Escape rate of Brownian motion on Riemannian manifolds

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Dedicated to the memory of my teacher E.M.Landis

Contents

1 Introduction

In this note, we are interested in the long time behaviour of the trajectories of the Brownian motion X_t on a Riemannian manifold M. We always assume that M is geodesically complete and non-compact.

The process X_t may be transient or recurrent. *Transience* of the process X_t means that X_t leaves eventually any geodesic ball $B(x,r)$, with probability 1. Otherwise, the process X_t is called *recurrent*. For example, the standard Brownian motion in $\mathbb{R}^n$ is transient if and only if $n > 2$.

One may wonder if the radius r of the ball $B(x,r)$ can be time-dependent. In other words, does there exist an increasing function $r(t)$ such that $X_t \notin B(x,r(t))$, for all t large enough, with probability 1? We call such function $r(t)$ a *lower radius* for X_t. Sphere $\partial B(x,r(t))$ can be regarded as a rear front of the process X_t.

There is a natural counterpart to a lower radius - *an upper radius*. An increasing function $R(t)$ is called an upper radius if, with probability 1, we have $X_t \in B(x,R(t))$ for all t large enough. Sphere $\partial B(x,R(t))$ can be regarded as a forefront of the process X_t.

A sharp estimate of a lower radius in $\mathbb{R}^n$ was obtained by Dvoretzky and Erdös [10]. Namely, a positive increasing function $r(t)$ is a lower radius for Brownian motion in $\mathbb{R}^n$, $n > 2$, if and only if

$$\int^{\infty} \left(\frac{r(t)}{\sqrt{t}}\right)^{n-2} \frac{dt}{t} < \infty, \tag{1.1}$$

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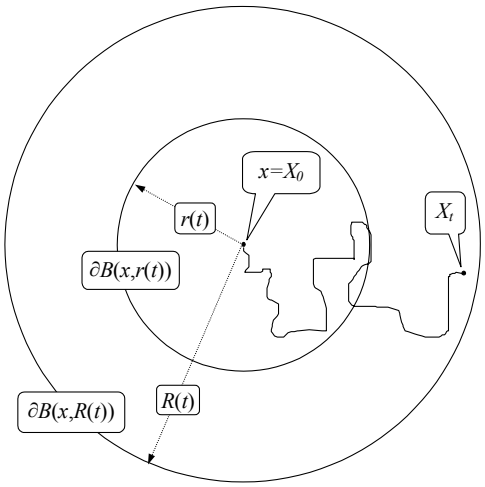

Figure 1: Lower radius $r(t)$ and upper radius $R(t)$

assuming that the function $r(t)/\sqrt{t}$ is decreasing. For example, the function

$$r(t) = \frac{C\sqrt{t}}{\log^{\frac{1+\varepsilon}{n-2}} t}, \quad C > 0, \tag{1.2}$$

is a lower radius if $\varepsilon > 0$ and is not if $\varepsilon \leq 0$. Note that the restriction $n > 2$ is essential, otherwise the process is recurrent, and there is no increasing lower radius.

The celebrated Khinchin's theorem - the law of the iterated logarithm - says that the function

$$R(t) = \sqrt{(2+\varepsilon)t \log\log t} \tag{1.3}$$

is an upper radius in $\mathbb{R}^n$ if $\varepsilon > 0$ and is not if $\varepsilon \leq 0$ (see [21, Section 4.12] and [5] for further results in this direction).

Let us state our result for an upper radius.

Theorem 1.1 (the law of the single logarithm) *Let M be a geodesically complete manifold. Assume that, for some $x_0 \in M$ and all r large enough,*

$$V(x_0, r) \leq \text{const}\, r^N,$$

with some $N > 0$. Then for any $\varepsilon > 0$, the function

$$R(t) = \sqrt{(N+\varepsilon)\, t \log t} \tag{1.4}$$

is an upper radius for the process X_t started at any point $x \in M$.

For the case when manifold M has in addition bounded geometry, Theorem 1.1 was proved in [16, Theorem 1.1]. There is a counterexample [17, Theorem 4], which shows that the function (1.4) cannot be in general improved to $\sqrt{Ct\,(\log t)^{1-\varepsilon}}$, however small is $\varepsilon > 0$. There is also another counterexample [4], for random walks on graphs though, for which even the function $\sqrt{ct \log t}$ is not an upper radius, provided $c > 0$ is small enough. It is very likely that a similar counterexample can be constructed in category of manifolds.

A more general statement, which allows a superpolynomial growth of $V(x_0, r)$ is considered in Section 4 (Theorem 4.1).

If a manifolds satisfies a priori a certain additional hypothesis, which is called *a relative Faber-Krahn inequality* (see Section 2.4 below for a detailed explanation), then the function (1.3) is also an upper

radius (see [16, Theorem 1.3] or Theorem 4.2 below). For example, if the Ricci curvature of the manifold M is non-negative then a relative Faber-Krahn inequality holds and, thus, Theorem 1.1 applies on such a manifold. However, the class of manifolds with a relative Faber-Krahn inequality is much wider than those with non-negative Ricci curvature. For example, if we take k copies of the same manifold satisfying a relative Faber-Krahn inequality, cut out a small hole in each of them and glue them together by a compact, then the resulting manifold satisfies a relative Faber-Krahn inequality again (see [18]).

For a lower radius, we have the following test:

Theorem 1.2 *Let M be geodesically complete, and let us assume that a relative Faber-Krahn inequality holds on M. Assume also that X_t is transient and denote, for a fixed $x_0 \in M$,*

$$\gamma(r) := \left(\int_r^\infty \frac{sds}{V(x_0, s)} \right)^{-1}. \tag{1.5}$$

Let $r(t)$ be an increasing positive function on $(0, \infty)$ such that

$$\int^\infty \frac{\gamma(r(t))}{V(x_0, \sqrt{t})} dt < \infty. \tag{1.6}$$

Then $r(t)$ is a lower radius for the process X_t started at point x_0.

The hypothesis of transience of X_t ensures that the integral (1.5) is convergent. Moreover, given the relative Faber-Krahn inequality, transience is equivalent to the convergence of this integral (see [15] or Section 2 below).

Let us consider some examples of function $V(x_0, r)$ in Theorem 1.2.

Example: Let $V(x, r) \asymp r^\nu$ for all r large enough, for some $\nu > 2$. We obtain from (1.5) $\gamma(r) \asymp r^{\nu-2}$, and (1.6) amounts to

$$\int^\infty \frac{r^{\nu-2}(t)dt}{t^{\nu/2}} < \infty. \tag{1.7}$$

Clearly, the Dvoretzky-Erdös condition (1.1) coincides with (1.7). Thus, Theorem 1.2 recovers the "sufficient" part of the Dvoretzky-Erdös theorem if we take $M = \mathbb{R}^\nu$. Note that we do not require that $r(t)/\sqrt{t}$ is decreasing.

Theorem 1.2 recovers also [16, Theorem 1.2], where a lower radius was constructed assuming a relative Faber–Krahn inequality and a polynomial growth of $V(x, r)$ of the order at least $\nu > 2$ (see Corollary 5.2 in Section 5).

Example: Let $V(x, r) \asymp r^2 \log^\nu r$, for all r large enough, with some $\nu > 1$. We find $\gamma(t) \asymp \log^{\nu-1} t$ for large t, and (1.6) acquires the form

$$\int^\infty \frac{\log^{\nu-1} r(t)}{t \log^\nu t} dt < \infty.$$

For example, we can put

$$r(t) = t^{(\log\log t)^{-\alpha}},$$

for any $\alpha > \frac{1}{\nu-1}$.

The question still remains open whether the lower radius provided by Theorem 1.2 is sharp in terms of the volume function $V(x, r)$ (except for the case $V(r) \asymp r^\nu$ which is covered by the "necessary" part of the theorem of Dvoretzky and Erdös).

See [1], [2] for other situations when the escape rate can be estimated.

Let us briefly discuss our approach to the proof. Given an increasing function $R(t)$, how to verify that it is an upper radius for X_t ? Fix the initial point x and consider sequences of times $\{t_k\} \uparrow \infty$ and of events:

$$\mathcal{A}_k = \{\exists t \in [t_{k-1}, t_k] : X_t \notin B(x, R(t))\}.$$

Necessary and sufficient condition that $R(t)$ be an upper radius is that, with probability 1, only finite number of events $\mathcal{A}_k$ can occur. The latter will be implied by the Borel–Cantelli lemma if we show that

$$\sum_k \mathbb{P}_x(\mathcal{A}_k) < \infty.$$

On the other hand, by the monotonicity argument,

$$\mathbb{P}_x\left(\mathcal{A}_k\right) \leq \mathbb{P}_x\left\{\exists t \leq t_k : X_t \notin B(x, R_{k-1})\right\} \tag{1.8}$$

where we use the notation $R_i = R(t_i)$. The right hand side of (1.8) is the probability that the process X_t started at x, hits the exterior of $B(x, R_{k-1})$ by time t_k.

Thus, the question amounts to estimating of a certain hitting probability. Similar argument works for the lower radius case and involves a different type of a hitting probability, namely

$$\mathbb{P}_x\left\{\exists t \geq t_{k-1} : X_t \in B(R_k))\right\}$$

(see Section 5). The approach of using the hitting probabilities to handle the upper and lower radii, is not new - see, for example, [4], [19]. Let us mention for comparison that the results of [16] cited above were obtained by using an entirely different technique based on the Kolmogorov inequality.

The main contribution of this paper is obtaining estimates of the hitting probabilities, which are good enough to run the above argument. While for the upper radius case we were very much helped by the existing upper bound of the hitting probability, which is due to Takeda [30], the key estimate, which serves the lower radius case, seems to be new and is of interest by itself - see Proposition 3.5 below. The method used in the proof of Proposition 3.5 is similar to that in [18].

The structure of the paper is the following. In Section 2, we collect all necessary preliminary material about *weighted manifolds*. This is a more general notion than Riemannian manifolds, and we use weighted manifold as the main underlying space. Among results cited in Section 2 are heat kernel upper bounds, estimates of capacities, mean-value inequality etc.

