The pointwise existence and properties of heat kernel

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Abstract. We consider a semigroup acting on the function space L^1 based a measure space. Assuming that the semigroup satisfies the L^1-L^∞ ultra-contractivity, we prove that it possesses an integral kernel that is defined pointwise and has some nice properties, including the joint measurability and the continuity in one variable. We apply this result to a heat semigroup associated with a regular Dirichlet form on the space L^2.

Contents

1. Introduction

In this paper we are concerned with the existence and properties of integral kernels of semigroups acting on the space $L^1 := L^1(M, \mu)$ where (M, μ) is a measure space. Assume that a semigroup $\{P_t\}_{t>0}$ is L^1-L^∞ *ultra-contractive,* that is, there exists a non-negative, measurable function ϕ on $(0, \infty)$ such that

$$\|P_t f\|_{L^\infty} \leq \phi(t)\|f\|_{L^1} \tag{1.1}$$

for all $f \in L^1$ and $t > 0$. The following two questions arise naturally.

(1) Does the semigroup $\{P_t\}_{t>0}$ possess an integral kernel that is a jointly measurable function $p_t(x, y)$ on $M \times M$ such that

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y), \tag{1.2}$$

for all $f \in L^1$, $t > 0$ and μ-almost all $x \in M$?

(2) Once the integral kernel exists, what further properties does it have?

The existence of the integral kernel was dealt with in many papers but a common drawback of most of them is that they do not address the joint measurability of $p_t(x, y)$ in (x, y) without which the notion of integral kernel is unusable. For example, this question

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was neglected in the widely-cited paper [3], where on-diagonal upper bounds of $p_t(x, y)$ were obtained. Later on, some efforts have been made in this direction in [2, Theorem 3.1], [7, Theorem 2.10], [6, Corollary 3.8], [14, Theorems 2 and 1], [1, Proposition 4.14]. Although the joint measurability was mentioned in the paragraph following [7, Theorem 2.10], it was based on the result of [2, Theorem 3.1] that had a gap in the proof.

In the setting of symmetric Dirichlet forms, the joint measurability of the integral kernel was proved in [6, Lemma 3.3, Corollary 3.8] but the proof uses a number of quite advanced tools, which makes it not self-contained.

In this paper, we fix the problem of a joint measurability in a more general setting. More precisely, we start with a semigroup $\{P_t\}_{t>0}$ in L^1 satisfying the L^1-L^∞ ultracontractivity, and show the existence of an integral kernel of the semigroup $\{P_t\}_{t>0}$, that is defined pointwise and possesses some other nice properties, see Theorem 2.1.

We apply our theorem to the most important and interesting class of L^1-semigroups that arise from Dirichlet forms and are referred to as *heat semigroups*. Their integral kernels are called *heat kernels*. We show in Theorem 2.2 the pointwise existence of the heat kernels. Heat kernels have been widely used in the literature for many purposes and in various settings, and our result provides a solid foundation for this concept.

Notation. The term "for *each* x" means "for a fixed but an arbitrary x". The term "for *all* (or *any*) x" means "for an arbitrary x" but the statement followed is *independent* of the choice of x. For simplicity, set $L^p := L^p(M, \mu)$ for $1 \le p \le \infty$ by omitting M, μ. The identities and inequalities between L^p-functions are understood μ-almost everywhere in M.

2. Main results

In this section, we state the main results of this paper. Let (M, μ) be a measure space. Recall that the support of a measure ν on M is the smallest closed set outside which ν vanishes:

$$\mathrm{supp}[\nu] := M \setminus \bigcup\{O \subset M : O \text{ is open with } \nu(O) = 0\}.$$

For a non-negative measurable function f, the *induced* measure $f.\nu$ is defined by

$$\mathrm{d}(f.\nu)(x) = f(x)\mathrm{d}\nu(x).$$

In particular, for a measurable subset F of M, we set $m := \mathbf{1}_F.\mu$. Then by definition

$$m(\Omega) = \int_\Omega dm = \int_\Omega \mathbf{1}_F \, d\mu = \int_{F\cap\Omega} d\mu = \mu(F \cap \Omega)$$

for any measurable subset Ω. A point x belongs to $\mathrm{supp}[\mathbf{1}_F.\mu] = \mathrm{supp}[m]$ if and only if there exists an open neighborhood U_x of x such that $m(U_x) = \mu(U_x \cap F) > 0$.

A subset F of M is said to be *regular* if

$$\mathrm{supp}[\mathbf{1}_F.\mu] = F,$$

that is, $\mu(U_x \cap F) > 0$ for any $x \in F$ and any open neighborhood U_x of x. A regular set excludes any unnecessary isolated point in M with zero measure (called non-atomic point). For a sequence of subsets $\{F_k\}_{k=1}^\infty$ of M, denote by

$$C(\{F_k\}) := \{u : u|_{F_k} \text{ is continuous for each } k\}.$$

An increasing sequence of closed subsets $\{F_k\}_{k=1}^\infty$ of M is called a *μ-nest* of M if

$$\lim_{k\to\infty} \mu(M \setminus F_k) = 0,$$

and is *regular* if each F_k is regular.

We say that a function $p_t(x, y)$ pointwise defined in $(0, \infty) \times M \times M$ satisfies *condition* (A_p) for some $1 \le p \le \infty$, if there exists a regular μ-nest $\{F_n\}_{n=1}^\infty$ of M such that the following properties are true: for each $t, s > 0$ and each x, y in M,

(1) measurability: $p_t(\cdot, \cdot)$ is jointly measurable in $M \times M$;

(2) continuity and integrability in one variable:

$$p_t(x,\cdot) \in C(\{F_n\}) \cap L^p \quad \text{and} \quad p_t(\cdot,y) \in C(\{F_n\}) \cap L^p; \tag{2.1}$$

(3) continuity in integral forms: for each $f \in L^p$,

$$\int_M p_t(\cdot,z)f(z)d\mu(z) \in C(\{F_n\}) \quad \text{and} \quad \int_M p_t(z,\cdot)f(z)d\mu(z) \in C(\{F_n\}); \tag{2.2}$$

(4) semigroup property:

$$p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)d\mu(z). \tag{2.3}$$

Let $T_0 \in (0,\infty]$. Recall that a family of linear bounded operators $\{P_t\}_{t\in(0,T_0)}$ from L^p to L^p for $1 \le p \le \infty$ is called a *semigroup* on L^p if for any $t_1, t_2 > 0$ with $t_1 + t_2 < T_0$,

$$P_{t_1+t_2} = P_{t_1}P_{t_2}, \tag{2.4}$$

where (2.4) is understood that for any $f \in L^p$, there is a null (or measure zero) set $\mathcal{N}_{t_1,t_2,f}$ depending on t_1, t_2, f such that

$$P_{t_1+t_2}f(x) = P_{t_1}(P_{t_2}f)(x)$$

holds for any point x outside the set $\mathcal{N}_{t,s,f}$. Note that the operator P_t may not be defined at $t = 0$. A jointly measurable function $p_t(x,y)$ on $(0,T_0)\times M \times M$ is said to be an *integral kernel* of a semigroup $\{P_t\}_{t\in(0,T_0)}$ on L^p for $1 \le p \le \infty$ if

$$P_tf(x) = \int_M p_t(x,y)f(y)d\mu(y) \tag{2.5}$$

for μ-almost all x in M when $f \in L^p$. An integral kernel may not be defined pointwise.

For a semigroup $\{P_t\}_{t\in(0,T_0)}$ on L^p for $1 \le p < \infty$, let $\{\widehat{P}_t\}_{t\in(0,T_0)}$ be a family of operators defined by

$$(P_tf, g) = (f, \widehat{P}_tg) \tag{2.6}$$

for any $f \in L^p, g \in L^q$, where $(\, , \,)$ is the usual inner product in the L^2 space, and $q := \frac{p}{p-1}$ is the conjugate of p. Clearly, each $\widehat{P}_t$ defined in this way is linear. It is also bounded from L^q to L^q:

$$\|\widehat{P}_t\|_{L^q\to L^q} := \sup_{\|g\|_{L^q}=1} \|\widehat{P}_tg\|_{L^q} \le \|P_t\|_{L^p\to L^p}\,, \tag{2.7}$$

since for any $f \in L^p, g \in L^q$

$$|(\widehat{P}_tg, f)| = |(P_tf,g)| \le \|P_tf\|_{L^p}\|g\|_{L^q} \le \|P_t\|_{L^p\to L^p}\|f\|_{L^p}\|g\|_{L^q}\,, \tag{2.8}$$

which gives that

$$\|\widehat{P}_tg\|_{L^q} = \sup_{\|f\|_{L^p}=1} |(\widehat{P}_tg, f)| \le \|P_t\|_{L^p\to L^p}\|g\|_{L^q}\,.$$

Moreover, $\{\widehat{P}_t\}_{t\in(0,T_0)}$ satisfies the semigroup property:

$$\widehat{P}_{t+s} = \widehat{P}_t\widehat{P}_s \qquad \text{for each } t,s > 0, \tag{2.9}$$

since for any $f \in L^p, g \in L^q$

$$(f, \widehat{P}_{t+s}g) = (P_{t+s}f, g) = (P_s(P_tf), g) = (P_tf, \widehat{P}_sg) = (f, \widehat{P}_t\widehat{P}_sg),$$

from which we see $\widehat{P}_{t+s}g = \widehat{P}_t\widehat{P}_sg$ almost everywhere in M for any $g \in L^q$. The family $\{\widehat{P}_t\}_{t>0}$ is called the *dual* semigroup of $\{P_t\}_{t\in(0,T_0)}$ on L^p for $1 \le p < \infty$.

We call the triple (M,d,μ) a *metric measure space*, if (M,d) is a locally compact, separable metric space, and μ is a Radon measure on M with full support. We will work on a semigroup $\{P_t\}_{t\in(0,T_0)}$ on the space $L^1(M,\mu)$.

THEOREM 2.1. *Let $T_0 \in (0,\infty]$ and $\{P_t\}_{t\in(0,T_0)}$ be a semigroup on $L^1(M,\mu)$ for a metric measure space (M,d,μ), and let $\{\widehat{P}_t\}_{t\in(0,T_0)}$ be its dual semigroup defined by (2.6) such that each $\widehat{P}_t$ $(t \in (0,T_0))$ is bounded from L^1 to L^1. Assume that there exist a countable family $\mathcal{S}$ of open sets with $M = \cup_{U\in\mathcal{S}} U$ and a function $\varphi : \mathcal{S}\times(0,T_0) \mapsto \mathbb{R}_+$ such that, for each $t \in (0,T_0)$, $U \in \mathcal{S}$ and each $f \in L^1$*

$$\|P_t f\|_{L^\infty(U)} \le \varphi(U,t)\|f\|_{L^1}, \tag{2.10}$$

$$\|\widehat{P}_t f\|_{L^\infty(U)} \le \varphi(U,t)\|f\|_{L^1}. \tag{2.11}$$

Then $\{P_t\}_{t\in(0,T_0)}$ possesses an integral kernel $p_t(x,y)$ pointwise defined in $(0,\infty)\times M\times M$ that satisfies condition (A_p) with $p=1$ for some regular μ-nest $\{F_n\}_{n=1}^\infty$ in M, and

$$p_t(x,y) = 0 \quad \text{for any } t > 0 \tag{2.12}$$

whenever one of points x, y lies outside $\cup_{n=1}^\infty F_n$. Moreover, for each $t \in (0,T_0)$ and each $x \in U$, $y \in M$

$$|p_t(x,y)| \le \varphi(U,t) \quad \text{and} \quad |p_t(y,x)| \le \varphi(U,t). \tag{2.13}$$

We will prove Theorem 2.1 in Section 4. Below we turn to consider an interesting class of semigroups in L^1, the heat semigroup $\{P_t\}_{t>0}$, whose integral kernel is called a *heat kernel*. We are concerned with the existence of a pointwise defined heat kernel.

A strongly continuous, contractive, symmetric, and sub-Markovian semigroup $\{P_t\}_{t>0}$ on L^2 is called a *heat* semigroup, that is, for any $t > 0$ and $f, g \in L^2$,

- *strongly continuous*:
$$\lim_{t\to 0} \|P_t f - f\|_{L^2} = 0; \tag{2.14}$$
- *contractive*:
$$\|P_t f\|_{L^2} \le \|f\|_{L^2}; \tag{2.15}$$
- *symmetric:* $(P_t f, g) = (f, P_t g)$;
- *sub-Markovian:* $P_t f \ge 0$ when $f \ge 0$, and $P_t f \le 1$ when $f \le 1$, where inequalities are understood in the sense of μ-almost everywhere.

Recall that a *Dirichlet form* $(\mathcal{E},\mathcal{F})$ on L^2 is a bilinear form satisfying that, for any $u, v \in \mathcal{F}$

- $\mathcal{F}$ is *dense* in L^2, and is *complete* in the norm of $\mathcal{E}_1^{1/2}$ where
$$\mathcal{E}_1(u) := \left(\|u\|_2^2 + \mathcal{E}(u)\right)^{1/2} \text{ with } \mathcal{E}(u) := \mathcal{E}(u,u);$$
- $(\mathcal{E},\mathcal{F})$ is *positive definite*: $\mathcal{E}(u) \ge 0$, and *symmetric*: $\mathcal{E}(u,v) = \mathcal{E}(v,u)$;
- the function $u_+ \wedge 1$ belongs to $\mathcal{F}$, and $\mathcal{E}(u_+ \wedge 1) \le \mathcal{E}(u)$.

A heat semigroup on L^2 and a Dirichlet form on L^2 are mutually corresponding (cf. [**5**, Theorem 1.4.1, p.25]). Any heat semigroup $\{P_t\}_{t>0}$ can be extended to be contractive both on L^1 (cf. [**5**, p.37]) and on L^∞, and therefore, is contractive on L^p for any $1 \le p \le \infty$ by using the Riesz-Thorin interpolation theorem. For simplicity, we still denote its extension by $\{P_t\}_{t>0}$.

A family of functions $p_t(x,y)$ on $(0,\infty)\times M\times M$ is called a *heat kernel* (or a symmetric *transition density*) if the following conditions are satisfied: for any $s, t > 0$ and μ-almost all $x, y \in M$,

(1) *measurability*: $p_t(\cdot,\cdot)$ is jointly *measurable* on $M \times M$;
(2) *Markov*: $p_t(x,y) \ge 0$ and
$$\int_M p_t(x,y)d\mu(y) \le 1;$$
(3) *symmetry*: $p_t(x,y) = p_t(y,x)$;
(4) *semigroup property*:
$$p_{s+t}(x,y) = \int_M p_s(x,z)p_t(z,y)d\mu(z).$$

(5) *identity approximation*: for any $f \in L^2$,

$$\int_M p_t(x,y)f(y)d\mu(y) \xrightarrow{L^2} f(x) \quad \text{as } t \to 0+.$$

For a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on L^2, an increasing sequence of closed subsets $\{F_k\}_{k=1}^\infty$ of M is called an $\mathcal{E}$-*nest* of M if

$$\lim_{k\to\infty} \operatorname{cap}(M \setminus F_k) = 0,$$

where $\operatorname{cap}(A)$ is the *capacity* of a measurable set A defined by

$$\operatorname{cap}(A) := \inf\{\mathcal{E}(u) + \|u\|_2^2 : u \in \mathcal{F},\ u \geq 1 \text{ a.e. on } A\}.$$

Note that by definition

$$\mu(A) \leq \operatorname{cap}(A) \tag{2.16}$$

for any measurable subset A of M.

