ON NONNEGATIVE SOLUTIONS OF THE INEQUALITY $\Delta u + u^\sigma \leq 0$ ON RIEMANNIAN MANIFOLDS

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Abstract. We study the uniqueness of a non-negative solution of the differential inequality

$$\Delta u + u^\sigma \leq 0 \tag{*}$$

on a complete Riemannian manifold, where $\sigma > 1$ is a parameter. We prove that if, for some $x_0 \in M$ and all large enough r,

$$\operatorname{vol} B(x_0, r) \leq C r^p \ln^q r,$$

where $p = \frac{2\sigma}{\sigma - 1}$, $q = \frac{1}{\sigma - 1}$ and $B(x, r)$ is a geodesic ball, then the only non-negative solution of $(*)$ is identical zero. We also show the sharpness of the above values of the exponents p, q.

1. Introduction

In this paper we are concerned with non-negative solutions of the differential inequality

$$\Delta u + u^\sigma \leq 0, \tag{1.1}$$

on a geodesically complete connected Riemannian manifold M, where Δ is Laplace-Beltrami operator on M, and $\sigma > 1$ is a given parameter. Clearly, (1.1) has always a trivial solution $u \equiv 0$. In $\mathbb{R}^n$ with $n \leq 2$ any non-negative solution of (1.1) is identical zero, that is, a non-negative solution is unique. It is well known that in $\mathbb{R}^n$ with $n > 2$ the uniqueness of a non-negative solution of (1.1) takes places if and only if $\sigma \leq \frac{n}{n-2}$ (cf. [6]).

A number of generalizations of this result to more general differential equations and inequalities in $\mathbb{R}^n$ has been obtained in a series of work of Mitidieri and Pohozaev [12, 13, 14] and more recently by Caristi and Mitidieri [4], [5]. These works are based on a method originating from [15] (see also [16]) that uses carefully chosen test functions for (1.1). However, when one tries to employ this method on a manifold M, one encounters the necessity to estimate the Laplacian of the distance function, which is only possible under certain curvature assumptions on M.

Inspired by [11], the first author and V. A. Kondratiev developed in [10] a variation of this method, that uses only the gradient of the distance function and volume of geodesic balls and, hence, is free from curvature assumptions. Fix some $\sigma > 1$ in (1.1) and set

$$p = \frac{2\sigma}{\sigma - 1}, \quad q = \frac{1}{\sigma - 1}. \tag{1.2}$$

Let $B(x, r)$ be the geodesic ball on M of radius r centered at x. It was proved in [10, Theorem 1.3] that if, for some $x_0 \in M$, $C > 0$, $\varepsilon > 0$ and all large enough r,

$$\mu(B(x_0, r)) \leq C r^p \ln^{q-\varepsilon} r, \tag{1.3}$$

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then the only non-negative solution to (1.1) on M is zero. The sharpness of the exponent p here is clear from the example of $\mathbb{R}^n$ where (1.3) holds with $p = n$ that by (1.2) corresponds to the critical value $\sigma = \frac{n}{n-2}$. The question of the sharpness of the exponent of $\ln r$ remained so far unresolved.

In this paper we show that in the critical case $\varepsilon = 0$ the uniqueness of non-negative solution of (1.1) holds as well. We also show that if $\varepsilon < 0$ then under the condition (1.3) there may be a positive solution of (1.1).

Solutions of (1.1) are understood in a weak sense. Denote by $W^1_{loc}(M)$ the space of functions $f \in L^2_{loc}(M)$ whose weak gradient ∇f is also in $L^2_{loc}(M)$. Denote by $W^1_c(M)$ the subspace of $W^1_{loc}(M)$ of functions with compact support.

Definition. A function u on M is called a *weak solution* of the inequality (1.1) if u is a non-negative function from $W^1_{loc}(M)$, and, for any non-negative function $\psi \in W^1_c(M)$, the following inequality holds:

$$-\int_M (\nabla u, \nabla \psi) d\mu + \int_M u^\sigma \psi d\mu \leq 0, \tag{1.4}$$

where $(\cdot, \cdot)$ is the inner product in $T_x M$ given by Riemannian metric.

Remark. Note that the first integral in (1.4) is finite by the compactness of $\operatorname{supp} \psi$. Therefore, the second integral in (1.4) is also finite, and hence, $u \in L^\sigma_{loc}$.

Our main result is the following theorem.

Theorem 1.1. *Let M be a connected geodesically complete Riemannian manifold. Assume that, for some $x_0 \in M$, $C > 0$, the following inequality*

$$\mu(B(x_0, r)) \leq C r^p \ln^q r, \tag{1.5}$$

holds for all large enough r, where p and q are defined by (1.2). Then any non-negative weak solution of (1.1) is identically equal to zero.

Theorem 1.1 is proved in Section 2. The main tool in the proof is a two-parameters family of carefully chosen test functions for (1.4), allowing to estimate the L^σ-norm of a solution u.

In Section 3 we give an example showing the sharpness of the exponents p and q. More precisely, if either $p > \frac{2\sigma}{\sigma-1}$ or $p = \frac{2\sigma}{\sigma-1}$ and $q > \frac{1}{\sigma-1}$ then there is a manifold satisfying (1.5) where the inequality (1.1) has a positive solution.