In Section 3, we prove the estimates of the hitting probabilities. In Sections 4 and 5, we prove our main results about the upper and lower radii respectively, which are somewhat more general than Theorems 1.1 and 1.2.

The dependencies of the statements are the following. The results, which are required for or are related to the upper radius case, are: Propositions 2.6, 3.6, Corollary 3.7 and Theorems 4.1, 4.2. The results related to the lower radius are: Propositions 2.2, 2.4, Corollary 2.5, Propositions 3.1-3.5 and Theorem 5.1.

NOTATION

The following list of notation is provided for convenience of a reader. See the next section for the detailed definitions.

- M - a Riemannian manifold of the dimension n;
- $\mathfrak{g}_{ij}$ - a Riemannian tensor on M; $\mathcal{G} := \det \|\mathfrak{g}_{ij}\|$ and $\left\|\mathfrak{g}^{ij}\right\| = \|\mathfrak{g}_{ij}\|^{-1}$;
- μ - a measure on M with the density σ against the Riemannian measure, where σ is a smooth positive function on M;
- μ' - the measure with density σ against the Riemannian measure of codimension 1 on any smooth hypersurface;
- (M, μ) - a weighted manifold;
- $\mathrm{div} F := \frac{1}{\sqrt{\mathcal{G}}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\mathcal{G}} F^i\right)$ where $F = \sum_i F^i \frac{\partial}{\partial x_i}$ is a vector field on M;
- $\nabla f := \mathfrak{g}^{ij} \frac{\partial f}{\partial x_j}$ where f is a function on M;
- $\Delta := \mathrm{div}\, \nabla$- the Laplace operator on M;
- $\Delta_\mu := \sigma^{-1} \mathrm{div}\left(\sigma \nabla\right)$ - the weighted Laplace operator on (M, μ);
- X_t - the Brownian motion on (M, μ), that is, the minimal diffusion on M with the generator Δ_μ;
- $p(t, x, y)$ - the heat kernel on (M, μ); alternatively, $p(t, x, y)$ is a density of the transition probability of the process X_t against measure μ;
- $p_\Omega(t, x, y)$ - the heat kernel in (Ω, μ) with the Dirichlet boundary condition on $\partial\Omega$; if one of the points x, y is outside Ω, then $p_\Omega(t, x, y) := 0$;

- $G(x,y)$ - the Green kernel on (M,μ), that is $G(x,y) := \int_0^\infty p(t,x,y)dt$;
- $G_\Omega(x,y)$ - the Green kernel in (Ω,μ);
- $\mathbb{P}_x$ - the probability measure in the space of paths X_t emanating from the point $x \in M$;
- $\psi_K(x)$ - the probability that the Brownian motion X_t visits a set $K \subset M$ ever assuming $X_0 = x$;
- $\psi_K(t,x) := \mathbb{P}_x \{X_s \in K \text{ for some } s \leq t\}$;
- $\Psi_K(t,x) := \mathbb{P}_x \{X_s \in K \text{ for some } s \geq t\}$;
- $d(x,y)$ - a geodesic distance between the points $x, y \in M$;
- $B(x,r)$ - a geodesic ball on M centred at the point $x \in M$ of the radius r;
- $V(x,r) := \mu(B(x,r))$;
- K_r - an open r-neighbourhood of a set K;
- C - a (large) positive constant, which may be different at different occurrences.

2 Analysis on weighted manifolds

The notion of weighted manifolds was introduced by Chavel and Feldman [7] and by Davies [8]. Basically, this is a Riemannian manifold endowed additionally with a Borel measure. Many facts from analysis on weighted manifold are very similar to those on Riemannian manifold. A further natural generalization would be to consider analysis on Dirichlet spaces. We do not do that solely to avoid technical complications, which arise from the non-smoothness of the space.

2.1 Weighted manifolds

We denote by M a Riemannian manifold of the dimension n and by $\mathfrak{g}_{ij}$ its Riemannian metric tensor. As is well known, a Riemannian metric induces a Riemannian distance, which will be denoted by $d(x,y)$, where $x, y \in M$. In particular, one can define a geodesic ball

$$B(x,r) = \{y \in M \mid d(x,y) < r\}.$$

Another notion linked to the Riemannian metric is *a gradient:* for any smooth enough function f defined in a coordinate chart $\{x_1, x_2, ...\}$, we denote by ∇f the following vector field

$$(\nabla f)^i = \sum_{j=1}^n \mathfrak{g}^{ij} \frac{\partial f}{\partial x_j},$$

where $\mathfrak{g}^{ij}$ are the entries of the inverse matrix $\|\mathfrak{g}_{ij}\|^{-1}$.

The next set of definitions relates to *a measure* on M. A Riemannian structure defines also volumes of all dimensions on M. However, we introduce a Borel measure μ on M, which is not necessarily the Riemannian volume. We require that μ has density $\sigma(x)$ against the Riemannian volume, where $\sigma(x)$ is a smooth positive function on M. We denote by μ' a measure on hypersurfaces in M which has density σ against the Riemannian area (=the volume of the codimension 1). A pair[1] (M,μ) is called *a weighted manifold.*

A weighted manifold possesses *a divergence* div_μ, which is a differential operator formally adjoint to ∇ with respect to μ. For any smooth enough vector field F, the divergence $\mathrm{div}_\mu F$ is a function which has the following expression in any coordinate chart:

$$\mathrm{div}_\mu F := \frac{1}{\sigma\sqrt{\mathcal{G}}} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sigma\sqrt{\mathcal{G}} F^i\right),$$

[1]Strictly speaking, we should consider a tripple $(M, \mathfrak{g}, \mu)$ but, for the sake of simplicity, we omit the Riemnnian metric $\mathfrak{g}$ from all notation.

where $\mathcal{G} := \det \|\mathfrak{g}_{ij}\|$. In particular if $\sigma \equiv 1$ then we obtain the Riemannian divergence

$$\mathrm{div} F := \frac{1}{\sqrt{\mathcal{G}}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\mathcal{G}} F^i \right).$$

We have the following *integration-by-part formula*:

$$\int_\Omega f \, \mathrm{div}_\mu F \, d\mu = - \int_\Omega (\nabla f, F) \, d\mu - \int_{\partial\Omega} f \, (F, \nu) \, d\mu', \tag{2.1}$$

where Ω is a precompact open set in M with smooth boundary, f and F are smooth enough function and a vector field, respectively, ν is an inward normal (in the sense of the Riemannian metric) vector field on $\partial\Omega$, and $(\cdot, \cdot)$ denotes the inner product of the vectors induced by the Riemannian tensor $\mathfrak{g}_{ij}$. If f and F have their supports in Ω then the last term in (2.1) disappears, and the smoothness of $\partial\Omega$ is no longer essential.

Finally, let us introduce *the weighted Laplace operator*

$$\Delta_\mu = \mathrm{div}_\mu \circ \nabla,$$

or, in a coordinate chart,

$$\begin{aligned} \Delta_\mu f &= \frac{1}{\sigma\sqrt{\mathcal{G}}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sigma \sqrt{\mathcal{G}} \mathfrak{g}^{ij} \frac{\partial f}{\partial x_j} \right) \\ &= \Delta f + \sigma^{-1} (\nabla \sigma, \nabla f). \end{aligned}$$

As follows from (2.1), we have the following *Green formula*:

$$\int_\Omega f \, \Delta_\mu g \, d\mu = \int_\Omega g \, \Delta_\mu f \, d\mu - \int_{\partial\Omega} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) d\mu', \tag{2.2}$$

where f and g are smooth enough functions in $\overline{\Omega}$.

2.2 Heat kernel

Given the weighted Laplace operator, we consider in $M \times (0, \infty)$ the associated heat equation

$$\frac{\partial u}{\partial t} = \Delta_\mu u \tag{2.3}$$

and denote by $p(t, x, y)$ *the heat kernel* - the smallest positive fundamental solution to (2.3). For example, if $M = \mathbb{R}^n$ and $\Delta_\mu = \Delta$, then

$$p(t, x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d^2(x, y)}{4t} \right).$$

It is known [9] that $p(t, x, y)$ always exists and possesses the following properties:

- $p(t, x, y) > 0$ and $p(t, x, y) \in C^\infty ((0, \infty) \times M \times M)$;
- $p(t, x, y)$ satisfies the heat equation (2.3) in the variables (x, t) and the initial condition

  $$p(t, \cdot, y) \longrightarrow \delta_y$$

  as $t \to 0+$;
- the semigroup identity: for all $x, y \in M$, $t > 0$ and $s \in (0, t)$,

  $$p(t, x, y) = \int_M p(s, x, z) p(t - s, z, y) d\mu(z); \tag{2.4}$$

- the symmetry:
$$p(t,x,y) = p(t,y,x);$$
- for all $t > 0$
$$\int_M p(t,x,y)d\mu(y) \le 1.$$

Any open region $\Omega \subset M$ can be considered itself as a weighted manifold, by restricting the Riemannian metric and the measure μ to Ω. Therefore, Ω has also the heat kernel which will be denoted by p_Ω. The minimality of p_Ω implies that p_Ω tends to 0 at $\partial\Omega$, provided $\partial\Omega$ is smooth. We always assume that $p_\Omega(t,x,y)$ is defined for all $x \in M$ and $y \in M$, by extending p_Ω by 0 outside Ω. The maximum principle implies that $p_\Omega \le p$. Moreover, for any exhaustion sequence $\{\Omega_k\}$, we have $p_{\Omega_k} \uparrow p$ as $k \to \infty$.