THEOREM 2.2. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(M,\mu)$ for a metric measure space (M,d,μ), and let $\{P_t\}_{t>0}$ be the associated heat semigroup on L^2. Fix $T_0 \in (0,\infty]$ and $1 \leq p \leq 2$. Assume that there exist a countable family $\mathcal{S}$ of open sets with $M = \cup_{U\in\mathcal{S}} U$ and a function $\varphi : \mathcal{S} \times (0,T_0) \mapsto \mathbb{R}_+$ such that, for each $t \in (0,T_0)$, $U \in \mathcal{S}$ and each $f \in L^p \cap L^2$*

$$\|P_t f\|_{L^\infty(U)} \leq \varphi(U,t)\|f\|_{L^p}. \tag{2.17}$$

Then $\{P_t\}_{t>0}$ possesses a heat kernel $p_t(x,y)$ pointwise defined in $(0,\infty) \times M \times M$ that satisfies condition (A_p) and some regular $\mathcal{E}$-nest $\{F_n\}_{n=1}^\infty$ of M, and

$$p_t(x,y) = 0 \ \text{ for any } t > 0 \tag{2.18}$$

whenever one of points x, y lies outside $\cup_{n=1}^\infty F_n$. Moreover, for each $t \in (0,T_0)$ and $x \in U$

$$\|p_t(x,\cdot)\|_{L^{p'}} \leq \varphi(U,t), \tag{2.19}$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p, and for any $1 \leq q \leq p'$.

$$\|p_t(x,\cdot)\|_{L^q} \leq (\varphi(U,t))^{(q-1)(p-1)}. \tag{2.20}$$

We prove Theorem 2.2 in Section 6.

3. Preliminaries

For a topological space X, let $\mathcal{B}(X)$ be a collection of all Borel sets of X and $\mathcal{X}$ a sigma-algebra on X. The following proposition shows the joint measurability of a function on a product space, which is a modification of [**9**, Lemma 9.2, p.122]. Similar results on measurability are addressed in [**11**], [**10**], [**13**]. We shall use this conclusion to prove the joint measurability of the integral kernel of a semigroup on L^1 (see the proofs of Theorem 2.1 and Theorem 2.2 below).

PROPOSITION 3.1. *Let $(X,d,\mathcal{X})$ be a separable metric space with $\mathcal{B}(X) \subset \mathcal{X}$ and $(Y,\mathcal{Y})$ a measurable space. For two sets $A \in \mathcal{X}$ and $B \in \mathcal{Y}$, assume that f is a real-valued function pointwise defined on $X \times Y$ satisfying the following conditions:*

(i) $f(x,y) = 0$ *for any* $(x,y) \in (A \times B)^c$*;*
(ii) $f(x,\cdot)$ *is measurable in* B *for each* $x \in A$*;*
(iii) $f(\cdot,y)$ *restricted on* A *is continuous for each* $y \in B$*.*

Then f is jointly measurable with respect to $(X \times Y, \mathcal{X} \times \mathcal{Y})$.

PROOF. Without loss of generality, we assume that $f(x,y) \geq 0$ for all $(x,y) \in X \times Y$. Otherwise, we shall consider $f_+ := f \vee 0$ and $f_- := (-f) \vee 0$ separately.

Since X is separable, we choose a countable set $\{x_n\}_{n=1}^{\infty} \subset A$ such that for each $k \geq 1$, $A \subset \cup_{n=1}^{\infty} B(x_n, \frac{1}{k})$, that is, A can be covered by balls $\{B(x_n, \frac{1}{k})\}_{n\geq 1}$ of the same radius $\frac{1}{k}$. For each $k \geq 1$, let $A_{1,k} := B(x_1, \frac{1}{k}) \cap A$ and

$$A_{n,k} := \Big(B(x_n, \frac{1}{k}) \setminus \cup_{i=1}^{n-1} B(x_i, \frac{1}{k})\Big) \cap A$$

for $n \geq 2$. Some of these sets may be empty, but the following argument will still work. Then the family of sets $\{A_{n,k}\}_{n=1}^{\infty}$ are disjoint, $A_{n,k} \subset B(x_n, \frac{1}{k})$, and

$$A = \bigsqcup_{n=1}^{\infty} A_{n,k}. \tag{3.1}$$

Define a family of functions f_k on $X \times Y$ by

$$f_k(x,y) := \sum_{n=1}^{\infty} f(x_n, y) \mathbf{1}_{A_{n,k}}(x), \quad x \in X, y \in Y, k \geq 1.$$

We claim that

1. For each k, the function f_k is jointly measurable;
2. For each $x \in X, y \in Y$, we have

$$\lim_{k\to\infty} f_k(x,y) = f(x,y). \tag{3.2}$$

Indeed, by assumptions (i), (ii), for each $n \geq 1$ and each $a \geq 0$,

$$\{y \in Y : f(x_n, y) \leq a\} = \{y \in B : f(x_n, y) \leq a\} \cup B^c \in \mathcal{Y},$$

which implies that $f(x_n, y)\mathbf{1}_{A_{n,k}}(x)$ is jointly measurable because both sets B^c and $\{y \in B : f(x_n, y) \leq a\}$ belong to the sigma-algebra $\mathcal{Y}$. Therefore, the function f_k, as the summation of products of two measurable functions $f(x_n, y)$ and $\mathbf{1}_{A_{n,k}}(x)$, is jointly measurable, thus proving our first claim.

To show our second claim, note that if $x \in A^c, y \in Y$, or if $x \in A, y \in B^c$, by the definition of f_k, we see that $f_k(x,y) = 0 = f(x,y)$, and thus (3.2) is obvious. On the other hand, let $x \in A, y \in B$. By assumption (iii), for each $\varepsilon > 0$, there exists some $K(x,y,\varepsilon) > 0$ such that for each $k \geq K(x,y,\varepsilon)$ and any $x' \in B(x, \frac{1}{k}) \cap A$,

$$|f(x', y) - f(x,y)| < \varepsilon.$$

Since $x \in A = \cup_{n=1}^{\infty} A_{n,k}$, we see that for each $k \geq K(x,y,\varepsilon)$, there exists some $n \geq 1$ such that $x \in A_{n,k}$, that is, $x \in B(x_n, \frac{1}{k}) \cap A$, which in turn gives that

$$x_n \in B(x, \frac{1}{k}) \cap A.$$

It follows that $|f(x_n, y) - f(x,y)| < \varepsilon$, and hence, by definition of f_k,

$$|f_k(x,y) - f(x,y)| = |f(x_n, y) - f(x,y)| < \varepsilon,$$

thus showing that (3.2) is also true. This proves our second claim.

Finally, by our claim, the function f, as a limit of jointly measurable functions f_k, is also jointly measurable on $X \times Y$. □

Let (M, μ) be a separable measure space. The following says that any μ-nest of M, after removing all the unnecessary points, each of which has an open neighborhood of zero measure, will become a *regular* μ-nest of M.

PROPOSITION 3.2. *Given a μ-nest $\{F_k\}$ of M, let $F'_k = \operatorname{supp}[\mathbf{1}_{F_k}.\mu]$ for each k. Then $F'_k \subset F_k$ for each $k \geq 1$, and $\{F'_k\}$ is a regular μ-nest.*

PROOF. Note that each F'_k is closed. For each k, the fact that $\mathbf{1}_{F_k}.\mu(F_k^c) = 0$ implies $F_k^c \subset \bigcup\{O \subset M : O \text{ is open with } \mathbf{1}_{F_k}.\mu(O) = 0\}$, thus showing that

$$F'_k = \mathrm{supp}[\mathbf{1}_{F_k}.\mu] = M \setminus \bigcup\{O \subset M : O \text{ is open with } \mathbf{1}_{F_k}.\mu(O) = 0\} \subset F_k.$$

For an open set $O \subset M$, if $\mathbf{1}_{F_{k+1}}.\mu(O) = 0$ then $\mathbf{1}_{F_k}.\mu(O) = 0$, and, hence, the set

$$\bigcup\{O \subset M : O \text{ is open with } \mathbf{1}_{F_{k+1}}.\mu(O) = 0\}$$

is contained in

$$\bigcup\{O \subset M : O \text{ is open with } \mathbf{1}_{F_k}.\mu(O) = 0\},$$

thus showing that

$$F'_{k+1} = \mathrm{supp}[\mathbf{1}_{F_{k+1}}.\mu] \supset \mathrm{supp}[\mathbf{1}_{F_k}.\mu] = F'_k. \tag{3.3}$$

On the other hand, since M is separable, by definition of F'_k, there exists a countable family of open sets $\{O_i\}_{i=1}^{\infty}$ with $\mu(F_k \cap O_i) = 0$, $i \geq 1$ such that $(F'_k)^c \subset \cup_{i=1}^{\infty} O_i$. It follows that

$$\begin{aligned}
\mu(F_k^c) &\leq \mu((F'_k)^c) \\
&= \mu(F_k^c \cap (F'_k)^c) + \mu(F_k \cap (F'_k)^c) \\
&\leq \mu(F_k^c) + \sum_{i=1}^{\infty} \mu(F_k \cap O_i) \\
&= \mu(F_k^c),
\end{aligned} \tag{3.4}$$

which gives that $\mu((F'_k)^c) = \mu(F_k^c) \downarrow 0$ as $k \uparrow \infty$. This together with (3.3) shows that $\{F'_k\}$ is a μ-nest of M.

It remains to prove that $\{F'_k\}$ is regular. Indeed, we have by (3.4)

$$\mu(F_k \setminus F'_k) = \mu((F'_k)^c) - \mu(F_k^c) = 0,$$

showing that for any open set O,

$$\mu(F_k \cap O) = \mu(F'_k \cap O) + \mu((F_k \setminus F'_k) \cap O) = \mu(F'_k \cap O).$$

Hence, we see that the following two sets are identical:

$$\bigcup\{O \subset M : O \text{ is open with } \mathbf{1}_{F_k}.\mu(O) = 0\}$$

and

$$\bigcup\{O \subset M : O \text{ is open with } \mathbf{1}_{F'_k}.\mu(O) = 0\},$$

which yields that $F'_k = \mathrm{supp}[\mathbf{1}_{F_k}.\mu] = \mathrm{supp}[\mathbf{1}_{F'_k}.\mu]$. Thus $\{F'_k\}$ is regular. $\square$

We introduce the notion of the μ-*quasi continuity* of a function.

DEFINITION 3.3. For a measure μ, we say that a function u on M is μ-*quasi continuous* (or *nearly* μ-*continuous*) if for any $\varepsilon > 0$ there is an open set $G \subset M$ such that $\mu(G) < \varepsilon$ and $u|_{M \setminus G}$ is finite continuous. Here $u|_{M\setminus G}$ is the restriction of u to $M \setminus G$.

PROPOSITION 3.4. *For a function u pointwise defined on M, the following two conditions are equivalent:*

(i). *u is μ-quasi continuous.*
(ii). *There is a (regular) μ-nest $\{F_k\}$ such that $u \in C(\{F_k\})$.*

PROOF. $(i) \Rightarrow (ii)$. Since u is μ-quasi continuous, for each $k \geq 1$, there is an open set G_k such that $\mu(G_k) < \frac{1}{k}$, and $u|_{M\setminus G_k}$ is continuous. Let

$$\widetilde{F}_k := (\bigcap_{j=1}^{k} G_j)^c = \bigcup_{j=1}^{k} G_j^c \quad \text{for } k \geq 1.$$

Then $\{\widetilde{F}_k\}_{k\geq 1}$ is increasing, and

$$\mu(M \setminus \widetilde{F}_k) = \mu(\bigcap_{j=1}^{k} G_j) \leq \mu(G_k) \leq \frac{1}{k} \to 0 \quad \text{as } k \uparrow \infty,$$

showing that $\{\widetilde{F}_k\}$ is a μ-nest on M. Note that the restriction of function u on each set

$$\widetilde{F}_k = \bigcup_{j=1}^{k} G_j^c$$

is continuous. Denote

$$F_k := \operatorname{supp}[\mathbf{1}_{\widetilde{F}_k}.\mu].$$

By Proposition 3.2, the μ-nest $\{F_k\}$ is regular. Note that $u \in C(\{F_k\})$, since u restricted on $\widetilde{F}_k(\supset F_k)$ is continuous.

$(ii) \Rightarrow (i)$. Assume $u \in C(\{F_k\})$ for a μ-nest $\{F_k\}$. For any $\varepsilon > 0$, choose k to be large enough such that $\mu(M \setminus F_k) < \varepsilon$. For such k, let $G := M \setminus F_k$. Then G is open, $\mu(G) < \varepsilon$, and $u|_{G^c} = u|_{F_k}$ is continuous. Thus u is μ-quasi continuous by definition. □

Lemma 3.5. *The following statements are true.*

- (i) *Let $S = \{u_l\}_{l\geq 1}$ be a countable family of μ-quasi continuous functions on M. Then there is a common regular μ-nest $\{F_k\}$ such that $S \subset C(\{F_k\})$.*
- (ii) *Let $\{F_k\}$ be a regular μ-nest and u belongs to $C(\{F_k\})$. If $u \geq 0$ μ-almost everywhere on an open set U, then $u(x) \geq 0$ for every point $x \in U \cap \left(\cup_{k=1}^{\infty} F_k\right)$.*

Lemma 3.5(i) says that any μ-quasi continuous function belongs to $C(\{F_k\})$ for some regular μ-nest $\{F_n\}$ of M, when S contains only one function, and at the same time, all countable μ-quasi continuous functions share a common regular μ-nest. Lemma 3.5(ii) says that an *almost everywhere* nonnegative function in a set is nonnegative *everywhere* in a slightly smaller subset.

Proof. (i). For each $l \geq 1$, since u_l is μ-quasi continuous, by Proposition 3.4, one can choose a μ-nest $\{F_k^{(l)}\}_{k=1}^{\infty}$ such that $u_l \in C(\{F_k^{(l)}\})$ and $\mu((F_k^{(l)})^c) < \frac{1}{2^l k}$. For $k \geq 1$, let

$$F_k := \bigcap_{l=1}^{\infty} F_k^{(l)}. \tag{3.5}$$

Clearly, each F_k is closed because so is $F_k^{(l)}$ for any $l, k \geq 1$. Moreover, $\{F_k\}_{k=1}^{\infty}$ is increasing because $F_k^{(l)} \subset F_{k+1}^{(l)}$ for any $k, l \geq 1$. Since by (3.5)

$$\mu(F_k^c) = \mu\big(\bigcup_{l=1}^{\infty} (F_k^{(l)})^c\big) \leq \sum_{l=1}^{\infty} \mu((F_k^{(l)})^c) \leq \sum_{l=1}^{\infty} \frac{1}{2^l k} = \frac{1}{k} \downarrow 0 \quad \text{as } k \uparrow \infty,$$

we see that $\{F_k\}$ is a μ-nest. Note that $S \subset C(\{F_k\})$ since $u_l|_{F_k^{(l)}}$ is continuous and $F_k \subset F_k^{(l)}$ for any $l \geq 1$ and $k \geq 1$. Let $\{F_k'\}$ be the regularization of $\{F_k\}$ as in Proposition 3.2. Clearly,

$$F_k' \subset F_k \subset F_k^{(l)} \quad \text{for any } k, l \geq 1. \tag{3.6}$$

Then $S \subset C(\{F_k'\})$, thus showing (i) by relabelling the notion F_k' by F_k.

(ii). Suppose that there is a point $x \in U \cap F_k$ such that $u(x) < 0$. Since $u|_{F_k}$ is continuous on each F_k, there is an open neighborhood $U_x \subset U$ of x such that $u(y) < 0$ for any point $y \in U_x \cap F_k$. Since $\{F_k\}$ is regular, we have $\mu(U_x \cap F_k) > 0$, thus showing that u is strictly negative in a subset of U with positive measure, a contraction to our assumption. □

REMARK 3.6. For a regular μ-nest $\{F_k\}$ of M and for a pointwise defined function u on M such that $u \in C(\{F_k\})$ and $u = 0$ outside $\cup_{k=1}^{\infty} F_k$, it is straightforward to see by Lemma 3.5(ii) that if $u \geq 0$ μ-almost everywhere in M, then $u \geq 0$ *pointwise* in M. We will frequently use this fact.