Note that if

$$\mu(B(x_0, r)) \leq C r^2 \ln r \tag{1.6}$$

for all large r then the manifold M is *parabolic*, that is, any non-negative superharmonic function on M is constant (cf. [3], [8]). For example, $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$. Since any positive solution of (1.1) is a superharmonic function, it follows that, on any parabolic manifold, in particular, under the condition (1.6), any non-negative solution of (1.1) is zero, for any value of σ. Obviously, our Theorem 1.1 is specific to the value of σ, and the value of p is always greater than 2, so that our hypothesis (1.5) is weaker than (1.6).

Notation. The letters $C, C', C_0, C_1, \ldots$ denote positive constants whose values are unimportant and may vary at different occurrences.

2. Proof of the main result

Proof of Theorem 1.1. We divide the proof into three parts. In Part 1, we prove that every non-trivial non-negative solution to (1.1) is in fact positive and, moreover, $\frac{1}{u} \in L^\infty_{loc}(M)$. In Part 2, we obtain the estimates (2.10) and (2.11) involving a test function and positive parameters. In Part 3, we choose in (2.10) and (2.11) specific test functions and parameters, which will allow us to conclude that $\int_M u^\sigma d\mu = 0$ and, hence, to finish the proof.

Part 1. We claim that if u is a non-negative solution to (1.1) and $\operatorname{essinf}_U u = 0$ for some non-empty precompact open set U, then $u \equiv 0$ on M. Let us cover U by a finite family $\{\Omega_j\}$ of charts. Then we must have $\operatorname{essinf}_{U\cap\Omega_j} u = 0$ for at least one value of j. Replacing U by $U \cap \Omega_j$, we can assume that U lies in a chart.

Note that by (1.1) the function u is (weakly) superharmonic function. Applying in U a strong minimum principle for weak supersolutions (cf. [7, Thm. 8.19]), we obtain $u = 0$ a.e. in U.

In order to prove that $u = 0$ a.e. on M, it suffices to show that $u = 0$ a.e. on any precompact open set V that lies in a chart on M. Let us connect U with V by a sequence of precompact open sets $\{U_i\}_{i=0}^n$ such that each U_i lies in a chart and

$$U_0 = U, \quad U_i \cap U_{i+1} \neq \emptyset, \quad U_n = V.$$

By induction, we obtain that $u = 0$ a.e. on U_i for any $i = 0, ..., n$. Indeed, the induction bases has been proved above. If it is already known that $u = 0$ a.e. on U_i then the condition $U_i \cap U_{i+1} \neq \emptyset$ implies that $\operatorname{essinf}_{U_{i+1}} u = 0$ whence as above we obtain $u = 0$ a.e.on U_{i+1}. In particular, $u = 0$ a.e. on V, which was claimed.

Hence, if u is a non-trivial non-negative solution to (1.1) then $\operatorname{essinf}_U u > 0$ for any non-empty precompact open set $U \subset M$. It follows that $\frac{1}{u}$ is essentially bounded on U, whence $\frac{1}{u} \in L^\infty_{loc}(M)$ follows.

In what follows we assume that u is a positive solutions of (1.1) satisfying the condition $\frac{1}{u} \in L^\infty_{loc}(M)$, and show that this assumption leads to contradiction.

Part 2. Fix some non-empty compact set $K \subset M$ and a Lipschitz function φ on M with compact support, such that $0 \le \varphi \le 1$ on M and $\varphi \equiv 1$ in a neighborhood of K. In particular, we have $\varphi \in W^1_c(M)$. We use the following test function for (1.4):

$$\psi(x) = \varphi(x)^s u(x)^{-t}, \tag{2.1}$$

where t, s are parameters that will be chosen to satisfy the conditions

$$0 < t < \min\left(1, \frac{\sigma-1}{2}\right) \quad \text{and} \quad s > \frac{4\sigma}{\sigma-1}. \tag{2.2}$$

In fact, s can be fixed once and for all as in (2.2), while t will be variable and will take all small enough values.

The function ψ has a compact support and is bounded, due to the local boundedness of $\frac{1}{u}$. Since

$$\nabla\psi = -tu^{-t-1}\varphi^s\nabla u + su^{-t}\varphi^{s-1}\nabla\varphi,$$

we see that $\nabla\psi \in L^2(M)$ and, consequently, $\psi \in W^1_c(M)$. We obtain from (1.4) that

$$t\int_M \varphi^s u^{-t-1}|\nabla u|^2 d\mu + \int_M \varphi^s u^{\sigma-t} d\mu \le s\int_M \varphi^{s-1}u^{-t}(\nabla u, \nabla\varphi)d\mu. \tag{2.3}$$

Using Cauchy-Schwarz inequality, let us estimate the right hand side of (2.3) as follows

$$\begin{aligned} s\int_M \varphi^{s-1}u^{-t}(\nabla u, \nabla\varphi)d\mu &= \int_M \left(\sqrt{t}u^{-\frac{t+1}{2}}\varphi^{\frac{s}{2}}\nabla u, \frac{s}{\sqrt{t}}u^{-\frac{t-1}{2}}\varphi^{\frac{s}{2}-1}\nabla\varphi\right)d\mu \\ &\le \frac{t}{2}\int_M u^{-t-1}\varphi^s\left\|\nabla u\right\|^2 d\mu \\ &\quad + \frac{s^2}{2t}\int_M u^{1-t}\varphi^{s-2}\left\|\nabla\varphi\right\|^2 d\mu. \end{aligned}$$