Given the heat kernel, one defines *the Green function* as follows: for all $x, y \in M$, $x \neq y$,
$$G(x,y) = \int_0^\infty p(t,x,y)dt. \tag{2.5}$$

The following dichotomy takes place: either $G(x,y) \equiv \infty$ for all x, y or $G(x,y) < \infty$ for all $x \neq y$. The manifold (M,μ) is called *parabolic* in the former case and non-parabolic in the latter one. For example, $\mathbb{R}^2$ is parabolic whereas $\mathbb{R}^n$ is not, for any $n > 2$. If M is non-parabolic then $G(x,y)$ is the smallest positive fundamental solution of the operator $-\Delta_\mu$, that is
$$\Delta_\mu G_\Omega(x,\cdot) = -\delta_x. \tag{2.6}$$

The heat kernel has a distinct probabilistic meaning: it is a density of the transition probability of the diffusion process X_t on M with the generator Δ_μ. The process X_t is called *the Brownian motion*[2] on (M,μ). Denote by $\mathbb{P}_x$ the probability measure on the space of trajectories emanating from a point $x \in M$. Then we have, for any Borel set $\Omega \subset M$,
$$\mathbb{P}_x\{X_t \in \Omega\} = \int_M p(t,y,x)d\mu(y).$$

The Brownian motion X_t may be either *recurrent* or *transient*. The former means that X_t visits any open set $\Omega \subset M$ at a sequence of times $\{t_k\} \to \infty$, with probability 1, and the latter means the opposite. It is known that recurrence of the Brownian motion on (M,μ) is *equivalent* to parabolicity of (M,μ) (see [22] or [15, Section 5]).

2.3 Capacity

Given an open set $\Omega \subset M$ and a precompact set $F \subset \Omega$, we define *the capacity of the capacitor* (F,Ω) as follows:
$$\operatorname{cap}(F,\Omega) := \inf_{\substack{\phi \in Lip_c(\overline{\Omega}) \\ \phi|_F = 1}} \int_M |\nabla\phi|^2 \, d\mu \tag{2.7}$$
where $Lip_c(\overline{\Omega})$ denotes a class of all Lipschitz functions on M, which have a compact support lying in $\overline{\Omega}$ (see [26] for various definitions of capacities). Clearly, capacity $\operatorname{cap}(F,\Omega)$ is increasing in F and decreasing in Ω.

If both F and Ω have smooth boundaries then the infimum in (2.7) is attained at a function ϕ, which is called *the capacity potential* of (F,Ω) and which is the smallest[3] positive solution to the following boundary value problem in $\Omega \setminus \overline{F}$:
$$\begin{cases} \Delta_\mu\phi = 0 \\ \phi|_{\partial F} = 1 \\ \phi|_{\partial\Omega} = 0\,. \end{cases} \tag{2.8}$$

[2]There are two alternative definitions of the Brownian motion: it may be a diffusion process with the generator $\frac{1}{2}\Delta_\mu$ or with the generator Δ_μ. One is reduced to another by a linear change of time. In Introduction, we used the former (classical) definition whereas in the main body of the paper, we adopt the latter definition, which is technically more convenient.

[3]If Ω is precompact then (2.8) has a unique solution and there is no need to select the smallest one. However, in general, (2.8) may have more than one solution.

If $\Omega = M$ then we write $\mathrm{cap}(F)$ for $\mathrm{cap}\,(F, \Omega)$. The capacity potential ϕ for capacitor (F, M) satisfies (2.8) as well but without the last line (see [31]). It has also the following representation, for any $x \notin F$,

$$\phi(x) = \int_{\partial D} \frac{\partial G_D(x, \cdot)}{\partial \nu} d\mu', \tag{2.9}$$

where $D := M \setminus \overline{F}$ and ν is the inward normal vector field on ∂D.

The following statement relates the notion of capacity to the parabolicity.

Proposition 2.1 *[11, Proposition 3] The manifold (M, μ) is parabolic if and only if $\mathrm{cap}\,(F) = 0$ for any compact $F \subset M$.*

For any set $A \subset M$, denote by A_ρ its open ρ-neighbourhood. We need the following upper bound of the capacity.

Proposition 2.2 *[29] If $A \subset M$ is an open precompact set and if, for some $\rho > 0$, set A_ρ is also precompact then*

$$\mathrm{cap}\,(A, A_\rho) \leq 2 \left(\int_0^\rho \frac{s ds}{\mu\,(A_s \setminus A)} \right)^{-1}.$$

In particular, if $A = B(x, R)$ and $A_\rho = B(x, R')$, where $R' = R + \rho$, then we obtain

$$\mathrm{cap}\,(B(x, R), B(x, R')) \leq 2 \left(\int_R^{R'} \frac{(r - R) dr}{V(x, r) - V(x, R)} \right)^{-1}. \tag{2.10}$$

Corollary 2.3 *([23], [32], [11], [29]) Let the manifold M be geodesically complete and assume that, for some point $x \in M$,*

$$\int^\infty \frac{r dr}{V(x, r)} = \infty.$$

Then (M, μ) is parabolic.

This follows immediately from Proposition 2.1 and the estimate (2.10) as we let $R' \to \infty$. The geodesic completeness is required to ensure that all geodesic balls are precompact.

2.4 Faber-Krahn inequality

For any open set $\Omega \subset M$, we denote

$$\lambda_1(\Omega) := \inf_{\phi \in C_0^\infty(\Omega)} \frac{\int |\nabla \phi|^2 d\mu}{\int \phi^2 d\mu}$$

assuming $\phi \not\equiv 0$. If Ω is precompact then $\lambda_1(\Omega)$ is the smallest Dirichlet eigenvalue of Δ_μ in $L^2(\Omega, \mu)$. If (M, μ) is the Euclidean space $\mathbb{R}^n$ with the Lebesgue measure μ, then the Faber-Krahn theorem says that, for any bounded open set Ω,

$$\lambda_1(\Omega) \geq c_n (\mu\Omega)^{-2/n},$$

where equality is attained if and only if Ω is a ball.

It turns out that some lower bounds for $\lambda_1(\Omega)$ in terms of the measure $\mu\Omega$ can be established on a wide class of manifolds, although they may be not that sharp as the Faber-Krahn theorem. Given a non-negative function $\Lambda(\cdot)$ on $(0, \infty)$ and a region $D \subset M$, we say that a Λ-Faber-Krahn inequality holds in D if, for any precompact open set $\Omega \subset D$, the following inequality holds:

$$\lambda_1(\Omega) \geq \Lambda(\mu\Omega).$$

If D is precompact, then the compactness argument implies that a Λ-Faber-Krahn inequality holds in D with the function

$$\Lambda(v) = a_D v^{-2/n}, \tag{2.11}$$

where a_D is a positive constant depending on the intrinsic geometry of D. We will refer to a_D as *a Faber-Krahn constant*[4] of D. Let us note that a Faber-Krahn constant relates in a certain way to the constants in the Sobolev and Nash inequalities (see [6] and [3]). It is known also that a Faber-Krahn inequality with function (2.11) follows from an isoperimetric inequality in D (see [27]).

In some situations, as in $\mathbb{R}^n$ or on Cartan–Hadamard manifolds, there exists a positive Faber-Krahn constant for the whole manifold M (see [20]). If this is not the case, one may still try and estimate Faber-Krahn constants in precompact regions. There is an important class of manifold (including those of non-negative Ricci curvature), on which a Faber-Krahn constant for *any ball* B can be written as follows:

$$a_B = \frac{c}{r^2} (\mu B)^{2/n}, \tag{2.12}$$

where r is the radius of B and the constant $c > 0$ is the same for all balls. For example, in $\mathbb{R}^n$ we have $\mu B = c_n r^n$, and r cancels in (2.12) so that a_B does depend on B. As was proved in [13], (2.12) holds also on any Riemannian manifold of non-negative Ricci curvature. On such a manifold, we have in general $\mu B \leq \text{const}\, r^n$, and a_B may become very small when r is big. For example, this is the case for any Riemannian manifold of the form $K \times R^m$, where K is a compact manifold of the dimension $n - m$. Indeed, on such a manifold $\mu B \asymp r^m$ for large r.

We say that *a relative Faber-Krahn inequality* holds on (M, μ), if for any geodesic ball $B(x, r) \subset M$ and for any precompact open set $\Omega \subset B(x, r)$, we have

$$\lambda_1 (\Omega) \geq \frac{c}{r^2} \left(\frac{V(x, r)}{\mu \Omega} \right)^{1/\alpha}, \tag{2.13}$$

where $V(x, r) := \mu (B(x, r))$ and α, c are positive constants (typically, but not necessary, $\alpha = n/2$, in which case (2.13) is equivalent to having a Faber-Krahn constant (2.12)).

We say that measure μ is *doubling* if, for all $x \in M$ and $r > 0$, we have

$$V(x, 2r) \leq C V(x, r),$$

where the constant C is the same for all x, r.

Importance of a relative Faber-Krahn inequality is explained by the following Proposition:

Proposition 2.4 *[13, Proposition 5.2] Given a geodesically complete weighted manifold (M, μ), the following statements are equivalent:*

(i) A relative Faber-Krahn inequality holds on (M, μ).

(ii) The measure μ is doubling and the following upper bound of the heat kernel holds, for all $x \in M$ and $t > 0$,

$$p(t, x, x) \leq \frac{C}{V(x, \sqrt{t})}. \tag{2.14}$$

(iii) The measure μ is doubling and, for any $\kappa > 4$, the following upper bound of the heat kernel holds

$$p(t, x, y) \leq \frac{C_\kappa}{V(x, \sqrt{t})} \exp \left(- \frac{d^2(x, y)}{\kappa t} \right), \tag{2.15}$$

for all $x, y \in M$ and $t > 0$.

Corollary 2.5 *Let (M, μ) be a geodesically complete weighted manifold and let a relative Faber-Krahn inequality hold on (M, μ). Then manifold (M, μ) is parabolic if and only if*

$$\int^\infty \frac{r dr}{V(x, r)} = \infty, \tag{2.16}$$

for some/all $x \in M$.

Indeed, if (2.16) holds then the manifold is parabolic by Corollary 2.3. If (2.16) does not hold, then it amounts to

$$\int^\infty \frac{dt}{V(x, \sqrt{t})} < \infty,$$

which implies, together with (2.15) and (2.5), that $G(x, y) < \infty$.