The following says that any function in the space L^p for $1 \leq p < \infty$ admits some μ-quasi continuous modification. Let $C(M)$ be the collection of all continuous functions on M, and $C_0(M)$ the collection of all continuous functions with compact supports on M.

LEMMA 3.7. *Let (M, d, μ) be a metric measure space. Any function u from the space L^p for $p \in [1, \infty)$ has a μ-quasi continuous modification $\widetilde{u}$, that is, there exists a regular μ-nest $\{F_k\}_{k=1}^{\infty}$ of M such that $\widetilde{u} \in C(\{F_k\})$ and $u = \widetilde{u}$ almost everywhere in M.*

PROOF. For any $v \in C(M) \cap L^p$ and any $\lambda > 0$, the set $G := \{x \in M : |v(x)| > \lambda\}$ is open and

$$\mu(G) \leq \int_G \frac{|v|^p}{\lambda^p} d\mu \leq \frac{\|v\|_{L^p}^p}{\lambda^p}. \tag{3.7}$$

Since $C_0(M)$ is dense in L^p, for any $u \in L^p$, we can choose a sequence of functions $\{u_k\} \subset C_0(M)$ such that $\|u_k - u\|_{L^p} \to 0$ as $k \uparrow \infty$. Without loss of generality, we assume that for each k,

$$\|u_{k+1} - u_k\|_{L^p}^p \leq 2^{-(p+1)k}.$$

Denote

$$G_k := \{x \in M : |u_{k+1}(x) - u_k(x)| > 2^{-k}\}.$$

Clearly, the set G_k is open. Using the inequality (3.7) with $G = G_k$, $\lambda = 2^{-k}$ and $v = u_{k+1} - u_k$, we obtain $\mu(G_k) \leq 2^{-k}$. For each k, let

$$E_k := \bigcap_{l=k}^{\infty} G_l^c.$$

Then $\{E_k\}$ is a μ-nest of M, since each E_k is closed, $\{E_k\}$ is increasing, and

$$\mu(E_k^c) = \mu(\bigcup_{l=k}^{\infty} G_l) \leq \sum_{l=k}^{\infty} \mu(G_l) \leq \sum_{l=k}^{\infty} 2^{-l} = 2^{-k+1} \to 0 \quad \text{as } k \to \infty.$$

Moreover, we have for any $x \in E_k$ and any $n > m \geq k+1$

$$|u_n(x) - u_m(x)| \leq \sum_{l=m}^{\infty} |u_{l+1}(x) - u_l(x)| = 2^{-m+1} \leq 2^{-k},$$

thus showing that the functions u_n uniformly converge as $n \to \infty$ on each E_k. Define

$$\widetilde{u}(x) := \lim_{n \to \infty} u_n(x) \quad \text{for } x \in \bigcup_{k=1}^{\infty} E_k.$$

Then $\widetilde{u} \in C(\{E_k\})$ and $u = \widetilde{u}$ almost everywhere in M. Finally, by regularization in Proposition 3.2, there is a regular μ-nest $\{F_k\}$ of M with $F_k \subset E_k$, and $\widetilde{u}$ is a μ-quasi continuous modification of u. The proof is complete. □

We remark that Lemma 3.7 is different from Lusin's theorem[1] in that μ here is not necessarily *finite*. If μ is finite on M, that is, if $\mu(M) < \infty$, then any measurable function u on M admits μ-quasi continuous modification by directly applying Lusin's theorem. In this case, Lusin's theorem is sharper than Lemma 3.7, as the function u in Lusin's theorem is assumed to be measurable (instead of $u \in L^p$).

[1]**Lusin's theorem**: Let M be a Hausdorff space, $\mathcal{A}$ a σ-algebra containing $\mathcal{B}(M)$, μ a regular measure on $\mathcal{A}$, and let u be an $\mathcal{A}$-measurable function on M. Let $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$. Then, for any $\varepsilon > 0$, there exists a compact set $K \subset A$ such that $\mu(A \setminus K) < \varepsilon$ and $u|_K$ is continuous.

Lemma 3.5 and Lemma 3.7 are respectively motivated by [5, Theorem 2.1.2, p.69] and [5, Theorem 2.1.3, p.71], wherein a regular Dirichlet form is assumed to exist in L^2. Here we do not assume the existence of a Dirichlet form.

4. Proof of Theorem 2.1

In this section, we prove Theorem 2.1.

Proof of Theorem 2.1. The proof is quite long. We divide the proof into four steps.

Step 1. We will extend the definitions of P_t and $\widehat{P}_t$ to all $t>0$ when $T_0<\infty$.

Indeed, we need only to consider P_t. Let us first extend the definition of P_t to all $t\in[T_0,2T_0)$. For $t\in[T_0,2T_0)$, let

$$P_tf=P_{t/2}P_{t/2}f \quad f\in L^1. \tag{4.1}$$

Note that the above definition of P_t is well defined since $t/2\in(0,T_0)$ and $P_{t/2}$ is bounded from L^1 to L^1. Moreover, $\|P_t\|_{L^1\mapsto L^1}\le(\|P_{t/2}\|_{L^1\mapsto L^1})^2<\infty$.

Let us verify that $\{P_t\}_{t\in(0,2T_0)}$ satisfies the semigroup property. Indeed, $\{P_t\}_{t\in(0,T_0)}$ satisfies the semigroup property, by (4.1), we have for any $t_1,t_2\in(0,2T_0)$ with $t_1+t_2<2T_0$ and $f\in L^1$,

$$\begin{aligned}P_{t_1}P_{t_2}f&=P_{t_1/2}P_{t_1/2}P_{t_2/2}P_{t_2/2}f=P_{t_1/2}P_{(t_1+t_2)/2}P_{t_2/2}f\\&=P_{t_1/2}P_{t_2/2}P_{t_1/2}P_{t_2/2}f=P_{(t_1+t_2)/2}P_{(t_1+t_2)/2}f=P_{t_1+t_2}f.\end{aligned}$$

Hence, we have extended the semigroup $\{P_t\}_{t\in(0,T_0)}$ on L^1 to the semigroup $\{P_t\}_{t\in(0,2T_0)}$. Repeating the above arguments, we can further extend it to $\{P_t\}_{t>0}$.

Let us extend (2.10). Indeed, by (2.10), we have for any $U\in\mathcal{S}$ and $t\ge T_0$,

$$\begin{aligned}\|P_tf\|_{L^\infty(U)}&=\|P_{T_0/2}P_{t-T_0/2}f\|_{L^\infty(U)}\\&\le\varphi(U,T_0/2)\|P_{t-T_0/2}f\|_{L^1}\\&\le\varphi(U,T_0/2)\|P_{t-T_0/2}\|_{L^1\mapsto L^1}\|f\|_{L^1},\quad f\in L^1,\end{aligned}$$

This shows that for any $t>0$,

$$\|P_tf\|_{L^\infty(U)}\le\widetilde{\varphi}(U,t)\|f\|_{L^1},\quad f\in L^1, \tag{4.2}$$

where

$$\widetilde{\varphi}(U,t):=\begin{cases}\varphi(U,t), & \text{for } t\in(0,T_0),\\ \varphi(U,T_0/2)\|P_{t-T_0/2}\|_{L^1\mapsto L^1} & \text{for } t\in[T_0,\infty).\end{cases}$$

Similarly, we can extend $\{\widehat{P}_t\}_{t\in(0,T_0)}$ to $\{\widehat{P}_t\}_{t>0}$, and obtain an inequality similar to (4.2).

Therefore, in the rest of the proof, it suffices to consider the case when $T_0=\infty$.

Step 2. We will construct a pointwise realization Q_tf (resp. $\widehat{Q}_tf$) for P_tf (resp. $\widehat{P}_tf$) when $t>0, f\in L^1$. In particular, we show that there exists a *common* regular μ-nest $\{F_n\}_{n=1}^\infty$ such that $Q_t=P_t$, $\widehat{Q}_t=\widehat{P}_t$ on L^1, and for *all* $t>0$ and *all* $f\in L^1$,

$$Q_tf\in C(\{F_n\}) \text{ and } \widehat{Q}_tf\in C(\{F_n\}). \tag{4.3}$$

Indeed, since L^1 is separable, there exists a countable family $\{f_k\}_{k=1}^\infty$ dense in L^1. Consider the countable family

$$\{P_sf_k,\widehat{P}_sf_k:k\ge1,s\in\mathbb{Q}_+\},$$

where $\mathbb{Q}_+$ is the set of all positive rational numbers. By Lemma 3.7, each P_sf_k, $\widehat{P}_sf_k$ has a μ-quasi continuous version, say, $h_{s,k},\widehat{h}_{s,k}$ respectively. Moreover, by Lemma 3.5(i), there is a common regular μ-nest $\{F_n\}_{n=1}^\infty$ such that

$$\left\{h_{s,k},\widehat{h}_{s,k}:k\ge1,s\in\mathbb{Q}_+\right\}\subset C(\{F_n\}).$$

Let
$$M_0 := \bigcup_{n=1}^{\infty} F_n \text{ and } \mathcal{N} = M \setminus M_0. \tag{4.4}$$
It is clear that $\mu(\mathcal{N}) = 0$ since $\{F_n\}$ is a μ-nest.

Therefore, it follows that for each $U \in \mathcal{S}$, $s \in \mathbb{Q}_+$ and each $k, j \geq 1$,
$$\begin{aligned}\sup_{x\in U\cap M_0} |h_{s,k}(x) - h_{s,j}(x)| &= \sup_{x\in U\setminus\mathcal{N}} |h_{s,k}(x) - h_{s,j}(x)| \\ &= \|h_{s,k} - h_{s,j}\|_{L^\infty(U)} \quad (\text{using } h_{s,k} - h_{s,j} \in C(\{F_n\}) \\ &\qquad\qquad \text{and Lemma 3.5}(ii)) \\ &= \|P_s f_k - P_s f_j\|_{L^\infty(U)} \\ &\leq \varphi(U, s)\|f_k - f_j\|_{L^1} \quad (\text{using (2.10)}).\end{aligned} \tag{4.5}$$
Since $\{f_k\}$ is dense in L^1, for each $f \in L^1$, there exists a sequence $\{f_{k_i}\}$ from the set $\{f_k\}$ such that
$$\|f_{k_i} - f\|_{L^1} \to 0 \text{ as } i \to \infty.$$
Thus by (4.5), for each $s \in \mathbb{Q}_+$ and $U \in \mathcal{S}$, the sequence $\{h_{s,k_i}\}_{i\geq1}$ converges uniformly to a function, say, $Q_s f$, in $U \cap M_0$. Clearly, the function $Q_s f$ is independent of the choice of $\{k_i\}$. Since $U \in \mathcal{S}$ is arbitrary, and $\mathcal{S}$ covers M, for any $s \in \mathbb{Q}_+$ and $f \in L^1$, we can define the function $Q_s f$ on M by
$$Q_s f(x) = \begin{cases} \lim_{i\to\infty} h_{s,k_i}(x) & \text{for } x \in M_0 = \bigcup_{U\in\mathcal{S}} (U \cap M_0), \\ 0 & \text{for } x \in \mathcal{N}. \end{cases} \tag{4.6}$$
Moreover, it follows that $Q_s f|_{F_n\cap U}$ is continuous for any $n \geq 1$ since each $h_{s,k_i}|_{F_n}$ is continuous, and hence,
$$Q_s f|_{F_n} = Q_s f|_{F_n\cap(\cup_{U\in\mathcal{S}} U)}$$
is continuous, that is,
$$Q_s f \in C(\{F_n\}) \text{ for } \textit{all } s \in \mathbb{Q}_+ \text{ and } \textit{all } f \in L^1. \tag{4.7}$$
Let us show that for each $s \in \mathbb{Q}_+$,
$$Q_s = P_s \ \text{ in } L^1, \tag{4.8}$$
that is, $Q_s f = P_s f$ almost everywhere in M when $f \in L^1$. Indeed, using the fact that P_s is $L^1 \mapsto L^1$ bounded, it follows that
$$\lim_{i\to\infty} \|h_{s,k_i} - P_s f\|_{L^1} = \lim_{i\to\infty} \|P_s f_{k_i} - P_s f\|_{L^1} = \lim_{i\to\infty} \|P_s(f_{k_i} - f)\|_{L^1} = 0,$$
which implies that h_{s,k_i} converges to $P_s f$ in measure as $i \to \infty$. By the definition (4.6), we obtain that $Q_s f \overset{\text{a.e.}}{=} P_s f$, which is (4.8). Here the sign $\overset{a.e.}{=}$ is understood in the sense of almost everywhere in M.

We will extend $Q_s f$ in (4.6) to any positive real number t (not only for positive rationals s) by using the semigroup property. To do this, we claim that for any fixed real number $t > 0$ and $f \in L^1$,
$$Q_s(P_{t-s} f)(x) = Q_{s'}(P_{t-s'} f)(x) \ \text{ for } \textit{every } x \in M \ \text{ and } s, s' \in (0, t) \cap \mathbb{Q}_+. \tag{4.9}$$
Indeed, we see by (4.6) that for any $t > 0, f \in L^1$ and $s, s' \in (0, t) \cap \mathbb{Q}_+$
$$Q_s(P_{t-s} f)(x) = 0 = Q_{s'}(P_{t-s'} f)(x) \ \text{ whenever } x \in \mathcal{N}, \tag{4.10}$$
since $P_{t-s} f, P_{t-s'} f \in L^1$. On the other hand, it follows from (4.8) and the semigroup property of $\{P_t\}$ that for any $t > 0, f \in L^1$ and $s, s' \in (0, t) \cap \mathbb{Q}_+$
$$Q_s(P_{t-s} f) \overset{\text{a.e.}}{=} P_s(P_{t-s} f) \overset{\text{a.e.}}{=} P_t f \overset{\text{a.e.}}{=} P_{s'}(P_{t-s'} f) \overset{\text{a.e.}}{=} Q_{s'}(P_{t-s'} f). \tag{4.11}$$
This together with (4.7) and Lemma 3.5(ii) that
$$Q_s(T_{t-s} f)(x) = Q_{s'}(T_{t-s'} f)(x) \ \text{ whenever } x \in M_0. \tag{4.12}$$

Combining (4.12) and (4.10), we obtain (4.9). Consequently, we can extend $Q_s f$ in (4.6) to any positive real number t by defining

$$Q_t f(x) = Q_s(P_{t-s}f)(x) \quad f \in L^1,\ x \in M, \tag{4.13}$$

where s is a positive rational smaller than t. Note that the above formula is consistent when t is rational.

Moreover, the family $\{Q_t\}_{t>0}$ possesses the following properties.