Substituting this inequality into (2.3), and cancelling out the half of the first term in (2.3), we obtain

$$\frac{t}{2}\int_M \varphi^s u^{-t-1}\|\nabla u\|^2 d\mu + \int_M \varphi^s u^{\sigma-t} d\mu \le \frac{s^2}{2t}\int_M u^{1-t}\varphi^{s-2}\|\nabla\varphi\|^2 d\mu. \tag{2.4}$$

Applying the Young inequality in the form

$$\int_M fg d\mu \le \varepsilon\int_M |f|^{p_1} d\mu + C_\varepsilon \int_M |g|^{p_2} d\mu,$$

where $\varepsilon > 0$ is arbitrary and

$$p_1 = \frac{\sigma - t}{1-t}, \quad \text{and} \quad p_2 = \frac{\sigma-t}{\sigma-1}$$

are Hölder conjugate, we estimate the right hand side of (2.4) as follows:

$$\begin{aligned}\frac{s^2}{2t}\int_M u^{1-t}\varphi^{s-2}\|\nabla\varphi\|^2 d\mu &= \int_M [u^{1-t}\varphi^{\frac{s}{p_1}}]\cdot[\tfrac{s^2}{2t}\varphi^{\frac{s}{p_2}-2}\|\nabla\varphi\|^2]d\mu \\ &\le \varepsilon\int_M u^{\sigma-t}\varphi^s d\mu \\ &\quad + C_\varepsilon\left(\frac{s^2}{2t}\right)^{\frac{\sigma-t}{\sigma-1}}\int_M \varphi^{s-2\frac{\sigma-t}{\sigma-1}}\|\nabla\varphi\|^{2\frac{\sigma-t}{\sigma-1}} d\mu. \end{aligned}\tag{2.5}$$

Choose here $\varepsilon = \frac{1}{2}$ and use in the right hand side the obvious inequalities

$$\left(\frac{s^2}{t}\right)^{\frac{\sigma-t}{\sigma-1}} \le \left(\frac{s^2}{t}\right)^{\frac{\sigma}{\sigma-1}} \quad \text{and} \quad \varphi^{s-2\frac{\sigma-t}{\sigma-1}} \le 1.$$

Combining (2.5) with (2.4), we obtain that

$$\frac{t}{2}\int_M \varphi^s u^{-t-1}\|\nabla u\|^2 d\mu + \frac{1}{2}\int_M \varphi^s u^{\sigma-t} d\mu \le Ct^{\frac{\sigma}{1-\sigma}}\int_M \|\nabla\varphi\|^{2\frac{\sigma-t}{\sigma-1}} d\mu, \tag{2.6}$$

where the value of s is absorbed into constant C.

Let us come back to (1.4) and use another test function $\psi = \varphi^s$, which yields

$$\begin{aligned}\int_M \varphi^s u^\sigma d\mu &\le s\int_M \varphi^{s-1}(\nabla u, \nabla\varphi) d\mu \\ &\le s\left(\int_M \varphi^s u^{-t-1}\|\nabla u\|^2 d\mu\right)^{1/2}\left(\int_M \varphi^{s-2}u^{t+1}\|\nabla\varphi\|^2 d\mu\right)^{1/2}. \end{aligned}\tag{2.7}$$

On the other hand, we obtain from (2.6) that

$$\int_M \varphi^s u^{-t-1}\|\nabla u\|^2 d\mu \le Ct^{-1-\frac{\sigma}{\sigma-1}}\int_M \|\nabla\varphi\|^{2\frac{\sigma-t}{\sigma-1}} d\mu.$$

Substituting into (2.7) yields

$$\begin{aligned}\int_M \varphi^s u^\sigma d\mu &\le C\left[t^{-1-\frac{\sigma}{\sigma-1}}\int_M \|\nabla\varphi\|^{2\frac{\sigma-t}{\sigma-1}} d\mu\right]^{1/2} \\ &\quad \times\left[\int_M \varphi^{s-2}u^{t+1}\|\nabla\varphi\|^2 d\mu\right]^{1/2}. \end{aligned}\tag{2.8}$$

Recall that $\varphi \equiv 1$ in a neighborhood of K so that $\nabla\varphi = 0$ on K. Applying Hölder inequality to the last term in (2.8) with the Hölder couple

$$p_3 = \frac{\sigma}{t+1}, \quad p_4 = \frac{\sigma}{\sigma-t-1},$$

we obtain

$$
\begin{aligned}
& \int_M \varphi^{s-2} u^{t+1} \|\nabla\varphi\|^2 \, d\mu \\
= & \int_{M\setminus K} \left(\varphi^{\frac{s}{p_3}} u^{t+1}\right) \left(\varphi^{\frac{s}{p_4}-2} \|\nabla\varphi\|^2\right) d\mu \\
\leq & \left(\int_{M\setminus K} \varphi^s u^\sigma d\mu\right)^{\frac{t+1}{\sigma}} \left(\int_{M\setminus K} \varphi^{s-\frac{2\sigma}{\sigma-t-1}} \|\nabla\varphi\|^{\frac{2\sigma}{\sigma-t-1}} \, d\mu\right)^{\frac{\sigma-t-1}{\sigma}} . \qquad (2.9)
\end{aligned}
$$

