the bilinear form B. Clearly, the equation (1.40) is equivalent to

$$(Au, v) = l(v) \quad \forall v \in H. \tag{1.43}$$

Again by Riesz representation theorem, there is $w \in H$ such that

$$(w, v) = l(v) \quad \forall v \in H.$$

Therefore, in order to solve (1.43) it suffices to find u so that $Au = w$.

Hence, the question of solving of (1.40) amounts to verifying that A is bijective, so that the equation $Au = w$ has a solution $u = A^{-1}w$.

Let us prove that A is bijective in the following few steps.

Step 1. Operator A is linear. Indeed, for any $u_1, u_2 \in H$ and for all $v \in H$ we have by (1.42)

$$(A(u_1 + u_2), v) = B(u_1 + u_2, v) = B(u_1, v) + B(u_2, v) = A(u_1, v) + A(u_2, v),$$

which implies $Au_1 + Au_2 = A(u_1 + u_2)$. The same argument shows that $A(\lambda u) = \lambda A(u)$ for any $\lambda \in \mathbb{R}$.

Step 2. Operator A is bounded. Indeed, it follows from (1.42) that

$$|(Au, v)| \leq C \|u\| \|v\|.$$

Setting here $v = Au$, we obtain

$$\|Au\|^2 \leq C \|u\| \|Au\|$$

whence $\|Au\| \leq C\|u\|$, which proves the claim.

Step 3. Operator A is injective. Indeed, setting $v = u$ in (1.42), we obtain

$$(Au, u) = B(u, u) \geq c\|u\|^2. \tag{1.44}$$

In particular, $Au = 0$ implies $u = 0$, that is, A is injective. Applying Cauchy-Schwarz inequality to the left hand side of (1.44), we obtain

$$\|Au\| \|u\| \geq c\|u\|^2$$

and, hence,

$$\|Au\| \geq c\|u\| \quad \forall u \in H. \tag{1.45}$$

Step 4. The image $\operatorname{Im} A$ is dense in H. Indeed, if $\overline{\operatorname{Im} A} \neq H$ then there is a non-zero vector u in H that is orthogonal to $\operatorname{Im} A$. In particular, $(Au, u) = 0$, which by (1.44) is not possible.

Step 5. Operator A is surjective, that is, $\operatorname{Im} A = H$. In the view of Step4, it suffices to verify that $\operatorname{Im} A$ is a closed set. Indeed, let $\{w_k\}$ be a sequence of elements from

$\operatorname{Im} A$ that converges to $w \in H$. Let us show that $w \in \operatorname{Im} A$. We have $w_k = Au_k$ for some $u_k \in H$. It follows from (1.45) that

$$\|w_k - w_l\| \geq c \|u_k - u_l\|,$$

which implies that the sequence $\{u_k\}$ is Cauchy. Hence, there exists the limit $u := \lim_{k\to\infty} u_k$. By the boundedness of A we obtain

$$Au = \lim_{k\to\infty} Au_k = \lim_{k\to\infty} w_k = w$$

and, hence, $w \in \operatorname{Im} A$.

Step 6. Finally, let us prove (1.41). Setting in (1.40) $v = u$ and using the coercive property of B, we obtain

$$c \|u\|^2 \leq B(u,u) = l(u) \leq \|l\| \|u\|,$$

whence $\|u\| \leq c^{-1} \|l\|$ follows. ■

1.4.5 Fredholm's alternative

Theorem 1.11 *Let K be a compact linear operator in a Hilbert space H. If the operator $I + K$ is injective then $I + K$ is surjective.*

Here I is the identity operator in H. In other words, either the equation $(I + K)x = 0$ has non-zero solution or the equation $(I + K)x = h$ has a solution $x \in H$ for any $h \in H$.

Note that in a finite dimensional Euclidean space H, any linear operator $A : H \to H$ has this property: if A is injective then A is surjective, because each of this properties is equivalent to $\det A \neq 0$. In infinite dimensional spaces this is not the case for arbitrary operators.

Proof. Denote $A = I + K$. Assuming that $\ker A = 0$, we will prove that $\operatorname{Im} A = H$. The proof consists of a few steps.

Step 1. Let us show that if $\{x_i\}$ is a bounded sequence of elements of H and if $\{Ax_i\}$ converges then $\{x_i\}$ has a convergent subsequence. Indeed, by the compactness of K, the sequence $\{Kx_i\}$ has a convergent subsequence $\{Kx_{i_k}\}$. Since $x_{i_k} + Kx_{i_k} = Ax_{i_k}$ converges, then also $\{x_{i_k}\}$ converges, which proves the claim.

Step 2. Let us prove that $\operatorname{Im} A$ is a closed subspace of H. The image of any linear operator is always a subspace, so we need to prove that $\operatorname{Im} A$ is closed. Let $\{y_i\}$ be a sequence of elements in $\operatorname{Im} A$ that converges to $y \in H$. Then $y_i = Ax_i$ for some $x_i \in H$.

Let us prove that $\{x_i\}$ is bounded. Indeed, if it is not the case then we can assume passing to a subsequence that $\|x_i\| \to \infty$. Setting $\widetilde{x}_i = \frac{x_i}{\|xi\|}$, we have

$$A\widetilde{x}_i = \frac{Ax_i}{\|x_i\|} = \frac{y_i}{\|x_i\|} \to 0 \text{ as } i \to \infty.$$

Since the sequence $\{\widetilde{x}_i\}$ is bounded and $A\widetilde{x}_i$ converges, we conclude by Step 1, that $\{\widetilde{x}_i\}$ has a convergent subsequence. Passing to this subsequence, we can assume that $\{\widetilde{x}_i\}$ converges, say, to $z \in H$. Clearly,

$$\|z\| = \lim_{i\to\infty} \|\widetilde{x}_i\| = 1$$

and

$$Az = \lim_{i\to\infty} A\widetilde{x}_i = 0$$

that is, $z \in \ker A$. Since $\ker A = 0$, we obtain $z = 0$ which contradicts $\|z\| = 1$.

Hence, the sequence $\{x_i\}$ is bounded. Since Ax_i converges, we conclude by Step 1, that the sequence $\{x_i\}$ contains a convergent subsequence. Denote it again by $\{x_i\}$ and set $x = \lim x_i$. Then we have

$$y = \lim y_i = \lim Ax_i = Ax \in \operatorname{Im} A,$$

which finishes the proof.

Step 3. Consider the sequence $\{V_k\}_{k=0}^{\infty}$ of subspaces $V_k := \operatorname{Im} A^k$, that is, $V_{k+1} = A(V_k)$. In particular, $V_0 = H$ and $V_1 = \operatorname{Im} A$. Clearly, we have $V_{k+1} \subset V_k$. By Step 2, V_1 is a closed subspace of V_0. In particular, V_1 is a Hilbert space. Since A can be considered as an operator in V_1, we conclude by Step 2 that $V_2 = A(V_1)$ is a closed subspace of V_1. Continuing by induction, we obtain that each V_{k+1} is a closed subspace of V_k.

Let us prove that $V_{k+1} = V_k$ for some k. Assume from the contrary that this is not the case, that is, $V_{k+1} \subsetneqq V_k$ for all $k \geq 0$. For any k, choose x_k from the orthogonal complement $V_{k+1}^{\perp}$ of V_{k+1} in V_k and so that $\|x_k\| = 1$. For all $i > j$ we have

$$Kx_i - Kx_j = -(x_i - x_j) + A(x_i - x_j) = x_j + (-x_i + Ax_i - Ax_j).$$

Since $i \geq j + 1$, we have

$$-x_i + Ax_i - Ax_j \in V_{j+1},$$

which implies, by the choice of $x_j \in V_{j+1}^{\perp}$ that

$$x_j \perp (-x_i + Ax_i - Ax_j).$$

Hence, by Pythagoras' Theorem,

$$\|Kx_i - Kx_j\|^2 = \|x_j\|^2 + \|(-x_i + Ax_i - Ax_j)\|^2 \geq 1.$$

Consequently, no subsequence of $\{Kx_i\}$ is a Cauchy sequence. On the other hand, the compactness of K implies that $\{Kx_i\}$ contains a convergent subsequence. This contradiction proves the claim.

Step 4. Finally, let us prove that if A is injective then $\operatorname{Im} A = H$. Let k be the minimal non-negative integer such that $V_{k+1} = V_k$. We need to prove that $k = 0$, which is equivalent to $\operatorname{Im} A = H$. Assume that $k \geq 1$ and consider the mapping $A : V_{k-1} \to V_k$. Note that $V_{k-1} = V_k \oplus V_k^{\perp}$ and the space $V_k^{\perp}$ is non-trivial by the assumption that $V_{k-1} \neq V_k$. The image of A on V_k coincides with V_k, by the assumption $A(V_k) = V_{k+1} = V_k$. However, $A\left(V_k^{\perp}\right)$ lies also in V_k, which implies that the operator $A : V_{k-1} \to V_k$ cannot be injective. This contradiction shows that $k = 0$, which finishes the proof. ∎

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1.4.6 Existence

Consider again an operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_i u\right) + \sum_{i=1}^{n} b_i\left(x\right) \partial_i u \tag{1.46}$$

in an open set $\Omega \subset \mathbb{R}^n$. As before, we assume that the coefficients a_{ij}, b_i are measurable functions, the second order part $\sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_i u\right)$ is uniformly elliptic divergence form operator, and that all functions b_i are bounded, that is, there is a constant b such that

$$\sum_{i=1}^{n} |b_i| \leq b \ \text{ a.e. in } \Omega.$$

Theorem 1.12 *If Ω is bounded and L is the operator* (1.46) *in Ω then the weak Dirichlet problem*

$$\begin{cases} Lu = f & in\ \Omega \\ u \in W_0^{1,2}\left(\Omega\right) \end{cases} \tag{1.47}$$

has a solution u for any $f \in L^2\left(\Omega\right)$.

Recall that by Theorem 1.3 the Dirichlet problem (1.47) has at most one solution, which together with Theorem 1.12 implies that (1.47) has exactly one solution.

Proof. Consider the following bilinear form on $W_0^{1,2}\left(\Omega\right)$:

$$[u, \varphi] = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \varphi \, dx - \int_\Omega \sum_{i=1}^{n} b_i \partial_i u \, \varphi \, dx.$$

As we know, the weak equation $Lu = f$ means that

$$[u, \varphi] = -\int_\Omega f \varphi dx \ \ \forall \varphi \in W_0^{1,2}\left(\Omega\right). \tag{1.48}$$

The bilinear form $[u, \varphi]$ is bounded as

$$|[u, \varphi]| \leq \left(\lambda + b\right) \|u\|_{W^{1,2}} \|\varphi\|_{W^{1,2}} \tag{1.49}$$

(cf. equation (1.10) in the proof of Lemma 1.1). If this form were coercive, that is, if for all $u \in W_0^{1,2}$

$$[u, u] \geq c \|u\|_{W^{1,2}}^2 \tag{1.50}$$

with some positive constant c, then we could conclude by the Lax-Milgram theorem that the equation (1.48) has a solution $u \in W_0^{1,2}\left(\Omega\right)$ that is, hence, is a solution of (1.47). However, the form $[u, \varphi]$ is not necessarily coercive. However, it still satisfies the following inequality:

$$\begin{aligned} [u, u] &= \int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i u \, dx - \int_\Omega \sum_{i=1}^{n} b_i \partial_i u \, u dx \\ &\geq \lambda^{-1} \int_\Omega |\nabla u|^2 \, dx - b \int_\Omega |\nabla u| \, |u| \, dx. \end{aligned}$$

Note that, for any $\varepsilon > 0$,

$$|\nabla u|\,|u| \leq \varepsilon\,|\nabla u|^2 + \frac{1}{\varepsilon}u^2,$$

so that

$$\begin{aligned}[u,u] &\geq \lambda^{-1}\int_\Omega |\nabla u|^2\,dx - \varepsilon b\int_\Omega |\nabla u|^2\,dx - \frac{b}{\varepsilon}\int_\Omega u^2dx \\ &= c\int_\Omega |\nabla u|^2\,dx - \frac{b}{\varepsilon}\int_\Omega u^2dx,\end{aligned}$$

where $c = \lambda^{-1} - b\varepsilon$.Choosing ε small enough, say $\varepsilon = \frac{1}{2}b^{-1}\lambda^{-1}$, we can ensure that $c > 0$. It follows that

$$\begin{aligned}[u,u] &\geq c\left(\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2dx\right) - \left(\frac{b}{\varepsilon} + c\right)\int_\Omega u^2dx \\ &\geq c\,\|u\|^2_{W^{1.2}} - C\,\|u\|^2_{L^2}\,,\end{aligned}$$

where $C = \frac{b}{\varepsilon} + c$. Rewrite this inequality as follows:

$$[u,u] + C\,\|u\|^2_{L^2} \geq c\,\|u\|^2_{W^{1,2}}\,, \tag{1.51}$$

which is of course weaker than (1.50). So, in general the form $[u,\varphi]$ is not coercive. However, a different bilinear form

$$[u,\varphi] + C\,(u,\varphi)$$

is bounded by (1.49) and is coercive by (1.51) (here $(\cdot,\cdot)$ is the inner product in L^2). Let us consider instead of (1.48) an auxiliary problem:

$$[u,\varphi] + C\,(u,\varphi) = -\int_\Omega f\varphi dx \quad \forall\varphi \in W_0^{1,2}(\Omega)\,. \tag{1.52}$$

By the Lax-Milgram theorem, the equation (1.52) has a unique solution $u \in W_0^{1,2}(\Omega)$. Moreover, for this solution we have

$$\|u\|_{W^{1,2}} \leq c^{-1}\,\|f\|_{L^2}\,, \tag{1.53}$$

because the norm of the functional $\varphi \mapsto \int_\Omega f\varphi$ in $W_0^{1,2}(\Omega)$ is bounded by $\|f\|_{L^2}$.

Denote by R the resolvent operator of (1.52), that is, the operator

$$\begin{aligned} L^2(\Omega) &\to W_0^{1,2}(\Omega) \\ f &\mapsto u.\end{aligned}$$

In other words, for any $f \in L^2(\Omega)$, we have $Rf = u$ where u is the unique solution of (1.52). Obviously, R is a linear operator. Moreover, R is a bounded operator because by (1.53)

$$\|Rf\|_{W^{1.2}} \leq c^{-1}\,\|f\|_{L^2}\,.$$

Now let us come back to the equation (1.48) and rewrite it in the equivalent form

$$[u,\varphi] + C\,(u,\varphi) = -\int_\Omega (f - Cu)\,\varphi dx \quad \forall\varphi \in W_0^{1,2}(\Omega)\,. \tag{1.54}$$

By the definition of the resolvent R, this equation is equivalent to

$$u = R(f - Cu)$$

that is, to

$$u + CRu = Rf. \tag{1.55}$$

Define the operator $K : L^2 \to L^2$ as composition of the following operators

$$L^2(\Omega) \overset{CR}{\to} W_0^{1,2}(\Omega) \overset{i}{\hookrightarrow} L^2(\Omega)$$

where i is the identical inclusion; that is,

$$K = i \circ (CR).$$

By the Compact Embedding Theorem, the operator i is compact. Since CR is bounded, we obtain that K is a compact operator. Setting $Rf = g$, let us rewrite (1.55) in the form

$$(I + K)u = g. \tag{1.56}$$

We consider this equation in the Hilbert space $L^2(\Omega)$, that is, both g and u are assumed to be in $L^2(\Omega)$. We claim that solving (1.56) for $u \in L^2(\Omega)$ is equivalent to solving (1.55) for $u \in W_0^{1,2}(\Omega)$. Indeed, the direction (1.55)$\Rightarrow$(1.56) is trivial because if $u \in W_0^{1,2}(\Omega)$ then $u \in L^2(\Omega)$. For the opposite direction observe that if $u \in L^2(\Omega)$ solves (1.56) with $g = Rf$ then

$$u = g - Ku = Rf - CRu \in W_0^{1,2}(\Omega)$$

by definition of the operator R.

Hence, it suffices to prove that the equation (1.56) has a solution $u \in L^2(\Omega)$ for any $g \in L^2(\Omega)$. By Fredholm's alternative, it suffices to prove that the operator $I + K$ is injective, that is, the equation

$$(I + K)u = 0$$

has the only solution $u = 0$. If $u \in L^2(\Omega)$ satisfies this equation then u satisfies (1.54) with $f = 0$, that is equivalent to

$$[u, \varphi] = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

By Theorem 1.3 we know that $u = 0$. Hence, $\ker(I + K) = 0$ and, by Fredholm's alternative we conclude that

$$\operatorname{Im}(I + K) = L^2(\Omega).$$

Therefore, the equation $(I + K)u = g$ has a solution $u \in L^2(\Omega)$ for any $g \in L^2(\Omega)$, which finishes the proof. ∎

1.5 Estimate of L^∞-norm of a solution

In this section we use the ∞-norm of a measurable function f in an open subset Ω of $\mathbb{R}^n$:

$$\|f\|_{L^\infty} := \operatorname*{esssup}_{\Omega} |f| .$$

The space $L^\infty(\Omega)$ consists of all measurable functions f on Ω with $\|f\|_{L^\infty} < \infty$. It is possible to prove that L^∞ is a linear space, $\|\cdot\|_{L^\infty}$ is a norm in $L^\infty(\Omega)$, and $L^\infty(\Omega)$ is a Banach space. The following extension of the Hölder inequality is obviously true:

$$\int_\Omega |fg|\, dx \le \|f\|_{L^\infty} \|g\|_{L^1} .$$

The Sobolev spaces $W^{k,p}(\Omega)$ are now defined by (1.4) also for $p = \infty$, as well as the spaces $W^{k,p}_{loc}(\Omega)$ (cf. (1.5)).

1.5.1 Operator without lower order terms

Theorem 1.13 *Let Ω be a bounded domain in $\mathbb{R}^n$ and let*

$$Lu = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_j u)$$

be a divergence form uniformly elliptic operator in Ω with measurable coefficients. If u solves the Dirichlet problem

$$\begin{cases} Lu = -f \ \text{weakly in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{1.57}$$

where $f \in L^2(\Omega)$, then

$$\|u\|_{L^\infty} \le C\,|\Omega|^{2/n} \|f\|_{L^\infty} , \tag{1.58}$$

where $C = C(n, \lambda)$ and λ is the ellipticity constant of L.

In the proof we use the following *Faber-Krahn inequality*: if $u \in W_0^{1,2}(\Omega)$ and

$$U = \{x \in \Omega : u(x) \neq 0\}$$

then

$$\int_\Omega |\nabla u|^2\, dx \ge c\,|U|^{-2/n} \int_\Omega u^2\, dx, \tag{1.59}$$

where $c = c(n) > 0$. This inequality is proved in Exercise 10 in the case $n \ge 2$, but it is also valid in the case $n = 1$. Indeed, in this case any function from $W_0^{1,2}$ is continuous, the set U is open and, hence, consists of disjoint union of open intervals, say $U = \sqcup_j I_j$. In each interval I_j, the function u vanishes at the endpoints, which implies then by Friedrichs' inequality that

$$\int_{I_j} |\nabla u|^2\, dx \ge |I_j|^{-2} \int_{I_j} u^2\, dx \ge |U|^{-2} \int_{I_j} u^2\, dx.$$

Summing up in all j, we obtain (1.59) with $n=1$ and $c=1$.

Denote by $\lambda_1(\Omega)$ the first (smallest) eigenvalue of the weak eigenvalue problem in Ω:

$$\begin{cases} \Delta v+\lambda v=0 \text{ in } \Omega \\ v\in W_0^{1,2}(\Omega) \end{cases}$$

By the *Rayleigh principle*, we have

$$\lambda_1(\Omega)=\inf_{u\in W_0^{1,2}(\Omega)\setminus\{0\}}\frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2dx}.$$

Since $|U|\leq|\Omega|$, it follows from (1.59) that

$$\lambda_1(\Omega)\geq c\,|\Omega|^{-2/n}. \tag{1.60}$$

This inequality is related to the following *Faber-Krahn theorem*: if Ω^* denotes a ball of the same volume as Ω then

$$\lambda_1(\Omega)\geq\lambda_1(\Omega^*). \tag{1.61}$$

In other words, among all domains with the same volume, the minimal value of λ_1 is achieved on balls. This is related to isoperimetric property of balls: among all domains with the same volume, the minimal boundary area is achieved on balls.

Observe that if $\Omega^*=B_R$ then

$$\lambda_1(\Omega^*)=\lambda_1(B_R)=\frac{c'}{R^2}$$

where $c'=c'(n)>0$. Since $|B_R|=c''R^n$, we obtain

$$\lambda_1(\Omega^*)=c\,|\Omega^*|^{-2/n},$$

which implies by (1.61) and $|\Omega^*|=|\Omega|$ that

$$\lambda_1(\Omega)\geq c\,|\Omega|^{-2/n}. \tag{1.62}$$

Of course, this looks the same as (1.60), except for the constant c in (1.62) is sharp and is achieved on balls, whereas the constant c in (1.60) was some positive constant. However, for our applications we do not need sharp constant c.

Proof of Theorem 1.13. If $\|f\|_{L^\infty}=\infty$ then (1.58) is trivially satisfied. If $\|f\|_{L^\infty}=0$ then by Theorem 1.2 we have $u=0$ and (1.58) holds. Let $0<\|f\|_{L^\infty}<\infty$. Dividing u and f by $\|f\|_{L^\infty}$, we can assume without loss of generality that $\|f\|_{L^\infty}=1$. 03.05.16

Fix $\alpha>0$ and consider a function $v=(u-\alpha)_+\in W_0^{1,2}(\Omega)$. By hypothesis that $Lu=-f$ weakly, we have the identity

$$\int_\Omega\sum_{i,j=1}^n a_{ij}\partial_j u\partial_i v\,dx=\int_\Omega fv\,dx.$$

Note that

$$\partial_j u\partial_i v=\partial_j v\partial_i v\ \text{ a.e.} \tag{1.63}$$

because, by Lemma 1.4, on the set $\{v=0\}=\{u\le\alpha\}$ we have $\partial_i v=0$ a.e., while on the set $\{v>0\}=\{u>\alpha\}$ we have $\partial_j v=\partial_j u$ a.e.. By (1.63) and the uniform ellipticity we obtain

$$\int_\Omega\sum_{i,j=1}^n a_{ij}\partial_j u\partial_i v\,dx=\int_\Omega\sum_{i,j=1}^n a_{ij}\partial_j v\partial_i v\,dx\ge\lambda^{-1}\int_\Omega|\nabla v|^2\,dx. \tag{1.64}$$

Consider the set

$$U_\alpha:=\{u>\alpha\}=\{v>0\}$$

and observe that, by the Faber-Krahn inequality,

$$\int_\Omega|\nabla v|^2\,dx\ge c\,|U_\alpha|^{-2/n}\int_\Omega v^2\,dx, \tag{1.65}$$

where $c=c(n)>0$. By $\|f\|_{L^\infty}=1$ and Cauchy-Schwarz inequality, we have

$$\int_\Omega fv\,dx\le\int_\Omega vdx=\int_{U_\alpha}1\cdot v\,dx\le|U_\alpha|^{1/2}\left(\int_\Omega v^2dx\right)^{1/2}.$$

Combining all the above inequalities, we obtain

$$c\lambda^{-1}\,|U_\alpha|^{-2/n}\int_\Omega v^2\,dx\le|U_\alpha|^{1/2}\left(\int_\Omega v^2dx\right)^{1/2},$$

whence

$$\left(\int_\Omega v^2dx\right)^{1/2}\le c^{-1}\lambda\,|U_\alpha|^{1/2+2/n}.$$

Let us rewrite this inequality in the form

$$\int_\Omega(u-\alpha)_+^2\,dx\le K\,|U_\alpha|^p. \tag{1.66}$$

where $K=\left(c^{-1}\lambda\right)^2$ and $p=1+4/n$.

Claim. *Assume that a measurable function u in Ω satisfies for any $\alpha>0$ the inequality (1.66) with some K and $p>1$. Then*

$$\operatorname*{esssup}_\Omega u\le C\,|\Omega|^{\frac{p-1}{2}}, \tag{1.67}$$

where $C=C(K,p)$.

In particular, if as above u is a solution of (1.57) with $\|f\|_{L^\infty}=1$ then (1.66) holds with $p=1+4/n$. Since $\frac{p-1}{2}=\frac{2}{n}$, we obtain by (1.67)

$$\operatorname*{esssup}_\Omega u\le C\,|\Omega|^{\frac{2}{n}}.$$

Applying the same argument to $-u$, we obtain the same estimate for $\operatorname{esssup}(-u)$, whence

$$\|u\|_{L^\infty}\le C\,|\Omega|^{2/n},$$

which coincides with (1.58) when $\|f\|_{L^\infty} = 1$.

Hence, it remains to prove the above Claim. Choose some $\beta > \alpha$ and consider the set $U_\beta = \{u > \beta\}$. Then we have

$$\int_\Omega (u-\alpha)_+^2 \, dx \geq \int_{U_\beta} (u-\alpha)_+^2 \, dx \geq (\beta-\alpha)^2 |U_\beta| ,$$

which together with (1.66) implies

$$(\beta-\alpha)^2 |U_\beta| \leq K |U_\alpha|^p ,$$

and, hence,

$$|U_\beta| \leq \frac{K}{(\beta-\alpha)^2} |U_\alpha|^p . \tag{1.68}$$

Fix $\alpha > 0$ to be chosen below, and consider a sequence $\{\alpha_k\}_{k=0}^\infty$ where $\alpha_k = \alpha\left(2-2^{-k}\right)$. This sequence is increasing, $\alpha_0 = \alpha$ and $\alpha_k \to 2\alpha$ as $k \to \infty$. Set

$$V_k = |\{u > \alpha_k\}|$$

and observe that by (1.68)

$$V_k \leq \frac{K}{(\alpha_k - \alpha_{k-1})^2} V_{k-1}^p .$$

Since $\alpha_k - \alpha_{k-1} = \alpha 2^{-k}$, it follows that

$$V_k \leq K\alpha^{-2} 4^k V_{k-1}^p = 4^k M V_{k-1}^p \tag{1.69}$$

where $M = K\alpha^{-2}$. Iterating this inequality, we obtain

$$\begin{aligned} V_k &\leq 4^k M V_{k-1}^p \leq 4^k M \left(4^{k-1} M V_{k-2}^p\right)^p = 4^{k+p(k-1)} M^{1+p} V_{k-2}^{p^2} \\ &\leq 4^{k+p(k-1)} M^{1+p} \left(4^{k-2} M V_{k-3}^p\right)^{p^2} = 4^{k+p(k-1)+p^2(k-2)} M^{1+p+p^2} V_{k-3}^{p^3} \\ &\leq \ldots \leq 4^{k+p(k-1)+\ldots+p^{k-1}} M^{1+p+p^2+\ldots+p^{k-1}} V_0^{p^k} . \end{aligned}$$

Let us use the identities

$$1 + p + p^2 + \ldots + p^{k-1} = \frac{p^k - 1}{p-1}$$

and

$$k + p(k-1) + p^2(k-2) + \ldots + p^{k-1} = \frac{p^{k+1} - (k+1)p + k}{(p-1)^2}$$

that are easily proved by induction. Then we have

$$\begin{aligned} V_k &\leq 4^{\frac{p^{k+1}-(k+1)p+k}{(p-1)^2}} M^{\frac{p^k-1}{p-1}} V_0^{p^k} \\ &= \left[4^{\frac{p}{(p-1)^2}} M^{\frac{1}{p-1}} V_0\right]^{p^k} 4^{\frac{-(k+1)p+k}{(p-1)^2}} M^{-\frac{1}{p-1}} . \end{aligned} \tag{1.70}$$

We would like to make sure that $V_k \to 0$ as $k \to \infty$. Since $V_0 \le |\Omega|$, it suffices to have for that

$$4^{\frac{p}{(p-1)^2}} M^{\frac{1}{p-1}} |\Omega| < 1,$$

that is,

$$4^{\frac{p}{(p-1)^2}} K^{\frac{1}{p-1}} \alpha^{-\frac{2}{p-1}} |\Omega| < 1.$$

For example, we can make the left hand side equal to $\frac{1}{2}$ by choosing α from the equation

$$4^{\frac{p}{(p-1)^2}} K^{\frac{1}{p-1}} \alpha^{-\frac{2}{p-1}} |\Omega| = \frac{1}{2}$$

that is,

$$\alpha = \left(2 \cdot 4^{\frac{p}{(p-1)^2}} K^{\frac{1}{p-1}} |\Omega|\right)^{\frac{p-1}{2}} = C_1 |\Omega|^{\frac{p-1}{2}}.$$

With this choice of α we have

$$|\{u > \alpha_k\}| \to 0 \text{ as } k \to \infty,$$

which implies that

$$|\{u \ge 2\alpha\}| = 0$$

and, hence,

$$\operatorname{esssup} u \le 2\alpha = 2C_1 |\Omega|^{\frac{p-1}{2}}, \tag{1.71}$$

which finishes the proof of (1.67) with $C = 2C_1$. ■

Theorem 1.13 provides a non-trivial estimate even in the case $L = \Delta$. Consider the following weak Dirichlet problem:

$$\begin{cases} \Delta u = -1 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.72}$$

We know that the solution $u(x)$ is a smooth function. In fact, it has the following probabilistic meaning: if $x \in \Omega$ is the starting point of Brownian motion $\{X_t\}$ in $\mathbb{R}^n$ then $u(x)$ is the *mean exit time* from Ω. In other words, if we define the first exist time τ_Ω from Ω by

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\},$$

then

$$u(x) = \mathbb{E}_x \tau_\Omega. \tag{1.73}$$

More generally, the Dirichlet problem

$$\begin{cases} \Delta u = -f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has solution

$$u(x) = \mathbb{E}_x \int_0^{\tau_\Omega} f(X_t)\,dt,$$

which implies (1.73) for $f = 1$.

Let u be the solution of (1.72). Then by Theorem 1.13 we have

$$\sup_\Omega u \le C |\Omega|^{2/n},$$

that is, the mean exit time from Ω is bounded from above by $C\,|\Omega|^{2/n}$. In particular, if $\Omega = B_R$ then $|\Omega| = c_n R^n$ and we obtain the estimate

$$\sup_{B_R} u \le C' R^2. \tag{1.74}$$

Note that the classical Dirichlet problem

$$\begin{cases} \Delta u = -1 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R \end{cases}$$

has an obvious solution

$$u\,(x) = \frac{R^2 - |x|^2}{2n}.$$

In particular, we see that

$$\sup_{B_R} u = u\,(0) = \frac{R^2}{2n},$$

which shows that the estimate (1.74) is optimal up to the value of the constant. Let us emphasize the following probabilistic meaning of the latter identity: the mean exit time from the center of the ball is equal to $\frac{R^2}{2n}$. In particular, it is proportional not to R as it would be in the case of a constant outward speed, but to R^2, which for large R means a significant slowdown in comparison with a constant speed movement. This happens because Brownian particle does not go away in radial direction but spends a lot of time for moving also in angular directions. For example, an observer staying at the origin and watching in the direction of the particle, will have to turn around all the times in order to keep the particle in the view.

1.5.2 Operator with lower order terms

Now we state and prove a more general version of Theorem 1.13. Consider in Ω a more general operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right) + \sum_{i=1}^{n} b_i \partial_i u \tag{1.75}$$

where the coefficients a_{ij} and b_i are measurable functions, the matrix (a_{ij}) is uniformly elliptic with the ellipticity constant λ, and all b_i are bounded, that is, there is a constant b such that

$$\sum_{i=1}^{n} |b_i| \le b \text{ a.e. in } \Omega. \tag{1.76}$$

We say that a function $u \in W^{1,2}_{loc}(\Omega)$ satisfies weakly in Ω the inequality $Lu \ge g$ where $g \in L^2_{loc}(\Omega)$ if, for any non-negative function $\varphi \in \mathcal{D}(\Omega)$,

$$-\int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \varphi \, dx + \int_\Omega \sum_{i=1}^{n} b_i \partial_i u \, \varphi \, dx \ge \int_\Omega g\varphi \, dx. \tag{1.77}$$

Similarly one defines the meaning of $Lu \le g$. If $u \in W^{1,2}(\Omega)$ and $g \in L^2(\Omega)$ then, as in the proof of Lemma 1.1, the test function φ in (1.77) can be taken from $W_0^{1,2}(\Omega)$.

Theorem 1.14 *Let Ω be a bounded domain in $\mathbb{R}^n$ and let L be the operator* (1.75). *Assume*

$$|\Omega| < \delta, \tag{1.78}$$

where $\delta = c_n \lambda^{-n} b^{-n}$ with some $c_n > 0$. If $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ satisfy

$$\begin{cases} Lu \geq -f \text{ weakly in } \Omega, \\ u_+ \in W_0^{1,2}(\Omega), \end{cases} \tag{1.79}$$

then, for any $q \in [2, \infty] \cap (n/2, \infty]$, the following estimate holds:

$$\operatorname*{esssup}_{\Omega} u \leq C |\Omega|^{\frac{2}{n} - \frac{1}{q}} \|f_+\|_{L^q} \tag{1.80}$$

with a constant $C = C(n, \lambda, q)$.

Theorem 1.14 extends Theorem 1.13 in three ways:

1. We allow in the operator L lower order terms.
2. We allow inequality $Lu \geq -f$ instead of equality.
3. We allows $u_+ \in W_0^{1,2}(\Omega)$ instead of $u \in W_0^{1,2}(\Omega)$
4. The main estimate in given in terms of $\|f_+\|_{L^q}$ instead of $\|f\|_{L^\infty}$, where q in particular can be ∞.

Let us explain why Theorem 1.14 contains Theorem 1.13. Indeed, if all $b_i = 0$ and, hence, $b = 0$ then $\delta = \infty$ and the restriction (1.78) on $|\Omega|$ is void. Assuming that $Lu = f$, $u \in W_0^{1,2}(\Omega)$ and applying (1.80) with $q = \infty$, we obtain

$$\operatorname*{esssup}_{\Omega} u \leq C |\Omega|^{\frac{2}{n}} \|f_+\|_{L^\infty} = C |\Omega|^{\frac{2}{n}} \operatorname{esssup} f_+. \tag{1.81}$$

Applying this inequality to function $-u$, we obtain

$$\operatorname*{essinf}_{\Omega} (-u) \leq C |\Omega|^{2/n} \operatorname{esssup} f_-,$$

whence it follows that

$$\operatorname*{esssup}_{\Omega} |u| \leq C |\Omega|^{2/n} \operatorname{esssup} |f|,$$

10.05.16 which is equivalent to (1.58).

Applying Theorem 1.14 with $f = 0$, we obtain the following the maximum principle: if $u_+ \in W_0^{1,2}(\Omega)$ and $Lu \geq 0$ weakly then $u \leq 0$ a.e. in Ω. The condition $u_+ \in W_0^{1,2}(\Omega)$ means that in some sense "$u_+ = 0$ on $\partial\Omega$", that is, "$u \leq 0$ on $\partial\Omega$".

Proof. Since f can be replaced in (1.79) by f_+, we can assume without loss of generality that $f \geq 0$. If $\|f\|_{L^q} = \infty$ then there is nothing to prove. If $0 < \|f\|_{L^q} < \infty$ then dividing f and u by $\|f\|_{L^q}$, we can assume that $\|f\|_{L^q} = 1$. Finally, the case $\|f\|_{L^q} = 0$ amounts to the previous case as follows. Indeed, if $Lu \geq 0$ then also $Lu \geq -\varepsilon$ for any $\varepsilon > 0$. Applying (1.80) with $f = \varepsilon$, we obtain

$$\operatorname*{esssup}_{\Omega} u \leq C |\Omega|^{\frac{2}{n} - \frac{1}{q}} \|\varepsilon\|_{L^q}.$$

Letting $\varepsilon \to 0$ we obtain (1.80) with $f = 0$.