- For all $t > 0, f \in L^1$,
$$Q_t f = Q_s(P_{t-s}f) \in C(\{F_n\}) \tag{4.14}$$
by using definition (4.13), (4.7), since $P_{t-s}f \in L^1$ when $f \in L^1$ and $s \in (0,t) \cap \mathbb{Q}_+$.
- For any $t > 0$
$$Q_t = P_t \ \text{ in } L^1 \tag{4.15}$$
by using (4.11), (4.13).
- For each $t > 0, x \in U \in \mathcal{S}$ and each $f \in L^1$, we have
$$|Q_t f(x)| \le \|P_t f\|_{L^\infty(U)} \le \varphi(U,t)\|f\|_{L^1}. \tag{4.16}$$
Indeed, when $x \in \mathcal{N}$, we see by (4.10) and (4.13) that $Q_t f(x) = 0$, and so (4.16) is true. On the other hand, when $x \in U \setminus \mathcal{N}$, we see that
$$\begin{aligned} |Q_t f(x)| &\le \sup_{z \in U \setminus \mathcal{N}} |Q_t f(z)| \\ &= \|Q_t f\|_{L^\infty(U)} \quad (\text{using (4.14) and Remark 3.6}) \\ &= \|P_t f\|_{L^\infty(U)} \quad (\text{using (4.15)}), \end{aligned}$$
and so (4.16) is also true.
- The $\{Q_t\}_{t>0}$ satisfies the semigroup property. More precisely, for any real $t_1, t_2 > 0, f \in L^1$ and any $x \in M$,
$$Q_{t_1+t_2} f(x) = Q_{t_1}(Q_{t_2} f)(x). \tag{4.17}$$
Indeed, we see by (4.15)
$$Q_{t_1+t_2} f \overset{a.e.}{=} P_{t_1+t_2} f \overset{a.e.}{=} P_{t_1}(P_{t_2} f) \overset{a.e.}{=} Q_{t_1}(Q_{t_2} f).$$
From this and (4.14), it follows that
$$0 \overset{a.e.}{=} Q_{t_1+t_2} f - Q_{t_1}(Q_{t_2} f) := u \in C(\{F_n\}).$$
Note that by (4.10),
$$Q_{t_1+t_2} f(x) = 0 = Q_{t_1}(Q_{t_2} f)(x)$$
for every $x \in \mathcal{N}$ and $f \in L^1$. Therefore, we conclude that (4.17) is true by Remark 3.6.

  Note that (4.17) also means that the pointwise realization in above way does not change the semigroup property of operators.
- Each operator from $\{Q_t\}_{t>0}$ is bounded. This immediately follows from (4.15):
$$\|Q_t\|_{L^1 \mapsto L^1} = \|P_t\|_{L^1 \mapsto L^1} < \infty. \tag{4.18}$$
- Each operator from $\{Q_t\}_{t>0}$ is linear as well. More precisely, for any $a, b \in \mathbb{R}$, $f, g \in L^1$ and any $x \in M$,
$$Q_t(af + bg)(x) = aQ_t f(x) + bQ_t g(x). \tag{4.19}$$
Indeed, for any $t > 0$ and any $f, g \in L^1$,
$$\begin{aligned} Q_t(af+bg) &\overset{a.e.}{=} P_t(af+bg) \quad (\text{using (4.15)}) \\ &\overset{a.e.}{=} aP_t f + bP_t g \quad (\text{using the linearity of } P_t) \\ &\overset{a.e.}{=} aQ_t f + bQ_t g \quad (\text{using (4.15) again}), \end{aligned}$$

that is, for μ-almost all $x \in M$

$$Q_t(af + bg)(x) = aQ_tf(x) + bQ_tg(x).$$

Since the functions on the both sides belong to $C(\{F_n\})$ by virtue of (4.14), this equality indeed is true for *every* point x in M_0 by using Lemma 3.5(ii). Clearly, (4.19) is also true for x in $\mathcal{N}$ by (4.10), since the both sides equal to zero in this case.

We turn to introduce a *pointwise* realization $\widehat{Q}_tf$ of function $\widehat{P}_tf$. Indeed, note that each $\widehat{h}_{s,k}$ is the μ-quasi continuous version of $\widehat{P}_tf_k$. Similar to (4.5), we have by (2.11) that for each $U \in \mathcal{S}$, $s \in \mathbb{Q}_+$ and each $k, j \geq 1$,

$$\sup_{x\in U\setminus\mathcal{N}} |\widehat{h}_{s,k}(x) - \widehat{h}_{s,j}(x)| \leq \varphi(U,s)\|f_k - f_j\|_{L^1}.$$

As before, let $\{f_{k_i}\}$ be a sequence from the set $\{f_k\}$ such that $\|f_{k_i} - f\|_{L^1} \to 0$ as $i \to \infty$. Hence, for any $s \in \mathbb{Q}_+$, $f \in L^1$, we can define a *pointwise* realization $\widehat{Q}_sf$ of function $\widehat{P}_sf$ by

$$\widehat{Q}_sf(x) = \begin{cases} \lim\limits_{i\to\infty} \widehat{h}_{s,k_i}(x) & \text{for } x \in M_0, \\ 0 & \text{otherwise.} \end{cases}$$

We similarly have that for any fixed real number $t > 0$ and $f \in L^1$,

$$\widehat{Q}_s(\widehat{P}_{t-s}f)(x) = \widehat{Q}_{s'}(\widehat{P}_{t-s'}f)(x) \quad \text{for } every\ x \in M \text{ and } s, s' \in (0,t) \cap \mathbb{Q}_+.$$

Consequently, for any x in M and any $f \in L^1$, we can extend $\widehat{Q}_sf(x)$ to any positive real number t by

$$\widehat{Q}_tf(x) = \widehat{Q}_s(\widehat{P}_{t-s}f)(x) \tag{4.20}$$

for some rational $s > 0$ strictly less than t. Clearly, for any $t > 0, f \in L^1$

$$\widehat{Q}_tf(x) = 0 \quad \text{whenever } x \in \mathcal{N}. \tag{4.21}$$

Then $\{\widehat{Q}_t\}_{t>0}$ possesses the following properties, whose proofs are the same to those for $\{Q_t\}_{t>0}$.

- For all $t > 0$ and all $f \in L^1$,
$$\widehat{Q}_tf \in C(\{F_n\}). \tag{4.22}$$
- For each $t > 0$,
$$\widehat{Q}_t = \widehat{P}_t \quad \text{in } L^1. \tag{4.23}$$
- For each $t > 0, x \in U \in \mathcal{S}$ and each $f \in L^1$,
$$|\widehat{Q}_tf(x)| \leq \|\widehat{P}_tf\|_{L^\infty(U)} \leq \varphi(U,t)\|f\|_{L^1}. \tag{4.24}$$
- The $\{\widehat{Q}_t\}_{t>0}$ satisfies the semigroup property: for any real $t_1, t_2 > 0, f \in L^1$ and any $x \in M$,
$$\widehat{Q}_{t_1+t_2}f(x) = \widehat{Q}_{t_1}(\widehat{Q}_{t_2}f)(x). \tag{4.25}$$
- Each operator from $\{\widehat{Q}_t\}_{t>0}$ is bounded:
$$\|\widehat{Q}_t\|_{L^1\mapsto L^1} = \|\widehat{P}_t\|_{L^1\mapsto L^1} < \infty \tag{4.26}$$
by using (4.23).
- Each operator from $\{\widehat{Q}_t\}_{t>0}$ is linear: for any $a, b \in \mathbb{R}$, $f, g \in L^1$ and any $x \in M$,
$$\widehat{Q}_t(af + bg)(x) = a\widehat{Q}_tf(x) + b\widehat{Q}_tg(x). \tag{4.27}$$
Moreover, for any $f, g \in L^1 \cap L^\infty$
$$(Q_tf, g) = (P_tf, g) = (f, \widehat{P}_tg) = (f, \widehat{Q}_tg) \tag{4.28}$$
by using (4.15), (2.6), and (4.23).

Step 3. We will work on the pointwise realization semigroups $\{Q_t\}_{t>0}$, $\{\widehat{Q}_t\}_{t>0}$, and show the existence of their integral kernels $q_t(x,y)$ and $\widehat{q}_t(x,y)$, respectively. The functions $q_t(x,y)$ and $\widehat{q}_t(x,y)$ will be used to construct the desired $p_t(x,y)$.

To do this, we have by (4.19), (4.16) that for any fixed $x \in M$, the map $L^1 \ni f \mapsto Q_t f(x)$ defines a bounded linear functional on L^1. Therefore, for each $t>0$, $x \in U \in \mathcal{S}$, there is a function $q_t(x,\cdot) \in L^\infty$ such that

$$Q_t f(x) = \int_M q_t(x,z) f(z) d\mu(z) \quad \text{for any } f \in L^1, \tag{4.29}$$

and by (4.16),

$$\|q_t(x,\cdot)\|_{L^\infty} = \sup_{\|f\|_{L^1}=1} |Q_t f(x)| \le \varphi(U,t). \tag{4.30}$$

Similarly, for each $t>0$, $y \in U \in \mathcal{S}$, we have by (4.24), (4.27) that the map $L^1 \ni f \mapsto \widehat{Q}_t f(y)$ defines a bounded linear functional on L^1. It follows that there is a function $\widehat{q}_t(y,\cdot) \in L^\infty$ such that each $t>0$, $y \in U \in \mathcal{S}$

$$\widehat{Q}_t f(y) = \int_M \widehat{q}_t(y,z) f(z) d\mu(z) \quad \text{for any } f \in L^1, \tag{4.31}$$

$$\|\widehat{q}_t(y,\cdot)\|_{L^\infty} \le \varphi(U,t). \tag{4.32}$$

Note that functions $q_t(x,\cdot)$ and $\widehat{q}_t(y,\cdot)$ in M are almost-everywhere defined for each $t>0$ and $x,y \in M$. Moreover, when $x,y \in \mathcal{N}, t>0$, we have

$$q_t(x,\cdot) = 0 = \widehat{q}_t(y,\cdot) \quad \mu\text{-a.e. in } M. \tag{4.33}$$

We show that for any $t>0$,

$$\sup_{x\in M} \|q_t(x,\cdot)\|_{L^1} \le \|\widehat{P}_t\|_{L^1\mapsto L^1}, \tag{4.34}$$

$$\sup_{y\in M} \|\widehat{q}_t(y,\cdot)\|_{L^1} \le \|P_t\|_{L^1\mapsto L^1}. \tag{4.35}$$

Indeed, by (2.7) with $p=1, q=\infty$, we have for each $t>0, f \in L^\infty$,

$$\|\widehat{P}_t f\|_{L^\infty} \le \|P_t\|_{L^1\mapsto L^1} \|f\|_{L^\infty}. \tag{4.36}$$

By duality, we also have for each $t>0, f \in L^\infty$,

$$\|P_t f\|_{L^\infty} \le \|\widehat{P}_t\|_{L^1\mapsto L^1} \|f\|_{L^\infty}. \tag{4.37}$$

It follows by Remark 3.6, (4.14), (4.15), and (4.37) that for any $f \in L^1 \cap L^\infty$ and $x \in M$,

$$|Q_t f(x)| \le \|Q_t f\|_{L^\infty} = \|P_t f\|_{L^\infty} \le \|\widehat{P}_t\|_{L^1\mapsto L^1} \|f\|_{L^\infty}. \tag{4.38}$$

For a compact subset K of M, consider the layer-cake decomposition of $|q_t(x,y)|$ over K :

$$|q_t(x,y)| = q_t(x,y)(\mathbf{1}_{K\cap V_t} - \mathbf{1}_{K\cap (V_t)^c}),$$

where the set

$$V_t := \{y \in M : q_t(x,y) \ge 0\}.$$

By (4.29), (4.38), we see for each $t>0, x \in M$,

$$\begin{aligned}\int_K |q_t(x,y)| d\mu(y) &= |Q_t(\mathbf{1}_{K\cap V_t} - \mathbf{1}_{K\cap (V_t)^c})(x)| \\ &\le \|\widehat{P}_t\|_{L^1\mapsto L^1} \|\mathbf{1}_{K\cap V_t} - \mathbf{1}_{K\cap (V_t)^c}\|_{L^\infty} \le \|\widehat{P}_t\|_{L^1\mapsto L^1}.\end{aligned}$$

Passing to the limit as $K \uparrow M$, we have (4.34). Similarly, inequality (4.35) also holds.

Step 4. We construct the desired $p_t(x,y)$ by using functions $q_t(x,y)$, $\widehat{q}_t(x,y)$. Indeed, for $t>0$ and $x,y \in M$, define

$$p_t(x,y) = \int_M q_{t/2}(x,z) \widehat{q}_{t/2}(y,z) d\mu(z). \tag{4.39}$$

Note that the integral in the right hand side of (4.39) is well defined by (4.32) and (4.34). We can rewrite (4.39) as follows:

$$\widehat{Q}_{t/2}q_{t/2}(x,\cdot)(y) = p_t(x,y) = Q_{t/2}\widehat{q}_{t/2}(y,\cdot)(x), \quad t>0,\ x,y\in M \tag{4.40}$$

by using (4.29), (4.31), (4.34), (4.35). The $p_t(x,y)$ is pointwise defined for $(t,x,y)\in(0,\infty)\times M\times M$.

Let us prove that the regular μ-nest $\{F_n\}$ and the function $p_t(x,y)$ defined above satisfy all the properties stated in Theorem 2.1. In fact, we have following.

- Property (2.12) is true whenever $x\in\mathcal{N}$ or $y\in\mathcal{N}$, by using definition (4.39) and (4.33).
- For $t>0$ and $x,y\in M$,
$$p_t(x,\cdot)\in C(\{F_n\}) \quad \text{and} \quad p_t(\cdot,y)\in C(\{F_n\}) \tag{4.41}$$
by using (4.40), (4.22), (4.35), and (4.14), (4.34). Consequently, for any $n\geq 1$, the function $p_t(\cdot,\cdot)\mathbf{1}_{F_n\times F_n}(\cdot,\cdot)$ is jointly measurable by Proposition 3.1. Hence, the joint measurability of $p_t(\cdot,\cdot)$ follows by noting that
$$p_t(x,y) = \lim_{n\to\infty} p_t(x,y)\mathbf{1}_{F_n\times F_n}(x,y), \quad x,y\in M.$$
- For $t>0$, $x\in M$ and $f\in L^1$,
$$Q_tf(x) = \int_M p_t(x,z)f(z)d\mu(z), \tag{4.42}$$
since we have
$$\begin{aligned}(p_t(x,\cdot),f) &= (\widehat{Q}_{t/2}q_{t/2}(x,\cdot),f) && \text{(by (4.40))}\\ &= (q_{t/2}(x,\cdot),Q_{t/2}f) && \text{(by (4.28))}\\ &= Q_{t/2}Q_{t/2}f(x) && \text{(by (4.29))}\\ &= Q_tf(x) && \text{(by (4.17))}.\end{aligned}$$
Similarly, for $t>0$, $y\in M$ and $f\in L^1$,
$$\widehat{Q}_tf(y) = \int_M p_t(z,y)f(z)d\mu(z) \tag{4.43}$$
by using (4.40), (4.28), (4.31), (4.25). Hence, property (2.2) follows from (4.42), (4.14) and (4.43), (4.22).
- For $t>0$ and $f\in L^1$,
$$P_tf \overset{\text{a.e.}}{=} Q_tf = \int_M p_t(\cdot,z)f(z)d\mu(z)$$
and
$$\widehat{P}_tf \overset{\text{a.e.}}{=} \widehat{Q}_tf = \int_M p_t(z,\cdot)f(z)d\mu(z),$$
by using (4.42), (4.15), and (4.43), (4.23). That is, the semigroups $\{P_t\}_{t>0}$ and $\{\widehat{P}_t\}_{t>0}$ possess the integral kernels $p_t(x,\cdot)$ and $p_t(\cdot,y)$, respectively.
- For any $t>0$ and $x\in U\in\mathcal{S}$,
$$\|p_t(x,\cdot)\|_{L^\infty}\leq\varphi(U,t)$$
by using (4.16) and (4.42). This, together with (4.41), Remark 3.6, and (2.12), yields that for any $t>0$ and $x\in U\in\mathcal{S}$, $y\in M$,
$$|p_t(x,y)|\leq\varphi(U,t).$$
Similarly, for any $t>0$ and $x\in U\in\mathcal{S}$, $y\in M$,
$$|p_t(y,x)|\leq\varphi(U,t)$$
by using (4.24), (4.43), (4.41), Remark 3.6, (2.12). Hence, property (2.13) follows.