By (2.2) we have $s-\frac{2\sigma}{\sigma-t-1}>0$ so that the term $\varphi^{s-\frac{2\sigma}{\sigma-t-1}}$ is bounded by 1. Substituting (2.9) into (2.8), we obtain

$$
\begin{aligned}
\int_M \varphi^s u^\sigma d\mu \leq \; & C_0 t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M \|\nabla\varphi\|^{2\frac{\sigma-t}{\sigma-1}} \, d\mu\right)^{\frac{1}{2}} \\
& \times \left(\int_{M\setminus K} \varphi^s u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}} \left(\int_M \|\nabla\varphi\|^{\frac{2\sigma}{\sigma-t-1}} \, d\mu\right)^{\frac{\sigma-t-1}{2\sigma}} . \qquad (2.10)
\end{aligned}
$$

Since $\int_M \varphi^s u^\sigma d\mu$ is finite due to Remark in Introduction, it follows from (2.10) that

$$
\begin{aligned}
\left(\int_M \varphi^s u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}} \leq \; & C_0 t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M \|\nabla\varphi\|^{2\frac{\sigma-t}{\sigma-1}} \, d\mu\right)^{\frac{1}{2}} \\
& \times \left(\int_M \|\nabla\varphi\|^{\frac{2\sigma}{\sigma-t-1}} \, d\mu\right)^{\frac{\sigma-t-1}{2\sigma}} . \qquad (2.11)
\end{aligned}
$$

Part 3. Set $r(x)=d(x,x_0)$, where x_0 is the point from the hypothesis (1.5). Fix some large $R>1$, set

$$
t=\frac{1}{\ln R}, \qquad K=B_R:=B(x_0,R),
$$

and consider the function

$$
\varphi(x)=\begin{cases} 1, & r(x)<R, \\ \left(\frac{r(x)}{R}\right)^{-t}, & r(x)\geq R. \end{cases} \qquad (2.12)
$$

Note that R will be chosen large enough so that t can be assumed to be sufficiently small, in particular, to satisfy (2.2).

We would like to use (2.11) with this function $\varphi(x)$. However, since $\operatorname{supp}\varphi$ is not compact, we consider instead a sequence $\{\varphi_n\}$ of functions with compact supports that is constructed as follows. For any $n=1,2,...$ define a cut-off function η_n by

$$
\eta_n(x)=\begin{cases} 1, & 0\leq r(x)\leq nR, \\ 2-\frac{r(x)}{nR}, & nR\leq r(x)\leq 2nR, \\ 0, & r(x)\geq 2nR. \end{cases} \qquad (2.13)
$$

Consider the function

$$
\varphi_n(x)=\varphi(x)\eta_n(x),
$$

so that $\varphi_n(x)\uparrow\varphi(x)$ as $n\to\infty$. Notice that

$$
|\nabla\varphi_n|^2 \leq 2\left(\eta_n^2|\nabla\varphi|^2+\varphi^2|\nabla\eta_n|^2\right), \qquad (2.14)
$$

which implies that, for any $a\geq 2$,

$$
|\nabla\varphi_n|^a \leq C_a\left(\eta_n^a|\nabla\varphi|^a+\varphi^a|\nabla\eta_n|^a\right). \qquad (2.15)
$$

We will consider only the values of a of the bounded range $a \leq 2p$ so that the constant C_a can be regarded as uniformly bounded.

Let us estimate the integral

$$I_n(a) := \int_M |\nabla \varphi_n|^a d\mu. \tag{2.16}$$

By (2.15), we have

$$\begin{aligned} I_n(a) &\leq C \int_M \eta_n^a |\nabla \varphi|^a d\mu + C \int_M \varphi^a |\nabla \eta_n|^a d\mu \\ &\leq C \int_{M \setminus B_R} |\nabla \varphi|^a d\mu + C \int_{B_{2nR} \setminus B_{nR}} \varphi^a |\nabla \eta_n|^a d\mu, \end{aligned} \tag{2.17}$$

where we have used that $\nabla \varphi = 0$ in B_R, and $\nabla \eta_n = 0$ outside $B_{2nR} \setminus B_{nR}$. Since $|\nabla \eta_n| \leq \frac{1}{nR}$, the second integral in (2.17) can be estimated as follows

$$\begin{aligned} \int_{B_{2nR} \setminus B_{nR}} \varphi^a |\nabla \eta_n|^a d\mu &\leq \frac{1}{(nR)^a} \int_{B_{2nR} \setminus B_{nR}} \varphi^a d\mu \\ &\leq \frac{1}{(nR)^a} \left(\sup_{B_{2nR} \setminus B_{nR}} \varphi^a \right) \mu(B_{2nR}) \\ &\leq \frac{C}{(nR)^a} \left(\frac{nR}{R} \right)^{-at} (2nR)^p \ln^q (2nR) \\ &= C' n^{p-a-at} R^{p-a} \ln^q (2nR), \end{aligned} \tag{2.18}$$

where we have used the definition (2.12) of the function φ and the volume estimate (1.5).