Hence, we assume in what follows that $f \geq 0$ and $\|f\|_{L^q} = 1$. As in the proof of Theorem 1.13, fix $\alpha > 0$ and consider a function

$$v := (u - \alpha)_+ = (u_+ - \alpha)_+ .$$

This function belongs to $W_0^{1,2}(\Omega)$ and is non-negative. By the hypothesis that $Lu \geq -f$ weakly, we have the inequality

$$-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i v \, dx + \int_\Omega \sum_{i=1}^n b_i \partial_i u \, v \, dx \geq -\int_\Omega f v \, dx,$$

that is,

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i v \, dx \leq \int_\Omega \sum_{i=1}^n b_i \partial_i u \, v \, dx + \int_\Omega f v \, dx. \tag{1.82}$$

We estimate the left hand side similarly to (1.64). Observe that

$$\partial_j u \partial_i v = \partial_j v \partial_i v \quad \text{a.e. in } \Omega$$

because on the set $\{v = 0\}$ we have $\partial_i v = 0$ a.e. (by Lemma 1.5), whereas on the set $\{v > 0\}$ we have

$$\partial_i u = \partial_i u_+ = \partial_i v,$$

by Exercise 14 and by Lemma 1.4. Hence, we have

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i v \, dx = \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j v \partial_i v \, dx \geq \lambda^{-1} \int_\Omega |\nabla v|^2 \, dx.$$

Now let us estimate the terms in the right hand side of (1.82). Using

$$\partial_i u \, v = \partial_i v \, v \quad \text{a.e. in } \Omega$$

and (1.76), we obtain, for any $\varepsilon > 0$,

$$\int_\Omega \sum_{i=1}^n b_i \partial_i u \, v \, dx \leq b \int_\Omega |\nabla v| \, |v| \, dx \leq \frac{b}{2} \int_\Omega \left(\varepsilon |\nabla v|^2 + \frac{1}{\varepsilon} v^2 \right) dx,$$

where we have use the inequality

$$XY \leq \frac{1}{2}\left(\varepsilon X^2 + \frac{1}{\varepsilon} Y^2 \right).$$

It follows that

$$\lambda^{-1} \int_\Omega |\nabla v|^2 \, dx \leq \frac{b\varepsilon}{2} \int_\Omega |\nabla v|^2 \, dx + \frac{b}{2\varepsilon} \int_\Omega v^2 dx + \int_\Omega f v dx.$$

Let us choose ε to satisfy the condition $b\varepsilon = \lambda^{-1}$, that is,

$$\varepsilon = \frac{1}{\lambda b}.$$

Then we obtain

$$\lambda^{-1}\int_\Omega |\nabla v|^2\,dx \le \frac{1}{2}\lambda^{-1}\int_\Omega |\nabla v|^2\,dx + \frac{\lambda b^2}{2}\int_\Omega v^2 dx + \int_\Omega fv dx,$$

whence

$$\int_\Omega |\nabla v|^2\,dx \le \lambda^2 b^2 \int_\Omega v^2 dx + 2\lambda \int_\Omega fv dx. \tag{1.83}$$

Using the Faber-Krahn inequality (1.65), we obtain from (1.83) that

$$c\,|U_\alpha|^{-2/n}\int_\Omega v^2\,dx \le \lambda^2 b^2 \int_\Omega v^2 dx + 2\lambda\int_\Omega fv dx, \tag{1.84}$$

where

$$U_\alpha = \{u > \alpha\} = \{v > 0\}.$$

We would like to have

$$c\,|U_\alpha|^{-2/n} > 2\lambda^2 b^2. \tag{1.85}$$

Since $|U_\alpha| \le |\Omega|$, it suffices to have

$$c\,|\Omega|^{-2/n} > 2\lambda^2 b^2,$$

which is equivalent to

$$|\Omega| < \left(\frac{c}{2}\right)^{n/2}\lambda^{-n} b^{-n},$$

which is equivalent to (1.78) with

$$\delta = \left(\frac{c}{2}\right)^{n/2}\lambda^{-n} b^{-n}. \tag{1.86}$$

Hence, (1.85) is satisfied, and (1.84) yields

$$\frac{1}{2}c\,|U_\alpha|^{-2/n}\int_\Omega v^2\,dx \le 2\lambda\int_\Omega fv dx. \tag{1.87}$$

Applying the Hölder inequality with the Hölder exponents q and $q' = \frac{q}{q-1}$ and using $\|f\|_{L^q} = 1$, we obtain

$$\int_\Omega fv dx \le \|f\|_{L^q}\|v\|_{L^{q'}} = \left(\int_{U_\alpha} v^{q'} dx\right)^{1/q'}$$

(note that if $q = \infty$ then $q' = 1$). Since $q \ge 2$ and, hence, $q' \le 2$, applying the Hölder inequality with one of the Hölder exponents $\frac{2}{q'}$, we obtain

$$\begin{aligned}\int_{U_\alpha} v^{q'} dx &\le \left(\int_{U_\alpha} 1 dx\right)^{1-\frac{q'}{2}}\left(\int_{U_\alpha}\left(v^{q'}\right)^{\frac{2}{q'}} dx\right)^{\frac{q'}{2}} \\ &= |U_\alpha|^{1-\frac{q'}{2}}\left(\int_\Omega v^2 dx\right)^{\frac{q'}{2}},\end{aligned}$$

whence

$$\int_\Omega fv dx \le |U_\alpha|^{\frac{1}{q'}-\frac{1}{2}}\left(\int_\Omega v^2 dx\right)^{\frac{1}{2}}.$$

Combining with (1.87), we obtain

$$\frac{1}{2}c\,|U_\alpha|^{-2/n}\int_\Omega v^2\,dx \le 2\lambda\,|U_\alpha|^{\frac{1}{q'}-\frac{1}{2}}\left(\int_{U_\alpha} v^2dx\right)^{\frac{1}{2}},$$

whence

$$\left(\int_\Omega v^2dx\right)^{\frac{1}{2}} \le 4c^{-1}\lambda\,|U_\alpha|^{\frac{1}{q'}-\frac{1}{2}+\frac{2}{n}},$$

and

$$\int_\Omega v^2dx \le K\,|U_\alpha|^{\frac{2}{q'}-1+\frac{4}{n}},$$

where $K = (4c^{-1}\lambda)^2$. Set

$$p = \frac{2}{q'} - 1 + \frac{4}{n}$$

and observe that

$$p = 2\left(1-\frac{1}{q}\right) - 1 + \frac{4}{n} = 1 - \frac{2}{q} + \frac{4}{n} > 1$$

because $q > \frac{n}{2}$. Hence, we have

$$\int_\Omega (u-\alpha)_+^2\,dx \le K\,|U_\alpha|^p$$

with $p > 1$. This inequality coincides with the inequality (1.66) from the proof of Theorem 1.13. Using the Claim from the proof of Theorem 1.13, we arrive at (1.71), that is,

$$\operatorname*{esssup}_\Omega u \le C\,|\Omega|^{\frac{p-1}{2}} = C\,|\Omega|^{\frac{2}{n}-\frac{1}{q}},$$

which finishes the proof of (1.80). ■

Let us discuss the restriction $|\Omega| < \delta$ that appears in the statement of Theorem 1.14. Consider the operator

$$L = \Delta + \sum_{i=1}^n b_i\partial_i u$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ and the Dirichlet problem

$$\begin{cases} Lu = -1 \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega)\,. \end{cases} \tag{1.88}$$

The estimate (1.80) of Theorem 1.14 yields, for $q = \infty$, that

$$u(x) \le C\,|\Omega|^{2/n} \text{ in } \Omega. \tag{1.89}$$

The function $u(x)$ has the following physical/probabilistic meaning. Operator L is the generator of a diffusion process with a drift $\overrightarrow{b} = (b_1, ..., b_n)$. In the case $\overrightarrow{b} \equiv 0$ this is Brownian motion, but in the case of non-zero $\overrightarrow{b}$ one can think of this diffusion process as Brownian motion in a media that moves at any point x with the velocity $\overrightarrow{b}(x)$ (in other words, media with convection). The function $u(x)$ that solves (1.88) gives the mean exit time of this diffusion from Ω assuming that the starting point is x. The estimate (1.89) provides an upper bound for the mean exit time, saying that exit on average occurs before time $C|\Omega|^{2/n}$.

However, if the drift $\overrightarrow{b}(x)$ is directed inwards the domain Ω, then one can imagine that the drift prevents the particle to escape from the domain, which may result in a longer exit time. As Theorem 1.14 says, this cannot happen if $|\Omega|$ is small enough, but as we will see in example below, this can happen if $|\Omega|$ is large enough (for large domains the effect of convection becomes dominating over diffusion).

Example. Consider one-dimensional example with $\Omega = (-R, R)$ and

$$Lu = u'' + bu'$$

where

$$b(x) = -\operatorname{sgn} x = \begin{cases} 1, & x < 0, \\ 0, & x = 0, \\ -1, & x > 0. \end{cases}$$

Let us solve explicitly the Dirichlet problem

$$\begin{cases} Lu = -1 \ \text{ in } \ (-R, R) \\ u(-R) = u(R) = 0. \end{cases}$$

It suffices to solve the problem

$$\begin{cases} Lu = -1 \ \text{ in } \ (0, R) \\ u'(0) = u(R) = 0 \end{cases} \tag{1.90}$$

and then extend u evenly to $(-R, 0)$, that is, by setting $u(-x) = u(x)$. Since u satisfies in $(0, R)$ the equation

$$u'' - u' = -1, \tag{1.91}$$

in $(-R, 0)$ it will satisfy

$$u'' + u' = -1.$$

Due to the the boundary condition $u'(0) = 0$, the function u is a weak solution of $Lu = -1$ on $(-R, R)$.

The ODE (1.91) has the general solution

$$u(x) = c_1 + c_2 e^x + x.$$

The boundary conditions $u'(0) = u(R) = 0$ give the following equations for c_1 and c_2:

$$\begin{aligned} c_2 + 1 &= 0 \\ c_1 + c_2 e^R + R &= 0 \end{aligned}$$

whence $c_2 = -1$ and $c_1 = e^R - R$. Hence, (1.90) has solution

$$u(x) = \left(e^R - R\right) - e^x + x.$$

In particular,

$$u(0) = e^R - R - 1.$$

We see that for small R

$$u(0) \approx \frac{R^2}{2}, \tag{1.92}$$

while for large R

$$u(0) \approx e^R. \tag{1.93}$$

Note that the estimate (1.80) with $q = \infty$ gives in this case

$$\|u\|_{L^\infty} \leq CR^2, \tag{1.94}$$

provided R is small enough, where the latter requirement is a consequence of (1.78). The estimate (1.94) agrees with (1.92), but (1.93) shows that (1.94) fails for large R, so that in general the restriction (1.78) cannot be dropped.

Chapter 2

Higher order derivatives of weak solutions

Recall the following property of the distributional Laplace operator in a domain of $\mathbb{R}^n$: if $u \in W^{1,2}_{loc}$ and $\Delta u \in L^2_{loc}$ then $u \in W^{2,2}_{loc}$. Moreover, if $\Delta u \in W^{k,2}_{loc}$ then $u \in W^{k+2,2}_{loc}$. In this Chapter we prove the same property for divergence form elliptic operators. The technique of Fourier series that worked for the Laplace operator, does not work for the operator with variable coefficients, so we use entirely different techniques based on
12.05.16 *difference operators*.

2.1 Existence of 2nd order weak derivatives

Consider the operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) \tag{2.1}$$

in a domain $\Omega \subset \mathbb{R}^n$. As before, we assume that this operator is uniformly elliptic and the coefficients a_{ij} are measurable. Recall that if $u \in W^{1,2}_{loc}(\Omega)$ and $f \in L^2_{loc}(\Omega)$ then the equation $Lu = f$ holds in a weak sense if, for any $\varphi \in \mathcal{D}(\Omega)$,

$$-\int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \varphi \, dx = \int_\Omega f \varphi \, dx. \tag{2.2}$$

Recall also that if $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ then the identity (2.2) can be extended to all $\varphi \in W^{1,2}_0(\Omega)$ (cf. Lemma 1.1).

Claim. *If $u \in W^{1,2}_{loc}(\Omega)$, $f \in L^2_{loc}(\Omega)$, and the identity (2.2) holds for all $\varphi \in \mathcal{D}(\Omega)$ then it also holds for all $\varphi \in W^{1,2}_c(\Omega)$.*

Proof. Fix a function $\varphi \in W^{1,2}_c(\Omega)$ and let U be a precompact open set such that $\operatorname{supp} \varphi \subset U$ and $\overline{U} \subset \Omega$. Clearly, the integration in (2.2) can be restricted to U. Since $u \in W^{1,2}(U)$, $f \in L^2(U)$, $\varphi \in W^{1,2}_0(U)$, we conclude that (2.2) holds by Lemma 1.1. ∎

Claim. *For any $u \in W^{1,2}_{loc}(\Omega)$ (and even for $u \in W^{1,1}_{loc}(\Omega)$) the expression Lu in (2.1) is well-defined in the distributional sense. The identity (2.2) is equivalent to the fact that $Lu = f$ holds in the distributional sense.*

Note that, for a general distribution $u \in \mathcal{D}'(\Omega)$ the expression Lu is not well-defined because the product $a_{ij}\partial_j u$ of a measurable function a_{ij} and a distribution $\partial_j u$ does not makes sense in general[1].

Proof. The function $\partial_j u$ belongs to $L^2_{loc}(\Omega)$ and, since a_{ij} are bounded, the function $a_{ij}\partial_j u$ belongs also to $L^2_{loc}(\Omega)$, in particular, to $\mathcal{D}'(\Omega)$. Hence, $\partial_i(a_{ij}\partial_j u)$ is defined as an element of $\mathcal{D}'(\Omega)$, where ∂_i is understood in distributional sense. Consequently, Lu is defined as an element of $\mathcal{D}'(\Omega)$.

By definition of distributional derivative, we have, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}(Lu, \varphi) &= \sum_{i,j=1}^{n} (\partial_i(a_{ij}\partial_j u), \varphi) = -\sum_{i,j=1}^{n} (a_{ij}\partial_j u, \partial_i\varphi) \\ &= -\sum_{i,j=1}^{n} \int_\Omega a_{ij}\partial_j u \partial_i \varphi \, dx.\end{aligned}$$

Hence, the identity (2.2) becomes

$$(Lu, \varphi) = (f, \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega),$$

which is equivalent to $Lu = f$.

For $u \in W^{1,1}_{loc}(\Omega)$ the proof is the same because L^2_{loc} can be replaced everywhere by L^1_{loc}. ■

Hence, from now on the expression Lu is well-defined as an element of $\mathcal{D}'(\Omega)$ for any $u \in W^{1,2}_{loc}$. Now we can state one of the main results of this Chapter.

Theorem 2.1 *Let L be the operator* (2.1) *and assume that all the coefficients a_{ij} of L are locally Lipschitz in Ω. If $u \in W^{1,2}_{loc}(\Omega)$ and $Lu \in L^2_{loc}(\Omega)$ then $u \in W^{2,2}_{loc}(\Omega)$.*

2.1.1 Lipschitz functions

A function f on a set $S \subset \mathbb{R}^n$ is called *Lipschitz* (or Lipschitz continuous) if there is a constant L such that

$$|f(x) - f(y)| \le L|x - y| \quad \forall x, y \in S.$$

The constant L is called a Lipschitz constant of f on S.

Let Ω be an open subset of $\mathbb{R}^n$. We say that a function $f : \Omega \to \mathbb{R}$ is *locally Lipschitz* if for any point $x \in \Omega$ there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \Omega$ and f is Lipschitz in $B_\varepsilon(x)$.

Let us list some simple properties of locally Lipschitz functions.

1. Any locally Lipschitz function is continuous.

[1] A product av of a distribution $v \in \mathcal{D}'(\Omega)$ and a function a on Ω makes sense only if $a \in C^\infty(\Omega)$. In this case av is defined as an element of $\mathcal{D}'(\Omega)$ as follows:

$$(av, \varphi) = (v, a\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega),$$

which makes sense because $a\varphi \in \mathcal{D}(\Omega)$.

2. If f, g are locally Lipschitz functions then $f+g$ and fg are locally Lipschitz. In particular, the set $Lip_{loc}(\Omega)$ of all locally Lipschitz functions in Ω is a vector space and even an algebra.

3. Any functions from $C^1(\Omega)$ is locally Lipschitz in Ω. Consequently, we have[2]
$$C^1(\Omega) \subset Lip_{loc}(\Omega) \subset C(\Omega). \tag{2.3}$$
In particular, Theorem 2.1 holds if all the coefficients a_{ij} belong to $C^1(\Omega)$.

4. If f is locally Lipschitz in Ω then f is Lipschitz on any compact subset of Ω.

Proof of the property 4.. Let K be a compact subset of Ω. We need to prove that there is a constant C such that, for any two points $x, y \in K$,
$$|f(x) - f(y)| \leq C|x-y|. \tag{2.4}$$
For any $x \in K$ there exists $\varepsilon = \varepsilon_x > 0$ such that the ball $B_{\varepsilon_x}(x)$ is contained in Ω and f is Lipschitz in $B_{\varepsilon_x}(x)$ with the Lipschitz constant L_x. The balls $\left\{B_{\frac{1}{2}\varepsilon_x}(x)\right\}_{x\in K}$ form an open covering of K, so choose a finite subcover $\left\{B_{\frac{1}{2}\varepsilon_{x_i}}(x_i)\right\}_{i=1}^N$ and set
$$\varepsilon := \min_i \varepsilon_{x_i} > 0, \quad L := \max_i L_{x_i} < \infty.$$
Let us now prove (2.4) if $x, y \in K$ are such that
$$|x-y| < \frac{1}{2}\varepsilon.$$
Indeed, the point x belongs to one of the balls $B_{\frac{1}{2}\varepsilon_{x_i}}(x_i)$. Since
$$|x_i - y| \leq |x_i - x| + |x-y| < \frac{1}{2}\varepsilon_{x_i} + \frac{1}{2}\varepsilon < \varepsilon_{x_i},$$
we see that $y \in B_{\varepsilon_{x_i}}(x_i)$. Hence, both x, y are contained in the same ball $B_{\varepsilon_{x_i}}(x_i)$, whence we obtain that
$$|f(x) - f(y)| \leq L_i|x-y| \leq L|x-y|.$$
Hence, (2.4) holds with $C = L$. Consider now the case
$$|x-y| \geq \frac{1}{2}\varepsilon.$$
Setting $M = \sup_K |f|$, we obtain
$$\frac{|f(x) - f(y)|}{|x-y|} \leq \frac{2M}{\frac{1}{2}\varepsilon}$$
so that (2.4) holds with $C = 4\varepsilon^{-1}M$. Hence, (2.4) holds for all $x, y \in K$ with $C = \max(L, 4\varepsilon^{-1}M)$. ■

[2] Both inclusions in (2.3) are strict. For example, function $|x|$ in $\mathbb{R}$ is Lipschitz but not C^1, whereas function $|x|^{1/2}$ is continuous but not locally Lipschitz.

2.1.2 Difference operators

For the proof of Theorem 2.1 we need the notion and properties of *difference* operators. Fix a unit vector $e \in \mathbb{R}^n$, a non-zero real number h and denote by ∂_e^h an operator that acts on any function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$\partial_e^h f(x) = \frac{f(x+he) - f(x)}{h}.$$

Obliviously, if f is differentiable then, for any $x \in \mathbb{R}^n$

$$\partial_e^h f(x) \to \partial_e f(x) \quad \text{as } h \to 0.$$

Unlike the differential operators, the difference operator ∂_e^h is defined on any function f. Moreover, if f belongs to a function space $\mathcal{F}$ that is translation invariant, then also $\partial_e^h f \in \mathcal{F}$. Note that all function spaces over $\mathbb{R}^n$ that we use: L^p, L^p_{loc}, $W^{k,p}$, $W^{k,p}_{loc}$, $W_0^{k,p}$ etc., are translation invariant.

In the next lemma we state and prove some simple properties of difference operators.

Lemma 2.2 (a) (Product rule) *For arbitrary functions f, g on $\mathbb{R}^n$ we have*

$$\partial_e^h (fg) = f(\cdot + he)\,\partial_e^h g + \left(\partial_e^h f\right) g. \tag{2.5}$$

(b) (Integration by parts) *If $f, g \in L^2(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} \left(\partial_e^h f\right) g\, dx = -\int_{\mathbb{R}^n} f\left(\partial_e^{-h} g\right) dx. \tag{2.6}$$

(c) (Interchangeability with ∂_i) *If $f \in L^1_{loc}(\mathbb{R}^n)$ and the distributional derivative $\partial_i f$ belongs to $L^1_{oc}(\mathbb{R}^n)$ then*

$$\partial_e^h (\partial_i f) = \partial_i \left(\partial_e^h f\right).$$

Proof. (a) We have

$$\begin{aligned}
\partial_e^h (fg)(x) &= \frac{1}{h}\left(f(x+he)\,g(x+he) - f(x)\,g(x)\right) \\
&= \frac{1}{h} f(x+he)\left(g(x+he) - g(x)\right) \\
&\quad + \frac{1}{h}\left(f(x+he) - f(x)\right) g(x) \\
&= f(x+he)\,\partial_e^h g(x) + \partial_e^h f(x)\, g(x),
\end{aligned}$$

which is equivalent to (2.5).

(b) Since all functions $f, \partial_e^h f, g, \partial_e^{-h} g$ are in L^2, the both integrals in (2.6) are convergent. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(\partial_e^h f\right) g\, dx &= \frac{1}{h}\int_{\mathbb{R}^n} \left(f(x+he) - f(x)\right) g(x)\, dx \\
&= \frac{1}{h}\int_{\mathbb{R}^n} f(x+he)\, g(x)\, dx - \frac{1}{h}\int_{\mathbb{R}^n} f(x)\, g(x)\, dx \\
&= \frac{1}{h}\int_{\mathbb{R}^n} f(x)\, g(x-he)\, dx - \frac{1}{h}\int_{\mathbb{R}^n} f(x)\, g(x)\, dx \\
&= -\int_{\mathbb{R}^n} f(x)\, \partial_e^{-h} g(x)\, dx.
\end{aligned}$$

(c) We have

$$\begin{aligned}\partial_i\left(\partial_e^h f\right) &= \partial_i \frac{f(x+he)-f(x)}{h} \\ &= \frac{1}{h}\left(\partial_i f(x+he)-\partial_i f(x)\right) \\ &= \partial_e^h\left(\partial_i f\right).\end{aligned}$$

■

Next Lemma provides an important tool for proving the existence of a partial derivative $\partial_e f$ in L^2.

Lemma 2.3 *If $f \in L^2(\mathbb{R}^n)$ and there is a constant K such that, for all small enough $|h|$,*

$$\left\|\partial_e^h f\right\|_{L^2} \leq K,$$

then the distributional derivative $\partial_e f$ belongs to $L^2(\mathbb{R}^n)$ and

$$\|\partial_e f\|_{L^2} \leq K.$$

Proof. Take any sequence $h_k \to 0$. The sequence $\left\{\partial_e^{h_k} f\right\}$ is bounded in L^2 by hypothesis. We use the fact that any bounded sequence in a Hilbert space contains a *weakly* convergent subsequence[3]. Hence, passing to a subsequence, we can assume that the sequence $\left\{\partial_e^{h_k} f\right\}$ converges weakly in L^2 to some function $g \in L^2$, that is,

$$\partial_e^{h_k} f \rightharpoonup g \quad \text{as } k \to \infty. \tag{2.7}$$

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Let us show that $\partial_e f = g$. By the weak convergence, we have, for any $\varphi \in L^2(\mathbb{R}^n)$

$$\left(\partial_e^{h_k} f, \varphi\right) \to (g, \varphi) \quad \text{as } k \to \infty, \tag{2.8}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\mathbb{R}^n)$. For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have by (2.6)

$$\left(\partial_e^{h_k} f, \varphi\right) = -\left(f, \partial_e^{-h_k}\varphi\right) \to -(f, \partial_e \varphi) \ \text{ as } k \to \infty, \tag{2.9}$$

because

$$\partial_e^{-h_k}\varphi \rightrightarrows \partial_e \varphi \quad \text{as } k \to \infty$$

and the integration in

$$\left(f, \partial_e^{-h_k}\varphi\right) = \int_{\mathbb{R}^n} f \ \partial_e^{-h_k}\varphi \, dx$$

can be reduced to a small neighborhood of $\operatorname{supp}\varphi$. The comparison of (2.8) and (2.9) yields

$$-(f, \partial_e\varphi) = (g, \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n.)$$

[3]Recall that a sequence $\{u_k\}$ of elements of a Hilbert space H converges weakly to $u \in H$ if

$$(u_k, \varphi) \to (u, \varphi) \quad \forall \varphi \in H.$$

The weak convergence is denoted by $u_k \rightharpoonup u$, and it is generally weaker that the strong (norm) convergence $u_k \to u$.

Considering now $(\cdot,\cdot)$ as pairing between distributions and test functions and recalling that the distributional derivative $\partial_e f$ is defined by

$$(\partial_e f, \varphi) = -(f, \partial_e \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n),$$

we conclude that $\partial_e f = g$. Consequently, we have $\partial_e f \in L^2(\mathbb{R}^n)$ and, by (2.7),

$$\partial_e^{h_k} f \rightharpoonup \partial_e f \text{ as } k \to \infty. \tag{2.10}$$

Since $\left\|\partial_e^{h_k} f\right\|_{L^2} \leq K$, we obtain that, for any $\varphi \in L^2(\mathbb{R}^n)$,

$$\left|\left(\partial_e^{h_k} f, \varphi\right)\right| \leq K \|\varphi\|_{L^2}$$

which implies by (2.10) that

$$|(\partial_e f, \varphi)| \leq K \|\varphi\|_{L^2}.$$

It follows that

$$\|\partial_e f\|_{L^2} = \sup_{\varphi \in L^2(\mathbb{R}^n)\setminus\{0\}} \frac{|(\partial_e f, \varphi)|}{\|\varphi\|_{L^2}} \leq K,$$

which finishes the proof. ∎

Corollary 2.4 (a) *If f is a Lipschitz function in $\mathbb{R}^n$ with compact support then $f \in W^{1,2}(\mathbb{R}^n)$. Moreover, $f \in W^{1,\infty}(\mathbb{R}^n)$.*

(b) *If f is a locally Lipschitz function in Ω then $f \in W^{1,2}_{loc}(\Omega)$. Moreover, $f \in W^{1,\infty}_{loc}(M)$.*

Proof. (a) Indeed, if L is the Lipschitz constant of f then for all x and all h we have $\left|\partial_e^h f(x)\right| \leq L$. Since $\partial_e^h f$ also has compact support, it follows that, for all $|h| < 1$, $\left\|\partial_e^h f\right\|_{L^2}$ is uniformly bounded, which implies by Lemma 2.3 that $\partial_e f \in L^2$ and, hence, $f \in W^{1,2}(\mathbb{R}^n)$.

Since f is continuous and has compact support, we see that f is bounded, that is, $f \in L^\infty(\mathbb{R}^n)$. Since $\left|\partial_e^h f(x)\right| \leq L$ pointwise, we have, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\left|\left(\partial_e^h f, \varphi\right)\right| \leq L \|\varphi\|_{L^1}.$$

By (2.10) we have the same property for $\partial_e f$, that is,

$$|(\partial_e f, \varphi)| \leq L \|\varphi\|_{L^1},$$

which implies that

$$\|\partial_e f\|_{L^\infty} = \sup_{\varphi \in \mathcal{D}(\mathbb{R}^n)\setminus\{0\}} \frac{|(\partial_e f, \varphi)|}{\|\varphi\|_{L^1}} \leq L.$$

Hence, $\partial_e f \in L^\infty(\mathbb{R}^n)$ and $f \in W^{1,\infty}(\mathbb{R}^n)$.

(b) Let U be a precompact open set such that $\overline{U} \subset \Omega$ and let φ be a cutoff function of U in Ω. Since φ is Lipschitz, it follows that $f\varphi$ is locally Lipschitz. Since $f\varphi$ has compact support, we conclude that $f\varphi$ is Lipschitz in a neighborhood of $\operatorname{supp}(f\varphi)$ and, hence, in $\mathbb{R}^n$. It follows by (a) that $f\varphi \in W^{1,2}(\mathbb{R}^n)$. Since $\varphi = 1$ in U, it follows that $f \in W^{1,2}(U)$ and, hence, $f \in W^{1,2}_{loc}(\Omega)$. Since also $f \in W^{1,\infty}(U)$, it follows that $f \in W^{1,\infty}_{loc}(\Omega)$. ∎

Lemma 2.5 *If* $f \in W_0^{1,2}(\mathbb{R}^n)$ *then*

$$\left\|\partial_e^h f\right\|_{L^2} \le \|\partial_e f\|_{L^2}. \tag{2.11}$$

Proof. It suffices to prove this for $f \in \mathcal{D}(\mathbb{R}^n)$, since the both sides of the inequality (2.11) are continuous functionals in $W^{1,2}(\mathbb{R}^n)$. We have

$$\begin{aligned}
\partial_e^h f(x) &= \frac{1}{h}\left(f(x+he) - f(x)\right) \\
&= \frac{1}{h}\int_0^h \frac{d}{dt}\left[f(x+te)\right] dt \\
&= \frac{1}{h}\int_0^h \partial_e f(x+te)\, dt
\end{aligned}$$

where we have used that

$$\frac{d}{dt}\left[f(x+te)\right] = \sum_{i=1}^n \partial_{x_i} f(x+te)\, e_i = \partial_e f(x+te).$$

It follows that

$$\begin{aligned}
\left|\partial_e^h f(x)\right|^2 &= \left(\frac{1}{h}\int_0^h \partial_e f(x+te)\, dt\right)^2 \\
&\le \frac{1}{h}\int_0^h \left|\partial_e f(x+te)\right|^2 dt
\end{aligned}$$

and, using Fubini's formula,

$$\begin{aligned}
\left\|\partial_e^h f\right\|_{L^2}^2 &\le \frac{1}{h}\int_{\mathbb{R}^n}\left(\int_0^h \left|\partial_e f(x+te)\right|^2 dt\right) dx \\
&= \frac{1}{h}\int_0^h\left(\int_{\mathbb{R}^n} \left|\partial_e f(x+te)\right|^2 dx\right) dt \\
&= \frac{1}{h}\int_0^h\left(\int_{\mathbb{R}^n} \left|\partial_e f(y)\right|^2 dy\right) dt \\
&= \frac{1}{h}\int_0^h \|\partial_e f\|_{L^2}^2\, dt = \|\partial_e f\|_{L^2}^2.
\end{aligned}$$

■

2.1.3 Proof of Theorem 2.1

We precede the proof by one more lemma. Consider in an open domain $\Omega \subset \mathbb{R}^n$ an operator

$$Lu = \sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right),$$

where the coefficients a_{ij} are measurable and L is uniformly elliptic.

Lemma 2.6 (Product rule for L) *If $u,v \in W^{1,2}_{loc}(\Omega)$ and $Lu, Lv \in L^2_{loc}(\Omega)$ then*

$$L(uv) = (Lu)v + 2\sum_{i,j=1}^{n} a_{ij}\partial_j u \partial_i v + uLv. \tag{2.12}$$

We will use in the proof the following product rule from Exercise 19: if $u,v \in W^{1,2}_{loc}(\Omega)$ then $uv \in W^{1,1}_{loc}(\Omega)$ and

$$\partial_j(uv) = (\partial_j u)v + u(\partial_j v). \tag{2.13}$$

In particular, since $uv \in W^{1,1}_{loc}(\Omega)$, the expression $L(uv)$ in (2.12) is well-defined as a distribution.

A simplified version of Lemma 2.6 Before the proof in full generality, let us prove the formula (2.12) in a simpler setting. Namely, let us first prove (2.12) assuming that $a_{ij} \in C^1(\Omega)$ and $u,v \in W^{2,2}_{loc}(\Omega)$. Then $a_{ij}\partial_j u \in W^{1,2}_{loc}(\Omega)$ and, hence, $\partial_i(a_{ij}\partial_j u) \in L^2_{loc}(\Omega)$. In particular, Lu and Lv belong to $L^2_{loc}(\Omega)$. Using (2.13) we obtain

$$\partial_i(a_{ij}\partial_j(uv)) = \partial_i(a_{ij}(\partial_j u)v) + \partial_i(a_{ij}u\partial_j v).$$

Since $a_{ij}\partial_j u$ and v belong to $W^{1,2}_{loc}(\Omega)$, we obtain by the product rule (2.13) that

$$\partial_i(a_{ij}(\partial_j u)v) = \partial_i(a_{ij}\partial_j u)v + a_{ij}\partial_j u\partial_i v.$$

Similarly, we have

$$\partial_i(a_{ij}u\partial_j v) = \partial_i(a_{ij}\partial_j v)u + a_{ij}\partial_i u\partial_j v.$$

Adding up in all i,j and using the symmetry of a_{ij}, we obtain that

$$L(uv) = (Lu)v + (Lv)u + 2\sum_{i,j=1}^{n} a_{ij}\partial_j u\partial_i v,$$

that is (2.13).

Note that under the weaker assumptions $u,v \in W^{1,2}_{loc}(\Omega)$ the above argument does not work since $a_{ij}\partial_j u$ can be claimed only to belong to $L^2_{loc}(\Omega)$. Hence, the term $\partial_i(a_{ij}\partial_j u)v$ is meaningless as a product of a distribution $\partial_i(a_{ij}\partial_j u)$ with a $W^{1,2}_{loc}$ function v.

Lemma 2.6 will be used in the proof of Theorem 2.1 where function u will be assumed in $W^{1,2}_{loc}(\Omega)$ and the fact that $u \in W^{2,2}_{loc}(\Omega)$ will have to be proved. Therefore, we need a full version of Lemma 2.6.

Proof of Lemma 2.6. Using the distributional definition of L and the product rule (2.13), we obtain, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}
(L(uv),\varphi) &= \sum_{i.j=1}^{n}(\partial_i(a_{ij}\partial_j(uv)),\varphi) \\
&= -\sum_{i.j=1}^{n}(a_{ij}\partial_j(uv),\partial_i\varphi) \\
&= -\sum_{i.j=1}^{n}(a_{ij}(\partial_j u)v,\partial_i\varphi) - \sum_{i.j=1}^{n}(a_{ij}u\partial_j v,\partial_i\varphi)
\end{aligned} \tag{2.14}$$

Using again the product rule

$$v\partial_i\varphi = \partial_i(v\varphi) - (\partial_i v)\varphi,$$

we obtain

$$\begin{aligned} -\sum_{i.j=1}^{n}\left(a_{ij}\left(\partial_j u\right)v,\partial_i\varphi\right) &= -\int_\Omega\sum_{i.j=1}^{n}a_{ij}\left(\partial_j u\right)v\partial_i\varphi\,dx \\ &= \int_\Omega\sum_{i.j=1}^{n}a_{ij}\partial_j u\,\partial_i v\,\varphi\,dx-\int_\Omega\sum_{i.j=1}^{n}a_{ij}\partial_j u\,\partial_i\left(v\varphi\right)dx. \end{aligned}$$

Next, recall that Lu satisfies the following identity:

$$-\int_\Omega\sum_{i,j=1}^{n}a_{ij}\partial_j u\partial_i\psi\,dx=\int_\Omega\left(Lu\right)\psi\,dx$$

for any $\psi\in W_c^{1,2}(\Omega)$. Since $v\varphi\in W_c^{1,2}(\Omega)$, setting here $\psi=v\varphi$, we obtain

$$-\int_\Omega\sum_{i.j=1}^{n}a_{ij}\partial_j u\,\partial_i\left(v\varphi\right)dx=\int_\Omega\left(Lu\right)v\varphi\,dx=\left(vLu,\varphi\right),$$

whence

$$-\sum_{i.j=1}^{n}\left(a_{ij}\left(\partial_j u\right)v,\partial_i\varphi\right)=\left(\sum_{i.j=1}^{n}a_{ij}\partial_j u\,\partial_i v\,,\varphi\right)+\left(vLu,\varphi\right). \tag{2.15}$$

Similarly, we have

$$-\sum_{i.j=1}^{n}\left(a_{ij}u\partial_j v,\partial_i\varphi\right)=\left(\sum_{i.j=1}^{n}a_{ij}\partial_j v\,\partial_i u\,,\varphi\right)+\left(uLv,\varphi\right). \tag{2.16}$$

Adding up (2.15) and (2.16), using $a_{ij}=a_{ji}$ and (2.14), we obtain

$$\left(L\left(uv\right),\varphi\right)=2\left(\sum_{i.j=1}^{n}a_{ij}\partial_j u\,\partial_i v\,,\varphi\right)+\left(vLu,\varphi\right)+\left(uLv,\varphi\right),$$

which is equivalent to (2.12). ∎

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Proof of Theorem 2.1. Set $f=Lu$. Consider first a special case when $u\in W_c^{1,2}(\Omega)$ and $f\in L^2(\Omega)$, and prove that in this case $u\in W^{2,2}(\Omega)$. It suffices to prove that all distributional derivatives $\partial_j(\partial_i u)$ belong to $L^2(\Omega)$.

Let us extend u to a function on $\mathbb{R}^n$ by setting $u=0$ in Ω^c. Then we have $u\in W_c^{1,2}(\mathbb{R}^n)$. We will prove that all second order derivatives $\partial_k(\partial_i u)$ are in $L^2(\mathbb{R}^n)$. Since $\partial_i u\in L^2(\mathbb{R}^n)$, by Lemma 2.3 it suffices to verify that, for any unit vector e, the norms $\left\|\partial_e^h(\partial_i u)\right\|_{L^2}$ are uniformly bounded for all small enough $|h|$. Since

$$\partial_e^h\left(\partial_i u\right)=\partial_i\left(\partial_e^h u\right),$$

it suffices to prove that, for some K and all small enough $|h|$,

$$\left\|\partial_i\left(\partial_e^h u\right)\right\|_{L^2}\leq K. \tag{2.17}$$

We are going to show that (2.17) holds with $K=\lambda\left(\|f\|_{L^2}+C\|\nabla u\|_{L^2}\right)$ where C depends on n and on the Lipschitz constant of the coefficients a_{ij} on $\operatorname{supp}u$.

Motivation. Before we start the proof of (2.17), let us explain an idea in a simpler situation. Assume that $u \in W_c^{3,2}(\Omega)$ and that $a_{ij} \in C^1(\Omega)$, and obtain an upper bound for the L^2-norm of the second derivatives of u. Set $v = \partial_k u$ for a fixed index k and obtain an upper bound for $\|\nabla v\|_{L^2}$ that would be analogous to (2.17).