- Property (2.1) is true. Indeed, it follows from (4.29), (4.42), and (4.34) that for $t > 0$ and $x \in M$,
$$\sup_{x\in M} \|p_t(x,\cdot)\|_{L^1} = \sup_{x\in M} \|q_t(x,\cdot)\|_{L^1} \le \|\widehat{P}_t\|_{L^1\mapsto L^1} < \infty.$$
Similarly, it follows from (4.31), (4.43), and (4.35) that for $t > 0$ and $x \in M$,
$$\sup_{y\in M} \|p_t(\cdot,y)\|_{L^1} = \sup_{y\in M} \|\widehat{q}_t(y,\cdot)\|_{L^1} \le \|P_t\|_{L^1\mapsto L^1} < \infty.$$
Consequently, property (2.1) follows from the above two formulas and (4.41).
- Finally, the semigroup property (2.3) is true. Indeed, note first that integral in the right hand side of (2.3) is well defined by (2.1) and (2.13). By (2.12), we only need to consider the case when $t > 0$ and $x, y \in M_0$.

    Since $p_t(\cdot,\cdot)$ is jointly measurable, we have by Fubini's Theorem that
$$\begin{aligned}\int_M p_{t+s}(x,y)f(y)d\mu(y) &= Q_{t+s}f(x) = Q_tQ_sf(x) \quad \text{(by (4.42) and (4.17))}\\ &= \int_M p_t(x,z)\left(\int_M p_s(z,y)f(y)d\mu(y)\right)d\mu(z) \quad \text{(by (4.42))}\\ &= \int_M \left(\int_M p_t(x,z)p_s(z,y)d\mu(z)\right)f(y)d\mu(y)\end{aligned}$$
for $t, s > 0$ and $x \in M_0$. Hence,
$$p_{t+s}(x,y) \overset{\text{a.e.}}{=} \int_M p_t(x,z)p_s(z,y)d\mu(z) \qquad \mu - a.a.\ y \in M.$$
This, together with (4.41), (2.2), (2.1), and Lemma 3.5(ii), gives that
$$p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)d\mu(z) \qquad y \in M_0.$$
Therefore, we obtain (2.3).

The proof is complete. □

Note that the *global* L^1-L^∞ ultra-contractivity *(1.1)* of a semigroup $\{P_t\}_{t>0}$ on L^1 will guarantee that conditions (2.10) and (2.11) are both true. Indeed, ultra-contractivity *(1.1)* is just (2.10) with $\varphi(U,t) = \varphi(t)$ whilst condition (2.11) is also true since
$$\|\widehat{P}_tf\|_{L^\infty} \le \varphi(t)\|f\|_{L^1} \tag{4.44}$$
for any $t > 0, f \in L^1$; this is because by (2.6), (1.1),
$$|(\widehat{P}_tf, g)| = |(P_tg, f)| \le \|P_tg\|_{L^\infty}\|f\|_{L^1} \le \varphi(t)\|g\|_{L^1}\|f\|_{L^1}$$
for any $g \in L^1$, and so
$$\|\widehat{P}_tf\|_{L^\infty} = \sup_{\|g\|_{L^1}=1} |(\widehat{P}_tf, g)| \le \sup_{\|g\|_{L^1}=1} \varphi(t)\|g\|_{L^1}\|f\|_{L^1} = \varphi(t)\|f\|_{L^1}.$$
Therefore, by Theorem 2.1, we have the following.

Corollary 4.1. *Let $\{P_t\}_{t\in(0,T_0)}$ be a semigroup on L^1, and let $\{\widehat{P}_t\}_{t\in(0,T_0)}$ be its dual semigroup defined by (2.6) such that each $\widehat{P}_t$ is bounded from L^1 to L^1, where $T_0 \in (0,\infty]$. If condition (1.1) holds, then $\{P_t\}_{t\in(0,T_0)}$ possesses an integral kernel $p_t(x,y)$ pointwise defined in $(0,\infty) \times M \times M$ that satisfies condition (A_p) with $p = 1$. Moreover, for each $t \in (0,T_0)$ and all $x, y \in M$,*
$$|p_t(x,y)| \le \varphi(t). \tag{4.45}$$

Corollary 4.2. *Assume that all the hypothesis of Theorem 2.1 are satisfied, and $p_t(x,y)$ is the corresponding integral kernel. Then the following statements are true.*

(1) *If in addition, the semigroup* $\{P_t\}_{t\in(0,T_0)}$ *is positive:* $P_t f \geq 0$ μ*-a.e. for any non-negative function* $f \in L^1$ *and* $t \in (0, T_0)$*, then*

$$p_t(x, y) \geq 0 \quad \text{for each } x, y \in M, t > 0.$$

(2) *If in addition, both* P_t *and* $\widehat{P}_t$ *have continuous versions for any* $t \in (0, T_0)$*, then* $p_t(x, \cdot)$ *and* $p_t(\cdot, x)$ *are continuous in* M *for any* $x \in M$ *and* $t > 0$*.*

(3) *If in addition,* $\{P_t\}$ *is symmetric:* $(P_t f, g) = (f, P_t g)$ *for any* $f, g \in L^1 \cap L^\infty, t \in (0, T_0)$*, then* $p_t(x, y)$ *is also symmetric:*

$$p_t(x, y) = p_t(y, x) \quad \text{for each } x, y \in M, t > 0.$$

PROOF. As shown in the proof of Theorem 2.1, it suffices to consider the case when $T_0 = \infty$.

(1). Let $\{F_n\}$ be the regular μ-nest as in Theorem 2.1. Fix $t > 0$. By (2.12), we only need to consider the case when $x, y \in M_0$ where $M_0 := \cup_{n=1}^\infty F_n$. Indeed, since P_t is positive, we obtain by (4.15) that for any $0 \leq f \in L^1$,

$$Q_t f \overset{\text{a.e.}}{=} P_t f \geq 0,$$

where Q_t is as in (4.13). This together with (4.14) and Lemma 3.5(ii) yields that

$$Q_t f(x) \geq 0 \quad x \in M_0.$$

Hence, we obtain by (4.42) that for any $x \in M_0$,

$$p_t(x, y) \geq 0 \quad \mu\text{-a.a. } y \in M_0.$$

Combining this and (2.1), and using Lemma 3.5(ii), we have that $p_t(x, y) \geq 0$ for any $x, y \in M_0$.

(2). Since P_t have continuous version for any $t > 0$, we can choose $Q_t = P_t$ in *Step 1* of the proof of Theorem 2.1. Thus the μ-nest $\{F_n\}$ can be taken as $F_n := M$ for $n \geq 1$, and the conclusion follows directly from (2.1).

(3). If $\{P_t\}$ is symmetric, we have $P_t = \widehat{P}_t$, and so $Q_t f(x) = \widehat{Q}_t f(x)$ for any $f \in L^1$ and $x \in M$. Hence,

$$\widehat{q}_t(x, \cdot) = q_t(x, \cdot) \quad \text{for each } x \in M, t > 0.$$

It follows from definition (4.39) that

$$p_t(x, y) = \int_M q_{t/2}(x, z)\widehat{q}_{t/2}(y, z)d\mu(z) = \int_M q_{t/2}(x, z)q_{t/2}(y, z)d\mu(z) = p_t(y, x)$$

for each $x, y \in M, t > 0$. □

The following result is an L^2 version of Theorem 2.1.

THEOREM 4.3. *Let* $\{P_t\}_{t\in(0,T_0)}$ *be a semigroup on* $L^2(M, \mu)$ *for a metric measure space* (M, d, μ)*, and let* $\{\widehat{P}_t\}_{t\in(0,T_0)}$ *be its dual semigroup defined by (2.6), where* $T_0 \in (0, \infty]$*. Assume that there exists a countable family* $\mathcal{S}$ *of open sets with* $M = \cup_{U\in\mathcal{S}} U$ *and a function* $\varphi : \mathcal{S} \times (0, T_0) \mapsto \mathbb{R}_+$ *such that, for each* $t \in (0, T_0)$*,* $U \in \mathcal{S}$ *and each* $f \in L^2$

$$\|P_t f\|_{L^\infty(U)} \vee \|\widehat{P}_t f\|_{L^\infty(U)} \leq \varphi(U, t)\|f\|_{L^2}. \tag{4.46}$$

Then $\{P_t\}_{t\in(0,T_0)}$ *possesses an integral kernel* $p_t(x, y)$ *pointwise defined in* $(0, \infty) \times M \times M$ *that satisfies condition* (A_p) *with* $p = 2$ *for some regular* μ*-nest* $\{F_n\}_{n=1}^\infty$ *in* M*, and*

$$p_t(x, y) = 0 \quad \text{for any } t > 0$$

whenever one of points x*,* y *lies outside* $\cup_{n=1}^\infty F_n$*. Moreover, for each* $t \in (0, T_0)$ *and any* $U \in \mathcal{S}$

$$\sup_{x\in U} (\|p_t(x, \cdot)\|_{L^2} \vee \|p_t(\cdot, x)\|_{L^2}) \leq \varphi(U, t). \tag{4.47}$$

PROOF. Note that by (2.7), for any $t \in (0, T_0)$,

$$\|\widehat{P}_t\|_{L^2 \mapsto L^2} = \|P_t\|_{L^2 \mapsto L^2} < \infty,$$

and so $\{\widehat{P}_t\}$ is also a semigroup on L^2.

The rest of the proof is similar to that of Theorem 2.1. We omit the detail. $\square$

REMARK 4.4. In Theorem 4.3, consider the following condition instead of the assumption (4.46): there exists a function $\varphi : M \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that, for any $t > 0$ and any ball $B := B(x_0, r)$,

$$\|P_t f\|_{L^\infty(B)} \vee \|\widehat{P}_t f\|_{L^\infty(B)} \leq \varphi(x_0, r, t)\|f\|_{L^2}, \quad f \in L^2.$$

Then (4.47) in Theorem 4.3 becomes that, for any $t > 0$ and any $B := B(x_0, r)$

$$\sup_{x \in B} \left(\|p_t(x, \cdot)\|_{L^2} \vee \|p_t(\cdot, x)\|_{L^2}\right) \leq \varphi(x_0, r, t).$$

In particular, by the semigroup property, the above gives an *on-diagonal upper estimate*:

$$\sup_{x \in B} |p_t(x, x)| \leq \varphi(x_0, r, t) \text{ for any } t > 0.$$

REMARK 4.5. The parallel statements on the additional properties of $p_t(x, y)$ (positivity, continuity, symmetry) in Corollary 4.2 corresponding to Theorem 4.3 for a semigroup $\{P_t\}_{t \in (0, T_0)}$ on the space L^2 (instead of on L^1) are also true.

5. Applications

In this section we give two applications of Theorem 2.1. One application is to consider a family of moving semigroups $\{P_t^B\}_{t>0}$ in L^1. We show the existence of pointwise integral kernel $p_t^B(x, y)$ and then look at its limit when B is expanding to the whole space M. The other application is to consider a non-symmetric Dirichlet form, and we obtain the existence of the non-symmetric heat kernel and its on-diagonal upper estimate.

Fix $T_0 \in (0, \infty]$. Let $\{P_t^B\}_{t \in (0, T_0)}$ be a semigroup in L^1 such that it vanishes outside B in the sense that

$$P_t^B f(x) = 0 \tag{5.1}$$

for any $t \in (0, T_0)$ and $f \in L^1$ and μ-almost all $x \in B^c$. A semigroup $\{P_t^B\}_{t \in (0, T_0)}$ in L^1 is *monotone increasing* in B if whenever $B_1 \subset B_2$, we have

$$P_t^{B_1} f \leq P_t^{B_2} f \quad \mu\text{-a.e.} \tag{5.2}$$

for any $0 \leq f \in L^1$ and $t \in (0, T_0)$, and is *positive* if

$$P_t^B f \geq 0 \quad \mu\text{-a.e.} \tag{5.3}$$

for any $0 \leq f \in L^1$ and $t \in (0, T_0)$.

The following is the first application of Theorem 2.1.

LEMMA 5.1. *Let $T_0 \in (0, \infty]$ and $\{P_t^B\}_{t \in (0, T_0)}$ be a semigroup on $L^1(M, \mu)$ for a separable metric space (M, d), which is monotone increasing in B, positive, and satisfies (5.1). Let $\{\widehat{P}_t^B\}_{t \in (0, T_0)}$ be defined by (2.6) such that it is a semigroup on L^1. Assume that there exists a function $\varphi : M \times (0, T_0) \mapsto \mathbb{R}_+$ such that, for any $t \in (0, T_0), f \in L^1$ and any metric ball B in M,*

$$\|P_t^B f\|_{L^\infty} \leq \varphi(B, t)\|f\|_{L^1}. \tag{5.4}$$

Then there exists a function $p_t(x, y)$ pointwise defined on $(0, \infty) \times M \times M$ satisfying the following.

1. *(*Measurability*) For any $t > 0$, function $p_t(\cdot, \cdot)$ is jointly measurable in $M \times M$.*
2. *(*Positivity*) For any $t > 0$ and any $x, y \in M$,*

$$p_t(x, y) \geq 0. \tag{5.5}$$

(3) *(*Semigroup property*) For any $t,s>0$ and any $x,y\in M$, we have*

$$p_{t+s}(x,y)=\int_M p_t(x,z)p_s(z,y)d\mu(z). \tag{5.6}$$

(4) *(*Limit kernel*) For any $t>0$ and any non-negative $f\in L^1$,*

$$\lim_{m\to\infty} P_t^{B_m} f=\int_M p_t(\cdot,y)f(y)d\mu(y), \tag{5.7}$$

where $\{B_m\}_{m=1}^{\infty}$ is any sequence of concentric balls tending to M.

In particular, if $\varphi(B,t)$ in (5.4) is further independent of B: $\varphi(B,t)=\varphi(t)$, then

$$p_t(x,y)\le\varphi(t) \tag{5.8}$$

for any $t\in(0,T_0)$ and any $x,y\in M$.

Proof. As shown in the proof of Theorem 2.1, it suffices to consider the case when $T_0=\infty$. We divide the proof into three steps.

Step 1. Fix a sequence of concentric metric balls $\{B_m\}_{m=1}^{\infty}$ such that $B_m\to M$ as $m\to\infty$. Consider the semigroup $\{P_t^{B_m}\}_{t>0}$ on L^1. By assumption (5.4), for any $t>0, f\in L^1$,

$$\|P_t^{B_m}f\|_{L^\infty}\le\varphi(B_m,t)\|f\|_{L^1}. \tag{5.9}$$

From this, we have for any $t>0, f\in L^1$

$$\begin{aligned}\|\widehat{P}_t^{B_m}f\|_{L^\infty}&=\sup_{\|g\|_{L^1}=1}|(\widehat{P}_t^{B_m}f,g)|=\sup_{\|g\|_{L^1}=1}|(f,P_t^{B_m}g)|\\&\le\sup_{\|g\|_{L^1}=1}\|P_t^{B_m}g\|_{L^\infty}\|f\|_{L^1}\le\varphi(B_m,t)\|f\|_{L^1}.\end{aligned} \tag{5.10}$$

Therefore, all the hypotheses in Theorem 2.1 are satisfied for the semigroup $\{P_t^{B_m}\}_{t>0}$ and its dual $\{\widehat{P}_t^{B}\}_{t>0}$. It follows that there exists a regular μ-nest $\{F_n^{(m)}\}$ of M and a function $p_t^{B_m}(x,y)$ pointwise defined on $(0,\infty)\times M\times M$ satisfying the following properties.

(1) For each $t>0$, the function $p_t^{B_m}(\cdot,\cdot)$ is jointly measurable in $M\times M$.