Before we estimate the first integral in (2.17), observe the following: if f is a non-negative decreasing function on $\mathbb{R}_+$ then, for large enough R,

$$\int_{M \setminus B_R} f(r(x)) \, d\mu(x) \leq C \int_{R/2}^{\infty} f(r) r^{p-1} \ln^q r dr, \tag{2.19}$$

which follows from (1.5) as follows:

$$\begin{aligned} \int_{M \setminus B_R} f d\mu &= \sum_{i=0}^{\infty} \int_{B_{2^{i+1}R} \setminus B_{2^i R}} f d\mu \\ &\leq \sum_{i=0}^{\infty} f(2^i R) \mu(B_{2^{i+1}R}) \\ &\leq C \sum_{i=0}^{\infty} f(2^i R)(2^{i+1} R)^p \ln^q (2^{i+1} R) \\ &\leq C' \sum_{i=0}^{\infty} f(2^i R)(2^{i-1} R)^{p-1} (2^{i-1} R) \ln^q (2^{i-1} R) \\ &\leq C' \int_{R/2}^{\infty} f(r) r^{p-1} \ln^q r dr. \end{aligned}$$

Hence, using $|\nabla\varphi| \le R^t t r^{-t-1}$, (2.19), and $R/2 > 1$, we obtain

$$\begin{aligned}\int_{M\setminus B_R} |\nabla\varphi|^a d\mu &\le C\int_{R/2}^{\infty} R^{at} t^a r^{-at-a} r^{p-1} \ln^q r dr \\ &\le CR^{at}t^a \int_1^\infty r^{-at-a+p}\ln^q r \frac{dr}{r} \\ &= CR^{at}t^a \int_0^\infty e^{-b\xi}\xi^q d\xi,\end{aligned}$$

where we have made the change $\xi = \ln r$ and set

$$b := at + a - p. \tag{2.20}$$

Assuming that $b > 0$ and making one more change $\tau = b\xi$, we obtain

$$\int_{M\setminus B_R} |\nabla\varphi|^a d\mu \le CR^{at}t^a b^{-q-1}\int_0^\infty e^{-\tau}\tau^q d\tau = C'R^{at}t^a b^{-q-1}, \tag{2.21}$$

where the value $\Gamma(q+1)$ of the integral is absorbed into the constant C'.

Substituting (2.18) and (2.21) into (2.17) yields

$$I_n(a) \le CR^{at}t^a b^{-q-1} + Cn^{-b}R^{p-a}\ln^q(2nR). \tag{2.22}$$

We will use (2.22) with those values of a for which $b > t$. Noticing also that $R^t = \exp(t\ln R) = e$, we obtain

$$I_n(a) \le Ce^a t^{a-q-1} + Cn^{-t}R^{p-a}\ln^q(2nR).$$

As we have remarked above, we will consider only the values of a in the bounded range $a \le 2p$. Hence, the term e^a in the above inequality can be replaced by a constant. Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} I_n(a) \le Ct^{a-q-1}. \tag{2.23}$$

Let us first use (2.23) with $a = \frac{2(\sigma-t)}{\sigma-1}$. Note that $a < p$, and for this value of a and for t as in (2.2), we have

$$\begin{aligned} b &= \frac{2(\sigma-t)}{\sigma-1}t + \frac{2(\sigma-t)}{\sigma-1} - \frac{2\sigma}{\sigma-1} \\ &= \frac{2t[(\sigma-1)-t]}{\sigma-1} > t\end{aligned}$$

and

$$a - q - 1 = \frac{2(\sigma-t)}{\sigma-1} - \frac{\sigma}{\sigma-1} = \frac{\sigma-2t}{\sigma-1}.$$

Hence, (2.23) yields

$$\limsup_{n\to\infty} I_n\left(\frac{2(\sigma-t)}{\sigma-1}\right) \le Ct^{\frac{\sigma-2t}{\sigma-1}}. \tag{2.24}$$

Similarly, for $a = \frac{2\sigma}{\sigma-t-1}$, we have by (2.2) $a < 2p$ and

$$b = \frac{2\sigma}{\sigma-t-1}t + \frac{2\sigma}{\sigma-t-1} - \frac{2\sigma}{\sigma-1} > t,$$

whence

$$\limsup_{n\to\infty} I_n\left(\frac{2\sigma}{\sigma-t-1}\right) \le Ct^{\frac{2\sigma}{\sigma-t-1}-\frac{\sigma}{\sigma-1}}. \tag{2.25}$$

The inequality (2.11) with function φ_n implies that

$$\left(\int_M \varphi_n^s u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}} \le J_n(t), \tag{2.26}$$

where

$$J_n(t) = C_0 t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} I_n\left(\frac{2(\sigma-t)}{\sigma-1}\right)^{\frac{1}{2}} I_n\left(\frac{2\sigma}{\sigma-t-1}\right)^{\frac{\sigma-t-1}{2\sigma}}.$$