[4]Note that a_D is not necessarily the best possible constant, so it is not unique.

2.5 Mean value inequality for subsolutions

Proposition 2.6 *Let $B(z,\rho)$ be a precompact ball on M. Assume that, for any region $\Omega \subset B(z,\rho)$,*

$$\lambda_1(\Omega) \geq a\,(\mu\Omega)^{-1/\alpha}\,, \tag{2.17}$$

with some positive a,α. Assume also that $u(x,s)$ is a smooth non-negative subsolution to the heat equation in cylinder $B(z,\rho)\times[t-\tau,t]$. Then

$$u^2(z,t) \leq \frac{C_\alpha a^{-\alpha}}{\min\left(\tau,\rho^2\right)^{\alpha+1}} \int\limits_{t-\tau}^{t}\int\limits_{B(z,\rho)} u^2(x,s)d\mu(x)ds \tag{2.18}$$

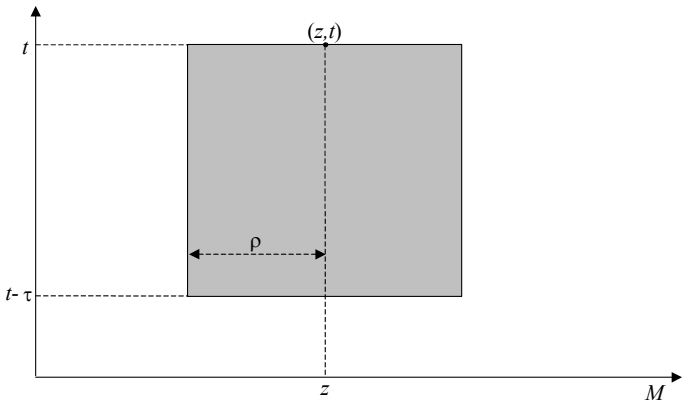

Figure 2: Cylinder $B(z,\rho)\times(t,t-\tau)$

This statement was proved by the author in [12, Theorem 3.1]. The constant a (which is an α-Faber-Krahn constant of the ball $B(z,\rho)$) is finitely proportional to a Sobolev constant (see [6]). It is well known and due to Moser [28] that a Sobolev inequality implies a mean value type inequality like (2.18), which gives another proof of Proposition 2.6.

We will use Proposition 2.6 in situation when the constant a depends on the ball $B(z,\rho)$ as follows:

$$a = a(z,\rho) = \frac{c}{\rho^2}V^{1/\alpha}(z,\rho). \tag{2.19}$$

For example, if (M,μ) satisfies a relative Faber-Krahn inequality (2.13), then this can be done for *any* ball $B(z,\rho)$ with the same constant c. In general, if we *fix* a ball $B(z,\rho)$ then the constant a can be represented in the form (2.19) just by choosing an appropriate c.

Given (2.19), the inequality (2.17) acquires form (2.13) (for $x=z$) and the mean value inequality (2.18) becomes

$$u^2(z,t) \leq \frac{C}{\min\left((\tau/\rho^2)^\alpha\,,\rho^2/\tau\right)}\frac{1}{\tau V(z,\rho)}\int\limits_{t-\tau}^{t}\int\limits_{B(z,\rho)} u^2(x,s)d\mu(x)ds. \tag{2.20}$$

Note that $\tau V(z,\rho)$ has is the volume of the cylinder $B(z,\rho)\times(t,t-\tau)$ assuming that the manifold $M\times(0,\infty)$ is equipped with the direct product measure.

Remark: Inequality (2.20) has a L^1 analogue. It is a general fact, which was observed in [24], that an L^2-mean value type inequality for subsolutions implies an L^1-mean value type inequality, too.

3 Hitting probabilities

3.1 Definition and basic facts

Given a closed set $K \in M$, we denote by $\psi_K(x)$ the probability that the process X_t started at x, will ever hit K, for some $t \geq 0$. We denote also

$$\psi_K(x,t) = \mathbb{P}_x \{X_s \in K, \quad \text{for some } s \in [0,t]\}$$

and

$$\Psi_K(x,t) = \mathbb{P}_x \{X_s \in K, \quad \text{for some } s \geq t\}.$$

Clearly, $\psi_K(x,t)$ is increasing in t, $\Psi_K(x,t)$ is decreasing in t and

$$\Psi_K(x,0) = \psi_K(x) = \psi_K(x,\infty).$$

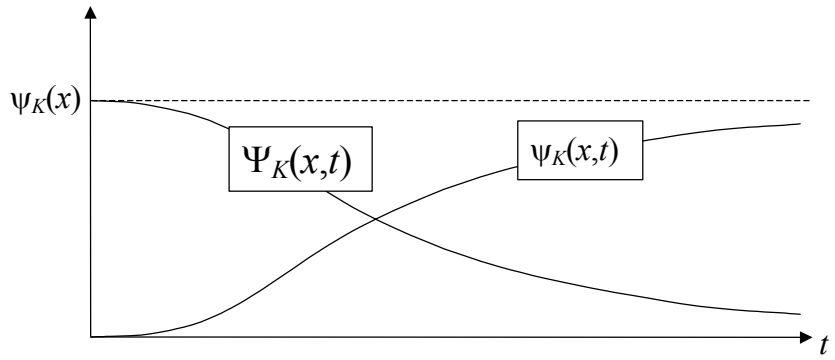

Figure 3: Functions $\psi_K(x,\cdot)$ and $\Psi_K(x,\cdot)$ on a non-parabolic manifold

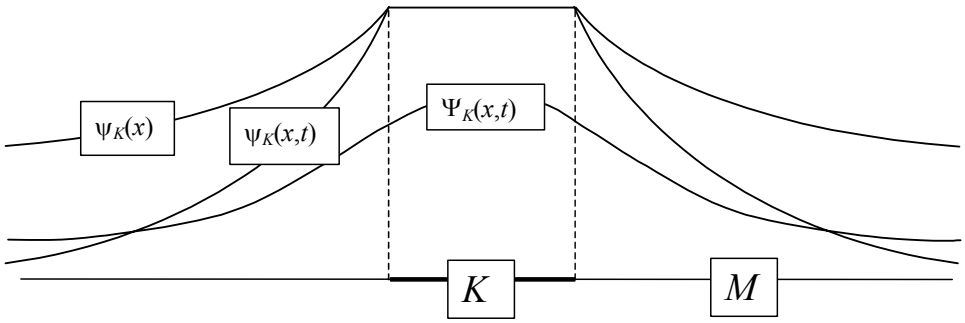

Figure 4: Functions $\psi_K(\cdot,t)$ and $\Psi_K(\cdot,t)$ on a non-parabolic manifold

Denote $\Omega = M \setminus K$. It is known (and due to Hunt) that $\psi_K(x)$ is the smallest positive solution to the equation $\Delta_\mu u = 0$ in Ω subject to the boundary condition $u|_K = 1$. In particular, if K is a compact then $\psi_K(x)$ is the capacity potential of K (see Section 2.3). Therefore,

$$\operatorname{cap}(K) = \int_M |\nabla \psi_K(x)|^2 \, d\mu(x).$$

Function $\psi_K(x,t)$ is known to be the smallest positive solution to the following initial boundary problem in $\Omega \times (0,\infty)$

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u \\ u|_{\partial\Omega} = 1 \\ u|_{t=0} = 0. \end{cases}$$

Function $\Psi_K(x,t)$ is the minimal positive solution in $M \times (0,\infty)$ of the following Cauchy problem in $M \times (0,\infty)$

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u \\ u|_{t=0} = \psi_K(x), \end{cases}$$

as will be shown below.

3.2 Estimates of $\Psi_K(x,t)$

The purpose of this section is to prove an upper bound of $\Psi_K(x,t)$ as stated in Proposition 3.5 below. We start with the following representation of $\Psi_K(x,t)$, which seems to be "well known" although we could not find its origin (its discrete analogue can be found in [19]).

Proposition 3.1 *We have the following identity, for all $x \in M$ and $t > 0$:*

$$\Psi_K(x,t) = \int_M p(t,x,z)\psi_K(z)d\mu(z).$$

Proof.

Indeed, the process X_t started at the point x, has at time t the law $p(t,x,\cdot)\mu(\cdot)$. The probability that X_s, started at x, visits K at some time $s \geq t$, is equal to the probability that X_τ, started with the initial distribution $p(t,x,\cdot)\mu(\cdot)$, visits K at some time $\tau \geq 0$.

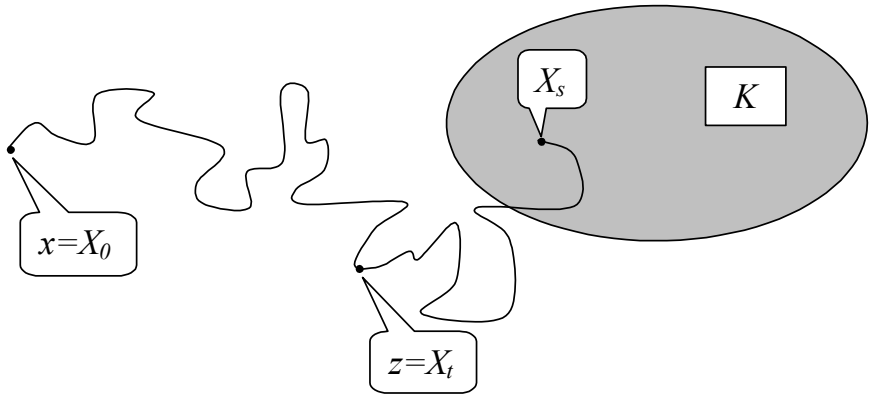

Figure 5: Trajectory X_s visits set K after time t.