By $Lu = f$ we have the identity

$$-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi \, dx = \int_\Omega f\varphi \, dx$$

that holds for all $\varphi \in W_0^{1,2}(\Omega)$. By the assumption $u \in W_c^{3,2}(\Omega)$, we have $\partial_{kk}u \in W_c^{1,2}(\Omega)$. Hence, we can use in the above identity the function $\varphi := \partial_{kk}u = \partial_k v$.

Since both functions $a_{ij}\partial_j u$ and $\partial_i v$ belong to $W_0^{1,2}(\Omega)$, we can use the integration by parts formula of Exercise 18 and obtain

$$\begin{aligned}
-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi \, dx &= -\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_k (\partial_i v)\, dx \\
&= \int_\Omega \sum_{i,j=1}^n \partial_k (a_{ij}\partial_j u)\, \partial_i v \, dx \\
&= \int_\Omega \sum_{i,j=1}^n (\partial_k a_{ij})\, \partial_j u \partial_i v \, dx + \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j v \partial_i v \, dx.
\end{aligned}$$

Hence, we have the identity

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j v \partial_i v \, dx = \int_\Omega f\partial_k v \, dx - \int_\Omega \sum_{i,j=1}^n (\partial_k a_{ij})\, \partial_j u \partial_i v \, dx.$$

Since all $\partial_k a_{ij}$ are bounded on $\operatorname{supp} u$, we obtain

$$\begin{aligned}
\left|\int_\Omega \sum_{i,j=1}^n (\partial_k a_{ij})\, \partial_j u \partial_i v \, dx\right| &\le \int_\Omega \sum_{i,j=1}^n |\partial_k a_{ij}|\, |\nabla u|\, |\nabla v| \, dx \\
&\le C\, \|\nabla u\|_{L^2} \|\nabla v\|_{L^2},
\end{aligned}$$

where $C = \sup_{\operatorname{supp} u} \sum_{i,j=1}^n |\partial_k a_{ij}|$. Also, we have

$$\int_\Omega f\partial_k v \, dx \le \|f\|_{L^2} \|\nabla v\|_{L^2}$$

and, by the uniform ellipticity condition,

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j v \partial_i v \ge \lambda^{-1} \|\nabla v\|_{L^2}^2 .$$

It follows that

$$\lambda^{-1} \|\nabla v\|_{L^2}^2 \le \|f\|_{L^2} \|\nabla v\|_{L^2} + C\, \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}$$

and

$$\|\nabla v\|_{L^2} \le \lambda \left(\|f\|_{L^2} + C\, \|\nabla u\|_{L^2}\right),$$

which is an analogous of (2.17).

Set $v = \partial_e^h u$. For simplicity of notations, we write $\partial^h \equiv \partial_e^h$. We always assume that $|h|$ is small enough, in particular, $|h|$ is much smaller that the distance from $\operatorname{supp} u$ to the boundary of Ω. Clearly, we have then $v \in W_c^{1,2}(\Omega)$ and $\partial^{-h} v \in W_c^{1,2}(\Omega)$. Since $Lu = f$, we have, for any $\varphi \in W_0^{1,2}(\Omega)$,

$$-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi \, dx = \int_\Omega f\varphi \, dx.$$

Setting here $\varphi = \partial^{-h} v = \partial^{-h}\left(\partial^h u\right)$, we obtain

$$-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i\left(\partial^{-h} v\right) dx = \int_\Omega f\left(\partial^{-h} v\right) dx.$$

On the left hand side, we apply the integration by parts formula[4] and the product rule for difference operators from Lemma 2.2:

$$\begin{aligned}
-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i\left(\partial^{-h} v\right) dx &= -\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial^{-h}\left(\partial_i v\right) dx \\
&= \int_\Omega \sum_{i,j=1}^n \partial^h\left(a_{ij}\partial_j u\right)\partial_i v\, dx \\
&= \int_\Omega \sum_{i,j=1}^n a_{ij}\left(x+eh\right)\partial^h\left(\partial_j u\right)\partial_i v\, dx \\
&\quad + \int_\Omega \sum_{i,j=1}^n \left(\partial^h a_{ij}\right)\partial_j u \partial_i v\, dx \\
&= \int_\Omega \sum_{i,j=1}^n a_{ij}\left(x+eh\right)\partial_j v \partial_i v\, dx \\
&\quad + \int_\Omega \sum_{i,j=1}^n \left(\partial^h a_{ij}\right)\partial_j u \partial_i v\, dx.
\end{aligned}$$

Hence, we obtain the identity

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\left(x+eh\right)\partial_j v \partial_i v\, dx = \int_\Omega f\left(\partial^{-h} v\right) dx - \int_\Omega \sum_{i,j=1}^n \left(\partial^h a_{ij}\right)\partial_j u \partial_i v\, dx.$$

Using the Cauchy-Schwarz inequality inequality and Lemma 2.5, we obtain

$$\left|\int_\Omega f\left(\partial^{-h} v\right) dx\right| \le \|f\|_{L^2}\left\|\partial^{-h} v\right\|_{L^2} \le \|f\|_{L^2}\|\nabla v\|_{L^2}.$$

Also we have

$$\begin{aligned}
\left|\int_\Omega \sum_{i,j=1}^n \left(\partial^h a_{ij}\right)\partial_j u \partial_i v\, dx\right| &\le \int_{\operatorname{supp} u} \sum_{i,j=1}^n \left|\partial^h a_{ij}\right| |\nabla u|\, |\nabla v|\, dx \\
&\le C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}
\end{aligned}$$

where

$$C := \sup_{\operatorname{supp} u} \sum_{i,j=1}^n \left|\partial^h a_{ij}\right|$$

[4] The integration by parts formula (2.6) of Lemma 2.2 was proved for functions $u, v \in L^2(\mathbb{R}^n)$. However, if both functions have compact supports in Ω then, for sufficiently small h, the integration in the both sides of (2.6) can be reduced to Ω.

is finite because a_{ij} are locally Lipschitz and $\operatorname{supp} u$ is compact. Hence, we obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x+eh)\,\partial_j v \partial_i v\,dx \le \left(\|f\|_{L^2} + C\|\nabla u\|_{L^2}\right)\|\nabla v\|_{L^2}.$$

On the other hand, by the uniform ellipticity we have

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x+eh)\,\partial_j v \partial_i v\,dx \ge \lambda^{-1}\|\nabla v\|_{L^2}^2$$

whence

$$\lambda^{-1}\|\nabla v\|_{L^2}^2 \le \left(\|f\|_{L^2} + C\|\nabla u\|_{L^2}\right)\|\nabla v\|_{L^2}$$

and

$$\|\nabla v\|_{L^2} \le \lambda\left(\|f\|_{L^2} + C\|\nabla u\|_{L^2}\right).$$

Since $v = \partial_e^h u$, we obtain (2.17) with $K = \lambda\left(\|f\|_{L^2} + C\|\nabla u\|_{L^2}\right)$.

Consider now a general case $u \in W^{1,2}_{loc}(\Omega)$ and $f \in L^2_{loc}(\Omega)$. In order to prove that $u \in W^{2,2}_{loc}(\Omega)$ it suffices to prove that $u \in W^{2,2}(U)$ for any precompact domain U such that $\overline{U} \subset \Omega$. Fix U and choose a cutoff function η of U in Ω. Consider function $w = u\eta$ that belongs to $W^{1,2}_c(\Omega)$. By Lemma 2.6 we have

$$Lw = (Lu)\,\eta + 2\sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \eta + u\,(L\eta).$$

A simple inspection shows that all the terms in the right hand side belong to $L^2(\Omega)$, which implies that $Lw \in L^2(\Omega)$. By the above special case, we conclude that $w \in W^{2,2}(\Omega)$, in particular, $w \in W^{2,2}(U)$. Since $u = w$ on U, it follows $u \in W^{2,2}(U)$, which finishes the proof. ■

Corollary 2.7 *Under the hypothesis of Theorem* 2.1, *in the expression*

$$Lu = \sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right)$$

each derivative ∂_j and ∂_i can be understood in $W^{1,2}_{loc}(\Omega)$.

If the both derivatives ∂_i and ∂_j in L are understood in $W^{1,2}_{loc}(\Omega)$ then one says that the operator L is understood in the *strong sense*. Recall that if $u \in W^{1,2}_{loc}(\Omega)$ then ∂_j acts in $W^{1,2}_{loc}(\Omega)$, whereas ∂_i acts in $\mathcal{D}'(\Omega)$; in this case we say that L is understood in the *weak sense*.[5]

Proof. By Theorem 2.1 we have $u \in W^{2,2}_{loc}(\Omega)$ and, hence, $\partial_j u \in W^{1,2}_{loc}(\Omega)$. Since a_{ij} are locally Lipschitz, we have also $a_{ij} \in W^{1,2}_{loc}$. Hence, by Exercise 19, we have $a_{ij}\partial_j u \in W^{1,1}_{loc}(\Omega)$ and

$$\partial_i\left(a_{ij}\partial_j u\right) = \left(\partial_i a_{ij}\right)\partial_j u + a_{ij}\partial_{ij} u.$$

[5]Recall for comparison that L is understood in the *classical sense* if the both operators ∂_i, ∂_j apply to C^1-functions, which is the case when $u \in C^2$ and $a_{ij} \in C^1$. If $a_{ij} \in C^\infty$ then operator L can be understood in the *distributional sense* for any $u \in \mathcal{D}'(\Omega)$.

Since a_{ij} is bounded, it follows that $a_{ij}\partial_j u \in L^2_{loc}(\Omega)$. Since $\partial_i a_{ij}$ is locally bounded (because $a_{ij} \in W^{1,\infty}_{loc}$), the above identity implies that $\partial_i (a_{ij}\partial_j u) \in L^2_{loc}(\Omega)$, whence $a_{ij}\partial_j u \in W^{1,2}_{loc}(\Omega)$. Hence, the operator ∂_i acts on a function from $W^{1,2}_{loc}(\Omega)$, which was to be proved. ■

Remark. In the course of the proof we have proved the following fact: the product of a function from $W^{1,2}_{loc}$ with a locally Lipschitz function belongs again to $W^{1,2}_{loc}$. Similarly one proves that the product of a function from $W^{1,2}$ with a Lipschitz function belongs to $W^{1,2}$.

2.2 Existence of higher order weak derivatives

As above, consider in a domain $\Omega \subset \mathbb{R}^n$ a uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij}\partial_j u). \tag{2.18}$$

Theorem 2.8 *Let $u \in W^{1,2}_{loc}(\Omega)$. If, for a non-negative integer k, we have $a_{ij} \in C^{k+1}(\Omega)$ and $Lu \in W^{k,2}_{loc}(\Omega)$ then $u \in W^{k+2,2}_{loc}(\Omega)$.*

For the proof we need the following lemma.

Lemma 2.9 *If $u \in W^{k,2}_{loc}(\Omega)$ and $v \in W^{k,\infty}_{loc}(\Omega)$ then $uv \in W^{k,2}_{loc}(\Omega)$.*

Proof. Induction in k. For $k = 0$ the claim is obvious: if $u \in L^2_{loc}(\Omega)$ and $v \in L^\infty_{loc}(\Omega)$ then $uv \in L^2_{loc}(\Omega)$. Assuming $k \geq 1$, let us make the inductive step from $k-1$ to k. Since $W^{k,\infty}_{loc} \subset W^{k,2}_{loc}$ and $k \geq 1$, the both functions u, v belong to $W^{1,2}_{loc}(\Omega)$. By Exercise 19, we conclude that $uv \in W^{1,1}_{loc}(\Omega)$ and

$$\partial_i (uv) = (\partial_i u)\, v + u\partial_i v.$$

Since $\partial_i u \in W^{k-1,2}_{loc}(\Omega)$ and $v \in W^{k-1,\infty}_{loc}(\Omega)$, we conclude by the inductive hypothesis that $(\partial_i u)\, v \in W^{k-1,2}_{loc}(\Omega)$. In the same way we obtain that $u\partial_i v \in W^{k-1,2}_{loc}(\Omega)$, whence it follows that $\partial_i (uv) \in W^{k-1,2}_{loc}(\Omega)$. Hence, $uv \in W^{k,2}_{loc}(\Omega)$, which was to be proved. ■

24.05.16 **Proof of Theorem 2.8.** Induction in k. The case $k = 0$ is covered by Theorem 2.1. Assuming $k \geq 1$, let us make inductive step from $k-1$ to k. Let

$$a_{ij} \in C^{k+1}(\Omega) \quad \text{and} \quad Lu \in W^{k,2}_{loc}(\Omega).$$

Then also $a_{ij} \in C^k(\Omega)$ and $Lu \in W^{k-1,2}_{loc}(\Omega)$, and the inductive hypothesis yields that

$$u \in W^{k+1,2}_{loc}(\Omega).$$

We need to prove that $u \in W^{k+2,2}_{loc}(\Omega)$, and for that it suffices to verify that any partial derivative $\partial_l u$ belongs to $W^{k+1,2}_{loc}(\Omega)$. We will show that

$$L(\partial_l u) \in W^{k-1,2}_{loc}(\Omega).$$

Since $\partial_l u \in W^{k,2}_{loc}(\Omega) \subset W^{1,2}_{loc}(\Omega)$, applying the inductive hypothesis to $\partial_l u$, we will conclude that $\partial_l u \in W^{k+1,2}_{loc}(\Omega)$ thus finishing the proof.

Hence, let us compute $L(\partial_l u)$. We have

$$L(\partial_l u) = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_j \partial_l u) = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_l \partial_j u).$$

Since both $\partial_j u$ and a_{ij} belong to $W^{1,2}_{loc}(\Omega)$, we have by the product rule in $W^{1,2}_{loc}(\Omega)$

$$\partial_l (a_{ij}\partial_j u) = a_{ij}\partial_l\partial_j u + (\partial_l a_{ij})\partial_j u,$$

whence

$$\begin{aligned} L(\partial_l u) &= \sum_{i,j=1}^n \partial_i (\partial_l (a_{ij}\partial_j u) - (\partial_l a_{ij})\partial_j u) \\ &= \partial_l (Lu) - \sum_{i,j=1}^n \partial_i (\partial_l a_{ij}\partial_j u). \end{aligned}$$

Note that $\partial_l (Lu) \in W^{k-1,2}_{loc}(\Omega)$. Since $\partial_j u \in W^{k,2}_{loc}(\Omega)$ and $\partial_l a_{ij} \in C^k(\Omega) \subset W^{k,\infty}_{loc}(\Omega)$, it follows by Lemma 2.9 that the product $(\partial_l a_{ij})\partial_j u$ belongs to $W^{k,2}_{loc}(\Omega)$ whence $\partial_i (\partial_l a_{ij}\partial_j u) \in W^{k-1,2}_{loc}(\Omega)$. Hence, $L(\partial_l u) \in W^{k-1,2}_{loc}(\Omega)$, which finishes the proof. ■

2.3 Operators with lower order terms

Here we extend the results of Theorems 2.1 and 2.8 to the operator with lower order terms. Consider in a domain $\Omega \subset \mathbb{R}^n$ the operator

$$Lu = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_j u) + \sum_{i=1}^n b_j \partial_j u + cu, \tag{2.19}$$

where the coefficients a_{ij}, b_j, c are measurable functions in Ω. For any $u \in W^{1,2}_{loc}(\Omega)$ the expression Lu is understood weakly, that is, the terms $a_{ij}\partial_j u$, $b_j\partial_j u$ and cu are elements of $L^2_{loc}(\Omega)$, while the terms $\partial_i (a_{ij}\partial_j u)$ are elements of $\mathcal{D}'(\Omega)$.

Theorem 2.10 *Let L be the operator* (2.19). *Assume that (a_{ij}) is uniformly elliptic and that the coefficients b_j, c are bounded in Ω. Let $u \in W^{1,2}_{loc}(\Omega)$.*

(*a*) *Assume that a_{ij} are locally Lipschitz. If $Lu \in L^2_{loc}(\Omega)$ then $u \in W^{2,2}_{loc}(\Omega)$.*

(*b*) *Let k be a non-negative integer. If $a_{ij} \in C^{k+1}(\Omega)$, $b_j, c \in C^k(\Omega)$ and $Lu \in W^{k,2}_{loc}(\Omega)$ then $u \in W^{k+2,2}_{loc}(\Omega)$.*

Proof. Consider the operator L_0 defined by

$$L_0 u := \sum_{i,j=1}^n \partial_i (a_{ij}\partial_j u) = Lu - \left(\sum_{j=1}^n b_j \partial_j u + cu \right).$$

(a) If $u \in W^{1,2}_{loc}(\Omega)$ then

$$\sum_{j=1}^{n} b_j \partial_j u + cu \in L^2_{loc}(\Omega),$$

which implies that $L_0 u \in L^2_{loc}(\Omega)$. By Theorem 2.1 we conclude that $u \in W^{2,2}_{loc}(\Omega)$.

(b) Induction in k. The inductive basis $k = 0$ is covered by part (a). Inductive step from $k-1$ to k. By the inductive hypothesis we already know that $u \in W^{k+1,2}_{loc}(\Omega)$. It follows from Lemma 2.9 that

$$\sum_{j=1}^{n} b_j \partial_j u + cu \in W^{k,2}_{loc}(\Omega),$$

and, hence, $L_0 u \in W^{k,2}_{loc}(\Omega)$. By Theorem 2.8 we conclude that $u \in W^{k+2,2}_{loc}(\Omega)$. ■

2.4 Existence of classical derivatives

Let us recall the following theorem.

Sobolev Embedding Theorem. *Let Ω be a domain in $\mathbb{R}^n$. If k, m are non-negative integers such that $k > m + \frac{n}{2}$ then $W^{k,2}_{loc}(\Omega) \hookrightarrow C^m(\Omega)$.*

Combining Theorem 2.10 with Sobolev embedding theorem, we obtain the following.

Corollary 2.11 *Under the hypotheses of Theorem 2.10(b), if*

$$k > m + \frac{n}{2} - 2,$$

where m is a non-negative integer, then $u \in C^m(\Omega)$. In particular, if $a_{ij}, b_j, c \in C^\infty(\Omega)$ and $Lu \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.

Proof. Indeed, by Theorem 2.10 we have $u \in W^{k+2,2}_{loc}(\Omega)$, and Sobolev Embedding Theorem yields $u \in C^m(\Omega)$. The second statement follows from the first one applied to any m. ■

2.5 Non-divergence form operator

Recall that for a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) + \sum_{j=1}^{n} b_j \partial_j u$$

in a domain $\Omega \subset \mathbb{R}^n$, the equation $Lu = f$ is understood in the weak sense if $u \in W^{1,2}_{loc}(\Omega)$ (and, hence, ∂_j acts on $W^{1,2}_{loc}$ while ∂_i acts on L^2_{loc}) and $Lu = f$ is understood in the strong sense if $u \in W^{2,2}_{loc}(\Omega)$ (and both ∂_j and ∂_i act in $W^{1,2}_{loc}$).

Consider now a *non-divergence* form elliptic operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{j=1}^{n} b_j \partial_j u \tag{2.20}$$

in a domain $\Omega \subset \mathbb{R}^n$. In this case the notion of a weak solution is not defined, while the notion of a strong solution makes sense as follows.

Definition. We say that the equation $Lu = f$ is satisfied in Ω in the *strong* sense if $u \in W^{2,2}_{loc}(\Omega)$ (so that $\partial_{ij}u$ and $\partial_j u$ belong to $L^2_{loc}(\Omega)$) and if $Lu(x) = f(x)$ holds for almost all $x \in \Omega$.

We say that the equation $Lu = f$ is satisfied in Ω in the *classical* sense if $u \in C^2(\Omega)$ and if $Lu(x) = f(x)$ holds for all $x \in \Omega$.

Example. Consider in $\mathbb{R}$ the function $u(x) = |x|$. Obviously, we have $u''(x) = 0$ for all $x \neq 0$, in particular, for almost all $x \in \mathbb{R}$. However, this function is not a strong solution of $u'' = 0$ because $u \notin W^{2,2}_{loc}(\Omega)$. Indeed, for distributional derivatives we have $u' = \operatorname{sgn} x \in L^2_{loc}$ and $u'' = 2\delta \notin L^2_{loc}$.

In fact, every strong solution of $\Delta u = 0$ in $\mathbb{R}^n$ is also a weak solution, and we obtain by Corollary 2.11 that $u \in C^\infty(\mathbb{R}^n)$.

Consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u \in W^{1,2}_0(\Omega), \end{cases} \tag{2.21}$$

where L is the operator (2.20) and the equation $Lu = f$ is understood in the strong or classical sense.

Theorem 2.12 *Let L be the operator* (2.20) *in a bounded domain $\Omega \subset \mathbb{R}^n$. Assume that (a_{ij}) is uniformly elliptic in Ω, a_{ij} are Lipschitz in Ω, b_j are bounded and measurable. Then, for any $f \in L^2(\Omega)$, the Dirichlet problem* (2.21) *has a unique strong solution.*

If in addition all the functions a_{ij}, b_j, f belong to $C^\infty(\Omega)$, then the solution u of (2.21) *belongs to $C^\infty(\Omega)$, and the equation $Lu = f$ is satisfied in the classical sense.*

Proof. By Corollary 2.4 we have $a_{ij} \in W^{1,2}_{loc}$. If $u \in W^{2,2}_{loc}(\Omega)$ then $\partial_j u \in W^{1,2}_{loc}$ and, by the product rule,

$$\partial_i(a_{ij}\partial_j u) = (\partial_i a_{ij})\partial_j u + a_{ij}\partial_{ij}u.$$

Therefore, for $u \in W^{2,2}_{loc}(\Omega)$, we have

$$\begin{aligned} Lu &= \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{j=1}^{n} b_j\partial_j u \\ &= \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) - \sum_{i,j=1}^{n}(\partial_i a_{ij})\partial_j u + \sum_{j=1}^{n} b_j \partial_j u \\ &= \widetilde{L}u, \end{aligned}$$

where $\widetilde{L}$ is a divergence form operator defined by

$$\widetilde{L}u = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{j=1}^{n} \widetilde{b}_j \partial_j u$$

with

$$\widetilde{b}_j = b_j - \sum_{i=1}^{n} \partial_i a_{ij}.$$

Since functions a_{ij} are Lipschitz in Ω, the weak derivatives $\partial_i a_{ij}$ are bounded in Ω (see Corollary 2.4(a) and Exercises). Since also b_j are bounded in Ω, we obtain that the coefficients $\widetilde{b}_j$ are bounded in Ω.

The above computation shows that $Lu = \widetilde{L}u$ for $u \in W_{loc}^{2,2}(\Omega)$. In particular, the strong Dirichlet problem (2.21) is equivalent to the strong Dirichlet problem

$$\begin{cases} \widetilde{L}u = f \ \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{2.22}$$

whose solution u is sought in the class $W_{loc}^{2,2}(\Omega)$. However, unlike the operator L, the divergence form operator $\widetilde{L}$ can be regarded also in a weak sense, that is, on functions from $W_{loc}^{1,2}(\Omega)$.

Hence, consider (2.22) first in the weak sense. By Theorem 1.12, the weak Dirichlet problem (2.22) has a solution u. Since a_{ij} are locally Lipschitz, we obtain by Theorem 2.10 that $u \in W_{loc}^{2,2}(\Omega)$ and by Corollary 2.7 that $\widetilde{L}u = f$ holds in the strong sense. Hence, the same function u is a strong solution of the Dirichlet problem (2.21), which proves the existence of solution of (2.21).

Since any strong solution u of (2.21) is a strong and, hence, a weak solution of
31.05.16 (2.22), we obtain by Theorem 1.3 the uniqueness of u.

If $a_{ij}, b_j, f \in C^\infty(\Omega)$ then by Corollary 2.11 the solution u of (2.22) belongs to C^∞ and, hence, $Lu = f$ is satisfied in the classical sense. ■

Remark. Theorem 1.14 yields the following estimate of the solution u of (2.22):

$$\|u\|_{L^\infty} \leq C\,|\Omega|^{\frac{2}{n}-\frac{1}{q}}\,\|f\|_{L^q} \tag{2.23}$$

with any $q \in [2,\infty] \cap (n/2,\infty]$, provided

$$|\Omega| < \delta,$$

where $\delta = c_n \lambda^{-n}\widetilde{b}^{-n}$ depends on the ellipticity constant λ of (a_{ij}) and on the constant

$$\widetilde{b} := \sup_\Omega \sum_{j=1}^{n} \left|\widetilde{b}_j\right| \leq \sup_\Omega \left(\sum_{j=1}^{n} |b_j| + \sum_{i,j=1}^{n} |\partial_i a_{ij}|\right) \leq b + n^2 K,$$

where $b = \sup_\Omega \sum_{j=1}^{n} |b_j|$ and K is a common Lipschitz constant of all a_{ij}. Hence, the same estimate holds for the solution u of (2.21). Note that $\widetilde{b}$ may be non-zero even if $b = 0$ because of $K \neq 0$.

Example. Let us give an example to show that the uniqueness statement of Theorem 2.12 fails if the coefficients a_{ij} are not Lipschitz. This implies that any the estimate of the type (2.23) cannot hold as it would imply the uniqueness.

Consider the operator $L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}$ in $\mathbb{R}^n$ with the coefficients

$$a_{ij}(x) = \begin{cases} \delta_{ij} + c\frac{x_i x_j}{|x|^2}, & x \neq 0, \\ \delta_{ij}, & x = 0, \end{cases}$$

where c is a positive constant. It is easy to verify that L is uniformly elliptic. Consider the following Dirichlet problem in a ball B_r:

$$\begin{cases} Lu = 0 \ \text{ in } B_r \\ u \in W_0^{1,2}(B_r) \end{cases} \tag{2.24}$$

where L is understood in the strong sense, that is, u has to be in $W_{loc}^{2,2}(B_r)$. If the coefficients a_{ij} were Lipschitz as in the statement of Theorem 2.12 then this problem would have a unique strong solution $u = 0$.

However, the coefficients a_{ij} are not Lipschitz near 0 (not even continuous), and the problem (2.24) can have a non-zero solution. Indeed, it is possible to prove that if $s \in (0,1)$, $n > 2(2-s)$ and $c = \frac{n-2+s}{1-s}$ then the function $u(x) = |x|^s - r^s$ belongs to $W^{2,2}(B_r) \cap W_0^{1,2}(B_r)$ and solves in B_r the equation $Lu = 0$ in the strong sense (see Exercise 31 for details). Hence, the uniqueness in the strong Dirichlet problem (2.24) fails. Consequently, the estimate (2.23) fails in this case, too.

Chapter 3

Holder continuity for equations in divergence form

In this Chapter we will consider again a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) \tag{3.1}$$

with measurable coefficients and will prove that any weak solution u of $Lu = 0$ is, in fact, a continuous function! Moreover, we will prove that weak solutions are *Hölder continuous.*

Definition. A function f on a set $S \subset \mathbb{R}^n$ is called *Hölder continuous* with the Hölder exponent $\alpha > 0$ if there is a constant C such that

$$|f(x) - f(y)| \leq C |x - y|^{\alpha}$$

for all $x, y \in S$.

For example, f is Lipschitz if and only if f is Hölder continuous with $\alpha = 1$.

Definition. Let S be a subset of $\mathbb{R}^n$. We say that a function f on S is *locally Hölder continuous* in S with the Hölder exponent $\alpha > 0$ if, for any point $x \in S$, there exists $\varepsilon > 0$ such that f is Hölder continuous in $B_\varepsilon(x) \cap S$ with the exponent α.

It is easy to prove that if f is locally Hölder continuous in S then f is Hölder continuous on any compact subset of S with the same Hölder exponent (the proof is the same as that in the case of Lipschitz functions). In particular, if S is compact then any locally Hölder continuous function on S is Hölder continuous.

The set of all locally Hölder continuous functions on S with the Hölder exponent $\alpha \in (0,1)$ will be denoted by $C^{\alpha}(S)$.

Theorem 3.1 (Theorem of de Giorgi) *If $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution of $Lu = 0$ in Ω then $u \in C^{\alpha}(\Omega)$ where $\alpha = \alpha(n, \lambda) > 0$ (where λ is the constant of ellipticity of L).*

In particular, weak solutions are always continuous functions. For comparison, let us observe that in order to obtain the continuity of a weak solution u by Corollary 2.11, we have to assume that $a_{ij} \in C^k$ with $k > \frac{n}{2} - 2$. Theorem 3.1 ensures the continuity of u *without* any assumption about a_{ij} except for uniform ellipticity and measurability.

Theorem 3.1 was proved by Ennio de Giorgi in 1957, which opened a new era in the theory of elliptic PDEs. A year later John Nash proved the Hölder continuity for solutions of parabolic equation $\partial_t u = Lu$, which contains the theorem of de Giorgi as a particular case for time-independent solutions.

We will prove Theorem 3.1 after a long preparatory work.

3.1 Mean value inequality for subsolutions

Let L be the operator (3.1) defined in a domain Ω of $\mathbb{R}^n$. We always assume that L is uniformly elliptic with the ellipticity constant λ and that the coefficients are measurable. Recall that if $u \in W^{1,2}(\Omega)$ then inequality $Lu \geq 0$ is satisfied in the weak sense in Ω if

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} \partial_j u \partial_i \varphi \, dx \leq 0 \tag{3.2}$$

for any non-negative function $\varphi \in W_0^{1,2}(\Omega)$ (Exercise 23). In this case we say that u is a subsolution of the equation $Lu = 0$. Similarly, if u satisfies $Lu \leq 0$, then u is called a supersolution.

Theorem 3.2 (The mean-value inequality for subsolutions) *Let $B_R \subset \Omega$ and let $u \in W^{1,2}(B_R)$ satisfy $Lu \geq 0$ in B_R in the weak sense. Then*

$$\operatorname*{esssup}_{B_{R/2}} u \leq \frac{C}{R^{n/2}} \left(\int_{B_R} u_+^2 dx \right)^{1/2}, \tag{3.3}$$

where $C = C(n, \lambda)$.

An equivalent form of (3.3) is

$$\operatorname*{esssup}_{B_{R/2}} u \leq C \left(\fint_{B_R} u_+^2 dx \right)^{1/2}, \tag{3.4}$$

where the constants C in (3.3) and (3.4) may be different (but both depend only on n and λ). The value

$$\left(\fint_{B_R} u_+^2 dx \right)^{1/2}$$

is called the *quadratic mean* of u_+ in B_R. Hence, $\operatorname{esssup}_{B_{R/2}} u$ is bounded by the quadratic mean of u_+ in B_R.

Recall that, for a harmonic function u in B_R, we have the mean value property

$$u(0) = \fint_{B_R} u dx.$$

Using the Cauchy-Schwarz inequality, we obtain

$$u(0) \leq \fint_{B_R} u_+ dx \leq \left(\fint_{B_R} u_+^2 dx \right)^{1/2} = \frac{1}{|B_R|^{1/2}} \left(\int_{B_R} u_+^2 dx \right)^{1/2}. \tag{3.5}$$

Fix a point $z \in B_{R/2}$. Applying (3.5) to the ball $B_{R/2}(z)$ instead of $B_R(0)$ and noticing that $B_{R/2}(z) \subset B_R(0)$, we obtain

$$u(z) \le \frac{1}{\left|B_{R/2}\right|^{1/2}} \left(\int_{B_{R/2}(z)} u_+^2 dx \right)^{1/2} \le \frac{C}{R^{n/2}} \left(\int_{B_R} u_+^2 dx \right)^{1/2},$$

which proved (3.3) for harmonic functions.

The proof of (3.3) for a general operator L is much more complicated because we do not have the mean value property in general. It is in some sense similar to the proof of Theorem 1.13.

Proof. Fix two values $0 < \alpha < \beta$ as well as $0 < r < \rho < R$ and set

$$a = \int_{B_\rho} (u - \alpha)_+^2 \, dx \quad \text{and} \quad b = \int_{B_r} (u - \beta)_+^2 \, dx. \tag{3.6}$$

Clearly, $b \le a$. The purpose of the first part of the proof to obtain a stronger inequality showing that b is essentially smaller than a. In the second part of the proof we will use an iteration procedure similar to that in the proof of Theorem 1.13.

Consider the function

$$v = (u - \beta)_+$$

that belongs to $W^{1,2}(B_R)$ (see Exercise 15). Consider also the function

$$\eta(x) = \begin{cases} 1, & |x| \le r, \\ \frac{\rho - |x|}{\rho - r}, & r < |x| < \rho, \\ 0, & |x| \ge \rho. \end{cases}$$

Clearly, η is continuous. Since $|x|$ is a Lipschitz function with Lipschitz constant 1, it follows that η is a Lipschitz function with the Lipschitz constant $\frac{1}{\rho - r}$.

Since η is bounded, it follows that η^2 is also a Lipschitz functions. Let us show that the function $\varphi = v\eta^2$ can be used as a test function in (3.2).

Claim. *If U is a bounded domain and if $f \in W^{1,2}(U)$ and $g \in W^{1,\infty}(U)$ then $fg \in W^{1,2}(U)$.*

Indeed, since $f \in L^2$ and $g \in L^\infty$, we see that $fg \in L^2$. Since $W^{1,\infty} \subset W^{1,2}$, the function g also belongs to $W^{1,2}$, and we obtain by the product rule that

$$\partial_i (fg) = (\partial_i f) g + f \partial_i g.$$

The right hand side belongs to L^2 because f and $\partial_i f$ belong to L^2 while g and $\partial_i g$ belong to L^∞. Hence, $\partial_i (fg) \in L^2$ and $fg \in W^{1,2}$ as claimed. Note that a similar argument was used in the proof of Corollary 2.7.

Since Lipschitz functions belong to $W^{1,\infty}$, it follows from this Claim that $v\eta^2$ belongs to $W^{1,2}(B_R)$. By construction of η, the function $v\eta^2$ is compactly supported in B_R, whence we obtain $v\eta^2 \in W_0^{1,2}(B_R)$. Finally, $v\eta^2 \ge 0$ so that $\varphi = v\eta^2$ can be used in (3.2).

Substituting $\varphi = v\eta^2$ unto (3.2) yields

$$\sum_{i,j=1}^n \int_{B_R} a_{ij} \partial_j u \, \partial_i \left(v\eta^2 \right) \, dx \le 0. \tag{3.7}$$

By the product rule, we have

$$\partial_i \left(v\eta^2\right) = \left(\partial_i v\right)\eta^2 + v\left(\partial_i \eta^2\right) = \left(\partial_i v\right)\eta^2 + 2v\eta\partial_i\eta. \tag{3.8}$$

Substituting (3.8) into (3.7), we obtain

$$\sum_{i,j=1}^{n}\int_{B_R} a_{ij}\partial_j u\,\partial_i v\,\eta^2 dx \le -2\sum_{i,j=1}^{n}\int_{B_R} a_{ij}\partial_j u\, v\eta\,\partial_i\eta\ dx. \tag{3.9}$$

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Recall that $\partial_j u\partial_i v = \partial_j v\partial_i v$ because on the set $\{u \le \beta\}$ we have $v = 0$ and, hence, $\partial_i v = 0$, while on the set $\{u > \beta\}$ we have $\partial_j u = \partial_j v$. Hence, the left hand side of (3.9) is equal to

$$\sum_{i,j=1}^{n}\int_{B_R} a_{ij}\partial_j v\,\partial_i v\,\eta^2 dx \ge \lambda^{-1}\int_{B_R}\left|\nabla v\right|^2\eta^2 dx.$$

Since $\partial_j u\, v = \partial_j v\, v$, the right hand side of (3.9) is equal to

$$\begin{aligned}
-2\sum_{i,j=1}^{n}\int_{B_R} a_{ij}\partial_j v\,\partial_i\eta\, v\eta\, dx &\le 2\lambda\int_{B_R}\left|\nabla v\right|\left|\nabla\eta\right| v\eta\, dx \\
&\le 2\lambda\left(\int_{B_R}\left|\nabla v\right|^2\eta^2 dx\right)^{1/2}\left(\int_{B_R}\left|\nabla\eta\right|^2 v^2 dx\right)^{1/2}.
\end{aligned}$$

Hence, (3.9) implies

$$\lambda^{-1}\int_{B_R}\left|\nabla v\right|^2\eta^2 dx \le 2\lambda\left(\int_{B_R}\left|\nabla v\right|^2\eta^2 dx\right)^{1/2}\left(\int_{B_R}\left|\nabla\eta\right|^2 v^2 dx\right)^{1/2},$$

whence

$$\int_{B_R}\left|\nabla v\right|^2\eta^2 dx \le 4\lambda^4\int_{B_R}\left|\nabla\eta\right|^2 v^2 dx. \tag{3.10}$$

Applying again the product, we obtain

$$\nabla\left(v\eta\right) = \eta\nabla v + v\nabla\eta$$

whence

$$\left|\nabla\left(v\eta\right)\right|^2 \le \left(\left|\eta\nabla v\right| + \left|v\nabla\eta\right|\right)^2 \le 2\left|\nabla v\right|^2\eta^2 + 2\left|\nabla\eta\right|^2 v^2.$$

Combining with (3.10) yields

$$\int_{B_R}\left|\nabla\left(v\eta\right)\right|^2 dx \le \left(8\lambda^4 + 2\right)\int_{B_R}\left|\nabla\eta\right|^2 v^2 dx.$$

Since $|\nabla\eta| = 0$ outside B_ρ and $|\nabla\eta| \le \frac{1}{\rho - r}$ in B_ρ, it follows that

$$\int_{B_R}\left|\nabla\left(v\eta\right)\right|^2 dx \le \frac{C}{\left(\rho - r\right)^2}\int_{B_\rho} v^2 dx \tag{3.11}$$

where $C = 8\lambda^4 + 2$.