Letting $n \to \infty$ and substituting the estimates (2.24) and (2.25), we obtain that

$$\limsup_{n\to\infty} J_n(t) \le C_0 t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} t^{\frac{\sigma-2t}{2(\sigma-1)}} t^{1-\frac{\sigma-t-1}{2(\sigma-1)}} = Ct^{-\frac{t}{2(\sigma-1)}}. \tag{2.27}$$

The main point of the above argument is that all the "large" exponents in the power of t have cancelled out, which in the end is a consequence of the estimate (2.21) based on the hypothesis (1.5). The remaining term $t^{-\frac{t}{2(\sigma-1)}}$ tends to 1 as $t \to 0$, which implies that the right hand side of (2.27) is a *bounded* function of t. Hence, there is a constant C_1 such that

$$\limsup_{n\to\infty} J_n(t) \le C_1, \tag{2.28}$$

for all small enough t. It follows from (2.26) that also

$$\int_M \varphi^s u^\sigma d\mu \le C, \tag{2.29}$$

for all small enough t. Since $\varphi = 1$ on B_R, it follows that

$$\int_{B_R} u^\sigma d\mu \le C,$$

which implies for $R \to \infty$ that

$$\int_M u^\sigma d\mu \le C. \tag{2.30}$$

Inequality (2.10) with function φ_n implies that

$$\int_M \varphi_n^s u^\sigma d\mu \le J_n(t)\left(\int_{M\setminus B_R} \varphi_n^s u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}. \tag{2.31}$$

Letting $n \to \infty$ and applying (2.28), we obtain

$$\int_M \varphi^s u^\sigma d\mu \le C_1\left(\int_{M\setminus B_R} \varphi^s u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}},$$

whence

$$\int_{B_R} u^\sigma d\mu \le C_1\left(\int_{M\setminus B_R} u^\sigma d\mu\right)^{\frac{t+1}{2\sigma}}, \tag{2.32}$$

Since by (2.30)

$$\int_{M\setminus B_R} u^\sigma d\mu \to 0 \text{ as } R \to \infty,$$

letting in (2.32) $R \to \infty$, we obtain

$$\int_M u^\sigma d\mu = 0,$$

which finishes the proof. ∎

3. An example

In this section, we will give an example that shows that the values of the parameters p and q in Theorem 1.1 are sharp and cannot be relaxed.

We will need the following statement.

Proposition 3.1. ([1], [10, Prop. 3.2]) *Let $\alpha(r)$ be a positive C^1-function on $(r_0, +\infty)$ satisfying*

$$\int_{r_0}^{\infty} \frac{dr}{\alpha(r)} < \infty. \tag{3.1}$$

Define the function $\gamma(r)$ on (r_0, ∞) by

$$\gamma(r) = \int_{r}^{\infty} \frac{ds}{\alpha(s)}. \tag{3.2}$$

Let $\beta(r)$ be a continuous function on (r_0, ∞) such that

$$\int_{r_0}^{\infty} \gamma(r)^{\sigma} |\beta(r)| dr < \infty. \tag{3.3}$$

Then the differential equation

$$(\alpha(r)y')' + \beta(r)y^{\sigma} = 0, \tag{3.4}$$

has a positive solution $y(r)$ in an interval $[R_0, +\infty)$ for large enough $R_0 > r_0$, such that

$$y(r) \sim \gamma(r) \quad \text{as } r \to \infty. \tag{3.5}$$

Given $\sigma > 1$, set as before $p = \frac{2\sigma}{\sigma-1}$ and choose some $q > \frac{1}{\sigma-1}$. We will construct an example of a manifold M satisfying the volume growth condition (1.5) with these values p, q and admitting a positive solution u of (1.1).

The manifold M will be $(\mathbb{R}^n, g)$ with the following Riemannian metric

$$g = dr^2 + \psi(r)^2 d\theta^2, \tag{3.6}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^n$ and $\psi(r)$ is a smooth, positive, increasing function on $(0, \infty)$ such that

$$\psi(r) = \begin{cases} r, & \text{for small enough } r, \\ \left(r^{p-1} \ln^q r\right)^{\frac{1}{n-1}}, & \text{for large enough } r. \end{cases} \tag{3.7}$$

It follows that, in a neighborhood of 0, the metric g is exactly Euclidean, so that it can be extended smoothly to the origin. Hence, $M = (\mathbb{R}^n, g)$ is a complete Riemannian manifold.

By (3.6), the geodesic ball $B_r = B(0, r)$ on M coincides with the Euclidean ball $\{|x| < r\}$. Denote by $S(r)$ the surface area of B_r in M. It follows from (3.6) that $S(r) = \omega_n \psi^{n-1}(r)$, that is

$$S(r) = \omega_n \begin{cases} r^{n-1}, & \text{for small enough } r, \\ r^{p-1} \ln^q r, & \text{for large enough } r, \end{cases} \tag{3.8}$$

where ω_n is the surface area of the unit ball in $\mathbb{R}^n$. The Riemannian volume of the ball B_r can be determined by

$$\mu(B_r) = \int_0^r S(\tau)\, d\tau,$$

whence it follows that, for large enough r,

$$\mu(B_r) \le C r^p \ln^q r. \tag{3.9}$$

Hence, the manifold M satisfied the volume growth condition of Theorem 1.1.