The latter is equal to

$$\int_M p(t,x,z)\psi_K(z)d\mu(z),$$

by the definition of $\psi_K(z)$ and by the Markov property of X_s. ■

Proposition 3.2 *Let K be a compact with smooth boundary and let K' be a precompact neighbourhood of K, also with smooth boundary. Assume that ϕ is a Lipschitz function on M such that $\phi \in C^2(K' \setminus K)$ and*

$$\begin{cases} 0 \leq \phi(x) \leq 1, \\ \phi|_K = 1, \\ \phi|_{M \setminus K'} = 0. \end{cases}$$

Then, for any $x \in M$,

$$\psi_K(x) = \int_{K' \setminus K} G_\Omega(x,y) \Delta_\mu \left(\phi^2(y)\right) d\mu(y) + \phi^2(x), \tag{3.1}$$

where $\Omega := M \setminus K$.

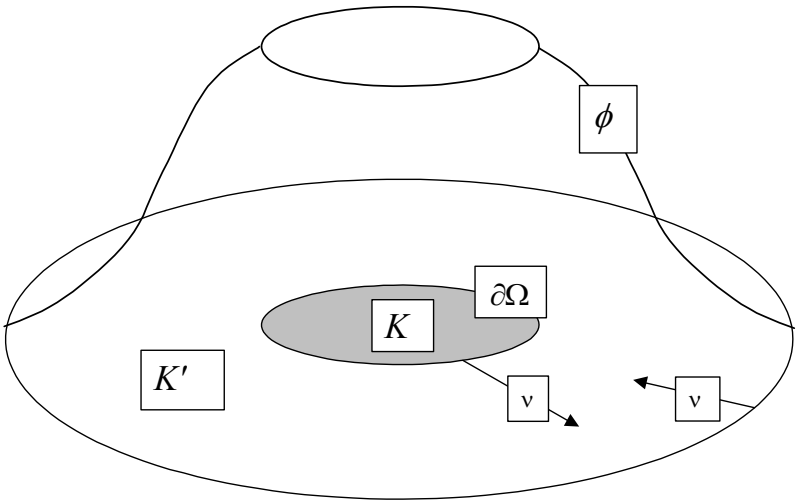

Figure 6: Sets K, K' and function ϕ.

Proof.

If $x \in K$ then $\psi_K(x) = \phi(x) = 1$ and $G_\Omega(x,y) = 0$, whence (3.1) follows immediately. Assume now that $x \notin K$. Then we have representation (2.9) for $\psi_K(x)$ as the capacity potential of K

$$\psi_K(x) = \int_{\partial\Omega} \frac{\partial G_\Omega(x,\cdot)}{\partial \nu} d\mu',$$

where ν is the inward unit normal vector field on $\partial\Omega$. The Green function $G_\Omega(x,\cdot)$ satisfies in Ω the equation (2.6)

$$\Delta_\mu G_\Omega(x,\cdot) = -\delta_x.$$

By multiplying it by ϕ^2 and integrating over $K' \setminus K$, we get

$$\int_{K' \setminus K} \Delta_\mu G_\Omega(x,\cdot)\phi^2 d\mu = -\phi^2(x).$$

The Green formula (2.2) yields

$$-\phi^2(x) = \int_{K' \setminus K} G_\Omega(x,\cdot)\Delta_\mu \phi^2 d\mu - \int_{\partial(K' \setminus K)} \left\{ \frac{\partial G_\Omega}{\partial \nu}\phi^2 - 2\phi \frac{\partial \phi}{\partial \nu} G_\Omega \right\} d\mu' \tag{3.2}$$

where ν is the inward normal vector field on $\partial(K' \setminus K)$. The boundary $\partial(K' \setminus K)$ consists of two parts: $\partial K'$ and ∂K. Since $\phi = 0$ on $\partial K'$, the second integral in (3.2) vanishes on $\partial K'$. On ∂K, we have $G_\Omega(x,\cdot) = 0$ and $\phi = 1$ whence

$$\int_{\partial K} \left\{ \frac{\partial G_\Omega}{\partial \nu}\phi^2 - 2\phi \frac{\partial \phi}{\partial \nu} G_\Omega \right\} d\mu' = \int_{\partial K} \frac{\partial G_\Omega}{\partial \nu} d\mu' = \psi_K(x).$$

By substituting this into (3.2), we obtain (3.1). ■

Proposition 3.3 *Let K be a compact with smooth boundary, K' be a precompact neighbourhood of K, also with smooth boundary and let ϕ be the capacity potential of capacitor (K, K'). Then, for any $x \in M$,*

$$\psi_K(x) \leq 2 \int_{K' \setminus K} G(x,y) \left|\nabla \phi(y)\right|^2 d\mu(y) + \phi^2(x). \tag{3.3}$$

Proof.

Since ϕ is the capacity potential, then $\Delta_\mu \phi = 0$ in $K' \setminus K$. Therefore,

$$\Delta_\mu \left(\phi^2\right) = \mathrm{div}_\mu \left(\nabla \phi^2\right) = 2\mathrm{div}_\mu \left(\phi \nabla \phi\right) = 2\phi \Delta_\mu \phi + 2\left|\nabla \phi\right|^2 = 2\left|\nabla \phi\right|^2.$$

By substituting into (3.1) and applying $G_\Omega \leq G$, we obtain (3.3). ■

Proposition 3.4 *Referring to Proposition 3.3, we have, for all $x \in M$ and $t > 0$,*

$$\Psi_K(x,t) \leq 2 \int_t^\infty \int_{K' \setminus K} p(s,x,y) \left|\nabla \phi(y)\right|^2 d\mu(y) ds + \int_{K'} p(t,x,y) \phi^2(y) d\mu(y). \tag{3.4}$$

Proof.

By using Propositions 3.1, 3.3 and the semigroup property of the heat kernel, we have

$$\begin{aligned}
\Psi_K(x,t) &= \int_M \psi_K(z) p(t,z,x) d\mu(z) \\
&\leq 2 \int_M \int_{K' \setminus K} G(z,y) \left|\nabla \phi(y)\right|^2 p(t,z,x) d\mu(y) d\mu(z) + \int_M p(t,z,x) \phi^2(z) d\mu(z) \\
&= 2 \int_M \int_{K' \setminus K} \int_0^\infty p(s,z,y) \left|\nabla \phi(y)\right|^2 p(t,z,x) ds d\mu(y) d\mu(z) + \int_M p(t,z,x) \phi^2(z) d\mu(z) \\
&= 2 \int_{K' \setminus K} \int_0^\infty p(t+s,x,y) \left|\nabla \phi(y)\right|^2 ds d\mu(y) + \int_M p(t,z,x) \phi^2(z) d\mu(z) \\
&= 2 \int_{K' \setminus K} \int_t^\infty p(s,x,y) \left|\nabla \phi(y)\right|^2 ds d\mu(y) + \int_M p(t,z,x) \phi^2(z) d\mu(z),
\end{aligned}$$

which was to be proved. ■

Proposition 3.5 *Let $K \subset M$ be a compact and K' be a precompact neighbourhood of K. Then, for any $x \in M$ and $t > 0$,*

$$\Psi_K(x,t) \leq 2\mathrm{cap}(K,K') \int_t^\infty \sup_{y \in K'} p(s,x,y) ds + \mu\left(K'\right) \sup_{y \in K'} p(t,x,y). \tag{3.5}$$

Proof.

Let us first assume that the boundaries of K and K' are smooth so that we can apply (3.4). Since ϕ is the capacity potential, we have

$$\int_{K' \setminus K} \left|\nabla \phi\right|^2 d\mu = \mathrm{cap}\left(K, K'\right),$$

and (3.5) follows immediately from (3.4) and from $\phi \leq 1$.

For arbitrary K and K', we approximate them first by the sets with smooth boundaries, for which we have already (3.5), and then pass to the limit by refining the approximation (see [26, 2.2.1 (iii)-(iv)] for continuity of capacity with respect to K and K').

■

3.3 Estimates of $\psi_K(x,t)$

The main result of this section is Corollary 3.7.

We say that a function $u(x,t)$ is a *subsolution* to the heat equation if $\partial u/\partial t - \Delta_\mu u \leq 0$.

Proposition 3.6 *Assume that $u(x,t)$ is a smooth subsolution to the heat equation in cylinder $A_r \times [0,T)$, where $A \subset M$ is a compact and r, T are arbitrary positive numbers. Suppose also that $0 \leq u(x,t) \leq 1$ and*

$$u(x,0) = 0 \quad in \ A_r.$$

Then, for any $t \in (0,T)$,

$$\int_A u^2(x,t) d\mu(x) \leq \mu(A_r) \max(1, \frac{r^2}{2t}) \exp\left(-\frac{r^2}{2t} + 1\right). \tag{3.6}$$

Remark: There is an L^1-version of (3.6), which reads as follows

$$\begin{aligned} \int_A u(x,t) d\mu(x) &\leq 16\mu(A_r) \int_r^\infty \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{\xi^2}{4t}\right) d\xi & (3.7) \\ &\leq \frac{16}{\sqrt{\pi}} \min(1, \frac{\sqrt{t}}{r}) \exp\left(-\frac{r^2}{4t}\right), & (3.8) \end{aligned}$$

see [30] and [25].

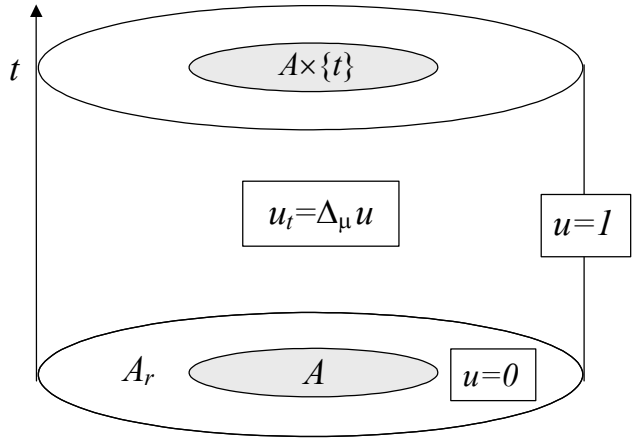

Figure 7: The maximal function u satisfying the hypotheses of Proposition 3.6

Remark: The maximal function $u(x,t)$ satisfying the hypotheses of Proposition 3.6, is one obtained by solving the following mixed problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u \quad \text{in} \quad A_r \times (0,T) \\ u|_{\partial A_r \times (0,T)} = 1 \\ u|_{A_r \times \{0\}} = 0. \end{cases} \tag{3.9}$$

In other words, $u(x,t) = \psi_{M \setminus A_r}(x,t)$. Inequality (3.7)-(3.8) gives, thus, an upper bound of the probability that X_t hits ∂A_r by the time t, provided the initial point X_0 is uniformly distributed in A. Probabilistic meaning of (3.6) is not that straightforward, but technically an L^2-estimate of u is more convenient for applications, in view of Proposition 2.6.