By the above Claim, the function $v\eta$ belongs to $W^{1,2}(B_R)$. Since $\operatorname{supp}(v\eta) \subset \overline{B}_\rho$, it follows that $v\eta \in W_0^{1,2}(B_{\rho'})$ for any $\rho' > \rho$. Applying the Faber-Krahn inequality 1.59, we obtain

$$\int_{B_{\rho'}} |\nabla(v\eta)|^2 \, dx \geq c\,|F|^{-2/n} \int_{B_{\rho'}} (v\eta)^2 \, dx, \tag{3.12}$$

where $c = c(n) > 0$ and

$$F := \{x \in B_{\rho'} : (v\eta)(x) > 0\}.$$

Since $\eta = 0$ outside B_ρ and $\eta > 0$ in B_ρ, we see that

$$F = \{x \in B_\rho : v(x) > 0\} = \{x \in B_\rho : u(x) > \beta\}.$$

For the same reason the integration over $B_{\rho'}$ can be replaced by that over B_ρ, so that

$$\int_{B_\rho} |\nabla(v\eta)|^2 \, dx \geq c\,|F|^{-2/n} \int_{B_\rho} (v\eta)^2 \, dx. \tag{3.13}$$

Combining with (3.11) and using that $\eta = 1$ on B_r, we obtain

$$|F|^{-2/n} \int_{B_\rho} (v\eta)^2 \, dx \leq \frac{C}{(\rho - r)^2} \int_{B_\rho} v^2 dx,$$

where we have absorbed c and C into a single constant C.

Since $\eta = 1$ on B_r, it follows that

$$\int_{B_r} v^2 dx \leq \frac{Cc}{(\rho - r)^2} |F|^{2/n} \int_{B_\rho} v^2 dx.$$

Finally, since $v = (u - \beta)_+ \leq (u - \alpha)_+$, we obtain

$$\int_{B_r} (u - \beta)_+^2 \, dx \leq \frac{C}{(\rho - r)^2} |F|^{2/n} \int_{B_\rho} (u - \alpha)_+^2 \, dx. \tag{3.14}$$

Let us estimate $|F|$ from above as follows. Since $u > \beta$ on F, we have

$$\int_{B_\rho} (u - \alpha)_+^2 \, dx \geq \int_F (u - \alpha)_+^2 \, dx \geq \int_F (\beta - \alpha)^2 \, dx = (\beta - \alpha)^2 |F|.$$

Recalling the notations a and b from (3.6), we rewrite this as

$$|F| \leq \frac{a}{(\beta - \alpha)^2},$$

and substitution into (3.14) inequality yields

$$b \leq \frac{C}{(\rho - r)^2} |F|^{2/n} a \leq \frac{C}{(\rho - r)^2 (\beta - \alpha)^{4/n}} a^{1+2/n}. \tag{3.15}$$

Consider now a sequence $\{R_k\}_{k=0}^{\infty}$ of radii where

$$R_k = \frac{1}{2}\left(1 + \frac{1}{2^k}\right) R.$$

Clearly, the sequence $\{R_k\}$ is monotone decreasing, $R_0 = R$ and $R_k \to \frac{R}{2}$ as $k \to \infty$. Also, fix some $\alpha > 0$ and consider a sequence $\{\alpha_k\}_{k=0}^{\infty}$ such that

$$\alpha_k = \left(2 - \frac{1}{2^k}\right)\alpha.$$

Clearly, the sequence $\{\alpha_k\}$ is monotone increasing, $\alpha_0 = \alpha$ and $\alpha_k \to 2\alpha$ as $k \to \infty$. Set

$$a_k = \int_{B_{R_k}} (u - \alpha_k)_+^2 \, dx.$$

Since the sequence $\{B_{R_k}\}$ of balls is shrinking and the sequence $\left\{(u - \alpha_k)_+\right\}$ of function is monotone decreasing, we see that the sequence $\{a_k\}$ is monotone decreasing.

Our aim is to choose α so that $a_k \to 0$ as $k \to \infty$. Since

$$\lim a_k = \int_{B_{R/2}} (u - 2\alpha)_+^2 \, dx,$$

in this case we will obtain that

$$\operatorname*{esssup}_{B_{R/2}} u \leq 2\alpha, \tag{3.16}$$

which will lead us to the desired estimate (3.3). Applying (3.15) to the pair a_{k-1}, a_k instead of a, b, we obtain

$$a_k \leq \frac{C}{(R_{k-1} - R_k)^2 (\alpha_k - \alpha_{k-1})^{4/n}} a_{k-1}^{1+2/n}.$$

Since $R_{k-1} - R_k = \frac{1}{2}\left(2^{-k}R\right)$ and $\alpha_k - \alpha_{k-1} = 2^{-k}\alpha$, it follows that

$$a_k \leq \frac{C 4^{(1+2/n)k}}{R^2 \alpha^{4/n}} a_{k-1}^{1+2/n}.$$

Setting

$$p = 1 + \frac{2}{n} \quad \text{and} \quad M = \frac{C}{R^2 \alpha^{4/n}}, \tag{3.17}$$

rewrite this inequality in the form

$$a_k \leq 4^{pk} M a_{k-1}^p. \tag{3.18}$$

This inequality is similar to the inequality (1.69) obtained in the proof of Theorem 1.13:

$$V_k \leq 4^k M V_{k-1}^p. \tag{3.19}$$

The difference between (3.19) and (3.18) is only that (3.18) uses 4^p instead of 4, which does not make any difference for the next argument. Indeed, iterating (3.19), we obtained in the proof of Theorem 1.13 the estimate (1.70), that is,

$$V_k \leq \left[4^{\frac{p}{(p-1)^2}} M^{\frac{1}{p-1}} V_0\right]^{p^k} 4^{\frac{-(k+1)p+k}{(p-1)^2}} M^{-\frac{1}{p-1}}.$$

Hence, iterating in the same way (3.18) and replacing everywhere 4 by 4^p, we obtain that

$$a_k \leq \left[4^{p\frac{p}{(p-1)^2}} M^{\frac{1}{p-1}} a_0\right]^{p^k} 4^{p\frac{-(k+1)p+k}{(p-1)^2}} M^{-\frac{1}{p-1}}. \tag{3.20}$$

We would like to derive from (3.20) that $a_k \to 0$ as $k \to \infty$. This will be the case whence the term in the square brackets is smaller than 1. Since

$$a_0 = \int_{B_R} (u-\alpha)_+^2 \, dx \le \int_{B_R} u_+^2 dx,$$

it suffices to have the following inequality

$$4^{\frac{p^2}{(p-1)^2}} M^{\frac{1}{p-1}} \int_{B_R} u_+^2 dx < 1.$$

Substituting M and p from (3.17), replace this inequality by the equality

$$4^{\frac{p^2}{(p-1)^2}} \left(\frac{C}{R^2 \alpha^{4/n}}\right)^{n/2} \int_{B_R} u_+^2 dx = \frac{1}{2},$$

which allows us to determine the desired value of α as follows:

$$\alpha^2 = \frac{C'}{R^n} \int_{B_R} u_+^2 dx.$$

Substituting into (3.16), we obtain

$$\operatorname*{esssup}_{B_{R/2}} u \le \frac{C''}{R^{n/2}} \left(\int_{B_R} u_+^2 dx\right)^{1/2},$$

which finishes the proof. ■

Corollary 3.3 *If $u \in W^{1,2}_{loc}(\Omega)$ solves $Lu = 0$ in Ω then $u \in L^\infty_{loc}(\Omega)$.*

Proof. Indeed, in any ball B_R such that $\overline{B}_R \subset \Omega$ we have $u \in L^2(B_R)$ and by Theorem 3.2

$$\operatorname*{esssup}_{B_{R/2}} u \le \frac{C}{R^{n/2}} \|u\|_{L^2(B_R)}.$$

Applying the same inequality to $-u$, we conclude that

$$\|u\|_{L^\infty(B_{B/2})} \le \frac{C}{R^{n/2}} \|u\|_{L^2(B_R)} < \infty.$$

Hence, $u \in L^\infty(B_{R/2})$ and $u \in L^\infty_{loc}(\Omega)$. ■

3.2 Weak Harnack inequality for positive supersolutions

Theorem 3.4 *Let $B_{3R} \subset \Omega$ and assume that $u \in W^{1,2}(B_{3R})$ is a non-negative weak supersolution of L in B_{3R}, that is, $Lu \le 0$ in B_{3R}. Choose some $a > 0$ and set*

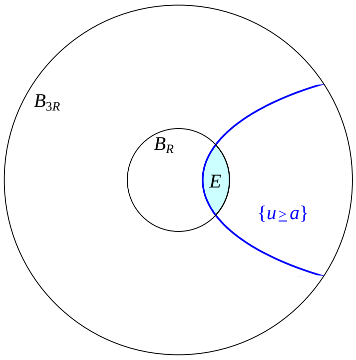
$$E = \{x \in B_R : u(x) \ge a\}.$$

For any $\varepsilon > 0$ there exists $\delta = \delta(n, \lambda, \varepsilon) > 0$ such that if

$$|E| \ge \varepsilon |B_R| \tag{3.21}$$

then

$$\operatorname*{essinf}_{B_R} u \ge \delta a. \tag{3.22}$$



Recall that any positive harmonic function u in a ball B_{3R} satisfies the Harnack inequality

$$\sup_{B_R} u \leq C \inf_{B_R} u,$$

where $C = C(n)$. In particular, for any $a \leq \sup_{B_R} u$, we have

$$\inf_{B_R} u \geq \delta a,$$

where $\delta = C^{-1}$, which looks similarly to (3.22). However, for the Harnack inequality we do not need to know that the measure of the set $E = \{u \geq a\} \cap B_R$ is positive – in fact, it suffices to know that this set is non-empty as the latter will imply $a \leq \sup_{B_R} u$. This is the reason why Theorem 3.4 is called a *weak* Harnack inequality. The word "weak" refers here not to "weak solution", but simply to a logically weaker statement. 07.06.16

Before the proof, let us derive from Theorem 3.4 the following mean value inequality for superharmonic functions.

Corollary 3.5 (Mean-value inequality for supersolutions) *Let $B_{3R} \subset \Omega$ and assume that $u \in W^{1,2}(B_{3R})$ is a non-negative weak supersolution of L in B_{3R}. Then*

$$\operatorname*{essinf}_{B_R} u \geq c \left(\fint_{B_R} u^{-1} dx \right)^{-1}, \tag{3.23}$$

where $c = c(n, \lambda) > 0$.

The value

$$\left(\fint_{\Omega} u^p dx \right)^{1/p}$$

is called the p-mean of function u in Ω. If $p = 1$ then this is the arithmetic mean, if $p = 2$ – the quadratic mean. For example, the quadratic mean was used in the mean-value inequality for subharmonic functions. If $p = -1$ as in (3.23) then the p-mean is called the *harmonic mean*. Hence, for a non-negative supersolution, $\operatorname{essinf}_{B_R} u$ is bounded from below by the harmonic mean of u in B_R.

Proof. If $\fint_{B_R} u^{-1} dx = \infty$ then (3.23) holds trivially. Assume that this integral is finite. For any $a > 0$, we have

$$|\{u < a\} \cap B_R| = \left| \left\{ \frac{1}{u} > \frac{1}{a} \right\} \cap B_R \right| \leq a \int_{B_R} \frac{1}{u} dx = a\mu(B) \fint_{B_R} \frac{1}{u} dx.$$

Choosing

$$a = \frac{1}{2}\left(\fint_{B_R} \frac{1}{u} dx\right)^{-1},$$

we obtain

$$|\{u < a\} \cap B_R| \le \frac{1}{2}\mu(B)$$

and, hence,

$$|\{u \ge a\} \cap B_R| \ge \frac{1}{2}\mu(B).$$

Applying Theorem 3.4 with $\varepsilon = 1/2$, we obtain

$$\operatorname*{essinf}_{B_R} u \ge \delta a = \frac{\delta}{2}\left(\fint_{B_R} \frac{1}{u} dx\right)^{-1},$$

which was to be proved. ∎

Proof of Theorem 3.4. Let us first observe that if the claim of Theorem 3.4 is proved under an additional assumption $\operatorname{essinf}_{B_{3R}} u > 0$, then it remains true also if $\operatorname{essinf}_{B_{3R}} u = 0$. Indeed, if the latter is the case, then consider the function $u + m$ for a positive m. Clearly, $L(u + m) = 0$. Observing that

$$u \ge a \Leftrightarrow u + m \ge a + m,$$

we can apply (3.22) to the function $u+m$ instead of u with the constant $a+m$ instead of a and obtain

$$\operatorname*{essinf}_{B_R}(u + m) \ge \delta(a + m).$$

Letting $m \to 0$, we obtain (3.22). Hence, in what follows we can assume without loss of generality that $\operatorname{essinf}_{B_{3R}} u > 0$.

Also, by replacing u by u/a, we can assume that $a = 1$. In this case we have

$$E = \{u \ge 1\} \cap B_R$$

and, assuming (3.21), we need to prove that

$$\operatorname*{essinf}_{B_R} u \ge \delta,$$

where $\delta = \delta(n, \lambda, \varepsilon) > 0$.

The main idea of the proof is to consider the function

$$v = \ln \frac{1}{u}.$$

In terms of this function, we have

$$E = \{v \le 0\} \cap B_R, \qquad |E| \ge \varepsilon |B_R|,$$

and we need to prove that

$$\operatorname*{esssup}_{B_R} v \le C = C(n, \lambda, \varepsilon). \tag{3.24}$$

The plan of the proof is as follows. We will first verify that v is a subsolution of L, which will imply by Theorem 3.2 that

$$\operatorname*{esssup}_{B_R} v \leq ... \int_{B_{2R}} v_+^2 dx.$$

Then, using a certain Poincaré inequality (similar to Friedrichs-Poincaré), we will deduce that

$$\int_{B_{2R}} v_+^2 dx \leq ... \int_{B_{2R}} |\nabla v|^2 .$$

Finally, using again specific properties of Lv, we will obtain an upper bound for

$$\int_{B_{2R}} |\nabla v|^2 ,$$

which together with the previous estimates will yield (3.24).

Now let us prove that v is a weak subsolution of L in B_{3R}. Firstly, let us verify that $v \in W^{1,2}(B_{3R})$. On the set $\{u \leq 1\}$ function v is non-negative. Since u is separated from 0, we see that in this case

$$0 \leq v < \text{const}\,.$$

On the set $\{u > 1\}$ function v is negative and

$$|v| = \ln u \leq u.$$

Hence, in the both cases

$$|v| \leq \text{const} + u,$$

which implies $v \in L^2(B_{3R})$. Since $\left(\ln \frac{1}{t}\right)' = -\frac{1}{t}$ is a bounded function outside a neighborhood of 0, that is, in the range of u, we obtain by the chain rule of Exercise 16, that

$$\partial_j v = \partial_j \ln \frac{1}{u} = -\frac{\partial_j u}{u} \in L^2(B_{3R})\,.$$

Hence, $v \in W^{1,2}(B_{3R})$. In the same way also the function $\frac{1}{u}$ belongs to $W^{1,2}(B_R)$, which will be used below. Indeed, $\frac{1}{u}$ is essentially bounded and, hence, is in $L^2(B_{3R})$, and by the same chain rule

$$\partial_j \left(\frac{1}{u}\right) = -\frac{\partial_j u}{u^2} \in L^2(B_{3R})\,.$$

Now let us verify that v is a subsolution of L, that is, $Lv \geq 0$ in B_{3R}. This is shown in Exercise 32 using the chain rule for L. Let us give a direct independent proof for that.

The motivation for $Lv \geq 0$ comes from the following observation: in the simplest case $n = 1$ and $L = \frac{d^2}{dx^2}$, if $u \in C^2$, $u > 0$ and $u'' \leq 0$ then we have

$$v'' = \left(\ln \frac{1}{u}\right)'' = \left(-\frac{u'}{u}\right)' = \frac{(u')^2 - u''u}{u^2} \geq \frac{(u')^2}{u^2} \geq 0.$$

If $n > 1$, $L = \Delta$, $u \in C^2$, $u > 0$ and $\Delta u \le 0$ then similarly

$$\begin{aligned}\Delta v &= \sum_{i=1}^{n} \partial_{ii} \ln \frac{1}{u} = \sum_{i=1}^{n} \frac{(\partial_i u)^2 - (\partial_{ii} u)\, u}{u^2} \\ &= \frac{|\nabla u|^2 - (\Delta u)\, u}{u^2} \ge \frac{|\nabla u|^2}{u^2} \ge 0.\end{aligned}$$

Noticing that $|\nabla v| = \left|\frac{\nabla u}{u}\right|$, we obtain from the above computation

$$\Delta v \ge |\nabla v|^2 . \tag{3.25}$$

In fact, the above computation shows that (3.25) is equivalent to $\Delta u \le 0$.

In the present general case, we have to verify that, for any non-negative test function $\varphi \in \mathcal{D}(B_{3R})$

$$-\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j v\, \partial_i \varphi\, dx \ge 0.$$

Since part of the following computation will also be used below for different purpose, we need to do it for a slightly more general class of φ, namely, assuming that φ is a non-negative Lipschitz function with compact support in B_{3R}. Since $\partial_j v = -\frac{\partial_j u}{u}$, we have

$$-\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j v\, \partial_i \varphi\, dx = \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \frac{\partial_j u}{u} \partial_i \varphi\, dx = \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u \frac{\partial_i \varphi}{u}\, dx. \tag{3.26}$$

Since $\varphi \in W^{1,\infty}$ and $1/u \in W^{1,2}$, the function φ/u belongs to $W^{1,2}$ (see Claim in the proof of Theorem 3.2) and by the product rule

$$\partial_i \left(\frac{\varphi}{u}\right) = \partial_i \left(\varphi \frac{1}{u}\right) = \frac{\partial_i \varphi}{u} - \varphi \frac{\partial_i u}{u^2}.$$

Hence, substituting

$$\frac{\partial_i \varphi}{u} = \partial_i \left(\frac{\varphi}{u}\right) + \varphi \frac{\partial_i u}{u^2}$$

into (3.26), we obtain

$$-\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j v\, \partial_i \varphi\, dx = \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u \left(\partial_i \left(\frac{\varphi}{u}\right) + \frac{\partial_i u}{u^2} \varphi\right) dx.$$

Since function φ has compact support in B_{3R}, we see that $\varphi/u \in W_c^{1,2}(B_{3R})$. Since also $\varphi/u \ge 0$, this function can be used as a test function in the weak inequality $Lu \le 0$, which leads to

$$\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u\, \partial_i \left(\frac{\varphi}{u}\right) dx \ge 0.$$

It follows that

$$\begin{aligned} -\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j v\, \partial_i \varphi\, dx &\ge \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u\, \frac{\partial_i u}{u^2} \varphi\, dx \\ &= \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j v\, \partial_i v\, \varphi\, dx \qquad (3.27) \\ &\ge 0, \end{aligned}$$

where we have used that $\partial_j u/u = -\partial_j v$ and the ellipticity of L. Hence, we have proved that $Lv \geq 0$.

Note that, in fact, we proved a stronger inequality (3.27) that is analogous of (3.25). Indeed, observing that the left hand side of (3.27) is equal to (Lv, φ) where Lv is regarded as distribution, we can rewrite (3.27) as follows:

$$Lv \geq \sum_{i,j=1}^{n} a_{ij}\partial_j v\, \partial_i v.$$

The inequality (3.27) will also be used below.

Applying the mean value inequality of Theorem 3.2 to a subsolution v, we obtain

$$\operatorname*{esssup}_{B_R} v \leq \frac{C}{R^{n/2}} \left(\int_{B_{2R}} v_+^2 dx \right)^{1/2}, \tag{3.28}$$

which completes the first step towards the proof of the bound (3.24).

In order to estimate further the integral in (3.28), we need the following fact.

Poincaré inequality *Let $v \in W^{1,2}(B_r)$ and consider the set*

$$H = \{x \in B_r : v(x) \leq 0\}.$$

Then

$$\int_{B_r} v_+^2 dx \leq C\frac{r^2 |B_r|}{|H|} \int_{B_r} |\nabla v_+|^2 dx \tag{3.29}$$

where $C = C(n)$.

Recall that the Friedrichs-Poincaré inequality says that if $v \in W_0^{1,2}(B_r)$ then

$$\int_{B_r} v^2 dx \leq Cr^2 \int_{B_r} |\nabla v|^2 dx. \tag{3.30}$$

For an arbitrary function $v \in W^{1,2}(B_r)$ this type of inequality cannot be true because by adding a large constant to v we can make $\int_{B_r} v^2 dx$ arbitrarily large, whereas $\int_{B_r} |\nabla v|^2 dx$ does not change. Assume for simplicity that $v \geq 0$. Then (3.29) amounts to

$$\int_{B_r} v^2 dx \leq C\frac{r^2 |B_r|}{|H|} \int_{B_r} |\nabla v|^2 dx,$$

where $H = \{v = 0\}$. Hence, if v vanishes on a large enough set (in the sense that $|H| \geq c|B_r|$), then we obtain again (3.30). As we see, the validity of (3.30) or similar inequalities depends on the property of v to vanish on certain sets.

The proof of (3.29) is non-trivial and will be given below (see Theorem 3.9 and Corollary 3.10).

Now let us apply (3.29) for the function $v = \ln \frac{1}{u}$ in the ball B_{2R}, that is, for $r = 2R$. Since

$$E = \{v \leq 1\} \cap B_R \subset \{v \leq 0\} \cap B_{2R} = H,$$

we have

$$|H| \geq |E| \geq \varepsilon |B_R| = \varepsilon 2^{-n} |B_{2R}|.$$

Then (3.29) yields

$$\int_{B_{2R}} v_+^2 dx \le C\frac{R^2}{\varepsilon}\int_{B_{2R}} |\nabla v_+|^2\, dx \le C\frac{R^2}{\varepsilon}\int_{B_{2R}} |\nabla v|^2\, dx$$

Combining with (3.28), we obtain

$$\operatorname*{essup}_{B_R} v \le \frac{C}{R^{n/2}}\left(\frac{R^2}{\varepsilon}\int_{B_{2R}} |\nabla v|^2\, dx\right)^{1/2}. \tag{3.31}$$

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The next step consists of estimating the integral $\int_{B_{2R}} |\nabla v|^2\, dx$. Consider the function

$$\eta(x) = \begin{cases} 1, & |x| \le r, \\ \frac{\rho-|x|}{\rho-r}, & r < |x| < \rho, \\ 0, & |x| \ge \rho, \end{cases}$$

where $r = 2R$ and $2R < \rho < 3R$, for example, we can take $\rho = \frac{5}{2}R$. Since $\varphi := \eta^2$ is a Lipschitz function with compact support in B_{3R}, we can use it in (3.27), which yields

$$\int_{B_{3R}} \sum_{i,j=1}^n a_{ij}\partial_j v\, \partial_i v\, \eta^2\, dx \le -\int_{B_{3R}} \sum_{i,j=1}^n a_{ij}\partial_j v\, \partial_i\left(\eta^2\right)\, dx. \tag{3.32}$$

Let us motivate the argument below first in the case $L = \Delta$. Recall that if $u > 0$ and $\Delta u \le 0$ then by (3.25) the function $v = \ln\frac{1}{u}$ satisfies the inequality

$$\Delta v \ge |\nabla v|^2.$$

It holds in the classical sense, which implies that in the weak sense, that is, for any non-negative function $\varphi \in W_c^{1,2}(B_{3R})$,

$$\int |\nabla v|^2 \varphi\, dx \le -\int \nabla v\cdot\nabla\varphi\, dx,$$

which is analogous of (3.27). Setting here $\varphi = \eta^2$ as above, rewrite it in the form

$$\int |\nabla v|^2 \eta^2\, dx \le -\int \nabla v\cdot\nabla\eta\,\eta\, dx \le \left(\int (|\nabla v|\,\eta)^2\, dx\right)^{1/.2}\left(\int |\nabla\eta|^2\, dx\right)^{1/2}$$

which implies

$$\int |\nabla v|^2\eta^2 dx \le \int |\nabla\eta|^2\, dx.$$

Since $\eta = 1$ on B_{2R}, $\eta = 0$ in B_{2R}^c, and $|\nabla\eta| \le \frac{1}{\rho-r}$, it follows that

$$\int_{B_{2R}} |\nabla v|^2\, dx \le \frac{|B_{3R}|}{(\rho-r)^2} = CR^{n-2}.$$

Using the uniform ellipticity of (a_{ij}), we estimate the left hand side of (3.32) as follows:

$$\int_{B_{3R}} \sum_{i,j=1}^n a_{ij}\partial_j v\, \partial_i v\, \eta^2\, dx \ge \lambda^{-1}\int_{B_{3R}} |\nabla v|^2\eta^2 dx,$$

while the right hand side of (3.32) is estimated as follows:

$$
\begin{aligned}
-\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij}\partial_j v\, \partial_i \left(\eta^2\right) \, dx &\leq \lambda \int_{B_{3R}} |\nabla v| \left|\nabla \eta^2\right| \\
&= 2\lambda \int_{B_{3R}} |\nabla v| \, |\nabla \eta| \, \eta.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\int_{B_{3R}} |\nabla v|^2 \eta^2 dx &\leq 2\lambda^2 \int_{B_{3R}} |\nabla v| \, |\nabla \eta| \, \eta \\
&\leq 2\lambda^2 \left(\int_{B_{3R}} (|\nabla v| \, \eta)^2 \, dx \right)^{1/2} \left(\int_{B_{3R}} |\nabla \eta|^2 \, dx \right)^{1/2},
\end{aligned}
$$

whence

$$
\int_{B_{3R}} |\nabla v|^2 \eta^2 dx \leq 4\lambda^4 \int_{B_{3R}} |\nabla \eta|^2 \, dx.
$$

Since $\eta = 1$ on B_{2R} and $|\nabla \eta| \leq \frac{1}{\rho - r}$, where $\rho - r = R/2$, we obtain

$$
\int_{B_{2R}} |\nabla v|^2 \, dx \leq 4\lambda^2 \frac{|B_{3R}|}{(\rho - r)^2} = CR^{n-2},
$$

where $C = C(n, \lambda)$. Finally, substituting this estimate into (3.31), we obtain

$$
\operatorname*{esssup}_{B_R} v \leq \frac{C}{R^{n/2}} \left(R^2 \varepsilon^{-1} C R^{n-2} \right)^{1/2} = C(n, \lambda, \varepsilon),
$$

which finishes the proof of (3.24). ■

3.3 Oscillation inequality and Theorem of de Giorgi

Define the oscillation of a function u in a domain D by

$$
\operatorname*{osc}_D u = \operatorname*{essup}_D u - \operatorname*{essinf}_D u.
$$

Observe that, for all real a, b,

$$
\operatorname*{osc}_D (au + b) = |a| \operatorname*{osc}_D u.
$$

Theorem 3.6 *(*Oscillation inequality*) Let $B_{3R} \subset \Omega$ and assume that $u \in W^{1,2}(B_{3R})$ is a weak solution of $Lu = 0$ in B_{3R}. Then*

$$
\operatorname*{osc}_{B_R} u \leq \gamma \operatorname*{osc}_{B_{3R}} u, \tag{3.33}
$$

where $\gamma = \gamma(n, \lambda) < 1$.

Proof. If $\operatorname{osc}_{B_{3R}} u = 0$ or ∞ then there is nothing to prove. If $0 < \operatorname{osc}_{B_{3R}} u < \infty$, then, by adding a constant to u and rescaling u, we can assume that

$$\operatorname*{essinf}_{B_{3R}} u = 0 \quad \text{and} \quad \operatorname*{esssup}_{B_{3R}} u = 2.$$

Consider the two sets

$$\{u \geq 1\} \cap B_R \quad \text{and} \quad \{u \leq 1\} \cap B_R. \tag{3.34}$$

One of these sets has measure $\geq \frac{1}{2}|B_R|$. Assume that this is the first set. Then by Theorem 3.4 with $a = 1$ and $\varepsilon = \frac{1}{2}$ we obtain that

$$\operatorname*{essinf}_{B_R} u \geq \delta = \delta\left(n, \lambda, \frac{1}{2}\right).$$

Hence,

$$\operatorname*{osc}_{B_R} u \leq 2 - \delta = \frac{2-\delta}{2} \operatorname*{osc}_{B_{3R}} u,$$

which proves (3.33) with $\gamma = \frac{2-\delta}{2} < 1$.

Assume now that the second set in (3.34) has measure at most $\frac{1}{2}|B_R|$. Consider the function $v = 2 - u$. For this function the oscillation in any domain is equal to that of u. Also we have $Lv = 0$ in B_{3R} and

$$u \leq 1 \Leftrightarrow v \geq 1.$$

Hence, the set $\{v \geq 1\} \cap B_R$ has measure $\geq \frac{1}{2}|B_R|$. Applying the same argument as above, we obtain that

$$\operatorname*{osc}_{B_R} v \leq \gamma \operatorname*{osc}_{B_{3R}} v,$$

which finishes the proof. ∎

Theorem 3.7 (Theorem of De Giorgi) *If $u \in W^{1,2}(\Omega)$ and $Lu = 0$ weakly in Ω then $u \in C^\alpha(\Omega)$ where $\alpha = \alpha(n, \lambda) > 0$. Moreover, for any compact subset K of Ω, we have*

$$\|u\|_{C^\alpha(K)} \leq C\|u\|_{L^2(\Omega)}$$

where

$$\|u\|_{C^\alpha(K)} := \sup_K |u| + \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \tag{3.35}$$

and $C = C(n, \lambda, \rho)$, $\rho = \operatorname{dist}(K, \partial\Omega)$.

Remark. Since K is compact, $C^\alpha(K)$ is the set of all Hölder continuous functions on K with the Hölder exponent α. Then the expression $\|u\|_{C^\alpha(K)}$, defined by (3.35), is finite for any $u \in C^\alpha(K)$, and is a norm in $C^\alpha(K)$. Moreover, one can show that $C^\alpha(K)$ with this norm is a Banach space. The expression

$$\sup_{\substack{x,y\in K\\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

is called a Hölder *seminorm* and, indeed, it is a seminorm in $C^{\alpha}(K)$.

Proof. Step 1. Let ρ be the distance from K to $\partial\Omega$, so that for any $x \in K$ the ball $B_{\rho}(x)$ is contained in Ω. Fix a point $z \in K$ and set

$$\rho_k = 3^{-k}\rho.$$

By Theorem 3.6 we have

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \leq \gamma \operatorname*{osc}_{B_{\rho_{k-1}}(z)} u, \tag{3.36}$$

which implies by induction that

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \leq \gamma^{k-1} \operatorname*{osc}_{B_{\rho_1}(z)} u \leq 2\gamma^{k-1} \operatorname*{esssup}_{B_{\rho_1}(z)} |u|.$$

Applying Theorem 3.2 to u and $-u$, we obtain that

$$\operatorname*{esssup}_{B_{\rho_1}(z)} |u| \leq \operatorname*{esssup}_{B_{\rho/2}(z)} |u| \leq C \|u\|_{L^2(B_\rho(z))} \leq C \|u\|_{L^2(\Omega)},$$

where $C = C(n, \lambda, \rho)$. Combining the above inequalities, we obtain

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \leq C\gamma^k \|u\|_{L^2(\Omega)}. \tag{3.37}$$

Note that without application of Theorem 3.2 we obtain

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \leq \gamma^k \operatorname*{osc}_{B_\rho(z)} u \leq 2\gamma^k \|u\|_{L^\infty(\Omega)}. \tag{3.38}$$

Step 2. Let us prove that, for almost all[1] $x, y \in K$ with

$$0 < |x - y| \leq \rho/2, \tag{3.39}$$

the following inequality holds

$$|u(x) - u(y)| \leq C |x - y|^{\alpha} \|u\|_{L^2(\Omega)} \tag{3.40}$$

where

$$\alpha = \log_3 \frac{1}{\gamma}$$

and $C = C(n, \lambda, \rho)$. For any couple x, y with $0 < |x - y| \leq \rho/2$ there is a non-negative integer k such that

$$\frac{1}{2}\rho_{k+1} < |x - y| \leq \frac{1}{2}\rho_k. \tag{3.41}$$

Hence, let us fix k and prove (3.40) for almost all x, y satisfying (3.41).[2]

[1] The expression "for almost all $x, y \in K$" has the following rigorous meaning: for almost all points $(x, y) \in K \times K$. Hence, here we use the Lebesgue measure in $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.

[2] Indeed, if we know already that the set S_k of points $(x, y) \in K \times K$ satisfying (3.41) and not satisfying (3.40) has measure 0 in $\mathbb{R}^{2n}$, then the set of points $(x, y) \in K \times K$ satisfying (3.39) and not satisfying (3.40) is $\bigcup_{k=0}^{\infty} S_k$, which also has measure zero.

The compact set K can be covered by a finite number of balls $B_{\frac{1}{2}\rho_k}(z_i)$ where $z_i \in K$. For any $x \in K$ there is z_i such that $x \in B_{\frac{1}{2}\rho_k}(z_i)$; then by (3.41) we have $y \in B_{\rho_k}(z_i)$. Hence, for any couple $x, y \in K$ satisfying (3.41) there is z_i such that $x, y \in B_{\rho_k}(z_i)$. Therefore, it suffices to prove (3.40) for almost all $x, y \in B_{\rho_k}(z)$ where $z = z_i$ is a fixed point on K.

By (3.37), we obtain that, for almost all $x, y \in B_{\rho_k}(z)$,

$$|u(x) - u(y)| \leq \operatorname*{osc}_{B_{\rho_k}(z)} u \leq C\gamma^k \|u\|_{L^2}. \tag{3.42}$$

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Let us express γ^k via $\rho_k = \rho 3^{-k}$. Setting

$$\alpha = \log_3 \frac{1}{\gamma} > 0,$$

we obtain $\gamma = 3^{-\alpha}$ and

$$\gamma^k = 3^{-\alpha k} = \left(\frac{\rho_k}{\rho}\right)^\alpha. \tag{3.43}$$

It follows from (3.42) that

$$|u(x) - u(y)| \leq C\rho_k^\alpha \|u\|_{L^2}.$$

This implies (3.40) because by (3.41)

$$\rho_k = 3\rho_{k+1} < 6|x - y|.$$

Alternatively, if we use (3.38) instead of (3.37) and (3.42), then we obtain, for almost all $x, y \in K$ with (3.39) that

$$|u(x) - u(y)| \leq C\left(\frac{|x-y|}{\rho}\right)^\alpha \|u\|_{L^\infty(\Omega)} \tag{3.44}$$

where $C = C(n, \lambda)$ does not depend on ρ.

Step 3. Now let us show that u has a C^α-version. It suffices to prove this for $u|_K$ where K is any compact subset of Ω. As above let ρ be the distance between K and $\partial\Omega$.

Choose a mollifier φ, that is, a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi \geq 0, \quad \operatorname{supp}\varphi \subset B_1 \quad \text{and} \quad \int_{\mathbb{R}^n} \varphi dx = 1.$$

Set for any positive integer k

$$\varphi_k(x) = k^n \varphi(kx), \tag{3.45}$$

so that

$$\operatorname{supp}\varphi_k \subset B_{1/k} \quad \text{and} \quad \int_{\mathbb{R}^n} \varphi_k dx = 1. \tag{3.46}$$

For any $u \in L^2(\mathbb{R}^n)$, consider its mollification, that is, the sequence of functions $\{u_k\}_{k=1}^\infty$ defined by

$$u_k(x) = u * \varphi_k(x) = \int_{\mathbb{R}^n} u(x-y)\varphi_k(y)\,dy.$$

It is known (cf. Exercise 4) that

$$u * \varphi_k \xrightarrow{L^2} u \quad \text{as } k \to \infty. \tag{3.47}$$

Let $u \in W^{1,2}(\Omega)$ be as above a solution of $Lu = 0$ in Ω. Extending u to $\mathbb{R}^n$ by setting $u = 0$ outside Ω, we obtain $u \in L^2(\mathbb{R}^n)$ and, hence, can define the mollification u_k as above. The idea of what follows is to show that the limit

$$\widetilde{u}(x) := \lim_{k \to \infty} u_k(x) \tag{3.48}$$

exists for all $x \in K$ and that $\widetilde{u} \in C^\alpha(K)$. Since K is arbitrary, this will imply that the limit (3.48) exists for all $x \in K$ and that $\widetilde{u} \in C^\alpha(\Omega)$. Since by (3.47) there is a subsequence $\{u_{k_i}\}$ such that

$$u_{k_i} \to u \text{ a.e.},$$

we will conclude that $\widetilde{u} = u$ a.e., which means that $\widetilde{u}$ is a C^α-version of u.

In order to prove the existence of $\lim_{k\to\infty} u_k(x)$ it suffices to prove that, for any $x \in K$, the sequence $\{u_k(x)\}$ is Cauchy. Since $\operatorname{supp} \varphi_k \subset B_{1/k}$, let us rewrite the definition of u_k in the form

$$u_k(x) = \int_{B_{1/k}(0)} u(x-y)\,\varphi_k(y)\,dy = \int_{B_{1/k}(x)} u(z)\,\varphi_k(x-z)\,dz. \tag{3.49}$$

Let $x \in K$. If $k > \rho^{-1}$ then $B_{1/k}(x) \subset \Omega$ so that the integration above is performed inside Ω.

