Stirling formula and Gauss integral by using Wallis product

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Abstract

The purpose of this note is to give elementary (accessible to 1st year students) proofs of three formulas mentioned in the title.

1 Wallis product

Theorem 1 *We have for* $n \in \mathbb{N}$

$$\frac{(2^n n!)^2}{(2n)!} \sim \sqrt{\pi n} \quad \text{as } n \to \infty. \tag{1}$$

An equivalent formulation:

$$\lim_{k\to\infty} \frac{(2\cdot 4\cdot \ldots \cdot 2k)^2}{(3\cdot 5\cdot \ldots \cdot (2k-1))^2 (2k+1)} = \frac{\pi}{2} \tag{2}$$

Proof. Set for all $n \in \mathbb{Z}_+$

$$I_n = \int_0^\pi \sin^n x \, dx.$$

We prove the following properties of I_n.

(a) $I_{n-1} \geq I_n > 0$. Since for $0 < x < \pi$ we have $0 < \sin x \leq 1$, it follows that

$$0 < \sin^n x \leq \sin^{n-1} x,$$

which implies $0 < I_n \leq I_{n-1}$.

(b) $I_0 = \pi$ and $I_1 = 2$. Indeed, we have

$$I_0 = \int_0^\pi dx = \pi \quad \text{and} \quad I_1 = \int_0^\pi \sin x \, dx = -\left[\cos x\right]_0^\pi = 2.$$

(c) $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$, which is proved by integrations by parts:

$$\begin{aligned}
I_n &= \int_0^\pi \sin^n x\,dx = -\int_0^\pi \sin^{n-1} x d\cos x \\
&= -\left[\sin^{n-1} x \cos x\right]_0^\pi + \int_0^\pi \cos x d \sin^{n-1} x \\
&= (n-1)\int_0^\pi \cos^2 x \sin^{n-2} x\,dx \\
&= (n-1)\int_0^\pi \left(1-\sin^2 x\right)\sin^{n-2} x\,dx \\
&= (n-1)\left(I_{n-2} - I_n\right),
\end{aligned}$$

whence $I_n = \frac{n-1}{n} I_{n-2}$ follows.

(d) $\lim_{n\to\infty} \frac{I_n}{I_{n-1}} = 1$. It follows from (c) and (a), that

$$\frac{n-1}{n} = \frac{I_n}{I_{n-2}} \leq \frac{I_n}{I_{n-1}} \leq 1.$$

Since $\frac{n-1}{n} \to 1$, we obtain that $\frac{I_n}{I_{n-1}} \to 1$ as $n \to \infty$.

(e) For any $k \in \mathbb{Z}_+$

$$I_{2k+1} = 2\frac{2\cdot 4\cdot \ldots \cdot (2k)}{3\cdot 5\cdot \ldots \cdot (2k+1)} \quad \text{and} \quad I_{2k} = \pi \frac{1\cdot 3\cdot \ldots \cdot (2k-1)}{2\cdot 4\cdot \ldots \cdot (2k)}\ .$$

Induction in k. For $k=0$ these identities are satisfied by (b).

Induction step from $k-1$ to k. Assuming that

$$I_{2k-1} = 2\frac{2\cdot 4\cdot \ldots \cdot (2k-2)}{3\cdot 5\cdot \ldots \cdot (2k-1)},$$

we obtain by (c)

$$I_{2k+1} = \frac{2k}{2k+1} I_{2k-1} = 2\frac{2\cdot 4\cdot \ldots \cdot (2k-2)\cdot (2k)}{3\cdot 5\cdot \ldots \cdot (2k-1)\cdot (2k+1)}.$$

The second identity for I_{2k} is proved similarly.