(2) For each $t>0$ and each x,y in M,

$$p_t^{B_m}(x,\cdot)\in C(\{F_n^{(m)}\})\cap L^1 \ \text{ and } \ p_t^{B_m}(\cdot,y)\in C(\{F_n^{(m)}\})\cap L^1. \tag{5.11}$$

(3) For each $t>0$ and each $f\in L^1$,

$$\int_M p_t^{B_m}(\cdot,z)f(z)d\mu(z)\in C(\{F_n^{(m)}\}) \ \text{ and } \ \int_M p_t^{B_m}(z,\cdot)f(z)d\mu(z)\in C(\{F_n^{(m)}\}). \tag{5.12}$$

(4) For each $t,s>0$ and each x,y in M

$$p_{t+s}^{B_m}(x,y)=\int_M p_t^{B_m}(x,z)p_s^{B_m}(z,y)d\mu(z). \tag{5.13}$$

(5) For each $t>0$, we have

$$p_t^{B_m}(x,y)=0 \tag{5.14}$$

whenever one of points x, y lies outside $\cup_{n=1}^{\infty}F_n^{(m)}$.

(6) For each $t>0$ and each $x,y\in M$,

$$|p_t^{B_m}(x,y)|\le\varphi(B_m,t). \tag{5.15}$$

(7) For each $t>0, f\in L^1$,

$$P_t^{B_m}f(\cdot)=\int_M p_t^{B_m}(\cdot,y)f(y)d\mu(y) \ \ \mu\text{-a.e. in } M. \tag{5.16}$$

(8) For each $t>0$ and each x,y in M,

$$p_t^{B_m}(x,y)\ge 0. \tag{5.17}$$

All the properties (1)-(7) above are proved in Theorem 2.1 except the property (8), which follows from Corollary 4.2(i).

Step 2**.** Without loss of generality, we assume that for all $m, n \geq 1$

$$\mu(M \setminus F_n^{(m)}) \leq \frac{1}{2^m n}.$$

Otherwise, we can take a subsequence satisfying this inequality. By Lemma 3.5(i) and its proof, we can construct a regular μ-nest $\{F_n\}$ such that

$$F_n \subset \bigcap_{m=1}^{\infty} F_n^{(m)}, \qquad n \geq 1. \tag{5.18}$$

The set F_n is not empty for large n, since the intersection $\cap_{m=1}^{\infty} F_n^{(m)}$ has almost the same measure as M for large n by using the fact that the measure of its complement is small:

$$\mu(\bigcup_{m=1}^{\infty} (F_n^{(m)})^c) \leq \sum_{m=1}^{\infty} \mu((F_n^{(m)})^c) \leq \sum_{m=1}^{\infty} \frac{1}{2^m n} = \frac{1}{n} \to 0 \ \text{ as } n \to \infty.$$

Since $C(\{F_n^{(m)}\}) \subset C(\{F_n\})$ for all $m \geq 1$, it follows from (5.11) that for any $m \geq 1$, $t > 0$ and $x, y \in M$,

$$p_t^{B_m}(x, \cdot) \in C(\{F_n\}) \ \text{ and } \ p_t^{B_m}(\cdot, y) \in C(\{F_n\}), \tag{5.19}$$

whilst by (5.12), for each $f \in L^1$,

$$\int_M p_t^{B_m}(\cdot, z) f(z) d\mu(z) \in C(\{F_n\}) \text{ and } \int_M p_t^{B_m}(z, \cdot) f(z) d\mu(z) \in C(\{F_n\}). \tag{5.20}$$

On the other hand, since $\{P_t^B\}_{t>0}$ is monotone increasing in B on L^1 by assumption, we have for any $0 \leq f \in L^1$

$$P_t^{B_m} f(x) \leq P_t^{B_{m+1}} f(x) \tag{5.21}$$

for each $t > 0$ and μ-almost all $x \in M$. By (5.16), (5.20), (5.21), and Lemma 3.5(ii), we obtain that for any $f \in L^1$

$$\int_M p_t^{B_m}(\cdot, z) f(z) d\mu(z) \leq \int_M p_t^{B_{m+1}}(z, \cdot) f(z) d\mu(z) \text{ for } x \in M_0 := \bigcup_{n=1}^{\infty} F_n.$$

Therefore, for each $t > 0$ and every $x \in M_0$,

$$p_t^{B_m}(x, y) \leq p_t^{B_{m+1}}(x, y) \quad \mu\text{-a.a. } y \in M.$$

By (5.19) and using Lemma 3.5(ii) again, this inequality holds for every $y \in M_0$.

We now define the function $p_t(x, y)$ for $t > 0$ by

$$p_t(x, y) = \begin{cases} \lim\limits_{m \to \infty} p_t^{B_m}(x, y) & \text{for } x, y \in M_0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.22}$$

Step 3. We verify that $p_t(x, y)$ defined by (5.22) satisfies all the properties in Lemma 5.1. In fact, the joint measurability of $p_t(x, y)$, being a limit of the jointly measurable functions $p_t^{B_m}(x, y)$, is obvious. The positivity (5.5) of $p_t(x, y)$ follows from (5.17) and definition (5.22).

Property (5.7) is also true by using (5.16) and the monotone convergence theorem.

We show the semigroup property (5.6). Note that if one point x or y lies outside the set M_0, then

$$p_{t+s}(x, y) = 0 = \int_M p_t(x, z) p_s(z, y) d\mu(z).$$

If both x and y belong to M_0, then the semigroup property follows from (5.13) and the monotone convergence theorem.

Finally, if $\varphi(B, t) = \varphi(t)$ for any ball B, we see by (2.13)

$$p_t^{B_m}(x, y) \leq \varphi(B_m, t) = \varphi(t)$$

for all $t > 0$ and all $x, y \in M_0$, thus (5.8) follows by taking $m \to \infty$. The proof is complete. □

Next we consider the *coercive closed* form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in L^2 introduced in [8, Definition 2.4 on p.16], and apply Theorem 2.1 to the semigroup corresponding to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Recall that a *coercive closed* form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in L^2 is a bilinear form $\mathcal{E}$ defined on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ satisfying that

- $\mathcal{D}(\mathcal{E})$ is *dense* in L^2, and is *complete* in the norm of $\mathcal{E}_1^{1/2}$ where
$$\mathcal{E}_1(u) := \left(\|u\|_2^2 + \mathcal{E}(u)\right)^{1/2}$$
with $\mathcal{E}(u) := \mathcal{E}(u, u)$.
- $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *positive definite*: $\mathcal{E}(u) \geq 0$ for any $u \in \mathcal{D}(\mathcal{E})$.
- The *weak sector condition* holds: there exists a constant K such that
$$|\mathcal{E}_1(u, v)| \leq K \mathcal{E}_1(u)^{1/2} \mathcal{E}_1(v)^{1/2}$$
for any $u, v \in \mathcal{D}(\mathcal{E})$.

For $u, v \in \mathcal{D}(\mathcal{E})$, we set
$$\widehat{\mathcal{E}}(u, v) = \mathcal{E}(v, u). \tag{5.23}$$
In particular, we see $\widehat{\mathcal{E}}(u) = \mathcal{E}(u)$ for any $u \in \mathcal{D}(\mathcal{E})$. Clearly, if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on L^2 then so is $(\widehat{\mathcal{E}}, \mathcal{D}(\mathcal{E}))$. A coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on L^2 is said to be *symmetric* if $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ for any $u, v \in \mathcal{D}(\mathcal{E})$, and to be *Markovian* if
$$\mathcal{E}(u_+ \wedge 1) \leq \mathcal{E}(u) \quad \text{for any } u \in \mathcal{D}(\mathcal{E}).$$
A coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on L^2 is called a *Dirichlet form* on L^2 if it is symmetric and Markovian, see [8, Definition 4.5 on p.34] or [5, on p.5]. It turns out that a coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in L^2 is uniquely corresponding to a strongly continuous contraction semigroup $\{P_t\}_{t>0}$ on L^2 (cf. [8, Diagram 2 on p.27]) by the relationship
$$\mathcal{E}(u, v) = \lim_{t \to 0} \left(\frac{u - P_t u}{t}, v \right) \tag{5.24}$$
for any $u, v \in \mathcal{D}(\mathcal{E})$. Clearly, the dual semigroup $\{\widehat{P}_t\}_{t>0}$ is unique corresponding to the coercive closed form $(\widehat{\mathcal{E}}, \mathcal{D}(\mathcal{E}))$, since for any $v, u \in \mathcal{D}(\mathcal{E})$,
$$\widehat{\mathcal{E}}(v, u) = \mathcal{E}(u, v) = \lim_{t \to 0} \left(\frac{u - P_t u}{t}, v \right) = \lim_{t \to 0} \left(\frac{v - \widehat{P}_t v}{t}, u \right). \tag{5.25}$$
Moreover, $\{\widehat{P}_t\}_{t>0}$ is also strongly continuous, contractive on L^2.

The following says that the *Nash inequality* associated with a coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on L^2 will imply conditions (2.10), (2.11) in Theorem 2.1. This conclusion was proved in [3, Theorem 2.1] when $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular conservative Dirichlet form in L^2. The following says that this result is still valid for a more general setting.

Lemma 5.2. *Let $\{P_t\}_{t>0}$ be a strongly continuous contraction semigroup on L^2 and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the corresponding coercive closed form on L^2 determined by (5.24). Assume that $\{P_t\}_{t>0}$, $\{\widehat{P}_t\}_{t>0}$ are bounded in the norm of L^1: for all $t > 0, f \in L^1$*
$$\|P_t f\|_{L^1} \leq a \|f\|_{L^1}, \tag{5.26}$$
$$\|\widehat{P}_t f\|_{L^1} \leq a \|f\|_{L^1} \tag{5.27}$$
for some constant $a > 0$. If $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the Nash inequality: there exist three constants $\lambda, \nu > 0$ and $\rho \geq 0$ such that
$$\|f\|_{L^2}^{2(1+\nu)} \leq \lambda \left(\mathcal{E}(f) + \rho \|f\|_{L^2}^2 \right) \|f\|_{L^1}^{2\nu} \quad \textit{for all } f \in \mathcal{D}(\mathcal{E}) \cap L^1, \tag{5.28}$$

then both $\{\widehat{P}_t\}_{t>0}$ and $\{P_t\}_{t>0}$ satisfies the L^1-L^∞ ultra-contractivity: for all $t>0,\ f\in L^1$

$$\max\{\|P_t f\|_{L^\infty}, \|\widehat{P}_t f\|_{L^\infty}\} \le a^2\left(\frac{\lambda}{\nu}\right)^{\frac{1}{\nu}} e^{\rho t} t^{-\frac{1}{\nu}}\|f\|_{L^1}. \tag{5.29}$$

Consequently, all the hypotheses in Theorem 2.1 are satisfied for $\{P_t\}_{t>0}$ and its dual $\{\widehat{P}_t\}_{t>0}$, and thus $\{P_t\}_{t>0}$ possesses an integral kernel $p_t(x,y)$ pointwise defined in $(0,\infty)\times M\times M$ satisfying condition (A_p) with $p=1$, and moreover,

$$|p_t(x,y)| \le a^2\left(\frac{\lambda}{\nu}\right)^{\frac{1}{\nu}} e^{\rho t} t^{-\frac{1}{\nu}} \tag{5.30}$$

for all $t>0$ and all x,y in M.

Proof. Let $\mathcal{A}$ be the domain of the infinitesimal generator of semigroup $\{P_t\}_{t>0}$ on L^2, that is, $\mathcal{A}$ is a subspace of L^2 that consists of all functions f such that there exists some $g\in L^2$ satisfying

$$\lim_{s\to 0}\|s^{-1}(f-P_s f)-g\|_{L^2}=0. \tag{5.31}$$

By the Hille-Yosida theorem, the space $\mathcal{A}$ is dense in L^2 since the semigroup $\{P_t\}_{t>0}$ on L^2 is strongly continuous. For each $t>0$, if $f\in\mathcal{A}$ then

$$P_t f\in\mathcal{A},$$

see for example [**8**, Exercise 1.9, p. 10].

For $t>0, f\in\mathcal{A}$, we have

$$\begin{aligned}
\frac{d}{dt}\|P_t f\|_{L^2}^2 &= \lim_{s\to 0} s^{-1}\left((P_{t+s}f, P_{t+s}f)-(P_t f, P_t f)\right)\\
&= \lim_{s\to 0} s^{-1}\left((P_{t+s}f-P_t f, P_{t+s}f-P_t f)+2(P_{t+s}f-P_t f, P_t f)\right) \qquad (5.32)\\
&= -2\mathcal{E}(P_t f, P_t f), \qquad (5.33)
\end{aligned}$$

since the first term on the right-hand side of (5.32) tends to zero, whilst for the second, we see by (5.24) and the fact that $P_t f\in\mathcal{A}$,

$$s^{-1}(P_{t+s}f-P_t f, P_t f) = -s^{-1}(P_t f - P_s(P_t f), P_t f)\to -\mathcal{E}(P_t f, P_t f) \quad \text{as } s\to 0.$$

Temporally fix $f\in\mathcal{A}$ with $\|f\|_{L^1}\le 1$. By (5.26)

$$\|P_t f\|_{L^1}\le a\|f\|_{L^1}\le a.$$

Let

$$u(t):=e^{-2\rho t}\|P_t f\|_{L^2}^2,\quad t>0.$$

Applying (5.28) with f being replaced by $P_t f$, we see by (5.33)

$$\begin{aligned}
-\frac{d}{dt}u(t) &= 2e^{-2\rho t}(\mathcal{E}(P_t f)+\rho\|P_t f\|_{L^2}^2)\\
&\ge 2e^{-2\rho t}\cdot\frac{1}{\lambda a^{2\nu}}\|P_t f\|_{L^2}^{2(1+\nu)}\\
&= \frac{2}{\lambda a^{2\nu}}e^{-2\rho t}(e^{2\rho t}u(t))^{1+\nu} = \frac{2}{\lambda a^{2\nu}}e^{2\nu\rho t}u(t)^{1+\nu}\ge\frac{2}{\lambda a^{2\nu}}u(t)^{1+\nu}.
\end{aligned}$$

Integrating this inequality over $(0,t)$, we obtain

$$u(t)^{-\nu}\ge\frac{2\nu}{\lambda a^{2\nu}}t.$$

Therefore,

$$\|P_t f\|_{L^2}^2 = e^{2\rho t}u(t)\le e^{2\rho t}\Big(\frac{2\nu}{\lambda a^{2\nu}}t\Big)^{-1/\nu}.$$

For a general non-zero $f \in \mathcal{A}$, we consider function $\frac{f}{\|f\|_{L^1}}$ and have

$$\|P_t f\|_{L^2} \le e^{\rho t}\Big(\frac{2\nu}{\lambda a^{2\nu}}t\Big)^{-1/(2\nu)}\|f\|_{L^1}$$

Since $\mathcal{A}$ is dense in L^2, we conclude that for any $t > 0$,

$$\|P_t\|_{L^1\to L^2} \le \Big(\frac{2\nu}{\lambda a^{2\nu}}t\Big)^{-1/(2\nu)} e^{\rho t}. \tag{5.34}$$

Similarly, we have for any $t > 0$,

$$\|\widehat{P}_t\|_{L^1\to L^2} \le \Big(\frac{2\nu}{\lambda a^{2\nu}}t\Big)^{-1/(2\nu)} e^{\rho t}. \tag{5.35}$$

We show (5.29) by using the semigroup property. Indeed, for any $f, g \in L^1 \cap L^2$

$$\begin{aligned}(P_{2t}f, g) &= (P_t(P_t f), g) = (P_t f, \widehat{P}_t g)\\ &\le \|P_t f\|_{L^2}\|\widehat{P}_t g\|_{L^2}\\ &\le \|P_t\|_{L^1\to L^2}\|f\|_{L^1}\cdot\|\widehat{P}_t\|_{L^1\to L^2}\|g\|_{L^1},\end{aligned}$$

which implies that

$$\begin{aligned}\|P_{2t}f\|_{L^\infty} &= \sup_{\|g\|_{L^1}\le 1}|(P_{2t}f, g)|\\ &\le \|P_t\|_{L^1\to L^2}\|f\|_{L^1}\cdot\|\widehat{P}_t\|_{L^1\to L^2},\end{aligned}$$

thus showing that

$$\|P_{2t}\|_{L^1\to L^\infty} \le \|P_t\|_{L^1\to L^2}\|\widehat{P}_t\|_{L^1\to L^2}. \tag{5.36}$$

Similarly, we have

$$\|\widehat{P}_{2t}\|_{L^1\to L^\infty} \le \|\widehat{P}_t\|_{L^1\to L^2}\|P_t\|_{L^1\to L^2}.$$

It follows from (5.36), (5.34), (5.35) that

$$\|P_{2t}\|_{L^1\to L^\infty} \le \Big(\frac{2\nu}{\lambda a^{2\nu}}t\Big)^{-1/\nu} e^{2\rho t},$$

which gives, after changing $2t$ by t, that

$$\|P_t\|_{L^1\to L^\infty} \le \Big(\frac{\nu}{\lambda a^{2\nu}}t\Big)^{-1/\nu} e^{\rho t}.$$

The same bound for $\|\widehat{P}_t\|_{L^1\to L^\infty}$ is true. Thus (5.29) follows.