For all $k, m > \rho^{-1}$ we have, using (3.49) and (3.46),

$$u_k(x) = \int_{B_{1/k}(x)} u(z)\,\varphi_k(x-z)\,dz = \int_{B_{1/m}(x)} \int_{B_{1/k}(x)} u(z)\,\varphi_k(x-z)\,dy\,\varphi_m(x-t)\,dt,$$

where $z \in B_{1/k}$, $t \in B_{1/m}$. Similarly, we have

$$u_m(x) = \int_{B_{1/m}} u(t)\,\varphi_m(x-t)\,dt = \int_{B_{1/k}(x)} \int_{B_{1/m}(x)} u(z)\,\varphi_m(x-z)\,dz\,\varphi_k(x-t)\,dt.$$

Using Fubini's theorem we obtain

$$u_k(x) - u_m(x) = \iint_{B_{1/k}(x)\times B_{1/m}(x)} (u(z) - u(t))\,\varphi_k(x-z)\,\varphi_m(x-t)\,dzdt. \tag{3.50}$$

Assume that $k, m > 2\rho^{-1}$ so that $\frac{1}{k}$ and $\frac{1}{m}$ are smaller than $\rho/2$. Then both balls $B_{1/k}(x)$ and $B_{1/m}(x)$ lie in the $\rho/2$-neighborhood of K. Denote the closed $\rho/2$-neighborhood of K by K'. Since K' is also a compact subset of Ω, we can apply the result of Step 2 to u on K', that is, for almost all $z, t \in K'$ such that

$$|z - t| < \rho/4,$$

we have

$$|u(z) - u(t)| \le C\,|z-t|^\alpha\,\|u\|_{L^2(\Omega)}.$$

If $z \in B_{1/k}(x)$ and $t \in B_{1/m}(x)$ then

$$|z-t| \leq \frac{1}{k} + \frac{1}{m}.$$

In particular, if k, m are large enough then this is smaller than $\rho/4$, and we obtain that, for almost all $(z,t) \in B_{1/k}(x) \times B_{1/m}(x)$

$$|u(z) - u(t)| \leq C \left|\frac{1}{k} + \frac{1}{m}\right|^{\alpha} \|u\|_{L^2(\Omega)}.$$

Substituting into (3.50) and using (3.46), we obtain

$$|u_k(x) - u_m(x)| \leq C \left(\frac{1}{k} + \frac{1}{m}\right)^{\alpha} \|u\|_{L^2(\Omega)} \to 0 \quad \text{as } k, m \to \infty.$$

Therefore, the sequence $\{u_k(x)\}$ is Cauchy for any $x \in K$ and, hence, has the limit

$$\widetilde{u}(x) := \lim_{k\to\infty} u_k(x).$$

Let us now show that $\widetilde{u} \in C^{\alpha}(K)$. For that let us estimate first $|u_k(x) - u_k(y)|$ for $x, y \in K$ assuming that

$$|x-y| < \rho/4.$$

Observe that

$$u_k(x) = \int_{B_{1/k}(x)} u(z)\,\varphi_k(x-z)\,dz = \int_{B_{1/k}(y)} \int_{B_{1/k}(x)} u(z)\,\varphi_k(x-z)\,dz\,\varphi_k(y-t)\,dt$$

and

$$u_k(y) = \int_{B_{1/k}(y)} u(t)\,\varphi_k(y-t)\,dt = \int_{B_{1/k}(x)} \int_{B_{1/k}(y)} u(t)\,\varphi_k(y-t)\,dt\,\varphi_k(x-z)\,dz.$$

Hence, using Fubini's theorem, we obtain

$$u_k(x) - u_k(y) = \iint_{B_{1/k}(x)\times B_{1/k}(y)} (u(z) - u(t))\,\varphi_k(x-z)\,\varphi_k(y-t)\,dz\,dt. \tag{3.51}$$

If k is large enough both balls $B_{1/k}(x)$ and $B_{1/k}(y)$ lie in K'. For all $z \in B_{1/k}(x)$ and $t \in B_{1/k}(y)$ we have by the triangle inequality

$$|z-t| < |x-y| + \frac{2}{k} < \rho/4,$$

provided k is large enough. Hence, by the result of Step 2 for K', we obtain, for almost all $(z,t) \in B_{1/k}(x) \times B_{1/k}(y)$ that

$$|u(z) - u(t)| \leq C \left(|x-y| + \frac{2}{k}\right)^{\alpha} \|u\|_{L^2(\Omega)},$$

whence by (3.51)

$$|u_k(x) - u_k(y)| \leq C \left(|x-y| + \frac{2}{k}\right)^{\alpha} \|u\|_{L^2(\Omega)}.$$

Letting $k \to \infty$ we obtain

$$\left|\widetilde{u}(x) - \widetilde{u}(y)\right| \leq C\left|x-y\right|^{\alpha}\|u\|_{L^2(\Omega)},$$

for all $x, y \in K$ such that $|x-y| < \rho/4$. The latter implies that $\widetilde{u}$ is Hölder continuous on K with the Hölder exponent α. Since $u = \widetilde{u}$ a.e., this means that u has a C^{α}-version, which was to be proved.

Step 4. It remains still to prove the estimate (3.35). Let us rename $\widetilde{u}$ back to u so that u is a continuous in Ω. By Theorem 3.2 we have, for any $x \in K$,

$$u(x) \leq \sup_{B_{\rho/2}(x)} u \leq C\|u\|_{L^2(B_\rho(x))} \leq C\|u\|_{L^2(\Omega)}.$$

Applying the same estimate to $-u$, we obtain

$$|u(x)| \leq C\|u\|_{L^2(\Omega)},$$

that is,

$$\sup_K |u| \leq C\|u\|_{L^2(\Omega)},$$

where $C = C(n, \lambda, \rho)$. By inequality (3.40) of Step 2 we have, for all $x, y \in K$ with

$$0 < |x-y| \leq \rho/2,$$

the following inequality

$$|u(x) - u(y)| \leq C|x-y|^{\alpha}\|u\|_{L^2(\Omega)}$$

(it was proved above for almost all x, y but now, due to the continuity of u, we obtain that it holds for all x, y). Hence, we obtain

$$\sup_{\substack{x,y\in K,\\ 0<|x-y|\leq\rho/2}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq C\|u\|_{L^2(\Omega)}.$$

Observe that

$$\sup_{\substack{x,y\in K,\\ |x-y|>\rho/2}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq 2\left(\frac{2}{\rho}\right)^{\alpha}\sup_K|u| \leq C\|u\|_{L^2(\Omega)}.$$

Finally, combining all these estimates, we obtain

$$\|u\|_{C^{\alpha}(K)} = \sup_K|u| + \sup_{\substack{x,y\in K,\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq C\|u\|_{L^2(\Omega)},$$

which finishes the proof of (3.35). ■

Corollary 3.8 *Under the hypotheses of Theorem* 3.7*, it is also true that, for any compact set $K \subset \Omega$ and for all $x, y \in K$ such that $|x-y| \leq \rho/2$,*

$$|u(x) - u(y)| \leq C\left(\frac{|x-y|}{\rho}\right)^{\alpha}\|u\|_{L^{\infty}(\Omega)}, \tag{3.52}$$

where $\rho = \operatorname{dist}(K, \partial\Omega)$ and the constant C depends only on n, λ (and does not depend on ρ).

Proof. Indeed, (3.52) was proved at the end of Step 2 for *almost all* x, y satisfying the above restrictions (see (3.44)). Since u is now continuous, the inequality (3.52) holds for *all* x, y. ■

3.4 Poincaré inequality

Theorem 3.9 *For any ball B_R in $\mathbb{R}^n$ and any $f \in W^{1,2}(B_R)$, the following inequality is true:*

$$\int_{B_R}\int_{B_R} (f(x) - f(y))^2 \, dxdy \leq CR^{n+2} \int_{B_R} |\nabla f|^2 \, dx, \tag{3.53}$$

where $C = C(n)$.

Dividing the both sides of (3.53) by $|B_R|^2$ and recalling that $|B_R| = c_n R^n$, we can rewrite it in the following form:

$$\fint_{B_R}\fint_{B_R} (f(x) - f(y))^2 \, dxdy \leq CR^2 \fint_{B_R} |\nabla f|^2 \, dx.$$

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Proof. Let us first prove (3.53) for $f \in C^1(B_R)$. For all $x, y \in B_R$ we have

$$\begin{aligned} f(y) - f(x) &= \int_0^1 \partial_t [f(x + t(y - x))] \, dt \\ &= \int_0^1 \nabla f(x + t(y - x)) \cdot (y - x) \, dt \\ &\leq \int_0^1 |\nabla f|(x + t(y - x)) \, |y - x| \, dt \\ &\leq 2R \int_0^1 |\nabla f|(x + t(y - x)) \, dt, \end{aligned}$$

whence by the Cauchy-Schwarz inequality

$$(f(y) - f(x))^2 \leq 4R^2 \int_0^1 |\nabla f|^2 (x + t(y - x)) \, dt.$$

It follows that

$$\int_{B_R}\int_{B_R} (f(x) - f(y))^2 \, dxdy \leq 4R^2 \int_{B_R}\int_{B_R}\int_0^1 |\nabla f|^2 (x + t(y - x)) \, dt \, dx \, dy \tag{3.54}$$

Set $F = |\nabla f|^2$ and extend F to the entire $\mathbb{R}^n$ by setting $F = 0$ outside B_R. In the view of (3.54), in order to prove (3.53) it remains to show that

$$\int_{B_R}\int_{B_R}\int_0^1 F(x + t(y - x)) \, dt \, dx \, dy \leq CR^n \int_{\mathbb{R}^n} F dx. \tag{3.55}$$

By Fubini's theorem, the integrations in the left hand side are all interchangeable. In the integral

$$\int_{B_R} F(x + t(y - x)) \, dy$$

let us make change $z = y - x$, so that

$$\int_{B_R} F(x + t(y - x)) \, dy = \int_{B_R(-x)} F(x + tz) \, dz \leq \int_{B_{2R}} F(x + tz) \, dz$$

and

$$\int_{B_R}\int_{B_R}\int_0^1 F\left(x+t\left(y-x\right)\right)dt\,dx\,dy \le \int_{B_{2R}}\int_{B_R}\int_0^1 F\left(x+tz\right)dt\,dx\,dz.$$

Then in the integral

$$\int_{B_R} F\left(x+tz\right)dx$$

let us make change $x' = x + tz$ so that

$$\int_{B_R} F\left(x+tz\right)dx = \int_{B_R(tz)} F\left(x'\right)dx' \le \int_{\mathbb{R}^n} F\left(x'\right)dx' = \int_{B_R} F\left(x'\right)dx'.$$

It follows that

$$\begin{aligned}
\int_{B_R}\int_{B_R}\int_0^1 F\left(x+t\left(y-x\right)\right)dxdydt &\le \int_{B_{2R}}\int_{B_R}\int_0^1 F\left(x'\right)dt\,dx'dz \\
&= 1\cdot\left|B_{2R}\right|\int_{B_R} F\left(x'\right)dx' \\
&= CR^n\int_{B_R} F\left(x\right)dx,
\end{aligned}$$

which finishes the proof of (3.55) for $f \in C^1\left(B_R\right)$.

Let now $f \in W^{1,2}\left(B_R\right)$. It suffices to prove that, for any $r < R$,

$$\int_{B_r}\int_{B_r}\left(f\left(x\right)-f\left(y\right)\right)^2 dxdy \le Cr^{n+2}\int_{B_r}\left|\nabla f\right|^2 dx, \tag{3.56}$$

and then let $r \to R$. Let ψ be a smooth cutoff function of B_r in B_R. Then $f\psi \in W_0^{1,2}\left(B_R\right)$ and, by setting $f\psi = 0$ outside B_R, we obtain that $f\psi \in W_0^{1,2}\left(\mathbb{R}^n\right)$. Since $f = f\psi$ in B_r, the function f in (3.56) can be replaced by $f\psi$. Hence, renaming $f\psi$ back into f, we can assume that $f \in W_0^{1,2}\left(\mathbb{R}^n\right)$.

Consider mollifications $f_k = f * \varphi_k$ where $\{\varphi_k\}$ is a sequence of mollifiers defined by (3.45). Then $f_k \in C^\infty\left(\mathbb{R}^n\right)$ and, hence, by the first part of the proof we have

$$\int_{B_r}\int_{B_r}\left(f_k\left(x\right)-f_k\left(y\right)\right)^2 dxdy \le Cr^{n+2}\int_{B_r}\left|\nabla f_k\right|^2 dx. \tag{3.57}$$

Since by Exercise 4

$$f_k \overset{W^{1,2}}{\to} f,$$

passing to the limit in (3.57) as $k \to \infty$, we obtain (3.56). ■

Now we can prove a version of the Poincaré inequality used in the proof of Theorem 3.4.

Corollary 3.10 *Let $v \in W^{1,2}\left(B_R\right)$ and consider the set*

$$H = \left\{x \in B_R : v\left(x\right) \le 0\right\}.$$

Then

$$\int_{B_R} v_+^2 dx \le C\frac{R^2\left|B_R\right|}{\left|H\right|}\int_{B_R}\left|\nabla v_+\right|^2 dx$$

where $C = C\left(n\right)$.

Proof. Note that $v_+ \in W^{1,2}(B_R)$. Renaming v_+ into v, we can assume that $v \geq 0$ and must prove that

$$\int_{B_R} v^2 dx \leq C\frac{R^2 |B_R|}{|H|} \int_{B_R} |\nabla v|^2 \, dx,$$

where $H = \{v = 0\}$. By (3.53) we have

$$\int_{B_R} \int_{B_R} (v(x) - v(y))^2 \, dx \, dy \leq CR^{n+2} \int_{B_R} |\nabla v|^2 \, dx.$$

Restricting the integration in the left hand side to $y \in H$ and noticing that $v(y) = 0$, we obtain

$$\int_H \int_{B_R} v(x)^2 \, dx \, dy \leq CR^{n+2} \int_{B_R} |\nabla v|^2 \, dx$$

whence

$$|H| \int_{B_R} v(x)^2 \, dx \ \leq CR^{n+2} \int_{B_R} |\nabla v|^2 \, dx.$$

Finally, it remains to observe that $R^{n+2} = cR^2 |B_R|$. ■

Remark. There is yet another form of the Poincaré inequality: for any ball B_R in $\mathbb{R}^n$ and for any $f \in W^{1,2}(B_R)$,

$$\int_{B_R} \left(f - \overline{f}\right)^2 dx \leq CR^2 \int_{B_R} |\nabla f|^2 \, dx, \tag{3.58}$$

where $C = C(n)$ and

$$\overline{f} := \fint_{B_R} f(x) \, dx$$

(see Exercise 38). In particular, if

$$\int_{B_R} f dx = 0$$

then (3.58) becomes

$$\int_{B_R} f^2 dx \leq CR^2 \int_{B_R} |\nabla f|^2 \, dx,$$

which has the same shape as the Friedrichs-Poincaré inequality.

3.5 Hölder continuity for inhomogeneous equation

As above, consider in a domain $\Omega \subset \mathbb{R}^n$ a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u)$$

with measurable coefficients.

Theorem 3.11 *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$, where $f \in L^q(\Omega)$ with $q \in [2, \infty] \cap (n/2, \infty]$. Then $u \in C^\beta(\Omega)$ where $\beta = \beta(n, \lambda, q) > 0$.*

Remark. If Ω is bounded then $f \in L^q(\Omega)$ implies $f \in L^2(\Omega)$. By Theorem 1.14, if u is a solution of the Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

with $f \in L^q(\Omega)$ then $u \in L^\infty(\Omega)$. Theorem 3.11 says that also $u \in C^\beta(\Omega)$.

Remark. Note that if $f \in L^q$ with $q < n/2$ then there may exist a solution $u \in W^{1,2}$ of $Lu = f$ that does not admit a continuous version (see Exercises).

Proof. Fix some compact set $K \subset \Omega$ and a point $z \in K$. It suffices to prove that, for small enough $r > 0$,

$$\operatorname*{osc}_{B_r(z)} u \le \operatorname{const} r^\beta, \tag{3.59}$$

where $\beta \in (0,1)$ and const may depends on $n, \lambda, \Omega, K, f, u$ but does not depend on z, r. This inequality is an analogous to the inequality (3.37) from Step 1 of the proof of Theorem 3.7. Arguing further as in the proof of Theorem 3.7, we will conclude that $u \in C^\beta(\Omega)$.

First we choose some positive $R < \operatorname{dist}(K, \partial\Omega)$ so that $B_R := B_R(z) \subset \Omega$. Let v be the solution of the Dirichlet problem in B_R:

$$\begin{cases} Lv = f \text{ weakly in } B_R \\ v \in W^{1,2}(B_R) \end{cases}$$

that exists by Theorem 1.2. Consider the difference $w = u - v$ that satisfies $Lw = 0$ in B_R. By Theorem 3.7, $w \in C^\alpha(B_R)$ where $\alpha = \alpha(n, \lambda) > 0$. Moreover, by Corollary 3.8, for any compact set $F \subset B_R$ and for all $x, y \in F$ such that $|x - y| < \rho/2$, we have

$$|w(x) - w(y)| \le C\left(\frac{|x-y|}{\rho}\right)^\alpha \|w\|_{L^\infty(B_R)}, \tag{3.60}$$

where $\rho = \operatorname{dist}(F, \partial B_R)$ and α, C depend only on n, λ. Take $F = B_{R/5}$ so that $\rho = \frac{4}{5}R$. Then, for all $x, y \in B_{R/5}$, we have

$$|x - y| < \frac{2}{5}R = \frac{1}{2}\rho.$$

Hence, (3.60) holds for all $x, y \in B_{R/5}$. 21.06.16

Fix some r such that $0 < r \le R/5$. Then (3.60) holds for all $x, y \in B_r$. Using that $\rho = \frac{4}{5}R$ and $|x - y| \le 2r$, we obtain from (3.60) that

$$\operatorname*{osc}_{B_r} w \le C\left(\frac{r}{R}\right)^\alpha \|w\|_{L^\infty(B_R)}.$$

Applying the same argument to $R/2$ instead of R, we obtain the following: if $0 < r \le R/10$ then

$$\operatorname*{osc}_{B_r} w \le C\left(\frac{r}{R}\right)^\alpha \|w\|_{L^\infty(B_{R/2})}.$$

By the mean value inequality of Theorem 3.2, we have

$$\|w\|_{L^\infty(B_{R/2})} \le \frac{C}{R^{n/2}} \|w\|_{L^2(B_R)}.$$

Using also that

$$\begin{aligned}
\|w\|_{L^2(B_R)} &\leq \|u\|_{L^2(\Omega)} + \|v\|_{L^2(B_R)} \\
&\leq \|u\|_{L^2(\Omega)} + CR^{n/2} \|v\|_{L^\infty(B_R)},
\end{aligned}$$

we obtain

$$\|w\|_{L^\infty\left(B_{R/2}\right)} \leq \frac{C}{R^{n/2}} \|u\|_{L^2(\Omega)} + C \|v\|_{L^\infty(B_R)}.$$

Hence,

$$\begin{aligned}
\operatorname*{osc}_{B_r} u &\leq \operatorname*{osc}_{B_r} v + \operatorname*{osc}_{B_r} w \\
&\leq 2 \|v\|_{L^\infty(B_R)} + C \left(\frac{r}{R}\right)^\alpha \left(\frac{C}{R^{n/2}} \|u\|_{L^2(\Omega)} + \|v\|_{L^\infty(B_R)}\right) \\
&\leq C \|v\|_{L^\infty(B_R)} + C \left(\frac{r}{R}\right)^\alpha \frac{1}{R^{n/2}} \|u\|_{L^2(\Omega)}. \qquad (3.61)
\end{aligned}$$

Since $f \in L^q(B_R)$, we obtain Theorem 1.14, that

$$\|v\|_{L^\infty(B_R)} \leq C |B_R|^{\frac{2}{n} - \frac{1}{q}} \|f\|_{L^q(B_R)},$$

which is equivalent to

$$\|v\|_{L^\infty(B_R)} \leq CR^{2-\frac{n}{q}} \|f\|_{L^q(\Omega)}. \qquad (3.62)$$

Substituting into (3.61), we obtain

$$\operatorname*{osc}_{B_r} u \leq CR^{2-\frac{n}{q}} \|f\|_{L^q(\Omega)} + C \left(\frac{r}{R}\right)^\alpha \frac{1}{R^{n/2}} \|u\|_{L^2(\Omega)}. \qquad (3.63)$$

So far r and R are arbitrary numbers such that

$$R < \operatorname{dist}(K, \partial\Omega) \quad \text{and} \quad 0 < r \leq R/10. \qquad (3.64)$$

Now, for any $r > 0$, we choose $R = R(r)$ so that

$$R^{2-n/q} = \left(\frac{r}{R}\right)^\alpha \frac{1}{R^{n/2}},$$

that is,

$$R = r^{\frac{\alpha}{2-n/q+\alpha+n/2}}.$$

Observe that

$$0 < \frac{\alpha}{2 - n/q + \alpha + n/2} < 1.$$

Therefore, if $r \to 0$ then $R \to 0$ and $R/r \to \infty$. Hence, if r is small enough (that is, $r \leq r_0$ where r_0 depends only on $\operatorname{dist}(K, \partial\Omega)$ and $\frac{\alpha}{2-n/q+\alpha+n/2}$), then the both conditions (3.64) are satisfied. For these values of r and R, we obtain from (3.63) that, for any $z \in K$,

$$\operatorname*{osc}_{B_r(z)} u \leq Cr^\beta \left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right), \qquad (3.65)$$

where

$$\beta = \frac{(2 - n/q)\,\alpha}{2 - n/q + \alpha + n/2} > 0,$$

thus proving (3.59) with $\beta = \beta(n, \lambda, q) > 0$. ∎

3.6 Applications to semi-linear equations

Consider a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u \right)$$

is a bounded domain Ω assuming that the coefficients are measurable. Given a function $f(x, v)$ on $\Omega \times \mathbb{R}$, consider the following *semi-linear* Dirichlet problem

$$\begin{cases} Lu = f(x, u) \ \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.66}$$

where the operator Lu is understood weakly as before. We assume that function f is such that the composition $f(x, u(x))$ belongs to $L^2(\Omega)$ whenever $u \in L^2(\Omega)$. Our goal is to investigate the solvability of the problem (3.66).

Fix first a function $v \in L^2(\Omega)$ and consider the following linear Dirichlet problem

$$\begin{cases} Lu = f(x, v) \ \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.67}$$

By Theorem 1.2, it has a unique solution u. Hence, we obtain the mapping

$$\begin{aligned} T \ : \ & L^2(\Omega) \to L^2(\Omega) \\ & Tv = u. \end{aligned}$$

The problem (3.66) amounts then to solving of the equation $Tu = u$. Hence, we face the problem of finding a *fixed point* of the mapping T.

3.6.1 Fixed point theorems

Let us discuss some fixed point theorems, that is, the statements that ensure the existence of a fixed point under certain hypotheses.

Theorem 3.12 *(*Fixed point theorem of Schauder*) Let K be a compact convex subset of a Banach space X. If $T : K \to K$ is a continuous mapping then T has a fixed point, that is, there exists a point $x \in K$ such that $Tx = x$.*

If $X = \mathbb{R}^n$ then then K can be any bounded closed convex subset of $\mathbb{R}^n$. In this case Theorem 3.12 is referred to as the fixed point theorem of Brouwer. In fact, theorem of Schauder is normally proved by using theorem of Brouwer.

Corollary 3.13 *Let K be a closed convex subset of a Banach space X and $T : K \to K$ is a continuous mapping such that the image $T(K)$ is precompact. Then T has a fixed point.*

Proof. Let C be the closed convex hull of $T(K)$. Then $C \subset K$ and C is compact. Clearly, T can be regarded as an operator from C to C, which implies by Theorem 3.12 that T has a fixed point. ■

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Definition. A mapping $T : X \to X$ is called *compact* if, for any bounded set $E \subset X$, the image $T(E)$ is precompact.

Note that if T is linear and compact then T is also bounded and, hence, continuous. However, in general a compact mapping T does not have to be continuous.

Theorem 3.14 *(*Fixed point theorem of Leray-Schauder*) Let $T : X \to X$ be a compact, continuous mapping. Assume that*

$$\text{the set } \{x \in X : x = \sigma T x \text{ for some } 0 < \sigma < 1\} \text{ is bounded.} \tag{3.68}$$

Then T has a fixed point.

Remark. The Leray-Schauder condition (3.68) can be regarded as a replacement of the contraction condition in the Banach fixed point theorem.

Example. Consider an affine mapping $Tx = x + a$ with some $a \in X$. The equation $x = \sigma T x$ is equivalent to $x = \sigma(x + a)$, that is, to

$$x = \frac{\sigma a}{1 - \sigma}.$$

This can be satisfied with any $\sigma \in (0, 1)$, and the norm of x is clearly unbounded. Hence, condition (3.68) fails. Obviously, T does not have a fixed point.

Example. Let $T(x)$ be a continuous function on $X = \mathbb{R}$. If the condition (3.68) holds then there is $R > 0$ such that any $x \in \mathbb{R}$ satisfying $x = \sigma T(x)$ with $\sigma \in (0, 1)$ admits the estimate $|x| < R$. We claim that in this case

$$T(R) \le R \quad \text{and} \quad T(-R) \ge -R. \tag{3.69}$$

Indeed, if $T(R) > R$ then we have $R = \sigma T(R)$ with some $\sigma \in (0, 1)$ and, hence, we should have $|R| < R$, which is wrong. In the same way, if $T(-R) < -R$ then we have $(-R) = \sigma T(-R)$ with some $\sigma \in (0, 1)$ and, hence, $|-R| < R$. This contradiction shows that (3.69) holds. Then the existence of the fixed point $x = T(x)$ follows from the intermediate point theorem, because the function $f(x) = x - T(x)$ is non-negative at $x = R$, non-positive at $x = -R$ and, hence, vanishes at some point $x \in [-R, R]$.

Proof. The condition (3.68) means that there $R > 0$ such that any x from the set (3.68) admits the estimate $\|x\| < R$. By dividing the norm in X by R, we can assume without loss of generality that $R = 1$. In other words, we assume that

$$\text{if } x = \sigma T x \text{ for some } 0 < \sigma < 1 \text{ then } \|x\| < 1. \tag{3.70}$$

Consider a mapping $S : X \to X$ defined by

$$Sx = \begin{cases} Tx, & \text{if } \|Tx\| \le 1 \\ \frac{Tx}{\|Tx\|}, & \text{if } \|Tx\| > 1. \end{cases} \tag{3.71}$$

We claim that S is continuous and compact. To see that, let use represent S in the form of composition

$$S = \Phi \circ T,$$

where $\Phi : X \to X$ is defined by

$$\Phi y = \begin{cases} y, & \text{if } \|y\| \le 1 \\ \frac{y}{\|y\|}, & \text{if } \|y\| > 1. \end{cases}$$

Then Φ is continuous because it can be represented in the form

$$\Phi y = \varphi(\|y\|)\, y$$

with the following function φ defined on $[0, \infty)$:

$$\varphi(t) = \begin{cases} 1 & t \le 1, \\ \frac{1}{t}, & t > 1. \end{cases}$$

Since φ is obviously continuous, we see that Φ is continuous, which implies that also S is continuous.

Since T is compact, for any bounded set $E \subset X$, the image $T(E)$ is precompact, that is, $\overline{T(E)}$ is compact. Since Φ is continuous, the set $\Phi\left(\overline{T(E)}\right)$ is compact, which implies that $S(E) = \Phi(T(E))$ is precompact. Hence, the mapping S is compact.

By construction, we have $\|Sx\| \le 1$ for all $x \in X$. Denote by B the closed unit ball of radius 1 in X. Then $S(X) \subset B$ and, in particular, $S(B) \subset B$. Hence, S can be regarded as a mapping from B to B. Since $S(B)$ is precompact, we obtain by Corollary 3.13 that S has a fixed point $x \in B$.

Let us verify that x is also a fixed point of T. Indeed, if $Tx \in B$ then $Tx = Sx$ and, hence, $Tx = x$. Assume now that $Tx \notin B$, that is, $\|Tx\| > 1$. In this case we obtain from (3.71) $\|Sx\| = 1$ and, hence, $\|x\| = 1$. On the other hand, (3.71) yields also

$$Tx = \|Tx\|\, Sx = \|Tx\|\, x$$

and $x = \sigma Tx$ where $\sigma = \frac{1}{\|Tx\|} < 1$. By (3.70) we must have $\|x\| < 1$, which contradicts $\|x\| = 1$. This contradiction shows that the second case is impossible, which finishes the proof. ■

3.6.2 A semi-linear Dirichlet problem

Consider a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u)$$

is a bounded domain Ω assuming that the coefficients are measurable, and the following *semi-linear* Dirichlet problem

$$\begin{cases} Lu = f(x, u) \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.72}$$

where the operator Lu is understood weakly as before. Function $f = f(x, u)$ is defined in $\Omega \times \mathbb{R}$, and we assume that it is Borel measurable. Then, for any measurable function u on Ω, the composite function $f(x, u(x))$ is also measurable.

We assume in addition that f satisfies the following two conditions:

$$|f(x,v)| \le C_1 (1 + |v|^\gamma), \tag{3.73}$$

for all $v \in \mathbb{R}$ and almost all $x \in \Omega$, and

$$|f(x,v_1) - f(x,v_2)| \le C_2 |v_1 - v_2| \tag{3.74}$$

for all $v_1, v_2 \in \mathbb{R}$ and almost all $x \in \Omega$, where γ, C_1, C_2 are positive constants.

Theorem 3.15 *Assume that the above hypotheses* (3.73) *and* (3.74) *hold with* $\gamma < 1$. *Then the following is true.*
(a) *The problem* (3.72) *has a solution* u.
(b) *If in addition* $|\Omega|$ *is small enough then the solution* u *is unique.*
(c) *If in addition* $\gamma < \frac{4}{n}$ *then* $u \in C^\beta(\Omega)$ *for some* $\beta = \beta(n, \lambda, \gamma) > 0$..

Remark. In part (b), without restriction on $|\Omega|$ there is no uniqueness for the problem (3.72). Indeed, even in the one dimensional case, the Dirichlet problem

$$\begin{cases} u'' = -u \\ u(0) = u(\pi) = 0 \end{cases}$$

has two solutions $u \equiv 0$ and $u(x) = \sin x$. Although the function $f(x,u) = -u$ does not satisfy (3.73), it is easy to modify it to satisfy (3.73) with any $\gamma > 0$:

$$f(x,u) := -\min(|u|, 1).$$

Then the problem

$$\begin{cases} u'' = f(x,u) \\ u(0) = u(\pi) = 0 \end{cases}$$

still has two solutions $u \equiv 0$ and $u(x) = \sin x$ because both solutions take values between 0 and 1, and for $u \in [0,1]$ we have $f(x,u) = -u$.

Similarly, if Ω is a bounded domain in $\mathbb{R}^n$ and u is an eigenfunction of the Laplace operator in Ω, that is,

$$\begin{cases} \Delta u = -\lambda u \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.75}$$

then we obtain again an example of non-uniqueness because $u \not\equiv 0$ and the problem (3.75) has also a solution $u \equiv 0$.

Remark. In part (c), the restriction $\gamma < 4/n$ is not optimal. In fact, if (3.73) holds with $\gamma \le 1$ then any solution u of (3.72) is Hölder continuous (see Exercise 53). In particular, all the eigenfunctions of L are Hölder continuous (see Exercise 49). On the other hand, if $\gamma > \frac{n}{n-4}$ then solution u does not have to be continuous (see Exercise 46).

Proof of Theorem 3.15. For any $v \in L^2(\Omega)$, the function

$$F_v(x) := f(x, v(x)) \tag{3.76}$$

belongs to $L^2(\Omega)$, because by (3.73) and $\gamma < 1$

$$|F_v(x)| \leq C(1 + |v|^\gamma) \leq C(2 + |v|) \in L^2(\Omega). \tag{3.77}$$

(a) For any $v \in L^2(\Omega)$, consider the following *linear* Dirichlet problem

$$\begin{cases} Lu = F_v \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{3.78}$$

that has a unique solution u by Theorem 1.2. Define the mapping $T : L^2(\Omega) \to L^2(\Omega)$ by $Tv = u$; that is, for any $v \in L^2(\Omega)$, the function Tv is defined as the solution u of (3.78) considered as an element of $L^2(\Omega)$. Clearly, if u solves (3.72) then

$$Tu = u.$$

Conversely, if $u \in L^2(\Omega)$ is a fixed point of T, then necessarily $u \in W_0^{1,2}(\Omega)$ because the range of T lies in $W_0^{1,2}(\Omega)$, and u solves the equation $Lu = F_u$, which is equivalent to (3.72).

Hence, the existence of solution of (3.72) is equivalent to the existence of a fixed point of the mapping $T : L^2(\Omega) \to L^2(\Omega)$. Let us first prove that T is continuous and compact. Clearly, T is the composition of the following mappings:

$$\begin{array}{ccccc} L^2(\Omega) & \to & L^2(\Omega) \to W_0^{1,2}(\Omega) \to L^2(\Omega) \\ v & \mapsto & F_v \quad\quad F_v \mapsto u \quad\quad\quad u \mapsto u \end{array}$$

where u is the solution of the Dirichlet problem (3.78). We know from the properties of the linear Dirichlet problem (3.78) that the mapping $F_v \mapsto u$ is linear and bounded:

$$\|u\|_{W^{1,2}(\Omega)} \leq C \|F_v\|_{L^2},$$

(cf. Exercise 20) and, hence, continuous. The mapping $v \mapsto F_v$, given by (3.76), is also continuous because by (3.74)

$$\|F_{v_1} - F_{v_2}\|_{L^2} \leq C \|v_1 - v_2\|_{L^2}. \tag{3.79}$$

Moreover, the mapping $v \mapsto F_v$ is bounded in the sense that image of any bounded set is bounded, because by (3.77)

$$\|F_v\|_{L^2} \leq C(1 + \|v\|_{L^2}).$$

Finally, the identical mapping $u \mapsto u$ from $W_0^{1,2}(\Omega)$ to L^2 is continuous and compact, the latter by the compact embedding theorem. Hence, we conclude that T is continuous and compact.

In order to apply Leray-Schauder theorem for existence of a fixed point of T, we need to prove that if $v = \sigma T v$ for some $0 < \sigma < 1$ then v is bounded. This equation implies that $v \in W_0^{1,2}(\Omega)$ and

$$Lv = \sigma L(Tv) = \sigma F_v,$$

that is, v solves the Dirichlet problem

$$\begin{cases} Lv = \sigma F_v \ \text{ in } \Omega. \\ v \in W_0^{1,2}(\Omega) \end{cases}$$

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Therefore, we have

$$\|v\|_{L^2} \leq \|v\|_{W^{1,2}} \leq C\|\sigma F_v\|_{L^2} \leq C\|F_v\|_{L^2}$$

(cf. Exercise 20), that is,

$$\int_\Omega v^2 dx \leq C \int_\Omega F_v^2 dx.$$

On the other hand, it follows from (3.77) that

$$\int_\Omega F_v^2 dx \leq C \int_\Omega \left(1+|v|^\gamma\right)^2 dx \leq C + C\int_\Omega |v|^{2\gamma} dx,$$

where the value of the constant C is changed at each occurrence. Hence, we obtain

$$\int_\Omega v^2 dx \leq C + C\int_\Omega |v|^{2\gamma} dx. \tag{3.80}$$

By Young's inequality, we have, for any $\varepsilon > 0$,

$$|v|^{2\gamma} = \frac{1}{\varepsilon}\varepsilon v^{2\gamma} \leq \frac{1}{\varepsilon^p} + \left(\varepsilon v^{2\gamma}\right)^q$$

where p, q is a pair of Hölder conjugate exponents. Choose $q = \frac{1}{\gamma}$ and, hence, $p = \frac{1}{1-\gamma}$, so that

$$|v|^{2\gamma} \leq \frac{1}{\varepsilon^p} + \varepsilon^q v^2$$

and

$$\int_\Omega |v|^{2\gamma} dx \leq C_\varepsilon + \varepsilon^q \int_\Omega v^2 dx.$$

Substitution into (3.80) yields

$$\int_\Omega v^2 dx \leq C_\varepsilon + C\varepsilon^q \int_\Omega v^2 dx.$$

Choosing ε so small that $\varepsilon^q \leq \frac{1}{2C}$, we obtain

$$\int_\Omega v^2 dx \leq 2C_\varepsilon,$$

that is, $\|v\|_{L^2}$ is bounded. By Theorem of Leray-Schauder we conclude that T has a fixed point and, hence, the Dirichlet problem (3.72) has a solution.