In what follows we prove the existence of a weak positive solution of $\Delta u + u^\sigma \leq 0$ on M. In fact, the solution u will depend only on the polar radius r, so that we can write $u = u(r)$. The construction of u will be done in two steps.

Step I. For a function $u = u(r)$, the inequality (1.1) becomes

$$u'' + \frac{S'}{S}u' + u^\sigma \leq 0 \tag{3.10}$$

(cf. [9, (3.93)]), that is

$$\left(Su'\right)' + Su^\sigma \leq 0. \tag{3.11}$$

For $r >> 1$, we have

$$\gamma(r) := \int_r^\infty \frac{d\tau}{S(\tau)} = \int_r^\infty \frac{d\tau}{\tau^{p-1}\ln^q \tau} \simeq \frac{1}{r^{p-2}\ln^q r},$$

and

$$\begin{aligned} \int_{r_0}^\infty \gamma(\tau)^\sigma S(\tau) d\tau &= \int_{r_0} \frac{\tau^p \ln^q \tau}{\tau^{\sigma(p-2)}\ln^{\sigma q}\tau}\frac{d\tau}{\tau} \\ &= \int_{r_0}^\infty \frac{1}{\tau^{\sigma(p-2)-p}\ln^{q(\sigma-1)}\tau}\frac{d\tau}{\tau} \\ &= \int_{r_0}^\infty \frac{1}{\ln^{q(\sigma-1)}\tau}\frac{d\tau}{\tau} \\ &< \infty, \end{aligned}$$

where we have used that $q > \frac{1}{\sigma-1}$.

Applying Proposition 3.1 with $\alpha(r) = \beta(r) = S(r)$, we obtain that there exists a positive solution u of (3.11) on $[R_0, +\infty)$ for some large enough R_0, such that

$$u(r) \sim \gamma(r) \simeq r^{-(p-2)}\ln^{-q} r \quad \text{ as } r \to \infty.$$

In particular, $u(r) \to 0$ as $r \to \infty$. By increasing R_0 if necessary, we can assume that $u'(R_0) < 0$.

Step II. Consider the following eigenvalue problem in a ball B_ρ of M:

$$\begin{cases} \Delta v + \lambda v = 0 \text{ in } B_\rho, \\ v|_{\partial B_\rho} = 0. \end{cases} \tag{3.12}$$

Denote by λ_ρ the principal (smallest) eigenvalue of this problem. It is known that $\lambda_\rho > 0$ and the corresponding eigenfunction v_ρ does not change sign in B_ρ (cf. [9, Thms 10.11, 10.22]). Normalizing v_ρ, we can assume that $v_\rho(0) = 1$ and, hence, $v_\rho > 0$ in B_ρ, while $v_\rho|_{\partial B_\rho} = 0$.

Since the principal eigenvalue λ_ρ is simple (cf. [9, Cor. 10.12]) and the Riemannian metric g is spherically symmetric, the eigenfunction v_ρ must also be spherically symmetric. Therefore, v_ρ can be regarded as a function of the polar radius r only. In terms of r, we can rewrite (3.12) as follows

$$v_\rho'' + \frac{S'}{S}v_\rho' + \lambda_\rho v_\rho = 0, \tag{3.13}$$

where $v_\rho(\rho) = 0$, $v_\rho(0) = 1$, $v_\rho'(0) = 0$, and $v_\rho > 0$ in $(0, \rho)$.

Multiplying (3.13) by S, we obtain

$$(Sv_\rho')' + \lambda_\rho S v_\rho = 0.$$

It follows that $(Sv_\rho')' \leq 0$, so that the function Sv_ρ' is decreasing. Since it vanishes at $r = 0$, it follows that $Sv_\rho'(r) \leq 0$ and, hence, $v_\rho'(r) \leq 0$ for all $r \in (0, \rho)$. Hence, the function

$v_\rho(r)$ is decreasing for $r<\rho$ which together with the boundary conditions implies that $0\le v_\rho\le 1$. It follows that v_ρ is a positive solution in B_ρ of the inequality

$$\Delta v_\rho+\lambda_\rho v^\sigma\le 0. \tag{3.14}$$

Let us show that $\lambda_\rho\to 0$ as $\rho\to\infty$. Indeed, it is known that

$$\lim_{\rho\to\infty}\lambda_\rho=\lambda_{\min}(M)$$

where $\lambda_{\min}(M)$ is the bottom of the spectrum of $-\Delta$ in $L^2(M,\mu)$, while by a theorem of Brooks

$$\lambda_{\min}(M)\le\frac14\left(\limsup_{\rho\to\infty}\frac{\ln\mu(B_\rho)}{\rho}\right)^2 \tag{3.15}$$

(cf. [2], [9, Thm 11.19]). The right hand side of (3.15) vanishes by (3.9), where we obtain that $\lim_{\rho\to\infty}\lambda_\rho=0$.