Finally, the upper bound of $p_t(x, y)$ in (5.30) follows directly from (2.13) where $U = M$,

$$\varphi(U, t) = \Big(\frac{\nu}{\lambda a^{2\nu}}t\Big)^{-1/\nu} e^{\rho t}.$$

The proof is complete. □

In order to apply Lemma 5.2, one needs to verify the contractivity (5.26), (5.27) of the semigroups $\{P_t\}_{t>0}$ and $\{\widehat{P}_t\}_{t>0}$ in the norm of L^1, respectively. The following provides a criterion in terms of the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Proposition 5.3 ([**12**, Theorem 1.1.5 on p.7]). *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a coercive closed form on L^2. Then the following statements are equivalent.*

(1) *For any $u \in \mathcal{D}(\mathcal{E})$, the function $u_+ \wedge 1 \in \mathcal{D}(\mathcal{E})$ and*

$$\mathcal{E}(u_+ \wedge 1, u - u_+ \wedge 1) \ge 0. \tag{5.37}$$

(2) *$\{P_t\}_{t>0}$ is sub-Markov: for any $f \in L^2$ with $0 \le f \le 1$ μ-a.e., we have $0 \le P_t f \le 1$ μ-a.e. for any $t > 0$.*

(3) *$\{\widehat{P}_t\}$ is positivity preserving and contractive in L^1: if $f \in L^1$ with $f \ge 0$ μ-a.e., then $\widehat{P}_t f \ge 0$ μ-a.e. and $\|\widehat{P}_t f\|_{L^1} \le \|f\|_{L^1}$.*

PROOF. Note that a coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a special *closed form* introduced in [**12**, on page 1] with $\alpha_0 = 0$. Proposition 5.3 follows immediately from [**12**, Theorem 1.1.5 on p.7] wherein the notions $\{T_t\}_{t>0}, \{\widehat{T}_t\}_{t>0}$ are used instead of $\{P_t\}_{t>0}, \{\widehat{P}_t\}_{t>0}$. □

We give an example where all the hypotheses in Lemma 5.2 are satisfied.

EXAMPLE 5.4. Consider the non-symmetric operator $\mathcal{L} = \Delta - b\cdot\nabla - c$ on $\mathbb{R}^n$ for $n \geq 3$, where the functions $b : \mathbb{R}^n \mapsto \mathbb{R}^n$ belong to the Kato class $K_{n,2}$:

$$K_{n,2} := \{b : \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \int_{|x-y|\leq r} \frac{|b(y)|}{|x-y|^{n-2}} dy = 0\},$$

and c is a positive constant. The Kato class $K_{n,2}$ is an extension of spaces L^p when p is large:

$$L^p \subset K_{n,2} \text{ if } \frac{n}{2} < p \leq \infty. \tag{5.38}$$

Indeed, if $u \in L^p$ with $\frac{n}{2} < p \leq \infty$, then by Hölder's inequality, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\int_{B(x,r)} \frac{|u(y)|}{|x-y|^{n-2}} dy &\leq \|u\|_p \left(\int_{B(x,r)} |x-y|^{-(n-2)q} dy\right)^{1/q} \qquad (\text{with } q = \frac{p}{p-1}) \\ &= \|u\|_p \left(\omega_{n-1} \int_0^r s^{-(n-2)q+n-1} ds\right)^{1/q} \\ &= C(n,p) r^{2-n/p} \|u\|_p \to 0 \quad \text{as } r \to 0\end{aligned}$$

since $2 - n/p > 0$, where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, thus showing (5.38).

Let $\mathcal{F} = W_0^{1,2}(\mathbb{R}^n)$ be the usual Sobolev space. The operator $\mathcal{L}$ determines a bilinear form on $\mathcal{F} \times \mathcal{F}$ by

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx + \int_{\mathbb{R}^n} b \cdot \nabla u \cdot v dx + c \int_{\mathbb{R}^n} uv dx, \quad u, v \in \mathcal{F}.$$

We claim that if the constant c is large enough, then $(\mathcal{E}, \mathcal{F})$ is coercive closed form on $L^2(\mathbb{R}^n)$.

To see this, applying [**4**, Theorem 3.25, p. 91], we have that there exists $c_0 = c_0(n,b) > 0$ such that for all $u \in \mathcal{F}$,

$$\int_{\mathbb{R}^n} |b(x)|^2 |u(x)|^2 dx \leq \frac{1}{4} \int_{\mathbb{R}^n} |\nabla u|^2 dx + c_0 \int_{\mathbb{R}^n} |u|^2 dx. \tag{5.39}$$

From this and using the elementary inequality $ab \leq \frac{1}{4}a^2 + b^2$ for $a, b \geq 0$, we have for any $u, v \in \mathcal{F}$

$$\begin{aligned}\left|\int_{\mathbb{R}^n} (b(x)\cdot\nabla u(x)) v(x) dx\right| &\leq \|\nabla u\|_2 \left(\int_{\mathbb{R}^n} |b(x)|^2 |v(x)|^2 dx\right)^{1/2} \\ &\leq \|\nabla u\|_2 \left(\frac{1}{4}\|\nabla v\|_2^2 + c_0 \|v\|_2^2\right)^{1/2} \\ &\leq \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{4}\|\nabla v\|_2^2 + c_0 \|v\|_2^2 .\end{aligned} \tag{5.40}$$

In particular, for any $u \in \mathcal{F}$

$$\left|\int_{\mathbb{R}^n} (b(x)\cdot\nabla u(x)) u(x) dx\right| \leq \frac{1}{2}\|\nabla u\|_2^2 + c_0 \|u\|_2^2 . \tag{5.41}$$

Denote by $\mathbb{D}_1(u,v) = \mathbb{D}(u,v) + (u,v)$ where

$$\mathbb{D}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx.$$

It follows by (5.41) that

$$\begin{aligned}\mathcal{E}(u,u) &= 2\mathbb{D}(u,u) + \int_{\mathbb{R}^n} (b(x)\cdot\nabla u(x))u(x)dx + c\|u\|_2^2 \\ &\geq 2\mathbb{D}(u,u) - \left(\mathbb{D}(u,u) + c_0\|u\|_2^2\right) + c\|u\|_2^2 \\ &= \mathbb{D}(u,u) + (c-c_0)\|u\|_{L^2}^2 \geq \mathbb{D}(u,u) \geq 0 \end{aligned} \tag{5.42}$$

whenever $c \geq c_0$. From this and using (5.41), we have

$$\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + \|u\|_2^2 \geq \mathbb{D}(u,u) + \|u\|_2^2 = \mathbb{D}_1(u,u). \tag{5.43}$$

On the other hand, it follows from (5.40) that for any $u, v \in \mathcal{F}$,

$$\begin{aligned}|\mathcal{E}(u,v)| &= \left| \int_{\mathbb{R}^n} \nabla u \nabla v dx + \int_{\mathbb{R}^n} (b(x)\cdot\nabla u(x))v(x)dx + c\int_{\mathbb{R}^n} uvdx \right| \\ &\leq \|\nabla u\|_2\|\nabla v\|_2 + \|\nabla u\|_2\left(\frac{1}{4}\|\nabla v\|_2^2 + c_0\|v\|_2^2\right)^{1/2} + c\|u\|_2\|v\|_2 \\ &\leq \|\nabla u\|_2\left(\|\nabla v\|_2 + \frac{1}{2}\|\nabla v\|_2 + \sqrt{c_0}\|v\|_2\right) + c\|u\|_2\|v\|_2 \\ &\leq c_1\mathbb{D}_1(u,u)^{1/2}\mathbb{D}_1(v,v)^{1/2} \end{aligned} \tag{5.44}$$

for some constant $c_1 > 0$. Combining this with (5.43), we obtain for all $u \in \mathcal{F}$

$$\mathbb{D}_1(u,u) \leq \mathcal{E}_1(u,u) \leq c_1\mathbb{D}_1(u,u) + \|u\|_2^2 \leq (c_1+1)\mathbb{D}_1(u,u) \tag{5.45}$$

whenever $c \geq c_0$. Therefore, $\mathcal{F}$ is complete in the norm of $\sqrt{\mathcal{E}_1(u,u)}$ if $c \geq c_0$, since $\mathcal{F}$ is complete in $\mathbb{D}_1$-norm. Clearly, $\mathcal{F}$ is dense in L^2, and the form $(\mathcal{E},\mathcal{F})$ is positive definite if $c \geq c_0$.

We need to verify the weak sector condition. Indeed, we have by (5.44), (5.45) that for all $u, v \in \mathcal{F}$,

$$\begin{aligned}|\mathcal{E}_1(u,v)| &= |\mathcal{E}(u,v) + (u,v)| \leq c_1\mathbb{D}_1(u,u)^{1/2}\mathbb{D}_1(v,v)^{1/2} + \|u\|_2\|v\|_2 \\ &\leq (c_1+1)\mathbb{D}_1(u,u)^{1/2}\mathbb{D}_1(v,v)^{1/2} \leq (c_1+1)\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}\end{aligned}$$

whenever $c \geq c_0$, thus proving the weak sector condition. Therefore, the form $(\mathcal{E},\mathcal{F})$ is a coercive closed form in $L^2(\mathbb{R}^n)$ when c is large enough.

In order to show the Nash inequality, note that for any $u \in \mathcal{F}\cap L^1(\mathbb{R}^n)$,

$$\|u\|_2^{2(1+2/n)} \leq c_2\mathbb{D}(u,u)\|u\|_1^{4/n}.$$

From this, we have by (5.42)

$$\|u\|_2^{2(1+2/n)} \leq c_2\mathcal{E}(u,u)\|u\|_1^{4/n},$$

thus showing that $(\mathcal{E},\mathcal{F})$ satisfies the Nash inequality (5.28) with $\lambda = c_2$, $\rho = 0$ and $\nu = \frac{2}{n}$.

We show that $\{\widehat{P}_t\}_{t>0}$ is contractive in $L^1(\mathbb{R}^n)$. Indeed, for any $u \in \mathcal{F}$, we have $u_+\wedge 1 \in \mathcal{F}$, and

$$u_+\wedge 1 = \begin{cases} 1, & \text{if } u > 1, \\ u, & \text{if } 0 < u \leq 1, \\ 0, & \text{if } u \leq 0, \end{cases} \qquad u - u_+\wedge 1 = \begin{cases} u-1, & \text{if } u > 1, \\ 0, & \text{if } 0 < u \leq 1, \\ u, & \text{if } u \leq 0, \end{cases}$$

$$\nabla(u_+\wedge 1) \overset{\text{a.e.}}{=} \begin{cases} \nabla u, & \text{if } 0 < u \leq 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \nabla(u - u_+\wedge 1) \overset{\text{a.e.}}{=} \begin{cases} 0, & \text{if } 0 < u \leq 1, \\ \nabla u, & \text{otherwise}. \end{cases}$$

If follows that

$$\mathcal{E}(u_+\wedge 1, u - u_+\wedge 1) = 0 + 0 + c\int_{\{u>1\}} (u-1)dx \geq 0,$$

and condition (5.37) is true. Thus, $\{\widehat{P}_t\}$ is contractive in L^1 by Proposition 5.3.

It remains to show that $\|P_t\|_{L^1\to L^1}$ is uniformly bounded in t. We will show that

$$\|P_t\|_{L^1\to L^1} \le 1 \text{ for all } t>0 \tag{5.46}$$

when the functions $b \in K_{n,2}$ further satisfies

$$\operatorname*{esup}_{\mathbb{R}^n} \operatorname{div} b \le c. \tag{5.47}$$

In fact, let $u := P_t f$ for any non-negative $f \in L^2$. Then

$$\frac{\partial}{\partial t}u = \mathcal{L}u = \Delta u - b\cdot\nabla u - cu,$$

where $\frac{\partial}{\partial t}u$ is understood the Fréchet derivative with respect to the inner product of L^2. Integrating over $\mathbb{R}^n$, we have by (5.47)

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} u(t,x)dx &= \int_{\mathbb{R}^n} (\Delta u(x) - b(x)\cdot\nabla u(x) - cu(x))dx \\ &= 0 + \int_{\mathbb{R}^n} (\operatorname{div} b(x) - c)u(x)dx \quad \text{(integration by parts)} \\ &\le 0.\end{aligned}$$

From this, we see for all $t>0$

$$\|P_t f\|_{L^1} = \int_{\mathbb{R}^n} u(t,x)dx \le \int_{\mathbb{R}^n} u(0,x)dx = \|f\|_{L^1},$$

thus showing (5.46).

Therefore, all the hypotheses in Lemma 5.2 are satisfied where $a=1$, and hence, the semigroup $\{P_t\}$ associated with the coercive closed form $(\mathcal{E},\mathcal{F})$ in L^2 possesses an integral kernel $p_t(x,y)$ satisfying condition (A_p) with $p=1$, and by (5.30) with $a=1$, $\lambda = c_2$, $\rho = 0$, $\nu = \frac{2}{n}$,

$$|p_t(x,y)| \le \left(\frac{c_2 n}{2}\right)^{n/2} t^{-\frac{n}{2}}$$

for all $t>0$ and all x,y in M.

There is a plenty of examples in which functions $b\in K_{n,2}$ satisfies (5.47), for instance, $b = (f_1, f_2, \cdots, f_n)$ where

$$f_i(x) = \frac{c}{n}\exp(-|x_i|) \quad \text{for } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \quad (1\le i\le n),$$

since each $f_i \in L^\infty \subset K_{n,2}$ by virtue of (5.38).

In the remainder of this section, we briefly state another application of Theorem 2.1 related with Green function.

Proposition 5.5. *Let (M,d,μ) be a metric measure space. Let $\{P_t\}_{t>0}$ be a semigroup on L^1 and $\widehat{P}_t = P_t$. Assume that $\{P_t\}_{t>0}$ possesses an integral kernel $q_t(x,y)$ such that for each $t>0$,*

$$|q_t(x,y)| \le \phi(t) \tag{5.48}$$

for μ-almost all x,y in M, where $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a measurable function. Then there exist a regular μ-nest $\{F_n\}_{n=1}^\infty$ and a pointwise defined version $p_t(x,y)$ of $q_t(x,y)$ in $(0,\infty)\times M\times M$, such that the following properties are true: for each $t,s>0$ and all $x,y\in M$,

1. *$p_t(\cdot,\cdot)$ is jointly measurable in $M\times M$;*
2. *$p_t(x,\cdot)$ and $p_t(\cdot,y)$ belong to $C(\{F_n\})$, and $|p_t(x,y)| \le \varphi(t)$;*
3. *$p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)d\mu(z)$;*
4. *$P_t f(x) = \int_M p_t(x,y)f(y)d\mu(y)$ for μ-almost all $x\in M$.*

Proof. By (5.48), we have for any $t>0$, $f\in L^1$,

$$\|\widehat{P}_t f\|_{L^\infty} = \|P_t f\|_{L^\infty} = \left\|\int q_t(\cdot,y)f(y)d\mu(y)\right\|_{L^\infty} \le \phi(t)\|f\|_{L^1}.$$

It follows that all the hypothesis of Theorem 2.1 are satisfied, and Proposition 5.5 follows. □

Proposition 5.5 has the following advantage. One may define the Green function $G(x,y)$ as the integral of a heat kernel $q_t(x,y)$ with respect to dt, that is,

$$G(x,y)=\int_0^\infty q_t(x,y)dt.$$

However, this integral may not be well defined, because $q_t(x,y)$ is defined for $\mu\times\mu$-almost all $(x,y)\in M\times M$ where the null set may depend on t. Proposition 5.5 says that one can use a pointwise defined version $p_t(x,y)$, instead of $q_t(x,y)$ itself, to define the Green function by

$$G(x,y)=\int_0^\infty p_t(x,y)dt, \tag{5.49}$$

since we have a common measurable set $\mathcal{N}$ independent of t with $\mu(\mathcal{N})=0$, and the integral in (5.49) makes sense in this way.