(b) Let us show that if $|\Omega|$ is small enough then the mapping T is a contraction in $L^2(\Omega)$. This will imply by the Banach fixed point theorem that T has a unique fixed point, that is, both uniqueness and existence. Let v_1 and v_2 be two functions from $L^2(\Omega)$, set $u_1 = Tv_1$ and $u_2 = Tv_2$. We need to prove that

$$\|u_1 - u_2\| \leq \theta \|v_1 - v_2\|$$

for some $\theta < 1$. Setting $u = u_1 - u_2$, we obtain

$$Lu = Lu_1 - Lu_2 = f(x, v_1) - f(x, v_2).$$

That is, for any $\varphi \in W_0^{1,2}(\Omega)$, we have

$$\int_\Omega \sum_{i,j=1}^n a_{ij} \partial_j u \, \partial_i \varphi \, dx = -\int_\Omega (f(x, v_1) - f(x, v_2)) \varphi dx. \tag{3.81}$$

By (3.74) we have

$$|f(x, v_1) - f(x, v_2)| \le C_2 |v_1 - v_2| \, .$$

Hence, setting in (3.81) $\varphi = u$ and using the uniform ellipticity of (a_{ij}), we obtain

$$\lambda^{-1} \int_\Omega |\nabla u|^2 \, dx \le C_2 \int_\Omega |v_1 - v_2| \, |u| \, dx.$$

On the other hand, by the Faber-Krahn inequality, we have

$$\int_\Omega |\nabla u|^2 \, dx \ge c_n |\Omega|^{-2/n} \int_\Omega u^2 dx.$$

Combining the two inequalities and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \int_\Omega u^2 dx &\le C |\Omega|^{2/n} \int_\Omega |v_1 - v_2| \, |u| \, dx \\ &\le C |\Omega|^{2/n} \left(\int_\Omega (v_1 - v_2)^2 \, dx \right)^{1/2} \left(\int_\Omega u^2 dx \right)^{1/2}, \end{aligned}$$

whence

$$\|u_1 - u_2\|_{L^2} \le C |\Omega|^{2/n} \|v_1 - v_2\|_{L^2} \, .$$

If $|\Omega|$ is small enough then $C |\Omega|^{2/n} < 1$, that is, T is a contraction, which was to be proved.

(c) By Theorem 3.11, a solution of (3.72) is Hölder continuous, provided $F_u \in L^q(\Omega)$ with

$$q \in [2, \infty] \cap (n/2, \infty]. \tag{3.82}$$

We have

$$\|F_u\|_{L^q} \le C \, \|1 + |u|^\gamma\|_{L^q} \le C' \left(1 + \||u|^\gamma\|_{L^q}\right).$$

Since $u \in L^2(\Omega)$ and

$$\||u|^\gamma\|_{L^q} = \left(\int_\Omega |u|^{\gamma q} \, dx \right)^{1/q},$$

we see that $\||u|^\gamma\|_{L^q} < \infty$ provided $\gamma q = 2$. Let us verify that $q := 2/\gamma$ satisfies (3.82). Indeed, we have $q > 2$ because $\gamma < 1$, and $q > n/2$ because $\gamma < 4/n$. Hence, q satisfies (3.82), and we obtain that $u \in C^\beta(\Omega)$ with $\beta = \beta(n, \lambda, \gamma) > 0$. ■

Chapter 4

Boundary behavior of solutions

Consider again in a bounded domain $\Omega \subset \mathbb{R}^n$ the weak linear Dirichlet problem

$$\begin{cases} Lu = f, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

We know that if $f \in L^q(\Omega)$ where

$$q \in [2, +\infty] \cap (n/2, \infty],$$

then $u \in C^\beta(\Omega)$ with $\beta > 0$, in particular, u is continuous in Ω. We can ask if u takes the boundary value in the classical sense, that is, if for a given point $x_0 \in \partial\Omega$,

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = 0.$$

The answer to this question depends in the properties of the boundary $\partial\Omega$ near x_0.

The aim of this Chapter is to prove the following: if $\partial\Omega$ is "good" enough in some sense then, in fact, $u \in C\left(\overline{\Omega}\right)$ and $u = 0$ on $\partial\Omega$ in the classical sense.

There are many different methods for investigation of the boundary behavior of solutions. We will use the method of continuation through the boundary, so that a boundary point $x_0 \in \partial\Omega$ becomes an interior point in a larger domain, so that the previous results about Hölder continuity in interior points can be used. We first consider a model case of a flat boundary.

4.1 Flat boundary

Consider an open set $\Omega_+ \subset \mathbb{R}^n_+$ such that a part of the boundary of Ω_+ lies on the hyperplane $H = \{x_n = 0\}$. Regarding H as $\mathbb{R}^{n-1}$, denote by Γ the interior of $\partial\Omega_+ \cap H$ considered as a subset of $\mathbb{R}^{n-1}$.

Let

$$L = \sum_{i,j=1}^n \partial_i \left(a_{ij} \partial_j u\right) \tag{4.1}$$

be a uniformly elliptic operator in Ω_+ with measurable coefficients. Let u be a solution of the following Dirichlet problem in Ω_+:

$$\begin{cases} Lu = f \\ u \in W_0^{1,2}(\Omega_+) \end{cases}$$

where so far $f \in L^2(\Omega_+)$. We would like to investigate the Hölder continuity of u up to Γ.

Define a *mirror reflection* in H as a mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

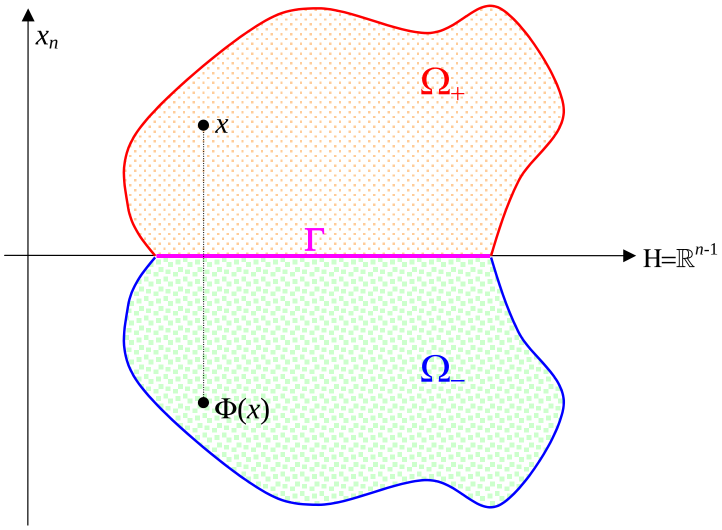
$$\Phi(x_1, ..., x_{n-1}, x_n) = (x_1, ..., x_{n-1}, -x_n).$$

Clearly, Φ is involution, that is, $\Phi^{-1} = \Phi$.

Let $\Omega_- = \Phi(\Omega_+)$ so that $\Omega_- \subset \mathbb{R}^n_-$. Observe that the set Γ belongs to the both boundaries $\partial\Omega_+$ and $\partial\Omega_-$. Consider the set

$$\Omega = \Omega_+ \cup \Omega_- \cup \Gamma$$

that is an open subset of $\mathbb{R}^n$ that is invariant for the mapping Φ. Note that all points of Γ are interior points of Ω. We are going to extends u, f, L from Ω_+ to Ω.



A function $v : \Omega \to \mathbb{R}$ is called *even* if

$$v(\Phi(x)) = v(x), \tag{4.2}$$

and *odd* if

$$v(\Phi(x)) = -v(x). \tag{4.3}$$

Any function $v : \Omega_+ \to \mathbb{R}$ allows obviously even and odd extensions to Ω, just by using (4.2) or (4.3), respectively (on Γ we set for simplicity $v = 0$).

Let us extend both functions u and f to Ω in the odd way, that is, by

$$u(\Phi(x)) = -u(x) \quad \text{and} \quad f(\Phi(x)) = -f(x)$$

for all $x \in \Omega_+$. 30.06.16

To extend the coefficients of L, we use the following notation:

$$\sigma_i = \begin{cases} 1, & i < n, \\ -1, & i = n. \end{cases}$$

Then set, for all $x \in \Omega_+$,

$$a_{ij}(\Phi(x)) = \sigma_i \sigma_j a_{ij}(x). \tag{4.4}$$

In other words,

- a_{ij} extends in the even way if $i, j < n$ or $i = j = n$;
- a_{ij} extends in the odd way if $i < n$, $j = n$ or $i = n$, $j < n$.

For $x \in \Gamma$ set $L = \Delta$. Hence, we obtain the extended operator L in Ω and the extended functions u and f in Ω.

Theorem 4.1 *Under the above conditions, the operator L is uniformly elliptic in Ω, $u \in W_0^{1,2}(\Omega)$ and $Lu = f$ in Ω.*

Proof. To prove that L is uniformly elliptic, it suffices to prove the following: if (a_{ij}) is a symmetric matrix such that, for any $\xi \in \mathbb{R}^n$,

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leq \lambda|\xi|^2 \tag{4.5}$$

then the same holds for the matrix $(\sigma_i\sigma_j a_{ij})$. We have

$$\sum_{i,j=1}^n (\sigma_i\sigma_j a_{ij})\,\xi_i\xi_j = \sum_{i,j=1}^n a_{ij}\eta_i\eta_j,$$

where $\eta_i = \sigma_i\xi_i$, that is, $\eta = (\xi_1, ..., \xi_{n-1}, -\xi_n)$. By (4.5) we have

$$\lambda^{-1}|\eta|^2 \leq \sum_{i,j=1}^n a_{ij}\eta_i\eta_j \leq \lambda|\eta|^2. \tag{4.6}$$

Since $|\eta| = |\xi|$, we obtain

$$\lambda^{-1}|\xi| \leq \sum_{i,j=1}^n (\sigma_i\sigma_j a_{ij})\,\xi_i\xi_j \leq \lambda|\xi|^2,$$

which proves the uniform ellipticity of $(\sigma_i\sigma_j a_{ij})$.

Since $u \in W_0^{1,2}(\Omega_+)$, we obtain that also $u\circ\Phi \in W_0^{1,2}(\Omega_-)$ and, hence, the extended u belongs to $W_0^{1,2}(\Omega_+ \cup \Omega_-)$. Since $\Omega \supset \Omega_+ \cup \Omega_-$, we obtain that also $u \in W_0^{1,2}(\Omega)$ (we use a general fact that if $\Omega' \subset \Omega''$ then $W_0^{1,2}(\Omega') \subset W_0^{1,2}(\Omega'')$ because $\mathcal{D}(\Omega') \subset \mathcal{D}(\Omega'')$).

Let us show that $Lu = f$ in Ω, that is, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega \left[\sum_{i,j=1}^n a_{ij}\partial_j u\,\partial_i\varphi + f\varphi\right] dx = 0. \tag{4.7}$$

For that we split the integral $\int_\Omega$ in the sum $\int_{\Omega_+} + \int_{\Omega_-}$, and in the integral $\int_{\Omega_-} ...dx$ we make change $x = \Phi(y)$ thus reducing it to an integral over Ω_+. In particular, we have

$$\int_{\Omega_-} f(x)\,\varphi(x)\,dx = \int_{\Omega_+} f(\Phi(y))\,\varphi(\Phi(y))\,|\det J_\Phi|\,dy,$$

where J_Φ is the Jacobi matrix of Φ. Obviously, $\det J_\Phi = -1$, whence

$$\int_{\Omega_-} f(x)\,\varphi(x)\,dx = \int_{\Omega_+} f(\Phi(y))\,\varphi(\Phi(y))\,dy.$$

Denoting

$$\psi(y) := \varphi(\Phi(y)) = \varphi(y_1, ..., y_{n-1}, -y_n)$$

and recalling that

$$f(\Phi(y)) = -f(y),$$

we obtain

$$\int_{\Omega_-} f(x)\varphi(x)\,dx = -\int_{\Omega_+} f(y)\psi(y)\,dy.$$

It follows that

$$\int_{\Omega} f\varphi dx = \int_{\Omega_+} f\varphi dx - \int_{\Omega_+} f\psi dx = \int_{\Omega_+} f\,(\varphi - \psi)\,dx.$$

Let us handle the term $a_{ij}\partial_j u\,\partial_i\varphi$. We have

$$(\partial_i\varphi)(\Phi(y)) = (\partial_i\varphi)(y_1, ..., y_{n-1}, -y_n) = \sigma_i\partial_i[\varphi(y_1, ..., y_{n-1}, -y_n)] = \partial_i\psi(y)$$

and similarly

$$(\partial_j u)(\Phi(y)) = (\partial_j u)(y_1, ..., y_{n-1}, -y_n) = \sigma_j\partial_j[u(y_1, ..., y_{n-1}, -y_n)] = -\sigma_j\partial_j u(y),$$

where we have used the fact that u is odd. Using also (4.4), we obtain

$$(a_{ij}\partial_j u\,\partial_i\varphi)(\Phi(y)) = -\sigma_i\sigma_j a_{ij}(y)\,\sigma_j\partial_j u(y)\,\sigma_i\partial_i\psi(y) = -(a_{ij}\partial_j u\partial_i\psi)(y),$$

as $\sigma_i^2 = \sigma_j^2 = 1$. Hence, we obtain

$$\int_{\Omega_-} (a_{ij}\partial_j u\,\partial_i\varphi)(x)\,dx = -\int_{\Omega_+} (a_{ij}\partial_j u\partial_i\psi)(y)\,dy,$$

which implies

$$\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}\partial_j u\,\partial_i\varphi\,dx = \int_{\Omega_+} \sum_{i,j=1}^{n} a_{ij}\partial_j u\,\partial_i(\varphi - \psi)\,dx.$$

It follows that

$$\int_{\Omega}\left[\sum_{i,j=1}^{n} a_{ij}\partial_j u\,\partial_i\varphi + f\varphi\right]dx = \int_{\Omega_+}\left[\sum_{i,j=1}^{n} a_{ij}\partial_j u\,\partial_i(\varphi-\psi) + f(\varphi-\psi)\right]dx. \tag{4.8}$$

Observe that the function $\varphi - \psi$ belongs to $W_0^{1,2}(\Omega_+)$ by Exercise[1] 30. Indeed, $\varphi - \psi$ belongs to $C^\infty\left(\overline{\Omega}_+\right)$ and, hence, it is in $W^{1,2}(\Omega_+)$ and it is continuous on $\partial\Omega_+$; moreover, $\varphi - \psi$ vanishes on $\partial\Omega_+$, because $\varphi - \psi = 0$ on Γ by construction of ψ, while $\varphi - \psi = 0$ on the rest of $\partial\Omega_+$ because φ and ψ vanish on $\partial\Omega$.

Since u solves $Lu = f$ in Ω_+, using $\varphi - \psi$ as a test function, we obtain that the right hand side of (4.8) vanishes, whence (4.7) follows. ■

[1]By Exercise 30, if g is a function on $\overline{\Omega}$ such that $g \in W^{1,2}(\Omega)$, g is continuous at any point of $\partial\Omega$, and $g = 0$ on $\partial\Omega$, then $g \in W_0^{1,2}(\Omega)$.

Corollary 4.2 *Let L be an operator in Ω_+ as above. Let u solves in Ω_+ the Dirichlet problem*

$$\begin{cases} Lu = f \text{ in } \Omega_+ \\ u \in W_0^{1,2}(\Omega_+) \end{cases}$$

where $f \in L^q(\Omega_+)$ with

$$q \in [2, \infty] \cap (n/2, \infty]. \tag{4.9}$$

Then $u \in C^\alpha(\Omega_+ \cup \Gamma)$ for some $\alpha = \alpha(n, \lambda, q) > 0$. In particular, u is continuous at any point of Γ and $u|_\Gamma = 0$.

Proof. Indeed, let us extend L, u, f to $\Omega = \Omega_+ \cup \Omega_- \cup \Gamma$ as in Theorem 4.1. By Theorem 4.1 we have $u \in W_0^{1,2}(\Omega)$ and $Lu = f$ in Ω. Since $f \in L^q(\Omega)$, we conclude by Theorem 3.11 that $u \in C^\alpha(\Omega)$. In particular, $u \in C^\alpha(\Omega_+ \cup \Gamma)$. Since u is continuous on Γ and u is odd with respect to the mirror reflection in Γ, we conclude that $u|_\Gamma = 0$. ■

4.2 Boundary as a graph

Let U be an open set in $\mathbb{R}^{n-1}$. Given a function $h : U \to \mathbb{R}$, consider its *graph*

$$\Gamma_h = \{(z, t) \in \mathbb{R}^n : z \in U, \ t = h(z)\}$$

and its *supergraph*:

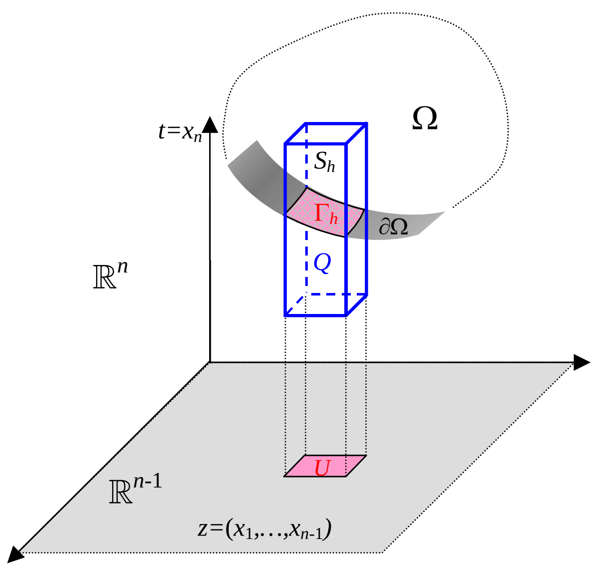
$$S_h = \{(z, t) \in \mathbb{R}^n : z \in U, \ t > h(z)\}.$$

Here $z \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, and we consider the pair (z, t) as the point $(z_1, ..., z_{n-1}, t)$ of $\mathbb{R}^n$.

A *cylinder* over U is any set $Q \subset \mathbb{R}^n$ of the form $U \times I$ where I is a non-empty open interval in $\mathbb{R}$.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with the following property: there is a cylinder Q over U such that

$$\Omega \cap Q = S_h \cap Q \quad \text{and} \quad \partial\Omega \cap Q = \Gamma_h.$$



Note that the set Ω_+ from the previous section with a piece Γ of a flat boundary is a particular case of the present construction with $h(z) \equiv 0$ and $U = \Gamma$. The following theorem generalizes Corollary 4.2.

Theorem 4.3 *Under the above conditions, assume that the function h belongs to $C^1(U)$. Consider a weak Dirichlet problem*

$$\begin{cases} Lu = f \ \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{4.10}$$

where L is the uniformly elliptic operator (4.1) *in Ω with measurable coefficients. If $f \in L^q(\Omega)$ with q as in* (4.9), *then $u \in C(\Omega \cup \Gamma_h)$ and $u|_{\Gamma_h} = 0$.*

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Proof. Choose an open subset V of U such that $\overline{V}$ is compact and $\overline{V} \subset U$. Let Γ be the graph of h restricted to V. It suffices to prove that $u \in C^\alpha(\Omega \cup \Gamma)$ for some $\alpha > 0$. We will see that the Hölder exponent α depends not only on λ, n, q but also on the sets Ω, U, V and on the function h.

Let us first extend the function h from U to $\mathbb{R}^{n-1}$ as follows. Choose first a constant c such that

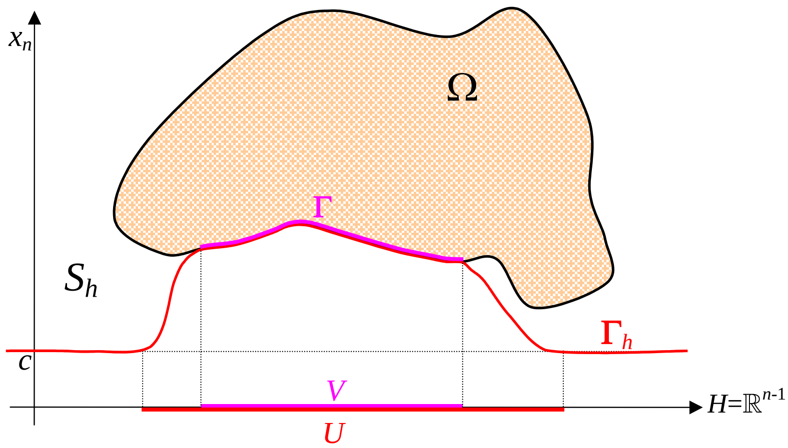
$$\Omega \subset \{x \in \mathbb{R}^n : x_n > c\}$$

and set $h = c$ in U^c. Then Ω is contained in the supergraph S_h of the extended function h. However, the so extended function h is not continuous on ∂U. On $U \setminus V$ let us redefine h to make is smaller and to have $h \in C^1(\mathbb{R}^{n-1})$.

Then we have $\sup_{\mathbb{R}^{n-1}} |\nabla h| < \infty$,

$$\Omega \subset S_h \quad \text{and} \quad \partial\Omega \supset \Gamma,$$

where, as above, Γ is the graph of $h|_V$ (see the picture below).



Let us consider the following mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$:

$$\Psi(x) = (x_1, ..., x_{n-1}, x_n - h(x_1, ..., x_{n-1})). \tag{4.11}$$

Clearly, Ψ is a C^1-diffeomorphism of $\mathbb{R}^n$, and the inverse mapping is given by

$$\Psi^{-1}(y) = (y_1 ..., y_{n-1}, y_n + h(y_1, ..., y_{n-1})). \tag{4.12}$$

Since

$$S_h = \{x \in \mathbb{R}^n : x_n > h(x_1, ..., x_{n-1})\}$$

and

$$\Gamma_h = \{x \in \mathbb{R}^n : x_n = h(x_1, ..., x_{n-1})\},$$

we obtain that

$$\Psi(S_h) = \mathbb{R}^n_+ \quad \text{and} \quad \Psi(\Gamma_h) = H := \mathbb{R}^{n-1},$$

as well as

$$\Psi(\Omega) \subset \mathbb{R}^n_+ \quad \text{and} \quad \Psi(\partial\Omega) \supset \Psi(\Gamma) = V.$$

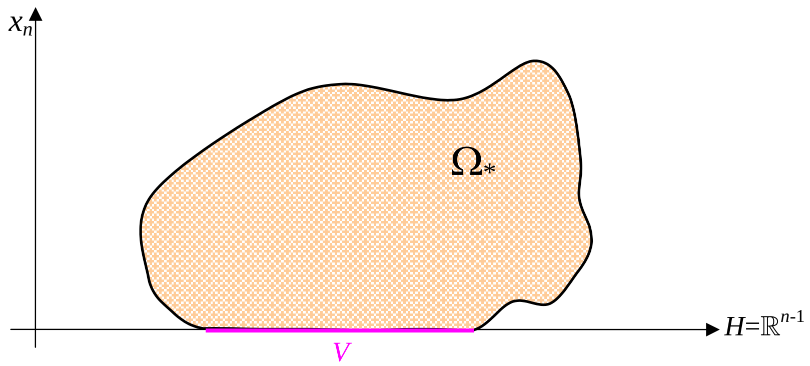
The mapping Ψ is called *straightening* as it straightens the piece Γ of the boundary $\partial\Omega$ into a flat piece V. Denote

$$\Omega_* = \Psi(\Omega),$$

so that

$$\Omega_* \subset \mathbb{R}^n_+ \quad \text{and} \quad \partial\Omega_* \supset V$$

(see the picture).



We can regard Ψ as a C^1-diffeomorphism between Ω and Ω_*. We denote an arbitrary point in Ω by x while that in Ω_* – by y, and write the mapping Ψ in the form $y = \Psi(x)$. We will need the Jacobi matrices of Ψ and Ψ^{-1}. Using (4.11) and (4.12), we find that

$$J_\Psi = \left(\frac{\partial y_k}{\partial x_i}\right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ -\partial_1 h & \cdots & -\partial_{n-1} h & 1 \end{pmatrix}$$

and

$$J_{\Psi^{-1}} = \left(\frac{\partial x_i}{\partial y_k}\right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ \partial_1 h & \cdots & \partial_{n-1} h & 1 \end{pmatrix}.$$

It is easy to verify that the matrices J_Ψ and $J_{\Psi^{-1}}$ are mutually inverse as they should be, and that

$$\det J_\Psi = \det J_{\Psi^{-1}} = 1.$$

Set
$$K=\max\left(1,\sup_{\mathbb{R}^{n-1}}|\nabla h|\right).$$
Then all the entries of the both matrices J_Ψ and $J_{\Psi^{-1}}$ are bounded by K.

Any function v on Ω can be *pushed forward* to a function v_* on Ω_* that is defined as follows:
$$v_*\left(\Psi\left(x\right)\right)=v\left(x\right)\quad\text{for all }x\in\Omega,$$
which is equivalent to
$$v_*\left(y\right)=v\left(\Psi^{-1}\left(y\right)\right)\quad\text{for all }y\in\Omega_*.$$
Let us prove some properties of push-forward.

(a) If $u\in L^p\left(\Omega\right)$ then $u_*\in L^p\left(\Omega_*\right)$. Indeed, changing $y=\Psi\left(x\right)$ in the integral, we obtain
$$\int_{\Omega_*}\left|u_*\left(y\right)\right|^p dy=\int_\Omega\left|u_*\left(\Psi\left(x\right)\right)\right|^p\left|J_\Psi\right|dx=\int_\Omega\left|u\left(x\right)\right|^p dx.$$
It follows also that
$$\|u\|_{L^p(\Omega)}=\|u_*\|_{L^p(\Omega_*)},$$
that is, push-forward is an isometry of $L^p\left(\Omega\right)$ and $L^p\left(\Omega_*\right)$.

(b) If $u\in W^{1,2}\left(\Omega\right)$ then $u_*\in W^{1,2}\left(\Omega_*\right)$. Indeed, observe that, by the chain rule,
$$\partial_{y_k}u_*\left(y\right)=\partial_{y_k}\left[u\left(\Psi^{-1}\left(y\right)\right)\right]=\sum_{i=1}^n\left(\partial_{x_i}u\right)_*\frac{\partial x_i}{\partial y_k}.$$
Since $\partial_{x_i}u\in L^2\left(\Omega\right)$, we obtain by (a) that $\left(\partial_{x_i}u\right)_*\in L^2\left(\Omega_*\right)$. Since all partial derivatives $\frac{\partial x_i}{\partial y_k}$ are bounded by K, we obtain that $\left(\partial_{x_i}u\right)_*\frac{\partial x_i}{\partial y_k}$ belongs to $L^2\left(\Omega_*\right)$, whence $\partial_{y_k}u_*\in L^2\left(\Omega_*\right)$. Hence, $u_*\in W^{1,2}\left(\Omega_*\right)$. It follows from this argument that
$$\left\|\partial_{y_k}u_*\right\|_{L^2}\leq K\sum_{i=1}^n\left\|\partial_{x_i}u\right\|_{L^2}\leq Kn\left\|\nabla u\right\|_{L^2},$$
$$\left\|\nabla u_*\right\|_{L^2}\leq\sum_{k=1}^n\left\|\partial_{y_k}u_*\right\|_{L^2}\leq Kn^2\left\|\nabla u\right\|_{L^2},$$
whence
$$\left(Kn^2\right)^{-1}\|u\|_{W^{1,2}(\Omega)}\leq\|u_*\|_{W^{1,2}(\Omega_*)}\leq Kn^2\|u\|_{W^{1,2}(\Omega)}.\tag{4.13}$$

(c) If $u\in W_0^{1,2}\left(\Omega\right)$ then $u_*\in W_0^{1,2}\left(\Omega_*\right)$. Observe that if $\varphi\in C_0^1\left(\Omega\right)$ then $\varphi_*\in C_0^1\left(\Omega_*\right)$. If $u\in W_0^{1,2}\left(\Omega\right)$ then u is the limit in $W^{1,2}\left(\Omega\right)$ of a sequence $\{\varphi_k\}$ of C_0^1 functions in Ω. By (4.13) we conclude that u_* is the limit in $W^{1,2}\left(\Omega_*\right)$ of the sequence $\{\left(\varphi_k\right)_*\}$. Since $\left(\varphi_k\right)_*\in C_0^1\left(\Omega_*\right)$, it follows that $u_*\in W_0^{1,2}\left(\Omega_*\right)$.

(d) By Exercise 3 we have the following property of push-forward. Let
$$L=\sum_{i,j=1}^n\partial_i\left(a_{ij}\partial_j\right)$$

be an operator in Ω and let $Lu = f$ hold weakly in Ω. Then

$$L_* u_* = f_* \text{ weakly in } \Omega, \tag{4.14}$$

where the operator L_* is given by

$$L_* = \frac{1}{\sqrt{D}} \sum_{i,k=1}^{n} \partial_{y_k} \left(b_{kl} \sqrt{D} \partial_{y_l} \right)$$

with

$$b_{kl}(y) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j}.$$

and

$$D = (\det J_\Psi)^{-2}.$$

Since $D = 1$, we have

$$L_* = \sum_{i,k=1}^{n} \partial_{y_k} \left(b_{kl} \partial_{y_l} \right).$$

Let us show that the operator L_* is uniformly elliptic in Ω_*. For any $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned} \sum_{k,l=1}^{n} b_{kl} \xi_k \xi_l &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \xi_k \xi_l \\ &= \sum_{i,j=1}^{n} a_{ij} \left(\sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i} \xi_k \right) \left(\sum_{l=1}^{n} \frac{\partial y_l}{\partial x_j} \xi_l \right). \end{aligned}$$

Set

$$\eta_i = \sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i} \xi_k \tag{4.15}$$

so that

$$\sum_{k,l=1}^{n} b_{kl} \xi_k \xi_l = \sum_{i,j=1}^{n} a_{ij} \eta_i \eta_j. \tag{4.16}$$

By the uniform ellipticity of (a_{ij}), we have

$$\lambda^{-1} |\eta|^2 \leq \sum_{i,j=1}^{n} a_{ij} \eta_i \eta_j \leq \lambda |\eta|^2. \tag{4.17}$$

Since the coefficients $\frac{\partial y_k}{\partial x_i}$ are bounded by K, we obtain from (4.15)

$$|\eta_i| \leq Kn |\xi|$$

and, hence,

$$|\eta| \leq Kn^2 |\xi|.$$

Since

$$\xi_k = \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_k} \eta_i,$$

and all the coefficients $\frac{\partial x_i}{\partial y_k}$ are also bounded by K, it follows that

$$|\xi| \le Kn^2 |\eta|,$$

whence

$$\left(Kn^2\right)^{-1} |\xi| \le |\eta| \le Kn^2 |\xi|.$$

Combining with (4.16) and (4.17), we obtain

$$\lambda_*^{-1} |\xi|^2 \le \sum_{k,l=1}^{n} b_{kl}\xi_k\xi_l \le \lambda_* |\xi|^2,$$

where $\lambda_* = \lambda\left(Kn^2\right)^2$. Hence, L_* is uniformly elliptic with the ellipticity constant λ_*.

Now let u solve the Dirichlet problem (4.10) with $f \in L^q(\Omega)$. By the above properties of push-forward, we obtain that u_* solves the Dirichlet problem

$$\begin{cases} L_* u_* = f_* \text{ weakly in } \Omega_* \\ u_* \in W_0^{1,2}(\Omega_*) \end{cases}$$

and $f_* \in L^q(\Omega_*)$. Since $\Omega_* \subset \mathbb{R}^n_+$ and the set V lies on $\partial\Omega_* \cap H$, we conclude by Corollary 4.2 that $u_* \in C^\alpha(\Omega_* \cup V)$ for some $\alpha = \alpha(n, \lambda_*, q) > 0$, and that $u_* = 0$ on V. It follows that also $u \in C^\alpha(\Omega \cup \Gamma)$, in particular, $u \in C(\Omega \cup \Gamma)$, and $u = 0$ on Γ, which finishes the proof. ■

Remark. Note that the exponent α depends via λ_* also on the constant K that bounds $|\nabla h|$. Since K depends on the extension of function h outside V, the value of α depends on V. Hence, we cannot claim that u is Hölder continuous on the full boundary $\partial\Omega$ inside Q.

Remark. The statement and proof of Theorem 4.3 (with necessary modifications) remain valid if h is a Lipschitz function rather than C^1.

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4.3 Domains with C^1 boundary

Given two sets $A \subset \mathbb{R}^{n-1}$ and $B \subset \mathbb{R}$, define the product $A \times_i B$ with respect to the coordinate x_i in $\mathbb{R}^n$ as follows:

$$A \times_i B = \{(x_1, ..., x_n) \in \mathbb{R}^n : (x_1, ...\hat{x}_i..., x_n) \in A,\ x_i \in B\},$$

where the notation $\hat{x}_i$ means that x_i is omitted, that is,

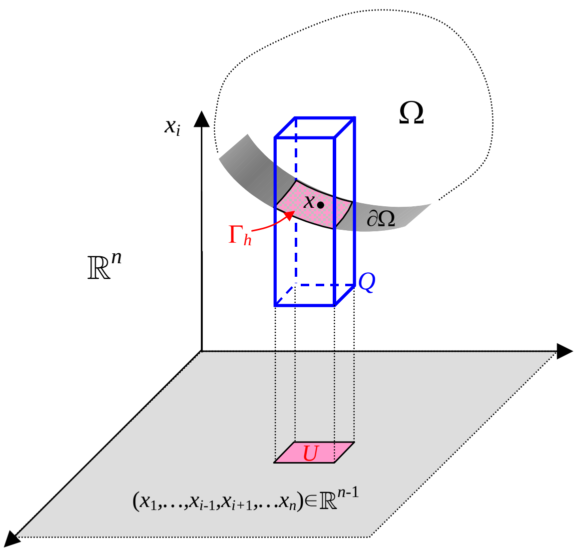
$$(x_1, ...\hat{x}_i..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

A (open) *cylinder* in $\mathbb{R}^n$ with respect to the coordinate x_i is any set Q of the form $Q = U \times_i I$ where U is an open subset of $\mathbb{R}^{n-1}$ and I is an open interval in $\mathbb{R}$.

Definition. Let Ω be an open subset of $\mathbb{R}^n$. We say that the boundary of Ω belongs to the class C^1 (or simply Ω belongs to C^1) if the following two conditions are satisfied:

(*i*) any open neighborhood of any point $x \in \partial\Omega$ has a non-empty intersection with $\overline{\Omega}^c$;

(ii) for any point $x \in \partial\Omega$ there exist a cylinder $Q = U \times_i I$ containing x and a C^1-function $h : U \to I$ such that $\partial\Omega \cap Q = \Gamma_h$ (in other words, $\partial\Omega$ is locally a C^1 graph).



Without loss of generality, we can assume that U (and, hence, Q) is connected.

Claim. *It follows from* (i) *and* (ii) *that* $\Omega \cap Q$ *coincides either with the supergraph of* h *in* Q *or with the subgraph of* h *in* Q.

Proof. Let S be the supergraph of h in Q and S' be the subgraph. Then

$$Q = S \sqcup S' \sqcup \Gamma_h.$$

Since S is an image of Q under a continuous mapping, it follows that S is connected. Since S is covered by the disjoint union $\Omega \sqcup \overline{\Omega}^c$ of open sets, it follows that $S \subset \Omega$ or $S \subset \overline{\Omega}^c$. The same argument applies also to S': either $S' \subset \Omega$ or $S' \subset \overline{\Omega}^c$.

However, S and S' cannot both be contained in the same of the two sets Ω or $\overline{\Omega}^c$. Indeed, if S and S' are both contained in Ω then any point x on Γ_h has in small enough neighborhoods no points from $\overline{\Omega}^c$, which contradicts (i). If S and S' are contained in $\overline{\Omega}^c$, and any point $x \in \Gamma_h$ has in small enough neighborhoods no points from Ω, which contradicts the definition of the boundary.

Hence, there remain only two possibilities:

- either $S \subset \Omega$ and $S' \subset \overline{\Omega}^c$
- or $S' \subset \Omega$ and $S \subset \overline{\Omega}^c$.

In the first case we have $\Omega \cap Q = S$, and in the second case $\Omega \cap Q = S'$. ∎

The next statement provides a large class of examples of domains with C^1 boundary. Recall that a bounded open set Ω is called a *region* if there exists a C^1 function F defined in an open neighborhood Ω' of $\overline{\Omega}$ such that

$$\Omega = \{x \in \Omega' : F(x) < 0\},$$

$$\partial\Omega = \{x \in \Omega : F(x) = 0\},$$

and

$$\nabla F \neq 0 \text{ on } \partial\Omega.$$

For example, a ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ is a region with function

$$F(x) = |x|^2 - R^2.$$

Lemma 4.4 *If Ω is a region then Ω has C^1 boundary.*

Proof. Fix some point $z \in \partial\Omega$. By the hypothesis $\nabla F(z) \neq 0$, the point z cannot be a local maximum of F. Since $F(z) = 0$, it follows that any neighborhood of z contains points x with $F(x) > 0$, that is, the points from $\overline{\Omega}^c$.