Let us show that there exists a sequence $\{\rho_k\}$ such that $v_{\rho_k}\to 1$, as $k\to\infty$, where the convergence is local in C^1. Indeed, let us first take that $\rho_k=k$. As v_k satisfies the equation $\Delta v_k+\lambda_k v_k=0$, the sequence $\{v_k\}$ is bounded, and $\lambda_k\to 0$, it follows by local elliptic regularity properties that there exists a subsequence $\{v_{k_i}\}$ that converges in C^∞_{loc} to a function v, and the latter satisfies $\Delta v=0$ (cf. [9, Thm 13.14]). The function v depends only on the polar radius and, hence, satisfies the conditions

$$\begin{cases} v''+\dfrac{S'}{S}v'=0,\\ v(0)=1.\end{cases}$$

Solving this ODE, we obtain a general solution

$$v(r)=C\int_0^r\frac{dr}{S(r)}+1.$$

Since $\int_0^r\frac{dr}{S(r)}$ diverges at 0, so the only bounded solution is $v\equiv 1$. We conclude that

$$v_{k_i}\xrightarrow{C^\infty_{loc}}1\quad\text{as } i\to\infty. \tag{3.16}$$

Choose ρ large enough so that $\rho>R_0$ and

$$\frac{v_\rho'}{v_\rho}(R_0)>\frac{u'}{u}(R_0), \tag{3.17}$$

where u is the function constructed in the first step. Indeed, it is possible to achieve (3.17) by choosing $\rho=k_i$ with large enough i because by (3.16)

$$\frac{v_{k_i}'}{v_{k_i}}(R_0)\to 0\ \text{ as } i\to\infty$$

whereas $\frac{u'}{u}(R_0)<0$ by construction.

Let us fix $\rho>R_0$ for which (3.17) is satisfied, and compare the functions $u(r)$ and $v_\rho(r)$ in the interval $[R_0,\rho)$. Set

$$m=\inf_{r\in[R_0,\rho)}\frac{u(r)}{v_\rho(r)}.$$

Since v_ρ vanishes at ρ and, hence,

$$\frac{u(r)}{v_\rho(r)}\to\infty\quad\text{as}\quad r\to\rho+,$$

the ratio $\frac{u}{v_\rho}$ attains its infimum value m at some point $\xi \in [R_0, \rho)$. We claim that $\xi > R_0$. Indeed, at $r = R_0$, we have by (3.17)

$$\left(\frac{u}{v_\rho}\right)'(R_0) = \frac{u'v_\rho - uv_\rho'}{v^2}(R_0) < 0,$$

so that u/v_ρ is strictly decreasing at R_0 and cannot have minimum at R_0. Hence, $\frac{u}{v_\rho}$ attains its minimum at an interior point $\xi \in (R_0, \rho)$, and at this point we have

$$\left(\frac{u}{v_\rho}\right)'(\xi) = 0.$$

It follows that

$$u(\xi) = mv_\rho(\xi) \quad \text{and} \quad u'(\xi) = mv_\rho'(\xi) \tag{3.18}$$

(see Fig. 1)

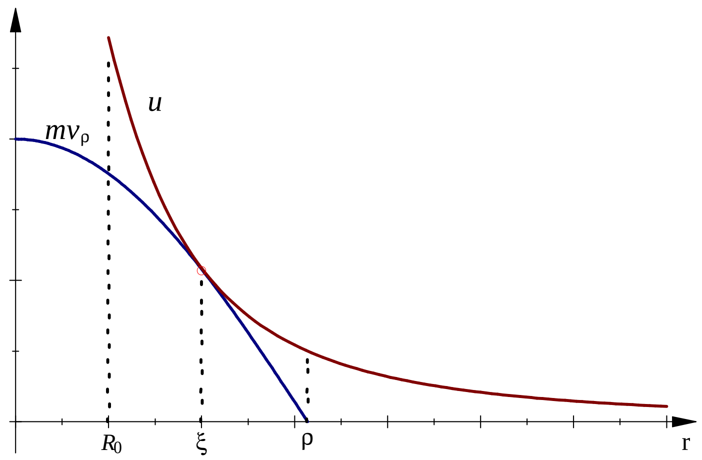

Figure 1. Functions u and mv_ρ

The function $u(r)$ has been defined for $r \geq R_0$, in particular, for $r \geq \xi$, whereas $v_\rho(r)$ has been defined for $r \leq \rho$, in particular, for $r \leq \xi$. Now we merge the two definitions by redefining/extending the function $u(r)$ for all $0 < r < \xi$ by setting $u(r) = mv_\rho(r)$.

It follows from (3.18) that $u \in C^1(M)$, in particular, $u \in W^1_{loc}(M)$. By (3.14), u satisfies the following inequality in B_ξ:

$$\Delta u + \frac{\lambda_\rho}{m^{\sigma-1}} u^\sigma \leq 0. \tag{3.19}$$

By (1.1), u satisfies the following inequality in $M \setminus B_{R_0}$:

$$\Delta u + u^\sigma \leq 0. \tag{3.20}$$

Combining (3.19) and (3.20), we obtain that u satisfies on M the following inequality

$$\Delta u + \delta u^\sigma \leq 0, \tag{3.21}$$

where $\delta = \min\{\lambda_\rho / m^{\sigma-1}, 1\}$. Finally, changing $u \mapsto cu$ where $c = \delta^{-\frac{1}{\sigma-1}}$ we obtain a positive solution to (1.1) on M, which concludes this example.

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