Proof of Proposition 3.6.

It follows immediately from the following inequality proved in [14, Theorem 3]

$$\int_A u^2(x,t) d\mu(x) \leq \mu(A_r \setminus A) \max(\frac{2t}{r^2}, \frac{r^2}{2t}) \exp\left(-\frac{r^2}{2t} + 1\right). \tag{3.10}$$

Indeed, if $\frac{2t}{r^2} \leq 1$, then (3.10) implies obviously (3.6). If $\frac{2t}{r^2} \geq 1$, then $\frac{r^2}{2t} \leq 1$, and (3.6) follows just from the fact that $|u| \leq 1$:

$$\int_A u^2(x,t) d\mu(x) \leq \mu(A) \leq \mu(A_r) \max(1, \frac{r^2}{2t}) \exp\left(-\frac{r^2}{2t} + 1\right).$$

■

Corollary 3.7 *Let $B(z,\rho)$ be a precompact ball on M and assume that a Faber-Krahn inequality*

$$\lambda_1(\Omega) \geq \frac{c}{\rho^2}\left(\frac{V(z,\rho)}{\mu\Omega}\right)^{\frac{1}{\alpha}}, \quad \forall \Omega \subset B(z,\rho), \tag{3.11}$$

holds for some $c>0$. Assume also that $u(x,t)$ is a smooth non-negative subsolution to the heat equation in cylinder $B(z,\rho+r)\times[0,T)$, such that $0\leq u(x,t)\leq 1$ and

$$u(x,0)=0 \quad \text{in } B(z,\rho+r).$$

Then, for all $t\in(0,T)$ and $\tau\in(0,t]$,

$$u(z,t)\leq \frac{C\max(1,r/\sqrt{t})}{\min\left(\left(\sqrt{\tau}/\rho\right)^{\alpha},\rho/\sqrt{\tau}\right)}\sqrt{\frac{V(z,\rho+r)}{V(z,\rho)}}\exp\left(-\frac{r^2}{4t}\right). \tag{3.12}$$

Indeed, by Proposition 2.6, we have (2.20). By Proposition 3.6, we have, for $A=B(z,\rho)$ and for any $s\in(0,t)$,

$$\begin{aligned}\int_{B(z,\rho)} u^2(x,s)d\mu(x) &\leq V(z,\rho+r)\max(1,\frac{r^2}{2s})\exp\left(-\frac{r^2}{2s}+1\right)\\ &\leq V(z,\rho+r)\max(1,\frac{r^2}{2t})\exp\left(-\frac{r^2}{2t}+1\right),\end{aligned}$$

where we have used in the last line the fact that the function $\max(1,\xi)\exp(-\xi)$ is monotone decreasing in ξ. By substituting this inequality into (2.20), we obtain (3.12). ■

Remark: Some improvement of (3.12) can be obtained in the following way. Instead of the L^2 estimate (3.6), one may use the sharper L^1 estimate (3.8) proved by Takeda [30] and Lyons [25]. Then one should prove and apply the L^1 mean value inequality mentioned at the end of Section 2.5. This is a longer way, but the improvement of (3.12) one gains does not make the final result better.

Remark: In the estimate (3.12), the most essential term on the right-hand side is the Gaussian factor.

4 Upper radius

We prove here the following theorem, which contains Theorem 1.1 from Introduction as a particular case.

Theorem 4.1 *Let M be a geodesically complete manifold, and let, for some point $z\in M$ and all $r>r_0$ (where r_0 is large enough),*

$$V(z,r)\leq v(r), \tag{4.1}$$

where $v(r)$ is a positive continuous increasing function on (r_0,∞) such that

(i) function $\frac{\log v(r)}{r^2}$ is strictly decreasing in r;

(ii) for some $\gamma>0$ and all r large enough,

$$v(r)\geq \log^{\gamma} r. \tag{4.2}$$

Let us define the function $\mathcal{R}(t)$ by the equation

$$t=\frac{\mathcal{R}^2(t)}{\log v(\mathcal{R}(t))}. \tag{4.3}$$

Then the function $R(t):=\mathcal{R}(\kappa t)$ is an upper radius for the process X_t started at z, for any $\kappa>2+4/\gamma$.

Remark: Hypothesis (i) mildly restricts from above the growth of $v(r)$ as $r \to \infty$. For example, it is satisfied for $v(r) = \exp\left(r^{2-\varepsilon}\right)$, $\varepsilon > 0$, and is not satisfied for $v(r) = \exp\left(r^2\right)$.

Remark: If the process X_i starts at a point x rather than at z, then one can still apply Theorem 4.1 with a slightly different function v, because (4.1) implies

$$V(x, r) \le v(r + d)$$

where $d = d(x, z)$.

Example: If, for r large enough,

$$V(z, r) \le Cr^{\nu}, \tag{4.4}$$

then the function $R(t) = \sqrt{(\nu + \varepsilon)\, t \log t}$ is an upper radius for the process started at z, for any $\varepsilon > 0$. Indeed, the function $v(r) = Cr^{\nu}$ satisfies both (i) (for any γ) and (ii), and (4.3) implies $\mathcal{R}(t) \sim \sqrt{\frac{\nu}{2} t \log t}$ as $t \to \infty$. Moreover, $R(t)$ is also an upper radius for the process started at any other point x because (4.4) yields, for r large enough, $V(x, r) \le C_x r^{\nu}$. This proves Theorem 1.1.

Example: If

$$V(z, r) \le C \log^{\nu} r,$$

then we obtain an upper radius $R(t) = \sqrt{ct \log \log t}$, for any $c > 2\nu + 4$.

Example: If

$$V(z, r) \le \exp\left(r^{\nu}\right), \quad \nu < 1,$$

then we obtain an upper radius $R(t) = Ct^{\frac{1}{2-\gamma}}$.

Proof of Theorem 4.1.

Given $R > 0$, denote $\psi_R(x, t) := \psi_{M \setminus B(z,R)}(x, t)$. Let ρ be any fixed small positive number. In the ball $B(z, \rho)$, we have always the Faber-Krahn inequality (3.11). Thus, we can apply Corollary 3.7, for $u = \psi_R$, to conclude

$$\psi_R(z, t) \le \frac{C \max(1, (R - \rho)/\sqrt{t})}{\min\left((\sqrt{\tau}/\rho)^{\alpha}, \rho/\sqrt{\tau}\right)} \sqrt{\frac{V(z, R)}{V(z, \rho)}} \exp\left(-\frac{(R - \rho)^2}{4t}\right), \tag{4.5}$$

for any $\tau \in (0, t]$. If $t > \rho^2$, then we may put here $\tau = \rho^2$ and obtain

$$\psi_R(z, t) \le C\sqrt{V(z, R)} \left(1 + \frac{R - \rho}{\sqrt{t}}\right) \exp\left(-\frac{(R - \rho)^2}{4t}\right), \tag{4.6}$$

where all factors depending only on ρ, have been absorbed into C. Note that inequality (4.6) does not depend on any geometric hypotheses except for the geodesic completeness and the structure of M in the ρ-neighbourhood of z.

Assuming that the process X_t starts at z, let us denote

$$\mathcal{M}(t) := \sup_{0 \le s \le t} d(X_0, X_s)$$

and introduce the series of events

$$\mathcal{A}_k := \{\mathcal{M}(t) \ge R(t) \quad \text{for some } t \in (t_k, t_{k+1}]\},$$

where

$$t_k := (1 + \varepsilon)^k, \quad k = 1, 2, ..., \tag{4.7}$$

for a positive ε to be chosen later. The function $R(t)$ is an upper radius if, with probability 1, only finite number of events $\mathcal{A}_k$ happen. By the lemma of Borel–Cantelli, this will follow from

$$\sum_k \mathbb{P}(\mathcal{A}_k) < \infty. \tag{4.8}$$

Let us estimate $\mathbb{P}(\mathcal{A}_k)$ from above. Since both $\mathcal{M}(t)$ and $R(t)$ are increasing in t, we have:

$$\mathbb{P}(\mathcal{A}_k) \le \mathbb{P}\{\mathcal{M}(t_{k+1}) \ge R(t_k)\} = \psi_{R_k}(z, t_{k+1}),$$

where we have denoted $R_k := R(t_k)$. By (4.6) and (4.1), we obtain

$$\mathbb{P}\left(\mathcal{A}_k\right) \leq C\sqrt{v(R_k)}\left(1+\frac{R_k}{\sqrt{t_{k+1}}}\right)\exp\left(-\frac{\left(R_k-\rho\right)^2}{4t_{k+1}}\right). \tag{4.9}$$

By (4.3) and $R(t) = \mathcal{R}(\kappa t)$, we have

$$\log v(R_k) = \frac{R_k^2}{\kappa t_k}. \tag{4.10}$$

The exponential term in (4.9) is estimated as follows, assuming that k is so large that $R_k - \rho \geq R_k/\left(1+\varepsilon\right)$:

$$\begin{aligned}
\exp\left(-\frac{\left(R_k-\rho\right)^2}{4t_{k+1}}\right) &= \exp\left(-\frac{(R_k-\rho)^2}{4(1+\varepsilon)t_k}\right) \\
&\leq \exp\left(-\frac{{R_k}^2}{4(1+\varepsilon)^2 t_k}\right) \\
&= \exp\left(-\frac{\kappa}{4(1+\varepsilon)^2}\frac{{R_k}^2}{\kappa t_k}\right) \\
&= \exp\left(-\frac{\kappa}{4(1+\varepsilon)^2}\log v(R_k)\right) \\
&= \left(v(R_k)\right)^{-\frac{\kappa}{4(1+\varepsilon)^2}}.
\end{aligned}$$