Since $\nabla F(z) \neq 0$, there is an index $i = 1, 2..., n$ such that $\partial_i F(z) \neq 0$. By the theorem of implicit function, the equation

$$F(x_1, x_2, ..., x_n) = 0$$

can be resolved in a neighborhood of z with respect to x_i as follows: there is a cylinder $Q = U \times_i I$ containing z and a C^1 function $f : U \to I$ such that, for all $x \in Q$,

$$F(x_1, ..., x_n) = 0 \Leftrightarrow x_i = f(x_1, ...\hat{x}_i...x_n).$$

Consequently, we have

$$\partial\Omega \cap Q = \Gamma_f,$$

and, hence, Ω is a domain with C^1 boundary. ∎

Theorem 4.5 *Assume that Ω is a bounded domain with C^1 boundary. Let L be a uniformly elliptic operator with measurable coefficients in Ω and let u solve the weak Dirichlet problem*

$$\begin{cases} Lu = f \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{4.18}$$

where $f \in L^q(\Omega)$ with $q \in [2, \infty] \cap (n/2, \infty]$. Then $u \in C^\alpha(\overline{\Omega})$ with some $\alpha > 0$ and $u|_{\partial\Omega} = 0$. Here α depends on n, λ, q and Ω.

Proof. By definition of C^1 boundary, for any point $x \in \partial\Omega$ there is a cylinder $Q_x = U_x \times_{i_x} I_x$ such that $\partial\Omega \cap Q_x$ is the graph of a C^1 function $h_x : U_x \to I_x$. Besides, by the above claim, $\Omega \cap Q_x$ is either supergraph or subgraph of h_x in Q_x.

As in the proof of Theorem 4.3, choose an open subset $V_x \subset U_x$ such that $x \in V_x$ and $\overline{V}_x$ is a compact subset of U_x. Let Γ_x be the graph of $h_x|_{V_x}$. By the proof of Theorem 4.3 we have $u \in C^{\alpha_x}(\Omega \cup \Gamma_x)$ where $\alpha_x > 0$, and $u = 0$ on Γ_x.

The family $Q'_x = V_x \times_{i_x} I_x$ of all cylinders Q'_x with $x \in \partial\Omega$ provides an open covering of $\partial\Omega$. Choose a finite subcover $\{Q'_{x_k}\}$, $k = 1, ..., N$, and set

$$\alpha := \min(\alpha_{x_1}, ..., \alpha_{x_N}) > 0.$$

Then we have $u \in C^\alpha(\Omega \cup \Gamma_{x_k})$ for any k. Since the union of all sets Γ_{x_k} over all k is $\partial\Omega$, we obtain that $u \in C^\alpha(\Omega \cup \partial\Omega)$ and $u = 0$ on $\partial\Omega$, which was to be proved. ∎

Remark. The statement and the proof of Theorem 4.5 remain valid if the boundary $\partial\Omega$ is Lipschitz rather than C^1.

4.4 Classical solutions

Now we can prove a result about existence of a classical solution.

Theorem 4.6 *Assume that Ω is a bounded domain with C^1 boundary and let k be an integer such that $k > n/2$. Consider in Ω a uniformly elliptic operator $L = \sum_{i,j=1}^{n} \partial_i (a_{ij}\partial_j)$ with coefficients $a_{ij} \in C^{k+1}\left(\overline{\Omega}\right)$. Then, for all $f \in C^k\left(\overline{\Omega}\right)$ and $g \in C^2\left(\overline{\Omega}\right)$, the classical Dirichlet problem*

$$\begin{cases} Lu = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases} \tag{4.19}$$

has exactly one solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$.

Remark. The assumptions of this theorem about functions a_{ij}, f, g are not quite optimal. They are to illustrate the method of obtaining classical solutions by means of weak solutions.

Proof. Consider first the weak Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } \Omega \\ u - g \in W_0^{1,2}(\Omega). \end{cases} \tag{4.20}$$

By Exercises 7 and 27, if $f \in L^2(\Omega)$ and $g \in W^{1,2}(\Omega)$ (which is the case under the present assumptions) then the problem (4.20) has a unique weak solution $u \in W^{1,2}(\Omega)$.

Since $f \in C^k\left(\overline{\Omega}\right)$, we have also $f \in W^{k,2}(\Omega)$. Since $a_{ij} \in C^{k+1}(\Omega)$, we obtain by Theorem 2.10(b) that

$$u \in W_{loc}^{k+2,2}(\Omega).$$

Since

$$k + 2 > \frac{n}{2} + 2,$$

the Sobolev embedding theorem implies that $u \in C^2(\Omega)$. Hence, u is a classical solution
12.07.16 of $Lu = f$ in Ω.

In order to investigate the behavior of u on $\partial\Omega$, let us rewrite (4.20) in terms of the function $v = u - g$ as follows:

$$\begin{cases} Lv = f - Lg \text{ weakly in } \Omega \\ v \in W_0^{1,2}(\Omega). \end{cases} \tag{4.21}$$

Since $g \in C^2\left(\overline{\Omega}\right)$ and $a_{ij} \in C^1\left(\overline{\Omega}\right)$, it follows that $Lg \in C\left(\overline{\Omega}\right)$, whence

$$f - Lg \in C\left(\overline{\Omega}\right) \subset L^\infty(\Omega).$$

In particular, the problem (4.21) has a unique weak solution v (this is an alternative proof of the existence and uniqueness of solution u of (4.20)). By Theorem 4.5 we obtain $v \in C^\alpha\left(\overline{\Omega}\right)$ with some $\alpha > 0$, and $v = 0$ on $\partial\Omega$. It follows that also $u \in C^\alpha\left(\overline{\Omega}\right)$ and $u = g$ on $\partial\Omega$, so that u satisfies the boundary value in the classical sense.

Hence, u is a classical solution of (4.19). Finally, the uniqueness of the classical solution of (4.19) in the class $C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ follows from the maximum principle of Exercise 1. ■

Recall from PDE the following result for the Laplace operator: let $f \in C^2(B_R)$ be bounded and let $g \in C(\partial B_R)$. Then the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } B_R \\ u = g & \text{on } \partial B_R \end{cases} \tag{4.22}$$

has exactly one classical solution $u \in C^2(B_R) \cap C(\overline{B}_R)$. Of course, the requirements here are much milder than those in Theorem 4.6. Of course, this is very special situation of $L = \Delta$ and $\Omega = B_R$ where one can expect better results than in general.

There is one more serious distinction between these two results. If u is the classical solution of (4.22), it may not be a weak solution in any sense, because, as we have seen on examples, the classical solution of (4.22) with arbitrary continuous function g on $\partial\Omega$ may have infinite energy:

$$\int_{B_R} |\nabla u|^2 \, dx = \infty,$$

and, hence, may be not in $W^{1,2}(B_R)$. Hence, for the methods based on weak solutions, one need to impose additional restriction on g.

Chapter 5

Harnack inequality

5.1 Statement of the Harnack inequality (Theorem of Moser)

Consider again in a domain $\Omega \subset \mathbb{R}^n$ a uniformly elliptic operator in divergence form

$$L = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j \right)$$

with measurable coefficients. Recall that if $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution of $Lu = 0$ in Ω then by Theorem 3.7 u is Hölder continuous in Ω.

Definition. We say that a function u is *L-harmonic* in Ω if u is the continuous version of a weak solution $u \in W^{1,2}_{loc}(\Omega)$ of $Lu = 0$ in Ω.

The main result of this Chapter is the following theorem.

Theorem 5.1 *If u is a non-negative L-harmonic function in a ball $B_{2R} \subset \Omega$ then*

$$\sup_{B_R} u \leq C \inf_{B_R} u \tag{5.1}$$

where $C = C(n, \lambda)$.

The inequality (5.1) is called the *Harnack inequality*, analogously to the classical Harnack inequality for harmonic functions that holds with the constant $C = 3^n$. This inequality for uniformly elliptic operators in divergence form with measurable coefficients was first proved by Jürgen Moser in 1961.

Recall the weak Harnack inequality of Theorem 3.4 that we now reformulate in the following form[1]:

Weak Harnack inequality *Let $B_{4R} \subset \Omega$ and assume that $u \in W^{1,2}(B_{4R})$ is L-harmonic in B_{4R}. Choose some $a > 0$ and set*

$$E = \{x \in B_R : u(x) \geq a\}.$$

If for some $\varepsilon > 0$

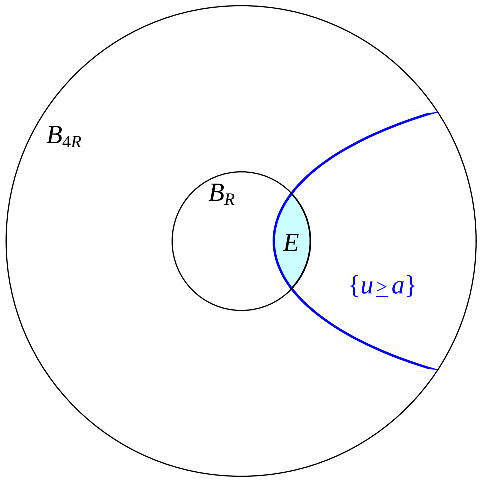
$$|E| \geq \varepsilon |B_R|,$$

[1] In comparison with Theorem 3.4, we replace B_{3R} by B_{4R} and supersolution by solution.

then

$$\inf_{B_R} u \geq \delta a, \tag{5.2}$$

where $\delta = \delta(n, \lambda, \varepsilon) > 0$.



The Harnack inequality (should it be already proved) implies the weak Harnack inequality as follows: if the set E has positive measure then we conclude that

$$a \leq \sup_{B_R} u,$$

and then (5.2) follows from (5.1).

However, in the proof of Theorem 5.1 we will use the weak Harnack inequality. Moreover, we will use only the following properties of L-harmonic functions (apart from continuity):

(i) the weak Harnack inequality;

(ii) if u is L-harmonic then also the function $au + b$ is L-harmonic for arbitrary $a, b \in \mathbb{R}$.

If these two properties hold for any other operator L then also the Harnack inequality holds for L.

The method of derivation of the Harnack inequality from the weak Harnack inequality was invented by Eugene Landis in 1970s as an alternative to a more complicated method of Moser that involved a difficult lemma of John-Nirenberg.

5.2 Lemmas of growth

For the proof of Theorem 5.1 we need some lemmas. The first lemma is an extension of the weak Harnack inequality.

Lemma 5.2 (Reiteration of the weak Harnack inequality) *Let u be a non-negative L-harmonic function in some ball $B_R(x)$. Consider a ball $B_r(y)$ where*

$$y \in B_{\frac{1}{9}R}(x) \quad \text{and} \quad r \leq \frac{2}{9}R.$$

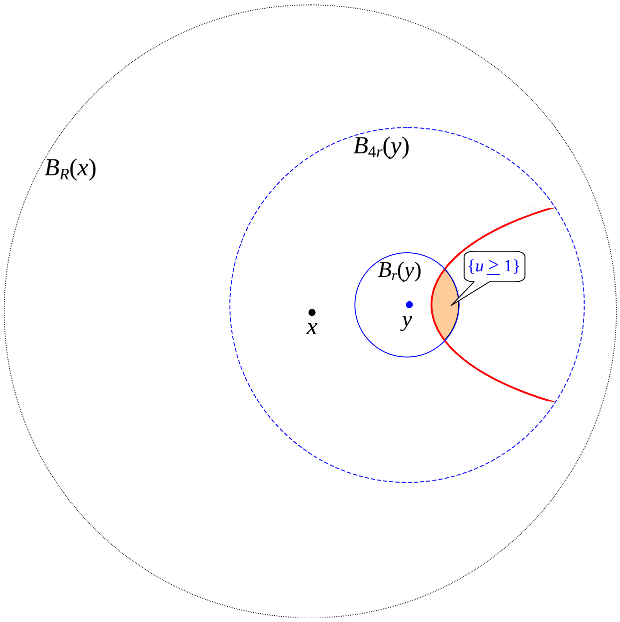
If

$$\frac{|\{u \geq 1\} \cap B_r(y)|}{|B_r(y)|} \geq \theta > 0 \tag{5.3}$$

then

$$u(x) \geq \left(\frac{r}{R}\right)^s \delta,$$

where $\delta = \delta(n, \lambda, \theta) > 0$ and $s = s(n, \lambda) > 0$.



Proof. Note that

$$B_{4r}(y) \subset B_R(x)$$

because

$$|x - y| + 4r < \frac{1}{9}R + \frac{8}{9}R = R.$$

Applying the weak Harnack inequality in $B_r(y)$ and using (5.3), we obtain that

$$\inf_{B_r(y)} u \geq \delta_1 := \delta(n, \lambda, \theta).$$

It follows that

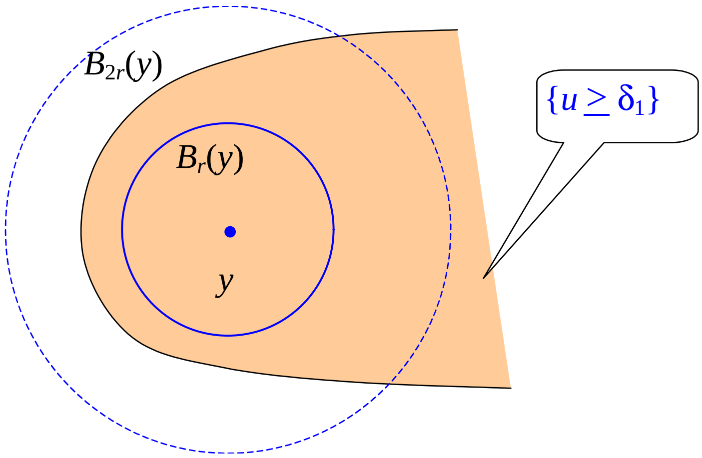
$$\frac{|\{u \geq \delta_1\} \cap B_{2r}(y)|}{|B_{2r}(y)|} \geq \frac{|B_r|}{|B_{2r}|} = 2^{-n}.$$

If $B_{8r}(y) \subset B_R(x)$ then applying the weak Harnack inequality in $B_{2r}(y)$, we obtain that

$$\inf_{B_{2r}(y)} u \geq \delta_1 \delta\left(n, \lambda, 2^{-n}\right) = \varepsilon \delta_1,$$

where

$$\varepsilon := \delta\left(n, \lambda, 2^{-n}\right).$$



It follows that

$$\frac{|\{u \geq \varepsilon\delta_1\} \cap B_{4r}(y)|}{|B_{4r}(y)|} \geq \frac{|B_{2r}|}{|B_{4r}|} = 2^{-n}.$$

Therefore, if $B_{16r}(y) \subset B_R(x)$ then

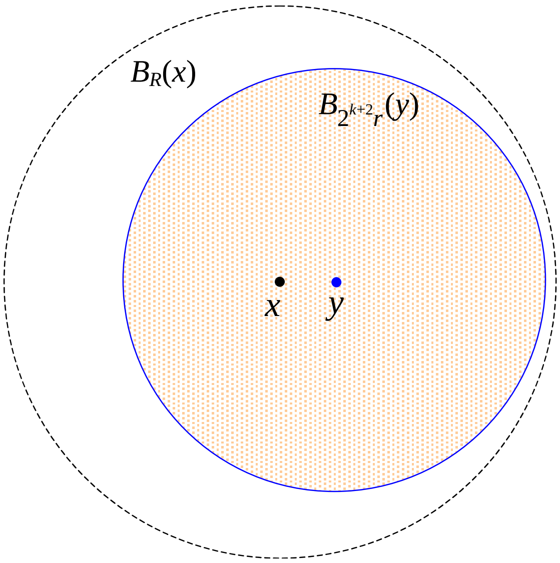
$$\inf_{B_{4r}} u \geq (\delta_1 \varepsilon) \varepsilon = \varepsilon^2 \delta_1.$$

We continue by induction and obtain the following statement for any positive integer k:

$$\text{if } B_{2^{k+2}r}(y) \subset B_R(x) \text{ then } \inf_{B_{2^k r}} u \geq \varepsilon^k \delta_1. \tag{5.4}$$

Let k be the maximal integer such that

$$B_{2^{k+2}r}(y) \subset B_R(x).$$



Then

$$2^{k+2}r + |x-y| \leq R$$

while

$$2^{k+3}r + |x-y| > R.$$

It follows that

$$2^k r > \frac{R - |x-y|}{8} \geq |x-y|$$

where we have used that $R > 9|x-y|$. Hence, for this value of k, we have

$$x \in B_{2^k r}(y).$$

Then by (5.4)

$$u(x) \geq \varepsilon^k \delta_1.$$

On the other hand, we have

$$2^k r < 2^{k+2} r + |x-y| \leq R$$

whence

$$k \leq \log_2 \frac{R}{r}.$$

It follows that

$$u(x) \geq \varepsilon^{\log_2 \frac{R}{r}} \delta_1 = \delta_1 2^{\log_2 \varepsilon \log_2 \frac{R}{r}} = \delta_1 \left(\frac{R}{r}\right)^{\log_2 \varepsilon} = \delta_1 \left(\frac{r}{R}\right)^s$$

with $s = \log_2 \frac{1}{\varepsilon} > 0$, which finishes the proof. ∎

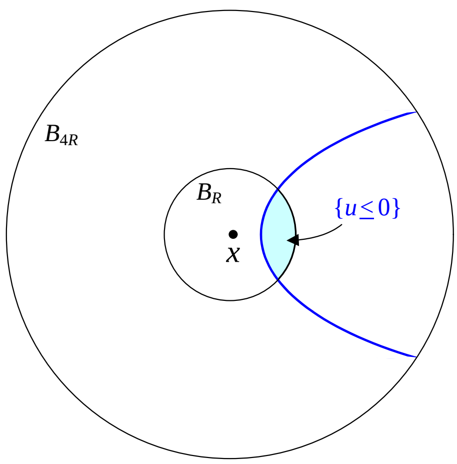
Lemma 5.3 (Alternative form of the weak Harnack inequality) *Let u be an L-harmonic function in some ball $B_{4R}(x)$. If*

$$\frac{|\{u \leq 0\} \cap B_R(x)|}{|B_R|} \geq \theta > 0,$$

then

$$\sup_{B_{4R}(x)} u \geq (1+\delta) u(x), \tag{5.5}$$

where $\delta = \delta(n, \lambda, \theta) > 0$ is the same as in the weak Harnack inequality.



Proof. If $u(x) \leq 0$ then (5.5) is trivially satisfied. Assume that $u(x) > 0$. By rescaling, we can assume also that

$$\sup_{B_{4R}(x)} u = 1.$$

Consider the function $v = 1 - u$ that is a non-negative L-harmonic function in $B_{4R}(x)$. Observe also, that

$$u \leq 0 \Leftrightarrow v \geq 1.$$

Hence, we obtain that

$$\frac{|\{v \geq 1\} \cap B_R(x)|}{|B_R|} \geq \theta.$$

By the weak Harnack inequality, we conclude that

$$\inf_{B_R(x)} v \geq \delta,$$

where $\delta = \delta(n, \lambda, \theta) > 0$. It follows that $v(x) \geq \delta$ and, hence

$$u(x) \leq 1 - \delta < \frac{1}{1+\delta} = \frac{1}{1+\delta} \sup_{B_{4R}} u,$$

which is equivalent to (5.5). ■

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Lemma 5.4 (Lemma of growth in a thin domain) *There exists $\varepsilon = \varepsilon(n, \lambda) > 0$ such that the following is true: if u is an L-harmonic function in a ball $B_R(x)$ and if*

$$\frac{|\{u > 0\} \cap B_R|}{|B_R|} \leq \varepsilon$$

then

$$\sup_{B_R} u \geq 4u(x).$$

Corollary 5.5 *Under the same assumptions, choose some $a \in \mathbb{R}$ and assume that*

$$\frac{|\{u > a\} \cap B_R|}{|B_R|} \leq \varepsilon.$$

Then

$$\sup_{B_R} u \geq a + 4(u(x) - a).$$

Proof. Indeed, just apply Lemma 5.4 to the L-harmonic function $v = u - a$. ■

Proof of Lemma 5.4. The value of ε will be determined later. So far consider ε as given. Consider any ball $B_r(y) \subset B_R(x)$ such that

$$\frac{|B_r|}{|B_R|} = 2\varepsilon,$$

which is equivalent to $\left(\frac{r}{R}\right)^n = 2\varepsilon$ and, hence, to

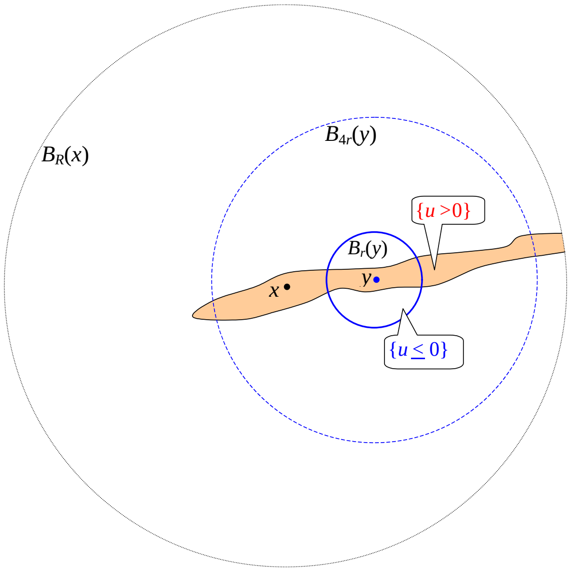
$$r = (2\varepsilon)^{1/n} R.$$

Then

$$\frac{|\{u > 0\} \cap B_r(y)|}{|B_r|} \leq \frac{|\{u > 0\} \cap B_R(x)|}{|B_R|} \frac{|B_R|}{|B_r|} \leq \varepsilon \frac{1}{2\varepsilon} = \frac{1}{2}.$$

It follows that

$$\frac{|\{u \leq 0\} \cap B_r(y)|}{|B_r|} \geq \frac{1}{2}.$$



If $B_{4r}(y) \subset B_R(x)$ then we can apply Lemma 5.3 and obtain that

$$\sup_{B_{4r}(y)} u \geq (1 + \delta) u(y),$$

where $\delta = \delta\left(n, \lambda, \frac{1}{2}\right) > 0$. By slightly reducing δ, we obtain the following claim.

Claim. *If $B_{4r}(y) \subset B_R(x)$ and $r = (2\varepsilon)^{1/n} R$ then there exists $y' \in B_{4r}(y)$ such that*

$$u(y') \geq (1 + \delta) u(y),$$

where $\delta > 0$ depends on n, λ.

Let us apply the Claim first for $y = x$. Assuming that ε is small enough, we obtain $4r < R$ and, hence, $B_{4r}(x) \subset B_R(x)$. Hence, we obtain by Claim a point $x_1 \in B_{4r}(x)$ such that

$$u(x_1) \geq (1 + \delta) u(x).$$

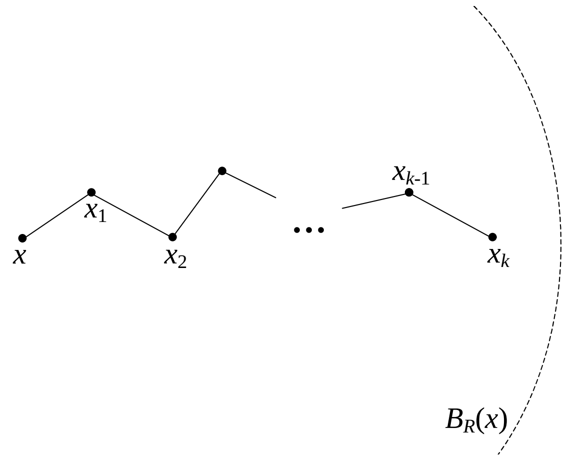
If $B_{4r}(x_1) \subset B_R(x)$ then we apply Claim again and obtain that there is $x_2 \in B_{4r}(x_1)$ such that

$$u(x_2) \geq (1 + \delta) u(x_1).$$

We continue construction of the sequence $\{x_k\}$ by induction: as long as $B_{4r}(x_k) \subset B_R(x)$, we obtain $x_{k+1} \in B_{4r}(x_k)$ such that

$$u(x_{k+1}) \geq (1 + \delta) u(x_k).$$

If, for some k, $B_{4r}(x_k)$ is not contained in $B_R(x)$ then we stop the construction.



By construction, if x_k exists then $x_k \in B_R(x)$ and

$$u(x_k) \geq (1+\delta)^k u(x_k). \tag{5.6}$$

Besides, we have

$$|x_{l+1} - x_l| < 4r \quad \text{for all } l \leq k-1,$$

which implies that

$$|x_k - x| < 4kr. \tag{5.7}$$

Let us prove by induction in k the following claim:

$$\text{if } 4kr < R \text{ then } x_k \text{ exists.}$$

We know already that x_1 exists. Let us prove the induction step, that is,

$$\text{if } 4(k+1)r < R \text{ then } x_{k+1} \text{ exists.}$$

Indeed, if $4(k+1)r < R$ then also $4kr < R$ and we obtain the inductive hypothesis that x_k exists. It follows from (5.7) that

$$B_{4r}(x_k) \subset B_{4(k+1)r}(x).$$

Since $4(k+1)r < R$, we see that $B_{4r}(x_k) \subset B_R(x)$, and this construction can be continued so that x_{k+1} exists, which finishes the inductive proof.

Let us choose the maximal integer k with $4kr < R$. Then we have

$$4(k+1)r \geq R$$

and, hence,

$$k \geq \frac{R}{4r} - 1 = \frac{1}{4(2\varepsilon)^{1/n}} - 1.$$

It follows from (5.6) that

$$u(x_k) \geq (1+\delta)^{\frac{1}{4(2\varepsilon)^{1/n}} - 1} u(x).$$

Finally, choosing ε small enough (depending only on δ and n, that is, on λ and n), we obtain

$$\sup_{B_R(x)} u \geq u(x_k) \geq 4u(x),$$

which was to be proved. ■

5.3 Proof of the Harnack inequality

Here we prove Theorem 5.1. Observe first that it suffices to prove the following version of the Harnack inequality: there exists a constant C, depending on n, λ and such that if u is a non-negative L-harmonic function on a ball $B_{KR}(x)$ (where $K = 18$) then

$$\sup_{B_R(x)} u \le Cu(x).$$

Without loss of generality, we can assume that

$$\sup_{B_R(x)} u = 2, \tag{5.8}$$

and we need to prove that

$$u(x) \ge c \tag{5.9}$$

for some positive constant $c = c(n, \lambda)$. Let us construct a sequence $\{x_k\}_{k\ge1}$ of points such that

$$x_k \in B_{2R}(x) \quad \text{and} \quad u(x_k) = 2^k. \tag{5.10}$$

A point x_1 with $u(x_1) = 2$ exists in $\overline{B}_R(x)$ by assumption (5.8). Assume that x_k satisfying (5.10) is already constructed. Then, for small enough $r > 0$, we have

$$\sup_{B_r(x_k)} u \le 2^{k+1}.$$

Set

$$r_k = \sup\left\{ r \in (0, R] : \sup_{B_r(x_k)} u \le 2^{k+1} \right\}.$$

If $r_k = R$ then we stop the process without constructing x_{k+1}. If $r < R$ then we necessarily have

$$\sup_{B_r(x_k)} u = 2^{k+1}$$

(note that $B_r(x_k) \subset B_R(x_k) \subset B_{4R}(x)$ so that u is defined in $B_r(x_k)$). Therefore, there exists $x_{k+1} \in \overline{B}_{r_{k+1}}(x_k)$ such that $u(x_{k+1}) = 2^{k+1}$.

If $x_{k+1} \in B_{2R}(x)$ then we keep x_{k+1} and go to the next step. If $x_{k+1} \notin B_{2R}(x)$ then we disregard x_{k+1} and stop the process.

Hence, we obtain a sequence of balls $\{B_{r_k}(x_k)\}$ such that

$$r_k \le R, \quad x_k \in B_{2R}(x), \quad u(x_k) = 2^k$$

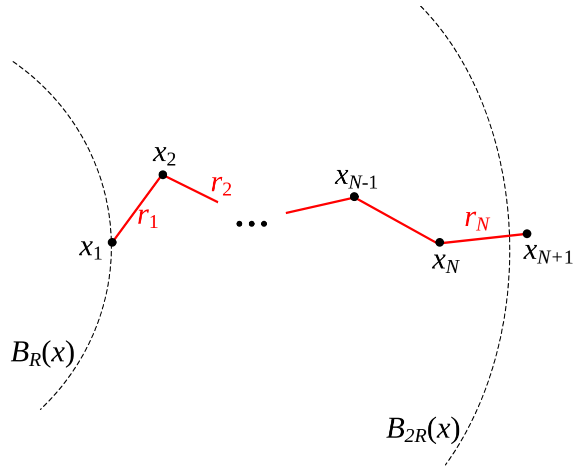
and

$$\sup_{B_{r_k}(x_k)} u \le 2^{k+1}. \tag{5.11}$$

Moreover, we have also

$$|x_{k+1} - x_k| \le r_k.$$

The sequence $\{x_k\}$ cannot be infinite because $u(x_k) \to \infty$ whereas u is bounded in $\overline{B}_{2R}(x)$ as a continuous function. Let N be the largest value of k in this sequence. Then we have either $r_N = R$ or $r_N < R$ and $x_{N+1} \notin B_{2R}(x)$ (where x_{N+1} is the disregarded point).



In the both cases we clearly have

$$r_1 + ... + r_N \geq R. \tag{5.12}$$

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In any ball $B_{r_k}(x_k)$ we have by (5.11)

$$\begin{aligned} \sup_{B_{r_k}(x_k)} u &\leq 2^{k+1} < 2^{k-1} + 4\left(2^k - 2^{k-1}\right) \\ &= a + 4\left(u\left(x_k\right) - a\right), \end{aligned}$$

where $a = 2^{k-1}$. By Corollary 5.5, we conclude that

$$\frac{|\{u > a\} \cap B_{r_k}(x_k)|}{|B_{r_k}|} > \varepsilon,$$

that is,

$$\frac{\left|\left\{u \geq 2^{k-1}\right\} \cap B_{r_k}(x_k)\right|}{|B_{r_k}|} \geq \varepsilon.$$

Now let us apply Lemma 5.2 with $B_r(y) = B_{r_k}(x_k)$. Since u is non-negative and L-harmonic in $B_{KR}(x)$, the following conditions need to be satisfied:

$$r_k \leq \frac{2}{9}KR \text{ and } |x_k - x| \leq \frac{1}{9}KR.$$

Since $r_k \leq R$ and $|x_k - x| \leq 2R$, the both conditions are satisfied if $K = 18$. By Lemma 5.2, we obtain that

$$u(x) \geq \left(\frac{r_k}{R}\right)^s \delta 2^{k-1}, \tag{5.13}$$

where $\delta = \delta(n, \lambda, \varepsilon) > 0$ and $s = s(n, \lambda) > 0$.

The question remains how to estimate $\left(\frac{r_k}{R}\right)^s 2^{k-1}$ from below, given the fact that we do not know much about the sequence $\{r_k\}$: the only available information is (5.12). The following trick was invented by Landis. The condition (5.12) implies that there exists $k \leq N$ such that

$$r_k \geq \frac{R}{k(k+1)}. \tag{5.14}$$

Indeed, if for all $k \leq N$ we have

$$r_k < \frac{R}{k(k+1)},$$

then it follows that

$$\sum_{k=1}^{N} r_k < \sum_{k=1}^{\infty} \frac{R}{k(k+1)} = R,$$

which contradicts (5.12). Hence, choose k that satisfies (5.14). For this k we obtain from (5.13) that

$$u(x) \geq \delta \left(\frac{r_k}{R}\right)^s 2^{k-1} \geq \delta \frac{2^{k-1}}{(k(k+1))^s}.$$

The next observation is that although we do not know the value of k, nevertheless we can obtain a lower bound of $u(x)$ independent of k because

$$m := \inf_{k \geq 1} \frac{2^{k-1}}{(k(k+1))^s} > 0.$$

Hence, we conclude that

$$u(x) \geq \delta m =: c,$$

which finishes the proof of (5.9).

Finally, let us prove that if u is non-negative and L-harmonic function in a ball B_{2R} then

$$\sup_{B_R} u \leq C \inf_{B_R} u.$$

Assume without loss of generality that the center of the ball B_R is 0. Let a be a point in $\overline{B}_R$ where u takes the maximal value and b be the point in $\overline{B}_R$ where u takes the minimal value. We need to prove that

$$u(a) \leq Cu(b)$$

for some $C = C(n, \lambda)$. It suffices to prove that

$$u(a) \leq Cu(0) \quad \text{and } u(0) \leq Cu(b).$$

Set $r = R/K$ (where $K = 18$ as above) and connect 0 and a by a sequence $\{x_j\}_{j=0}^{K}$ of points such that

$$x_0 = 0, \quad x_K = a, \quad |x_j - x_{j+1}| \leq r.$$

For that, it suffices to choose all x_k on the interval $[0, a]$ dividing this interval into K equal parts.

Since $x_j \in \overline{B}_R$, the ball $B_{Kr}(x_j) = B_R(x_j)$ is contained in $B_{2R}(0)$. By the form of the Harnack inequality that we proved above, we conclude that

$$\sup_{B_r(x_j)} u \leq Cu(x_j).$$

Since $x_{j+1} \in \overline{B}_r(x_j)$, it follows that

$$u(x_{j+1}) \leq Cu(x_j)$$

and, hence,

$$u(a) \leq C^K u(0).$$

The inequality for $u(b)$ is proved in the same way.

5.4 * Some applications of the Harnack inequality

5.4.1 Convergence theorems

Theorem 5.6 *Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of L-harmonic functions in a domain $\Omega \subset \mathbb{R}^n$. If*

$$u_k \overset{L^2_{loc}(\Omega)}{\longrightarrow} u \quad \text{as } k \to \infty$$

then the function u is also L-harmonic in Ω. Moreover, the sequence $\{u_k\}$ converges to u locally uniformly.

Proof. Let us show that the sequence $\{u_k\}$ converges also in $W^{1,2}_{loc}(\Omega)$. For that it suffices to show that the sequence of $\{\nabla u_k\}$ is Cauchy in $L^2\left(B_{R/2}\right)$ in any ball $B_{R/2}$ such that $\overline{B}_R \subset \Omega$. For that we use the inequality (3.10) from the proof of Theorem 3.2:

$$\int_{B_R} |\nabla v|^2 \eta^2 dx \le 4\lambda^4 \int_{B_R} |\nabla \eta|^2 v^2 dx, \tag{5.15}$$

where v is any L-harmonic function[2] in Ω and η is any Lipschitz function with compact support in B_R; in particular, choose η to be the following bump function:

$$\eta(x) = \begin{cases} 1, & |x| \le r, \\ \frac{\rho - |x|}{\rho - r}, & r < |x| < \rho, \\ 0, & |x| \ge \rho. \end{cases} \tag{5.16}$$

where $0 < r < \rho < R$. Take $r = \frac{1}{2}R$ and $\rho = \frac{3}{4}R$. Then it follows from (5.15) that

$$\int_{B_{R/2}} |\nabla v|^2 dx \le \frac{C}{R^2} \int_{B_R} v^2 dx. \tag{5.17}$$

Let us apply this inequality to $v = u_k - u_l$. Since

$$\|u_k - u_l\|_{L^2(B_R)} \to 0 \ \text{ as } k, l \to \infty,$$

it follows from (5.17) that

$$\|\nabla u_k - \nabla u_l\|_{L^2\left(B_{R/2}\right)} \to 0 \ \text{ as } k, l \to \infty.$$

Hence, ∇u_k converges in $L^2_{loc}(\Omega)$, which implies that $u \in W^{1,2}_{loc}$ and $u_k \to u$ in $W^{1,2}_{loc}(\Omega)$.

Since each u_k satisfies the identity

$$\int_\Omega \sum_{i,j=1}^n a_{ij} \partial_j u_k \partial_i \varphi = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$, passing to the limit as $k \to \infty$, we obtain the same identity for u, whence $Lu = 0$ follows.

[2] In fact, (5.15) was proved for $v = u_+$ where u is L-harmonic function. Applying (5.15) also to $v = u_-$, we obtain the same inequality with $v = u$.

The last claim follows from Theorem 3.2 that implies that, for any ball $\overline{B}_R \subset \Omega$,

$$\sup_{B_{R/2}} |u - u_k| \leq \frac{C}{R^{n/2}} \|u - u_k\|_{L^2(B_R)}.$$

Since $\|u - u_k\|_{L^2(B_R)} \to 0$ as $k \to \infty$, it follows that also

$$\sup_{B_{R/2}} |u - u_k| \to 0,$$

which means that $u_k \to u$ locally uniformly. ■

Theorem 5.7 *Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of L-harmonic functions in a connected domain $\Omega \subset \mathbb{R}^n$. Assume that this sequence is monotone increasing, that is, $u_{k+1}(x) \geq u_k(x)$ for all $k \geq 1, x \in \Omega$. Then the function*

$$u(x) := \lim_{k \to \infty} u_k(x)$$

is either identically equal to ∞ in Ω, or it is an L-harmonic function in Ω. Moreover, in the latter case the sequence $\{u_k\}$ converges to u locally uniformly.

Proof. By replacing u_k with $u_k - u_1$, we can assume that all functions u_k are non-negative. Consider the sets

$$F = \{x \in \Omega : u(x) < \infty\}$$

and

$$I = \{x \in \Omega : u(x) = \infty\}$$

so that $\Omega = F \sqcup I$. Let us prove that both F and I are open sets.

Indeed, take a point $x \in F$ and show that also $B_\varepsilon(x) \in F$ for some $\varepsilon > 0$. Choose ε so that $B_{2\varepsilon}(x) \subset \Omega$. By the Harnack inequality, we have

$$\sup_{B_\varepsilon(x)} u_k \leq C \inf_{B_\varepsilon(x)} u_k \leq C u_k(x).$$

By passing to the limit as $k \to \infty$, we obtain

$$\sup_{B_\varepsilon(x)} u \leq C u(x).$$

Since $u(x) < \infty$, we obtain that also $\sup_{B_\varepsilon(x)} u < \infty$ and, hence, $B_\varepsilon(x) \subset F$. Hence, F is open.