6. Proof of Theorem 2.2

In this section we prove Theorem 2.2. We shall use the following results in the proof.

Proposition 6.1 ([5, Lemma 2.1.3 on p.69]). *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on L^2. Given an $\mathcal{E}$-nest $\{F_k\}$ of M, let $F_k' = \mathrm{supp}[\mathbf{1}_{F_k}.\mu]$ for each k. Then $F_k'\subset F_k$ for each $k\ge 1$, and $\{F_k'\}$ is a regular $\mathcal{E}$-nest.*

For a Dirichlet form $(\mathcal{E},\mathcal{F})$, a function u is *$\mathcal{E}$-quasi continuous* if for any $\varepsilon>0$, there is an open set $G\subset M$ such that $\mathrm{cap}(G)<\varepsilon$ and $u|_{M\setminus G}$ is finite continuous (cf. [5, on p.69]).

Lemma 6.2 ([5, Theorem 2.1.2 on p.69]). *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on L^2. The following statements are true.*

- (i) *Let $S=\{u_l, l\ge 1\}$ be a countable family of $\mathcal{E}$-quasi continuous functions on M. Then there is a common regular $\mathcal{E}$-nest $\{F_k\}$ of M such that $S\subset C(\{F_k\})$.*
- (ii) *Let $\{F_k\}$ be a regular $\mathcal{E}$-nest on M and u belongs to $C(\{F_k\})$. If $u\ge 0$ μ-almost everywhere on an open set U, then $u(x)\ge 0$ for every point $x\in U\cap\left(\cup_{k=1}^\infty F_k\right)$.*

Lemma 6.3 ([5, Theorem 2.1.3 on p.71]). *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on L^2. Then each function u in $\mathcal{F}$ has an $\mathcal{E}$-quasi continuous modification $\widetilde{u}$, that is, function $\widetilde{u}$ is $\mathcal{E}$-quasi continuous and $u=\widetilde{u}$ almost everywhere in M.*

We need to assume that $(\mathcal{E},\mathcal{F})$ is regular in Lemma 6.3. We begin to prove Theorem 2.2.

Proof of Theorem 2.2. As shown in the proof of Theorem 2.1, it suffices to consider the case when $T_0=\infty$. We sketch the proof, since the argument is similar to that for Theorem 2.1. In fact, one needs only to replace μ-nest in the proof of Theorem 2.1 by $\mathcal{E}$-nest here, and the rest argument keeps the same but much simpler since $\{P_t\}_{t>0}$ is symmetric: $P_t=\widehat{P}_t$. Let $p\in[1,2]$, and note that $\{P_t\}$ is contractive on L^q ($q\in[1,\infty]$), that is,

$$\|P_t f\|_{L^p}\le\|f\|_{L^q}\quad t>0,\ f\in L^q.$$

Step 1**.** We show that there exists a pointwise realization $Q_t f$ for $P_t f$ when $f\in L^p$, $t>0$, and also a *common* regular $\mathcal{E}$-nest $\{F_n\}_{n=1}^\infty$ of M such that for *all* $t>0$ and *all* $f\in L^p$,

$$Q_t f\in C(\{F_n\}). \tag{6.1}$$

Indeed, note that if $f \in L^2 \cap L^p(M)$, then

$$P_t f \in \mathcal{F} \quad \text{for any } t > 0 \tag{6.2}$$

(cf. [5, Lemma 1.3.3(i) on p.23]). Since $(\mathcal{E}, \mathcal{F})$ is regular, the space $\mathcal{F} \cap C_0(M)$ is dense in $C_0(M)$ in the supremum norm. Using the fact that $C_0(M)$ is dense in L^p, we see that $\mathcal{F} \cap C_0(M)$ is dense in L^p. Since L^p is separable, there exists a sequence $\{f_k\}_{k=1}^{\infty}$ from $\mathcal{F} \cap C_0(M)$ dense in L^p. It follows from (6.2) that the function $P_t f_k \in \mathcal{F}$ for each $t > 0, k \geq 1$, and thus it has an $\mathcal{E}$-quasi-continuous version $h_{t,k}$ by using Lemma 6.3. Consider the countable family

$$\{h_{s,k} : s \in \mathbb{Q}_+, k \geq 1\},$$

where $\mathbb{Q}_+$ is the set of all positive rational numbers as before. By Lemma 6.2(i), there exists a common regular $\mathcal{E}$-nest $\{F_n\}_{n=1}^{\infty}$ such that

$$h_{s,k} \in C(\{F_n\}) \quad \text{for all } s \in \mathbb{Q}_+, k \geq 1. \tag{6.3}$$

Set $M_0 := \cup_{n=1}^{\infty} F_n$ and $\mathcal{N} = M \setminus M_0$. Clearly, $\mu(\mathcal{N}) = \mathrm{cap}(\mathcal{N}) = 0$.

We will extend (6.3) to any function f in L^p (not only for f_k) by using assumption (2.17), and then continue to extend it to any real positive t (not only for rationals s) by using the semigroup property. To do this, similar to (4.5), we have for each $x \in U$ in $\mathcal{S}$ and $k, j \geq 1$,

$$\begin{aligned}
\sup_{x \in U \cap M_0} |h_{s,k}(x) - h_{s,j}(x)| &= \sup_{x \in U \setminus \mathcal{N}} |h_{s,k}(x) - h_{s,j}(x)| \\
&= \|h_{s,k} - h_{s,j}\|_{L^\infty(U)} \quad (\text{using } h_{s,k} - h_{s,j} \in C(\{F_n\}) \\
&\qquad\qquad\qquad\qquad\qquad \text{and Lemma 6.2}(ii)) \\
&= \|P_s f_k - P_s f_j\|_{L^\infty(U)} \quad (\text{using } h_{s,k} \overset{\text{a.e.}}{=} P_s f_k \ \forall k) \\
&\leq \varphi(U, s) \|f_k - f_j\|_{L^p} \quad (\text{using } (2.17)).
\end{aligned} \tag{6.4}$$

For any $f \in L^p$, there is a sequence $\{f_{k_i}\}_{i \geq 1}$ from $\{f_k\}_{k \geq 1}$ such that $\|f_{k_i} - f\|_{L^p} \to 0$ as $i \to \infty$. Thus by (6.4), for each $s \in \mathbb{Q}_+$ and $U \in \mathcal{S}$, the sequence $\{h_{s,k_i}\}_{i \geq 1}$ converges uniformly to a function, say, $Q_s f$, in $U \cap M_0$. Since $U \in \mathcal{S}$ is arbitrary, and $\mathcal{S}$ covers M, for any $s \in \mathbb{Q}_+$ and $f \in L^p$, we can define the function $Q_s f$ on M by

$$Q_s f(x) = \begin{cases} \lim_{i \to \infty} h_{s,k_i}(x) & \text{for } x \in M_0 = \bigcup_{U \in \mathcal{S}} (U \cap M_0), \\ 0 & \text{for } x \in \mathcal{N}. \end{cases} \tag{6.5}$$

It follows by (6.3) that

$$Q_s f \in C(\{F_n\}) \tag{6.6}$$

for all $s \in \mathbb{Q}_+, k \geq 1$ and all $f \in L^p$.

Similarly to (4.9), we can prove that

$$Q_s(P_{t-s} f)(x) = Q_{s'}(P_{t-s'} f)(x) \quad \text{for } \textit{every } x \in M \text{ and } s, s' \in (0, t) \cap \mathbb{Q}_+.$$

Consequently, we can extend $Q_s f$ in (6.5) to any positive real number t by defining

$$Q_t f(x) = Q_s(P_{t-s} f)(x), f \in L^p, \ x \in M, \tag{6.7}$$

where s is a positive rational smaller than t. Note that the above formula is consistent when t is rational.

Then $\{Q_t\}_{t>0}$ satisfies the following properties.

- For each $t > 0$, we have

  $$Q_t = P_t \text{ on } L^p.$$

  Moreover, for any $x \in \mathcal{N}$ and any $f \in L^p$,

  $$Q_t f(x) = 0.$$

- For each $t>0, x\in U\in\mathcal{S}$ and each $f\in L^p$, we have
$$|Q_tf(x)|\le\|P_tf\|_{L^\infty(U)}\le\varphi(U,t)\|f\|_{L^p}.$$
- For *all* $t>0$ and *all* $f\in L^p$, we have
$$Q_tf\in C(\{F_n\}).$$
- $\{Q_t\}_{t>0}$ satisfies the semigroup property: for any real $t_1,t_2>0, f\in L^p$ and any $x\in M$,
$$Q_{t_1+t_2}f(x)=Q_{t_1}(Q_{t_2}f)(x).$$
- Q_t is bounded and linear: for each $t>0$
$$\|Q_t\|_{L^p\mapsto L^p}=\|P_t\|_{L^p\mapsto L^p}\le 1<\infty,$$
$$Q_t(af+bg)(x)=aQ_tf(x)+bQ_tg(x)$$
for all $x\in M$, $a,b\in\mathbb{R}$ and $f,g\in L^p$, and
$$(Q_tf,g)=(P_tf,g)=(f,P_tg)=(f,Q_tg).$$

Step 2. We show that the semigroup $\{Q_t\}_{t>0}$ possesses an integral kernel $q_t(x,y)$. More precisely, let $p'=\frac{p}{p-1}\in[2,\infty]$ be the Hölder conjugate of p. Then, for each $t>0$ and $x\in U\in\mathcal{S}$, there exists a function $q_t(x,\cdot)$ in $L^{p'}$ such that for any $f\in L^p$,
$$Q_tf(x)=\int_M q_t(x,y)f(y)d\mu(y),\tag{6.8}$$
$$\|q_t(x,\cdot)\|_{L^{p'}}\le\varphi(U,t),\tag{6.9}$$
and
$$\sup_{x\in M}\|q_t(x,\cdot)\|_{L^1}\le 1.\tag{6.10}$$
Consequently, by (6.9), (6.10) and Hölder inequality, we obtain for any $q\in[1,p']$,
$$\sup_{x\in U}\|q_t(x,\cdot)\|_{L^q}\le(\varphi(U,t))^{(q-1)(p-1)}.\tag{6.11}$$
In particular, for any $t>0$, $q_t(x,\cdot)\in L^p$ since $p\in[1,p']$.

Note that the function $q_t(x,\cdot)$ is defined for each $t>0$ and each $x\in M$, and
$$q_t(x,\cdot)=0\text{ in }M\tag{6.12}$$
whenever $x\in\mathcal{N}$ and $t>0$.

Step 3. We construct the desired $p_t(x,y)$ by using function $q_t(x,y)$. Indeed, we can define $p_t(x,y)$ for any $t>0$ and any $x,y\in M$ by
$$p_t(x,y)=\int_M q_{t/2}(x,z)q_{t/2}(y,z)d\mu(z).\tag{6.13}$$
Note that the integral in the right hand side of (6.13) is well defined by (6.9) and the fact that $q_t(x,\cdot)\in L^p$. Similar to (4.42), we have for any $t>0$ and any $x\in M$
$$Q_tf(x)=\int_M p_t(x,z)f(z)d\mu(z).\tag{6.14}$$

Finally, we verify that the function $p_t(x,y)$ defined by (6.13) is a heat kernel. We only need to verify the symmetry and positivity of $p_t(x,y)$, and (2.20). Other properties can be verified as in *Step* 3 of the proof of Theorem 2.1.

Indeed, symmetry follows directly from the definition (6.13). Positivity can be verified by the similar arguments in Corollary 4.2(1). It remains to prove (2.20). By (6.8) and (6.14), we obtain that for any $t>0$ and $x\in M$,
$$p_t(x,\cdot)\overset{\text{a.e.}}{=}q_t(x,\cdot).$$
This together with (6.11) yields (2.20).

The proof is complete. □

Remark 6.4. Consider the special case when $\mathcal{S} = \{M\}$, $p = 1$ and $T_0 = \infty$ in Theorem 2.2. In this case, by the fact that $p_t(x, \cdot)$ is quasi-continuous, the inequality (2.19) becomes the *diagonal upper estimate*: for any $t > 0$

$$p_t(x, y) \le \varphi(M, t) \quad x, y \in M.$$

This result in this special case was already addressed in [**2**, Theorem 3.1]. However, the authors used the joint measurability of the function $p_0(t, x, y)$ (whose counterpart is $q_t(x, y)$ in (6.8) in our paper) in (x, y) without proof; see formula [**2**, (3.5)] and formulas following it, neither did they prove the joint measurability of $p_t(x, y)$ in (x, y) for any fixed $t > 0$. Note also that the function $M(t)$ in [**2**, Theorem 3.1] was assumed to be left continuous. While, the function $\varphi(M, t)$ in Theorem 2.2 is not assumed to be left continuous.

Remark 6.5. Under the assumption in Theorem 2.2, when $\mathcal{S} = \{M\}$, one can prove by duality that condition (2.17) with $p = 1$ is equivalent to that with $p = 2$. While in the present settings, it is not clear that whether they are equivalent or not. Roughly speaking, if we denote the the function in condition (2.17) by φ_p, then condition (2.17) with $p = 1$ implies that the heat kernel satisfies the *diagonal upper estimate*:

$$p_t(x, y) \le \varphi_1(U, t) \quad t \in (0, T_0),\ x \in U \in \mathcal{S},\ y \in M. \tag{6.15}$$

While, (2.17) with $p = 2$ implies that the heat kernel satisfies the *on-diagonal upper estimate*:

$$p_t(x, x) = \|p_{t/2}(x, \cdot)\|_{L^2}^2 \le \varphi_2(U, t/2)^2 \quad t \in (0, T_0),\ x \in U \in \mathcal{S}. \tag{6.16}$$

Clearly, (6.15) implies (6.16) with $\varphi_2(U, t) := \sqrt{\varphi_1(U, 2t)}$. While, it is not clear whether (6.16) implies (6.15) with some function φ_1 determined by φ_2.

Remark 6.6. Note that the number p in Theorem 2.2 is assumed to be in $[1, 2]$. In fact, under the assumption of Theorem 2.2 but with $p \in (2, \infty)$, we can follow the arguments of Lemma 5.1 to obtain all the results of Theorem 2.2 except that $p_t(x, \cdot) \in C(\{F_n\})$. The reason is as follows. In the case when $p \in (2, \infty)$, we have $p' = \frac{p}{p-1} < 2 < p$, then the combination of (6.9) and (6.10) can not guarantee that $q_{t/2}(x, \cdot) \in L^p$ as done in the case when $p \in [1, 2]$. Consequently, by (6.13), the function $p_t(x, \cdot) = Q_{t/2} q_{t/2}(x, \cdot)(\cdot)$ may not be in $C(\{F_n\})$.

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