Since

$$\frac{R_k^2}{t_{k+1}} = \frac{R_k^2}{(1+\varepsilon)t_k} = \frac{\kappa}{(1+\varepsilon)}\log v\left(R_k\right) \to \infty, \tag{4.11}$$

as $k \to \infty$, the factor $\left(1+\frac{R_k}{\sqrt{t_{k+1}}}\right)$ in (4.9) can be majorized by an arbitrarily small power of the exponential term in (4.9). Hence, we can write, for k large enough,

$$\mathbb{P}\left(\mathcal{A}_k\right) \leq \left(v(R_k)\right)^{\frac{1}{2}-\frac{\kappa}{4(1+\varepsilon)^3}}. \tag{4.12}$$

Given $\kappa > 2 + 4/\gamma$, which is equivalent to

$$\frac{1}{2} - \frac{\kappa}{4} < -\frac{1}{\gamma},$$

we choose ε to be so small that

$$\frac{1}{2} - \frac{\kappa}{4(1+\varepsilon)^3} < -\frac{1}{\gamma}.$$

Since (4.11) implies $R_k > \sqrt{t_k}$ (for all k large enough), we obtain from (4.12), for some $\eta > 1$,

$$\mathbb{P}\left(\mathcal{A}_k\right) \leq \left(v(\sqrt{t_k})\right)^{-\eta/\gamma}.$$

Therefore, by (4.2) and (4.7),

$$\mathbb{P}\left(\mathcal{A}_k\right) \leq \frac{C}{\left(\log t_k\right)^\eta} = \frac{C'}{k^\eta},$$

whence (4.8) follows. ■

The approach adopted in the proof of Theorem 4.1, can be used to reprove the following theorem:

Theorem 4.2 *[16, Theorem 1.3] Let M be a geodesically complete manifold. Assume that a relative Faber-Krahn inequality holds on (M,μ), that is, for any ball $B(x,r) \subset M$ and for any region $\Omega \subset B(x,r)$,*

$$\lambda_1\left(\Omega\right) \geq \frac{c}{r^2}\left(\frac{V(x,r)}{\mu\Omega}\right)^{\frac{1}{\alpha}}, \tag{4.13}$$

where $c, \alpha > 0$. Then the process X_i has an upper radius

$$R(t) = \sqrt{\kappa t \log\log t},$$

for any $\kappa > 4$.

Proof.

The proof follows the same line as above except for the choice of ρ. Whereas in Theorem 4.1 we had to choose ρ to be small enough, in the current setting ρ can be taken arbitrary, due to the hypothesis that (4.13) holds for *all* r. Thus, we may put in (4.5) $\rho = \delta R$, for some $\delta \in (0,1)$, and $\tau = t$, to obtain

$$\psi_R(z,t) \leq \frac{C \max(1, R/\sqrt{t})}{\min\left(\left(\sqrt{t}/R\right)^{\alpha}, R/\sqrt{t}\right)} \sqrt{\frac{V(z,R)}{V(z,\delta R)}} \exp\left(-(1-\delta)^2 \frac{R^2}{4t}\right),$$

where z is now any point on M.

By Proposition 2.4, hypothesis (4.13) implies the doubling property $V(z,2r) \leq CV(z,r)$, uniformly in $z \in M$ and $r > 0$. Therefore, the ratio $\frac{V(z,R)}{V(z,\delta R)}$ is bounded from above by a constant depending only on δ, and we obtain

$$\psi_R(z,t) \leq C\left(\left(\frac{R}{\sqrt{t}}\right)^{\alpha+1} + \frac{\sqrt{t}}{R}\right) \exp\left(-(1-\delta)^2 \frac{R^2}{4t}\right). \tag{4.14}$$

We claim that, in fact,

$$\psi_R(z,t) \leq C \exp\left(-(1-\delta)^3 \frac{R^2}{4t}\right), \tag{4.15}$$

for all positive R and t (inequality (4.15) was also proved in [18]). Indeed, if $\frac{R}{\sqrt{t}}$ is large enough then it follows directly from (4.14), by absorbing the polynomial terms by the exponential one, at the expense of the additional factor $(1-\delta)$ at the exponent. Otherwise, (4.15) follows trivially from $\psi_R(z,t) \leq 1$.

Therefore, in the notation of the previous proof and by letting

$$R(t) := \sqrt{(4+8\varepsilon)\, t \log\log t},$$

we obtain, instead of (4.9),

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_k) &\leq C \exp\left(-(1-\delta)^3 \frac{R_k^2}{4t_{k+1}}\right) \\
&= C \exp\left(-\frac{(1-\delta)^3}{(1+\varepsilon)} \frac{R_k^2}{4t_k}\right) \\
&= C \exp\left(-\frac{(1-\delta)^3 (1+2\varepsilon)}{(1+\varepsilon)} \log\log t_k\right) \\
&\leq C' k^{-(1-\delta)^3 \frac{1+2\varepsilon}{1+\varepsilon}}.
\end{aligned}$$

Therefore, for δ small enough, we conclude

$$\sum_k \mathbb{P}(\mathcal{A}_k) < \infty,$$

which was to be proved. ■

5 Lower radius

We prove here a slightly improved version of Theorem 1.2.

Theorem 5.1 *Let M be geodesically complete, and let us assume that, for some point $z \in M$, the following holds*

(i) the doubling property: for all r large enough,

$$V(z,2r) \leq CV(z,r); \tag{5.1}$$

(ii) upper bound of the heat kernel: for all $y \in M$ and for all t large enough,

$$p(t,z,y) \leq \frac{C}{V(z,\sqrt{t})}. \tag{5.2}$$

Assume also that (M,μ) is non-parabolic and denote

$$\gamma(r) := \left(\int_r^\infty \frac{sds}{V(z,s)} \right)^{-1}. \tag{5.3}$$

Let $\mathcal{R}(t)$ be an increasing positive function on $(0,\infty)$ such that

$$\int^\infty \frac{\gamma(\mathcal{R}(t))}{V(z,\sqrt{t})} dt < \infty. \tag{5.4}$$

Then $\mathcal{R}(t)$ is a lower radius for the process X_t started at z.

Remark: Let us compare the hypotheses (5.1) and (5.2) with the assumption of Theorem 1.2. The latter is a relative Faber-Krahn inequality, which is equivalent, by Proposition 2.4, to inequalities (5.1) and (5.2) for *all* points $z \in M$ and for all r and t. Therefore, Theorem 1.2 follows from Theorem 5.1.

Remark: Given (5.1) and (5.2), the non-parabolicity of (M,μ) is *equivalent* to

$$\int^\infty \frac{sds}{V(z,s)} < \infty, \tag{5.5}$$

which is proved similarly to Corollary 2.5.

Remark: If the process X_t starts at another point x then one can still apply Theorem 5.1, by replacing z by x in (5.3) and (5.4) (but still keeping z in (5.1) and (5.2)). Indeed, we have, for r large enough,

$$V(z,r) \asymp V(x,r),$$

whence the condition (5.1) holds for the point x, too. Denote $d = d(x,z)$. Then, for all t large enough, we have

$$p(t,x,y) \leq C(x,z)\, p(t+d^2,z,y),$$

uniformly in y (which follows from the local Harnack inequality in a compact containing x and z; the point y does not play any role because we consider in this context the heat kernel as a function of (t,x), which solves the heat equation). Thus, the condition (5.2) holds for the point x as well.

Proof. Let us denote for simplicity $B(R) = B(z,R)$, $v(R) = V(z,R)$ and introduce the function $I(r)$ as follows

$$I(r) := \int_r^\infty \frac{\xi d\xi}{v(\xi)} = \frac{1}{2} \int_{r^2}^\infty \frac{ds}{v(\sqrt{s})}. \tag{5.6}$$

Hypothesis (5.4) is equivalent to

$$\int^\infty \frac{dt}{I(\mathcal{R}(t)) v(\sqrt{t})} < \infty. \tag{5.7}$$

Our aim is to prove that $\mathcal{R}(t)$ is a lower radius for the process X_t, conditioned by $X_0 = z$, that is

$$\mathbb{P}_z \left\{ X_t \notin B(\mathcal{R}(t)) \quad \text{for all } t \text{ large enough} \right\} = 1. \tag{5.8}$$

Following [19], let us consider a numerical sequence $\{t_k\} \uparrow \infty$ (as $k \to \infty$) and a sequence of events

$$\mathcal{A}_k = \left\{ \exists t \in [t_{k-1}, t_k] : X_t \in B(\mathcal{R}(t)) \right\}.$$

Then (5.8) means exactly that, with probability 1, only finite number of $\mathcal{A}_k$ occurs. By the lemma of Borel-Cantelli, (5.8) will follow from

$$\sum_k \mathbb{P}\left(\mathcal{A}_k\right) < \infty. \tag{5.9}$$

We estimate $\mathbb{P}(\mathcal{A}_k)$ as follows, using the monotonicity of $\mathcal{R}(t)$:

$$\begin{aligned}\mathbb{P}(\mathcal{A}_k) &\leq \mathbb{P}\{\exists t \in [t_{k-1}, t_k] : X_t \in B(\mathcal{R}(t_k))\} \\ &\leq \mathbb{P}\{\exists t \geq t_{k-1} : X_t \in B(\mathcal{R}(t_k))\} \\ &= \Psi_{B(R_k)}(z, t_{k-1}),\end{aligned}$$

where we have set $R_k := \mathcal{R}(t_k)$. Therefore, (5.9) will follow from

$$\sum_k \Psi_{B(R_k)}(z, t_{k-1}) < \infty. \tag{5.10}$$

In order to prove (5.10), we apply the upper bound (3.5) of function Ψ_K proved in Section 3.2. By letting $K = B(R)$ and $K' = B(R')$ (where $R' > R$ is to be chosen later) and by using (5.2), we obtain, for t large enough,

$$\Psi_{B(R)}(z, t) \leq C\mathrm{cap}(B(R), B(R')) \int_t^\infty \frac{ds}{v(\sqrt{s})} + C\frac{v(R')}{v(\sqrt{t})}. \tag{5.11}$$

By the doubling property (5.1) of $v(r)$, we have, for R large enough,

$$I(R) - I(2r) = \int_R^{2R} \frac{rdr}{v(r)} = \frac{1}{4}\int_{2R}^{4R} \frac{sds}{v(s/2)} \leq \frac{C}{4}\int_{2R}^{4R} \frac{sds}{v(s)} \leq C'I(2R)$$

whence

$$I(2R) \geq \delta I(R), \tag{5.12}$$

where $\delta = (1 + C')^{-1}$.