In the same way one proves that

$$\inf_{B_\varepsilon(x)} u \geq C^{-1} u(x),$$

which implies that I is open.

Since Ω is connected and $\Omega = F \sqcup I$, it follows that either $I = \Omega$ or $F = \Omega$. In the former case we have $u \equiv \infty$ in Ω, in the latter case: $u(x) < \infty$ for all $x \in \Omega$.

Let us prove that in the latter case u is L-harmonic. For that, we first show that the convergence $u_k \to u$ is locally uniform, that is, for any $x \in \Omega$ there is $\varepsilon > 0$ such that

$$u_k \rightrightarrows u \text{ in } B_\varepsilon(x) \quad \text{as } k \to \infty.$$

Then the L-harmonicity of u will follow by Theorem 5.6.

Choose again $\varepsilon > 0$ so that $B_{2\varepsilon}(x) \subset \Omega$. For any two indices $k > l$, apply the Harnack inequality to the non-negative L-harmonic function $u_k - u_l$:

$$\sup_{B_\varepsilon(x)} (u_k - u_l) \leq C (u_k - u_l)(x).$$

Since $(u_k - u_l)(x) \to 0$ as $k, l \to \infty$, it follows that

$$u_k - u_l \rightrightarrows 0 \text{ in } B_\varepsilon(x) \text{ as } k, l \to \infty.$$

Hence, the sequence $\{u_k\}$ converges uniformly in $B_\varepsilon(x)$. Since $\{u_k\}$ convergence pointwise to u, it follows that

$$u_k \rightrightarrows u \text{ in } B_\varepsilon(x) \text{ as } k \to \infty,$$

which finishes the proof. ■

Theorem 5.8 *If $\{u_k\}$ is a sequence of L-harmonic functions in Ω that is bounded in $L^2(\Omega)$, then there is a subsequence $\{u_{k_i}\}$ that converges to an L-harmonic function locally uniformly.*

Proof. Consider any ball $\overline{B}_R \subset \Omega$. Let us apply the inequality (3.11) from the proof of Theorem 3.2 that says the following: v is L-harmonic in Ω then

$$\int_{B_R} |\nabla (v\eta)|^2 \, dx \leq \frac{C}{(\rho - r)^2} \int_{B_\rho} v^2 dx$$

where we take $0 < r < \rho < R$ and function η is defined by (5.16). Taking $r = \frac{1}{2}R$ and $\rho = \frac{3}{4}R$, and applying this to $v = u_k$,

$$\int_{B_R} |\nabla (u_k \eta)|^2 \, dx \leq \frac{C}{R^2} \int_{B_R} u_k^2 dx.$$

Since the right hand side is uniformly bounded for all k, so is the left hand side. Therefore, the sequence $\{u_k \eta\}_{k=1}^\infty$ is bounded in $W^{1,2}(B_R)$. Since $u_k \eta \in W_0^{1,2}(B_R)$, we obtain by the compact embedding theorem that this sequence has a convergent subsequence in $L^2(B_R)$. Since $\eta = 1$ on $B_{R/2}$, it follows that $\{u_k\}$ has a convergence subsequence in $L^2\left(B_{R/2}\right)$.

Covering Ω by a countable family of the balls and using the diagonal process, we conclude that $\{u_k\}$ has a subsequence that converges in $L^2_{loc}(\Omega)$ to some function u. By Theorem 5.6 we conclude that u is L-harmonic and the convergence is locally uniform.
■

5.4.2 Liouville theorem

Theorem 5.9 *If u is a non-negative L-harmonic function in $\mathbb{R}^n$ then $u \equiv \mathrm{const}$.*

Proof. By subtracting from u the constant $\inf_{\mathbb{R}^n} u$, we can assume without loss of generality that $\inf_{\mathbb{R}^n} u = 0$. We can apply the Harnack inequality to u in any ball B_R because u is L-harmonic and non-negative in B_{2R} for any $R > 0$. Hence, we obtain

$$\sup_{B_R} u \leq C \inf_{B_R} u,$$

where C does not depend on R. Letting $R \to \infty$, we see that the right hand side goes to 0. Hence, the left hand side also goes to 0, and we conclude that $u \equiv 0$. ∎

5.4.3 Green function

We state the next theorem without proof.

Theorem 5.10 *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists a function $G(x, y)$ on $\Omega \times \Omega$ with the following properties:*

1. *$G(x, y)$ is jointly continuous in $(x, y) \in \Omega \times \Omega \setminus \mathrm{diag}$.*

2. *$G(x, y) \geq 0$.*

3. *$G(x, y) = G(y, x)$.*

4. *For any function $f \in L^2(\mathbb{R}^n)$, the following function*

$$u(x) = \int_\Omega G(x, y) f(y)\, dy$$

is a weak solution of the Dirichlet problem

$$\begin{cases} Lu = -f \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

5. *Assume $n > 2$. Then, for any compact set $K \subset \Omega$, there are positive constants $c_1, c_2 > 0$ such that*

$$c_1 |x - y|^{2-n} \leq G(x, y) \leq c_2 |x - y|^{2-n} \tag{5.18}$$

for all $x, y \in K$.

This theorem was proved by Walter Littman, Guido Stampacchia, and Hans Weinberger in 1963. The Harnack inequality of Theorem 5.1 was used to prove the estimate (5.18).

5.4.4 Boundary regularity

Let Ω be a bounded domain in $\mathbb{R}^n$ and consider the following Dirichlet problem in Ω:

$$\begin{cases} Lu = 0 \text{ in } \Omega \\ u - g \in W_0^{1,2}(\Omega) \end{cases} \tag{5.19}$$

where $g \in C^1\left(\overline{\Omega}\right)$ is a given function.

Definition. We say that a point $z \in \partial\Omega$ is *regular* for (5.19) if, for any $g \in C^1\left(\overline{\Omega}\right)$, the (continuous version of the) solution u of (5.19) satisfies

$$\lim_{\substack{x \to z \\ x \in \Omega}} u(x) = g(z). \tag{5.20}$$

Fix a point z on the boundary $\partial\Omega$ and, for any integer $k \geq 1$, consider the following sets:

$$E_k(z) = B_{2^{-k}}(z) \cap \Omega^c.$$

Theorem 5.11 *Assume $n > 2$. Then a point $z \in \partial\Omega$ is regular for* (5.19) *if and only if*

$$\sum_{k=1}^{\infty} 2^{k(n-2)} \operatorname{cap}(E_k(z)) = \infty. \tag{5.21}$$

This theorem was proved by W.Littman, G.Stampacchia, and H.F.Weinberger in 1963 using their estimate (5.18) of the Green function. For the case $L = \Delta$, Theorem 5.11 was first proved by Norbert Wiener in 1924. The condition (5.21) for regularity is called *Wiener's criterion.*

One of the consequences of Theorem 5.11 is that the notion of regularity of $z \in \partial\Omega$ does not depend on the choice of the operator L as long as it in the divergence form and uniformly elliptic.

Chapter 6

* Equations in non-divergence form

6.1 Strong and classical solutions

Consider in a domain $\Omega \subset \mathbb{R}^n$ a non-divergence form operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij} u$$

with measurable coefficients $a_{ij} \in C^\infty(\Omega)$. Assume that L is uniformly elliptic with the ellipticity constant λ. Given a function $f \in L^p_{loc}(\Omega)$, where $p \geq 1$, we say that u is a *strong* solution of $Lu = f$ in Ω if $u \in W^{2,p}_{loc}(\Omega)$ and the equation

$$\sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij} u(x) = f(x) \tag{6.1}$$

is satisfied for almost all $x \in \Omega$. Here $\partial_{ij} u$ is the weak derivative of u that obviously belongs to $L^p_{loc}(\Omega)$. Here we consider only strong solutions of the class $W^{2,n}_{loc}$, that is, $p = n$. By the Sobolev embedding theorem, we have

$$W^{2,n}_{loc}(\Omega) \hookrightarrow C(\Omega),$$

so that all strong solutions are continuous functions.

Assume now that the coefficients a_{ij} are continuous in Ω. Given a function $f \in C(\Omega)$, we say that u is a *classical* solution of $Lu = f$ in Ω if $u \in C^2(\Omega)$ and the equation (6.1) is satisfied for all $x \in \Omega$. Of course, any classical solution is also strong.

If u is a solution of $Lu = 0$ (either strong or classical) then we refer to u as an *L-harmonic function.*

6.2 Theorem of Krylov-Safonov

The main results of this Chapter are stated in the next two theorems that were proved by Nikolai Krylov and Michail Safonov in 1980 based on the previous work of Eugene Landis.

Theorem 6.1 (Estimate of the Hölder norm) *If u is an L-harmonic function in Ω then $u \in C^{\alpha}(\Omega)$ with some $\alpha = \alpha(n, \lambda) > 0$. Moreover, for any compact set $K \subset \Omega$,*

$$\|u\|_{C^{\alpha}(K)} \leq C \|u\|_{C(\Omega)}, \tag{6.2}$$

where $C = C(n, \lambda, \operatorname{dist}(K, \partial\Omega))$.

Of course, if u is a classical solution then $u \in C^2(\Omega)$ and, hence, $u \in C^{\alpha}(\Omega)$ with any $\alpha < 1$. However, even in this case the estimate (6.2) of the Hölder norm is highly non-trivial, because α and C do not depend on a particular solution u.

Theorem 6.2 (The Harnack inequality) *If u is a non-negative L-harmonic function in a ball $B_{2R} \subset \Omega$ then*

$$\sup_{B_R} u \leq C \inf_{B_R} u$$

where $C = C(n, \lambda)$.

In this Chapter we will prove restricted versions of Theorems 6.1 and 6.2 assuming that $a_{ij} \in C^{\infty}(\Omega)$ and that the L-harmonic functions are classical solutions of $Lu = 0$. Passage from C^{∞} coefficients to the general case can be done by using approximation techniques that we do not consider here.

6.3 Weak Harnack inequality

From now on we assume that $a_{ij} \in C^{\infty}(\Omega)$ and that any L-harmonic function u is classical, that is, belongs to $C^2(\Omega)$. In fact, by Corollary 2.11, we have $u \in C^{\infty}(\Omega)$.

As in the case of the divergence form operator, we will concentrate on the proof of the weak Harnack inequality for L-harmonic functions. Then both Theorems 6.1 and 6.2 follow in the same way as for the divergence form case. Hence, our main goal is the following theorem.

Theorem 6.3 (Weak Harnack inequality for non-divergence form operator) *Let u be a non-negative L-harmonic function in a ball $B_{4R} \subset \Omega$. Choose any $a > 0$ and consider the set*

$$E = \{u \geq a\} \cap B_R.$$

If, for some $\theta > 0$,

$$|E| \geq \theta |B_R|,$$

then

$$\inf_{B_R} u \geq \delta a,$$

where $\delta = \delta(n, \lambda, \theta) > 0$.

We present here the proof devised by E.Landis shortly after Krylov and Safonov announced the proofs of Theorems 6.1 and 6.2. This proof has advantage that it is in many ways similar to the proof in the divergence form case.

However, there is a crucial distinction between the two cases. In the present case of a non-divergence form operator, the proof uses a highly non-trivial theorem of Alexandrov-Pucci that we state below and that provides an estimate of solution of the corresponding Dirichlet problem. We precede it by the statement of the existence result that we also need.

Theorem 6.4 *Let $B_R \subset \Omega$ and $f \in C^\infty\left(\overline{B}_R\right)$. Then the classical Dirichlet problem*

$$\begin{cases} Lu = f \text{ in } B_R \\ u = 0 \text{ on } \partial B_R \end{cases} \tag{6.3}$$

has a solution $u \in C^2(B_R) \cap C\left(\overline{B}_R\right)$.

Approach to the proof. Rewrite the operator L in the form

$$\begin{aligned} Lu &= \sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right) - \sum_{j=1}^n \left(\sum_{i=1}^n \partial_i a_{ij}\right)\partial_j u \\ &= \sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right) + \sum_{j=1}^n b_j \partial_j u, \end{aligned}$$

where

$$b_j = \sum_{i=1}^n \partial_i a_{ij}.$$

Then we need the classical solvability of the Dirichlet problem for the divergence form operator with lower order terms and with smooth coefficients.

Since L has now a divergence form, we can consider first the weak Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } B_R, \\ u \in W_0^{1,2}(B_R). \end{cases}$$

By Theorem 2.12, this problem has a solution $u \in C^\infty(B_R)$, that is hence a classical solution of $Lu = f$.

We need still to ensure the boundary condition $u = 0$ in the classical sense. For the operators without lower order terms b_j the corresponding result is contained in Theorem 4.6. With the terms b_j one basically has to repeat all the theory of Hölder regularity (both interior and up to the boundary) and then to arrive to a version of Theorem 4.6 for the operator with lower order terms. We skip this part. ■

Theorem 6.5 (Theorem of Alexandrov – Pucci) *If u is a classical solution of the Dirichlet problem* (6.3) *with $f \in C\left(\overline{\Omega}\right)$ then the following estimate is true:*

$$\sup_{B_R} |u| \le CR \|f\|_{L^n(B_R)},$$

where $C = C(n, \lambda)$.

The proof of this theorem will be given later. In the next section we prove three lemmas needed for the proof of the weak Harnack inequality.

6.4 Some lemmas

Lemma 6.6 *Let u be an L-harmonic function in Ω and assume that $u \geq 0$ in a ball $B_{4R}(z) \subset \Omega$. Choose any $a > 0$ and consider the set*

$$E = \{u \geq a\} \cap B_R(z).$$

If the set E contains a ball $B_r(y)$ then

$$\inf_{B_R(z)} u \geq c \left(\frac{r}{R}\right)^s a,$$

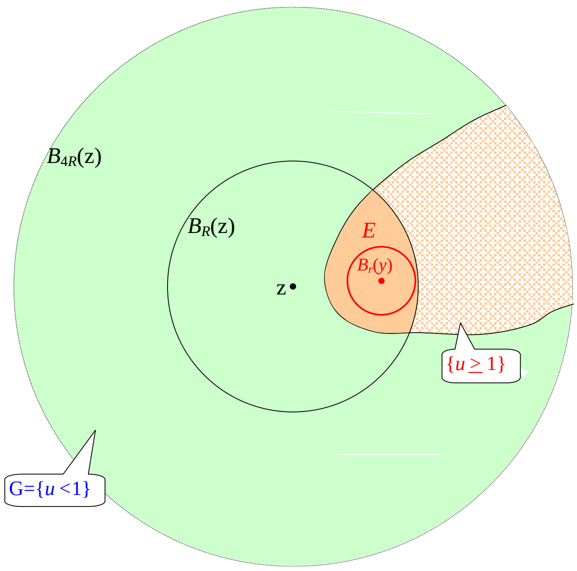
where $s = s(n, \lambda) > 0$ and $c = c(n, \lambda) > 0$.

Proof. Without loss of generality, we can take $a = 1$, so that

$$E = \{u \geq 1\} \cap B_R(z).$$

Assume also for simplicity that y is the origin of $\mathbb{R}^n$. Consider the set

$$G = \{u < 1\} \cap B_{4R}(z).$$



Fix some $s > 0$ to be chosen later, and consider the following function

$$w(x) = \left(\frac{1}{|x|^s} - \frac{1}{(3R)^s}\right) r^s$$

Since the origin is at y, outside the ball $B_r(y)$ we have $|x| \geq r$, whence

$$w(x) \leq 1 \text{ outside } B_r(y).$$

Since by hypotheses $B_r(y) \subset E$ and hence $B_r(y) \cap G = \emptyset$, it follows that

$$w(x) \leq 1 \text{ on } \overline{G}.$$

Since on $\partial B_{4R}(z)$ we have $|x| \geq 3R$, it follows that

$$w(x) \leq 0 \text{ on } \partial B_{4R}(z).$$

Recall that by Exercise 5 we have in $\mathbb{R}^n \setminus \{0\}$

$$L|x|^{-s} > 0$$

provided $s > n\lambda^2 - 2$. Choose one of such values of s, for example, $s = n\lambda^2$. Since $G \subset \mathbb{R}^n \setminus \{0\}$, we obtain

$$Lw > 0 \text{ in } G.$$

As we have seen above, the values of w on ∂G are as follows:

$$\begin{array}{lll} w & \leq & 1 \text{ on } \partial G \cap B_{4R}(z) \\ w & \leq & 0 \text{ on } \partial G \cap \partial B_{4R}(z). \end{array}$$

Let us compare w with u in G. The function u satisfies

$$Lu = 0 \text{ in } G$$

and the boundary conditions:

$$\begin{array}{lll} u & \geq & 1 \text{ on } \partial G \cap B_{4R}(z), \\ u & \geq & 0 \text{ on } \partial G \cap \partial B_{4r}(z). \end{array}$$

Using the comparison principle of Exercise 2, we conclude that

$$u \geq w \text{ in } G.$$

It follows that

$$\inf_{B_R(z)} u = \inf_{B_R(z) \cap G} u \geq \inf_{B_R(z) \cap G} w \geq \inf_{B_R(z)} w.$$

Since in $B_R(z)$ we have $|x| \leq 2R$, it follows that in $B_R(z)$

$$w(x) \geq \left(\frac{1}{(2R)^s} - \frac{1}{(3R)^s} \right) r^s = c \left(\frac{r}{R} \right)^s,$$

where $c = 2^{-s} - 3^{-s} > 0$. We conclude that

$$\inf_{B_R(z)} u \geq c \left(\frac{r}{R} \right)^s,$$

which was to be proved. ■

Lemma 6.7 (Lemma of growth in a thin domain) *Let u be a non-negative L-harmonic function in a ball $B_R \subset \Omega$. There exists $\varepsilon = \varepsilon(n, \lambda) > 0$ with the following property: if for some $a > 0$*

$$\frac{|\{u < a\} \cap B_R|}{|B_R|} \leq \varepsilon,$$

then

$$\inf_{B_{R/4}} u \geq \frac{1}{2} a.$$

Restating this lemma in terms of the function $v = a - u$ with $a = \sup_{B_R} u$ yields the following: if v is L-harmonic in B_R and

$$\frac{|\{v > 0\} \cap B_R|}{|B_R|} \leq \varepsilon$$

then

$$\sup_{B_R} u \geq 2 \sup_{B_R/4} u.$$

This formulation matches that of Lemma 5.4 for the divergence form operators (except for the value 2 instead of 4, which is unimportant).

Proof. Assume that the ball B_R is centered at the origin. Without loss of generality set $a = 1$, and consider the set

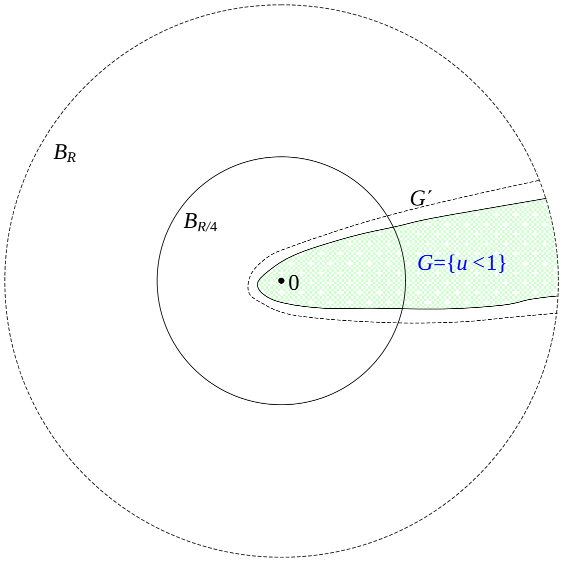
$$G = \{u < 1\} \cap B_R.$$

Since $|G| < \varepsilon |B_R|$, there exists an open set G' in B_R such that

$$\overline{G} \cap B_R \subset G'$$

and

$$|G'| < 2\varepsilon |B_R| \tag{6.4}$$



Choose a function $f \in C^\infty\left(\overline{B}_R\right)$ such that

$$0 \leq f \leq 1, \quad f = 1 \text{ on } G, \quad f = 0 \text{ outside } G'.$$

By Theorem 6.4, the following Dirichlet problem

$$\begin{cases} Lv = -f \ \text{ in } B_R \\ v = 0 \text{ on } \partial B_R \end{cases}$$

has a classical solution $v \in C^2(B_R) \cap C\left(\overline{B}_R\right)$. Since $Lv \leq 0$, it follows by the minimum principle that $v \geq 0$ in B_R. By Theorem 6.5 of Alexandrov and Pucci,

$$\sup_{B_R} v \leq CR \|f\|_{L^n(B_R)} \leq CR |G'|^{1/n} \leq C' R^2 \varepsilon^{1/n}, \tag{6.5}$$

where we have also used (6.4). Consider now the function

$$w(x) = c_1 - c_2 |x|^2 - c_3 v(x)$$

where c_1, c_2, c_3 are positive constant to be chosen. We would like w to satisfy the same conditions as in the previous proof:

(i) $Lw \geq 0$ in G

(ii) $w \leq 1$ in $\overline{G}$

(iii) $w \leq 0$ on ∂B_R

We have in G

$$\begin{aligned} Lw &= -c_2 L |x|^2 - c_3 Lv \\ &= -2c_2 \sum_{i=1}^{n} a_{ii}(x) + c_3 f \\ &\geq -2c_2 \lambda n + c_3 f \\ &\geq -2c_2 \lambda n + c_3, \end{aligned}$$

where we have used that $f = 1$ on G. Hence, in order to satisfy (i), the constants c_2 and c_3 should satisfy

$$c_3 \geq 2c_2 \lambda n.$$

In G we have $w(x) \leq c_1$; hence, (ii) is satisfied if

$$c_1 \leq 1.$$

Finally, on ∂B_R we have $|x| = R$ and, hence,

$$w(x) \leq c_1 - c_2 R^2.$$

Hence, to satisfy (iii) we should have

$$c_1 \leq c_2 R^2.$$

Therefore, we choose c_1, c_2, c_3 as follows:

$$\begin{aligned} c_1 &= 1 \\ c_2 &= R^{-2} \\ c_3 &= 2c_2 \lambda n = \frac{2\lambda n}{R^2}. \end{aligned}$$

Comparing w with u as in the previous proof, we obtain again that $u \geq w$ in G. Hence, we have

$$\inf_{B_{R/4}} u = \inf_{B_{R/4} \cap G} u \geq \inf_{B_{R/4} \cap G} w \geq \inf_{B_{R/4}} w.$$

In $B_{R/4}$ we have, using (6.5),

$$\begin{aligned} w(x) &\geq c_1 - c_2 (R/4)^2 - c_3 \sup v \\ &\geq c_1 - c_2 (R/4)^2 - c_3 C' R^2 \varepsilon^{1/n} \\ &= 1 - \frac{1}{16} - 2\lambda n C' \varepsilon^{1/n}. \end{aligned}$$

Choosing ε small enough depending on λ and n, we obtain

$$\inf_{B_{R/4}} w \geq \frac{1}{2},$$

which finishes the proof. ■

Lemma 6.8 *Under conditions of Lemma* 6.7, *if*

$$\frac{\left|\{u < a\} \cap B_{R/4}\right|}{\left|B_{R/4}\right|} \leq \varepsilon$$

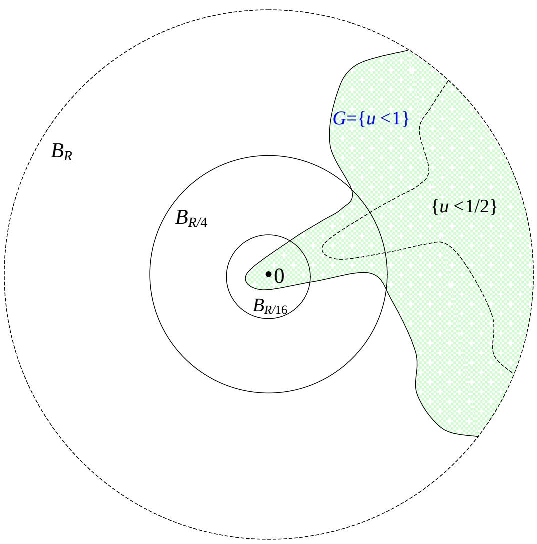
then

$$\inf_{B_{R/4}} u \geq \gamma a,$$

where $\gamma = \gamma(n, \lambda) > 0$.

Proof. Let $a = 1$ and let ε be from Lemma 6.7. Applying Lemma 6.7 to the ball $B_{R/4}$ instead of B_R, we obtain that

$$\inf_{B_{R/16}} u \geq \frac{1}{2}.$$



Hence, the set $\left\{u \geq \frac{1}{2}\right\} \cap B_{R/4}$ contains the ball $B_{R/16}$. Applying Lemma 6.6, we obtain

$$\inf_{B_{R/4}} u \geq c\left(\frac{R/16}{R/4}\right)^s \frac{3}{4} = c4^{-s}\frac{1}{2} =: \gamma,$$

which finishes the proof. ■

6.5 Proof of the weak Harnack inequality

Set without loss of generality $a = 1$. Let u be a non-negative L-harmonic function in a ball $B_{4R} \subset \Omega$. Assuming that the set

$$E = \{u \geq 1\} \cap B_R$$

satisfies the condition

$$|E| \geq \theta \, |B_R| \, ,$$

where $\theta > 0$, we need to prove that

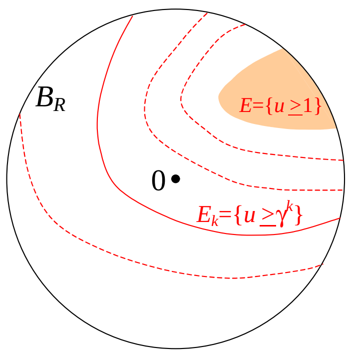
$$\inf_{B_R} u \geq \delta,$$

where $\delta = \delta\,(n, \lambda, \theta) > 0$.

Consider for any non-negative integer k the set

$$E_k = \left\{u \geq \gamma^k\right\} \cap B_R,$$

where $\gamma \in (0, 1)$ is the constant from Lemma 6.8.



The main part of the proof is contained in the following claim.

Claim. *There exist $\beta = \beta\,(n, \lambda) > 0$ and a positive integer $l = l\,(n, \lambda, \theta)$ such that, for any $k \geq 0$ the following dichotomy holds:*

(i) *either*

$$|E_{k+1}| \geq (1 + \beta)\, |E_k|$$

(ii) *or*

$$E_{k+l} = B_R.$$

Let us first show how this Claim allows to finish the proof. Since the function u in B_R is bounded, the case (1) cannot holds for all k. Let N be the minimal value of k such that (i) does not holds for $k = N$. In other words, (i) holds for $k = 0, ..., N - 1$ but does not holds for $k = N$. Hence, (ii) holds for $k = N$.

It follows that

$$|E_N| \geq (1+\beta) |E_{N-1}| \geq ... \geq (1+\beta)^N |E_0| .$$

Since $|E_N| \leq |B_R|$ and $|E_0| = |E| \geq \theta |B_R|$, it follows that

$$(1+\beta)^N \leq \frac{1}{\theta}$$

whence

$$N \leq \frac{\ln \frac{1}{\theta}}{\ln (1+\beta)}.$$

On the other hand, applying (ii) for $k = N$, we obtain

$$E_{N+l} = B_R$$

that is,

$$\inf_{B_R} u = \inf_{E_{N+l}} u \geq \gamma^{N+l} \geq \gamma^{\frac{\ln \frac{1}{\theta}}{\ln(1+\beta)}+l} =: \delta,$$

which finished the proof of the weak Harnack inequality.

Now let us prove the above Claim. It suffices to prove it for the special case $k = 0$, that is,

(i) either $|E_1| \geq (1+\beta) |E_0|$

(ii) or $E_l = B_R$.

Indeed, if it is proved for $k = 0$, then for a general k consider the function $v = u/\gamma^k$. Consider the sets

$$\widetilde{E}_j = \left\{v \geq \gamma^j\right\} \cap B_R$$

where j is a non-negative integer. Clearly, we have

$$E_{k+j} = \left\{u \geq \gamma^{k+j}\right\} \cap B_R = \left\{v \geq \gamma^j\right\} \cap B_R = \widetilde{E}_j.$$

In particular, $E_k = \widetilde{E}_0$ and $E_{k+1} = \widetilde{E}_1$. Hence, applying the special case of the Claim to function v, we obtain the general case of the Claim for function u.

Hence, let us prove the above special case $k = 0$. Let us reformulate it in the following equivalent way:

(i) either $|E_1| \geq (1+\beta) |E_0|$

(ii) or $\inf_{B_R} u \geq \delta$, where $\delta = \delta(n, \lambda, \theta) > 0$.

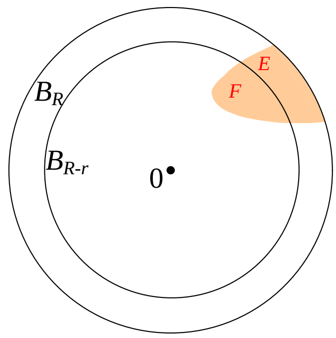
Indeed, if the latter condition holds then we find l such that $\gamma^l \leq \delta$, and obtain $E_l = B_R$.

Choose $r < R$ such that

$$|E \cap B_{R-r}| = \frac{1}{2} |E| \tag{6.6}$$

and set

$$F := E \cap B_{R-r} = \{u \geq 1\} \cap B_{R-r}.$$

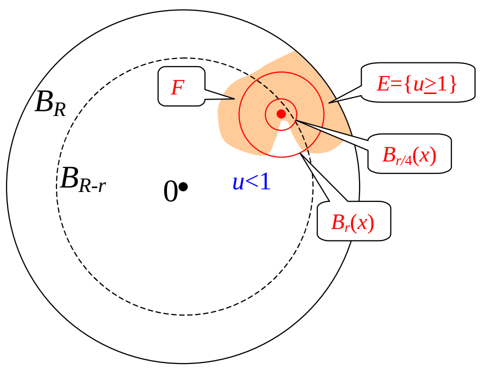


Consider two cases.

Case 1. Assume that there exists $x \in F$ such that

$$\frac{|\{u<1\} \cap B_r(x)|}{|B_r|} \leq \varepsilon,$$

where $\varepsilon = \varepsilon(n, \lambda) > 0$ is the constant from Lemma 6.7.



Then by Lemma 6.7 we have

$$\inf_{B_{r/4}(x)} u \geq \frac{1}{2}.$$

Note that $B_{r/4}(x) \subset B_R$. Hence, in B_R there is a ball $B_{r/4}(x)$ where $u \geq \frac{1}{2}$. Applying Lemma 6.6, we conclude that

$$\inf_{B_R} u \geq c \left(\frac{r/4}{R}\right)^s \frac{1}{2}.$$

From (6.6) we have

$$|B_R| - |B_{R-r}| = |B_R \setminus B_{R-r}| \geq |E \setminus B_{R-r}| = \frac{1}{2}|E| \geq \frac{1}{2}\theta |B_R|$$

which implies after division by B_R that

$$1-\left(\frac{R-r}{R}\right)^n \geq \frac{1}{2}\theta.$$

It follows that

$$\frac{r}{R} \geq 1-\left(1-\frac{1}{2}\theta\right)^{1/n}.$$

Hence, we obtain

$$\inf_{B_R} u \geq \frac{c}{2}4^{-s}\left(1-\left(1-\frac{1}{2}\theta\right)^{1/n}\right)^s =: \delta > 0,$$

which means that the alternative (ii) takes places.

Case 2 (main). Assume that, for any $x \in F$, we have

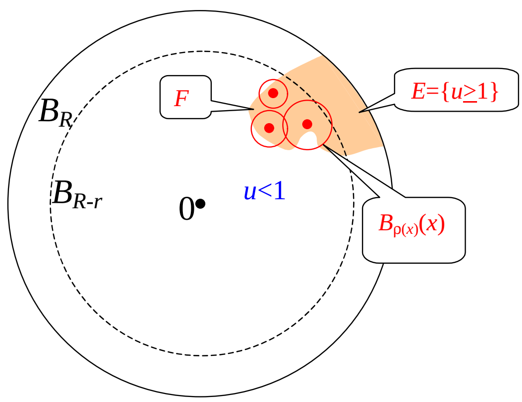
$$\frac{|\{u<1\} \cap B_r(x)|}{|B_r|} \geq \varepsilon.$$

For any $x \in F$ and $\rho > 0$ consider the quotient:

$$\frac{|\{u<1\} \cap B_\rho(x)|}{|B_\rho|}.$$

As $\rho \to 0$, this quotient goes to 0 for almost all $x \in F$ because in F we have $u \geq 1$. On the other hand, for $\rho = r$, this quotient is $\geq \varepsilon$. Hence, for almost all $x \in F$, there exists $\rho(x) \in (0, r)$ such that

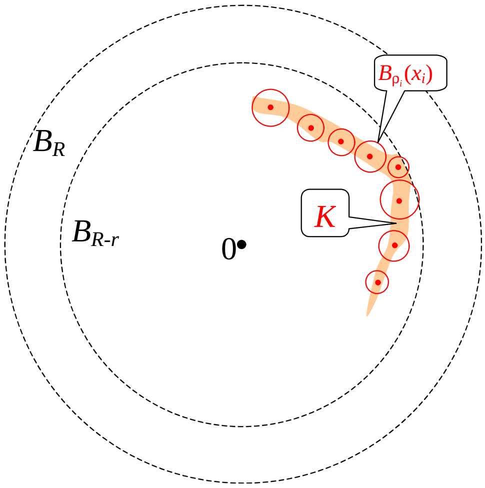
$$\frac{\left|\{u<1\} \cap B_{\rho(x)}(x)\right|}{\left|B_{\rho(x)}\right|} = \varepsilon. \tag{6.7}$$



Denote this set of points x by F', so that $F' \subset F$ and $|F'| = |F|$. By the property of the Lebesgue measure, there is a compact set $K \subset F'$ such that

$$|K| \geq \frac{1}{2}|F'| = \frac{1}{2}|F| = \frac{1}{4}|E|.$$

The family of ball $\left\{B_{\rho(x)}\left(x\right)\right\}_{x\in K}$ forms an open covering of K. Choose a finite subcover $\left\{B_{\rho_i}\left(x_i\right)\right\}$ where $\rho_i=\rho\left(x_i\right)$. By the standard ball covering argument, we can pass to a subsequence and, hence, assume that the balls $\left\{B_{\rho_i}\left(x_i\right)\right\}$ are disjoint while $\left\{B_{3\rho_i}\left(x_i\right)\right\}$ cover K.



Observe that $x_i\in B_{R-r}$, whence

$$|x_i|+4\rho_i\leq R-r+4\rho_i\leq R+3r\leq R+3R=4R.$$

Therefore, $B_{4\rho_i}\left(x_i\right)\subset B_{4R}$. We can apply in $B_{4\rho_i}\left(x_i\right)$ Lemma 6.8 because by (6.7)

$$\frac{\left|\{u<1\}\cap B_{\rho_i}\left(x_i\right)\right|}{\left|B_{\rho_i}\right|}=\varepsilon, \tag{6.8}$$

which yields

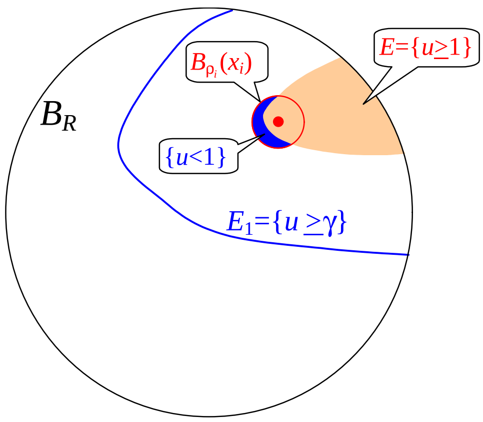
$$\inf_{B_{\rho_i}(x_i)}u\geq\gamma. \tag{6.9}$$

By construction, all balls $B_{\rho_i}\left(x_i\right)$ are contained B_R, which implies by (6.9) that

$$(E_1\setminus E)\cap B_{\rho_i(x_i)}=\{\gamma\leq u<1\}\cap B_{\rho_i(x_i)}=\{u<1\}\cap B_{\rho_i}\left(x_i\right).$$

Combining with (6.8), we obtain

$$\left|(E_1\setminus E)\cap B_{\rho_i(x_i)}\right|=\varepsilon\left|B_{\rho_i}\left(x_i\right)\right|.$$



Adding up in i and using that all balls $B_{\rho_i}(x_i)$ are disjoint, we obtain

$$\begin{aligned} |E_1 \setminus E| &\geq \sum_i \varepsilon \left|B_{\rho_i}(x_i)\right| \\ &= 3^{-n} \sum_i \varepsilon \left|B_{3\rho_i}(x_i)\right| \\ &\geq 3^{-n} \varepsilon |K| \geq 3^{-n} \frac{\varepsilon}{4} |E|, \end{aligned}$$

whence

$$|E_1| \geq \left(1 + 3^{-n} \frac{\varepsilon}{4}\right) |E|,$$

thus proving the alternative (i) with $\beta = 3^{-n} \frac{\varepsilon}{4}$.