For any $R > 0$, let us choose R' so that

$$I(R'/2) = \frac{\delta}{2} I(R). \tag{5.13}$$

Clearly, (5.12) implies $R'/2 > 2R$ (at least, for R large enough) and

$$R' > 4R. \tag{5.14}$$

We apply now the upper bound (2.10) of the capacity, which says

$$\mathrm{cap}(B(R), B(R')) \leq 2\left(\int_R^{R'} \frac{(r-R)dr}{v(r) - v(R)}\right)^{-1}. \tag{5.15}$$

It implies, by using (5.1), (5.14) and (5.13), that

$$\begin{aligned}\mathrm{cap}(B(R), B(R'))^{-1} &\geq \frac{1}{2}\int_{2R}^{R'} \frac{(r-R)dr}{v(r)} \\ &\geq \frac{1}{4}\int_{2R}^{R'} \frac{rdr}{v(r)} \\ &= \frac{1}{16}\int_R^{R'/2} \frac{sds}{v(s/2)} \\ &\geq \frac{1}{16}\int_R^{R'/2} \frac{sds}{v(s)} \\ &= \frac{1}{16}\left(I(R) - I(R'/2)\right) \\ &= \frac{1 - \delta/2}{16} I(R),\end{aligned}$$

and

$$\mathrm{cap}(B(R), B(R')) \leq \frac{C}{I(R)}. \tag{5.16}$$

Thus, we obtain from (5.11)

$$\Psi_{B(R)}(x,t) \le C\frac{I(\sqrt{t})}{I\,(R)} + C\frac{v(R')}{v(\sqrt{t})},$$

for all R and t large enough. In order to prove (5.10), we have to verify that

$$\sum_k \frac{I\,(\sqrt{t_{k-1}})}{I(R_k)} < \infty \tag{5.17}$$

and

$$\sum_k \frac{v(R'_k)}{v(\sqrt{t_{k-1}})} < \infty. \tag{5.18}$$

We choose the sequence $\{t_k\}$ so that

$$I(\sqrt{t_{k+1}}) = \frac{1}{2} I(\sqrt{t_k}),$$

which is equivalent to

$$\int_{t_{k+1}}^{\infty} \frac{ds}{v(\sqrt{s})} = \frac{1}{2}\int_{t_k}^{\infty} \frac{ds}{v(\sqrt{s})}.$$

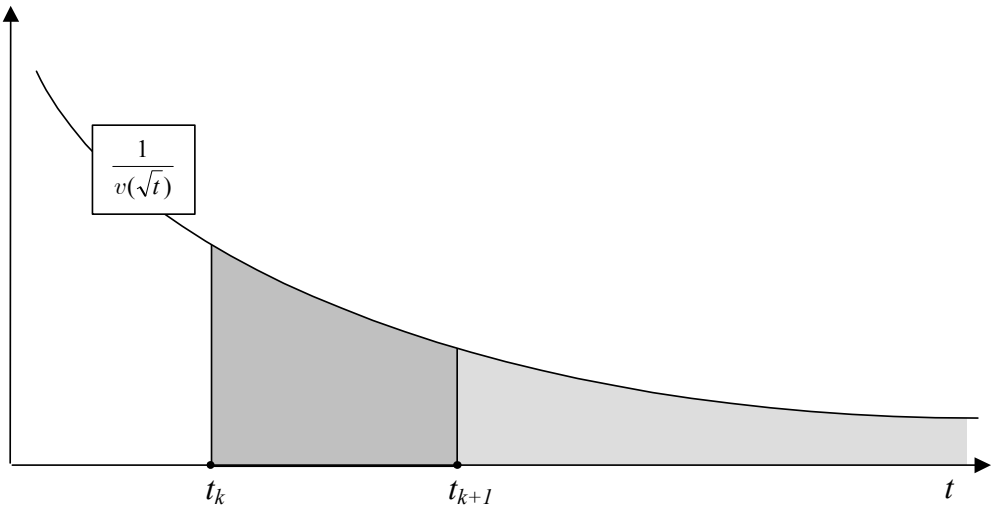

Figure 8: Choice of t_{k+1}: the shaded areas are equal.

Then

$$I(\sqrt{t_{k-1}}) = 2I(\sqrt{t_k}) = 4\left(I(\sqrt{t_k}) - I(\sqrt{t_{k+1}})\right) = 2\int_{t_k}^{t_{k+1}} \frac{ds}{v(\sqrt{s})},$$

and (5.17) amounts to

$$\sum_k \frac{1}{I(R_k)} \int_{t_k}^{t_{k+1}} \frac{ds}{v(\sqrt{s})} < \infty,$$

which follows from (5.7).

Let us now prove (5.18). By using (5.12) and (5.13), we see that

$$I(R') \ge \delta I(R'/2) = \frac{\delta^2}{2} I(R). \tag{5.19}$$

In the view of (5.17) and (5.19), it suffices to show that, for all k large enough,

$$\frac{v(R'_k)}{v(\sqrt{t_{k-1}})} \le C\frac{I\,(\sqrt{t_{k-1}})}{I(R'_k)} \tag{5.20}$$

Denote for simplicity $a_k = R'_k$ and $b_k = \sqrt{t_{k-1}}$. Then (5.20) can be rewritten as

$$\frac{v(a_k)}{v(b_k)} \leq C \frac{I(b_k)}{I(a_k)}. \tag{5.21}$$

In order to prove (5.21), we observe that, by(5.17) and (5.19),

$$\sum_k \frac{I(b_k)}{I(a_k)} < \infty.$$

In particular, $a_k < b_k$, for all k large enough. Therefore, we have

$$\begin{aligned} I(a_k) - I(b_k) &= \int_{a_k}^{b_k} \frac{sds}{v(s)} \\ &\leq \frac{b_k^2}{v(a_k)} \\ &= \frac{2}{3} \frac{v(b_k)}{v(a_k)} \int_{b_k}^{2b_k} \frac{sds}{v(b_k)} \\ &\leq C \frac{v(b_k)}{v(a_k)} \int_{b_k}^{2b_k} \frac{sds}{v(s)} \\ &\leq C \frac{v(b_k)}{v(a_k)} I(b_k) \end{aligned}$$

whence

$$I(a_k) \leq I(b_k) \left(1 + C \frac{v(b_k)}{v(a_k)}\right) \leq I(b_k) (1 + C) \frac{v(b_k)}{v(a_k)}$$

and

$$\frac{I(b_k)}{I(a_k)} \geq (1 + C)^{-1} \frac{v(a_k)}{v(b_k)},$$

which was to be proved. ■

Corollary 5.2 *Referring to Theorem 5.1, let us assume in addition that, for all large enough r and $R > r$, the following inequality holds*

$$\frac{V(z, R)}{V(z, r)} \geq c \left(\frac{R}{r}\right)^{\nu}, \tag{5.22}$$

with some $\nu > 2$ and $c > 0$. Let $r(t)$ be an increasing positive function such that $r(t)/\sqrt{t}$ is non-increasing. If

$$\int^{\infty} \frac{r^{\nu-2}(t)}{t^{\nu/2}} dt < \infty \tag{5.23}$$

then $r(t)$ is a lower radius.

Remark: Note that $\nu > 2$ implies automatically the non-parabolicity of (M, μ).

Proof.

It suffices to show that (5.23) implies (5.4). This will follow from the inequality

$$\frac{\gamma(r(t))}{v\left(\sqrt{t}\right)} \leq C \frac{r^{\nu-2}(t)}{t^{\nu/2}}, \tag{5.24}$$

which should be true for all t large enough (we use again the notation $v(R) = V(z, R)$). By using definition (5.3) of γ, we rewrite (5.24) as follows

$$\frac{t^{\nu/2}}{v\left(\sqrt{t}\right)} \leq C r^{\nu-2} \int_r^{\infty} \frac{sds}{v(s)}, \tag{5.25}$$

where $r = r(t)$. The fact that $r(t)/\sqrt{t}$ is non-increasing together with (5.23) implies that $r(t)/\sqrt{t} \to 0$ as $t \to \infty$. Therefore, we may assume $t > 4r^2$ in (5.25). We have

$$r^{\nu-2} \int_r^{\infty} \frac{sds}{v(s)} \geq r^{\nu-2} \int_r^{2r} \frac{sds}{v(s)} \geq \frac{r^{\nu}}{v(2r)}$$

so that (5.25) will follow from

$$\frac{t^{\nu/2}}{v(\sqrt{t})} \leq C \frac{(2r)^{\nu}}{v(2r)}. \tag{5.26}$$

We are left to observe that (5.26) is true by (5.22) and $\sqrt{t} > 2r$. ■

Example: The following function satisfies (5.23)

$$r(t) = \frac{C\sqrt{t}}{\log^{\frac{1}{\nu-2}} t \, (\log\log t)^{\frac{1+\varepsilon}{\nu-2}}},$$

for any $\varepsilon > 0$. Let us note that a slightly worse lower radius (with $2+\varepsilon$ in the exponent instead of $1+\varepsilon$) was obtained in a similar setting in [16, Theorem 1.2].

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