(f) $I_{n-1} I_n = \frac{2\pi}{n}$. It follows from (e), that

$$I_{2k} I_{2k+1} = \pi \frac{1\cdot 3\cdot \ldots \cdot (2k-1)}{2\cdot 4\cdot \ldots \cdot (2k)} \cdot 2\frac{2\cdot 4\cdot \ldots \cdot (2k)}{3\cdot 5\cdot \ldots \cdot (2k+1)} = \frac{2\pi}{2k+1},$$

which proves $I_{n-1} I_n = \frac{2\pi}{n}$ for $n = 2k+1$. For the case $n = 2k$ we have similarly

$$I_{2k-1} I_{2k} = 2\frac{2\cdot 4\cdot \ldots \cdot (2k-2)}{3\cdot 5\cdot \ldots \cdot (2k-1)} \cdot \pi \frac{1\cdot 3\cdot \ldots \cdot (2k-1)}{2\cdot 4\cdot \ldots \cdot (2k)} = \frac{2\pi}{2k}.$$

(g) $I_n \sim \sqrt{\frac{2\pi}{n}}$. It follows from (f), that

$$I_n^2 = I_n I_{n-1} \frac{I_n}{I_{n-1}} = \frac{2\pi}{n}\frac{I_n}{I_{n-1}},$$

whence

$$\frac{I_n^2}{2\pi/n} = \frac{I_n}{I_{n-1}} \to 1,$$

and $I_n^2 \sim \frac{2\pi}{n}$. It follows that

$$I_n = \int_0^\pi \sin^n x dx \sim \sqrt{\frac{2\pi}{n}}. \tag{3}$$

Finally, let us prove (1). For $n = 2k+1$ we have by (g)

$$2\frac{2\cdot 4\cdot ...\cdot (2k)}{3\cdot 5\cdot ...\cdot (2k+1)} \sim \sqrt{\frac{2\pi}{2k+1}},$$

whence

$$\frac{2\cdot 4\cdot ...\cdot (2k)}{3\cdot 5\cdot ...\cdot (2k-1)} \sim \sqrt{\frac{\pi}{2}(2k+1)} \sim \sqrt{\pi k}.$$

The left hand side is equal to

$$\frac{(2\cdot 4\cdot ...\cdot (2k))^2}{3\cdot 5\cdot ...\cdot (2k-1)\cdot 2\cdot 4\cdot ...\cdot 2k} = \frac{\left(2^k k!\right)^2}{(2k)!}$$

whence (1) follows. ■

2 Stirling formula

Theorem 2 *We have for $n \in \mathbb{N}$*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad as \quad n\to\infty. \tag{4}$$

Proof. The asymptotic identity (4) is equivalent to

$$\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!} \to 1 \ \text{ as } n\to\infty,$$

which is equivalent to

$$\ln\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!} \to 0 \ \text{ as } n\to\infty,$$

that is

$$\left(\frac{1}{2}\ln n + n\ln\frac{n}{e}\right) - (\ln 1 + \ln 2 + ... + \ln n) \to \ln\frac{1}{\sqrt{2\pi}} \ \text{ as } n\to\infty.$$

We first prove that

$$\lim_{n\to\infty}\left(n\ln\frac{n}{e} - \left(\ln 2 + \ln 3 + ... + \ln(n-1) + \frac{1}{2}\ln n\right)\right) \tag{5}$$

exists and is finite, and then we compute the value of the limit.

Consider the function $f(x)$ on $[1,\infty)$ defined by the following conditions:

1. $f(n) = \ln n$ for all $n \in \mathbb{N}$
2. for $x \in [n, n+1]$, $f(x)$ is a linear function.

By the concavity of $\ln x$, we see that

$$f(x) \le \ln x \quad \text{for all } x \ge 1.$$

It follows that

$$a_n := \int_1^n (\ln x - f(x))\, dx, \quad n \in \mathbb{N},$$

is a non-negative monotone increasing sequence. We have

$$\int_1^n \ln x\, dx = n \ln n - n + 1 = n \ln \frac{n}{e} + 1$$

and

$$\begin{aligned} \int_1^n f(x)\, dx &= \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\, dx = \sum_{k=1}^{n-1} \frac{f(k) + f(k+1)}{2} \\ &= \ln 2 + \dots + \ln(n-1) + \frac{1}{2} \ln n\ . \end{aligned}$$

Hence, we obtain

$$a_n = n \ln \frac{n}{e} + 1 - \left(\ln 2 + \dots + \ln(n-1) + \frac{1}{2} \ln n \right).$$

Hence, the existence of the limit (5) is equivalent to existence of the limit $\lim_{n\to\infty} a_n$. Since $\{a_n\}$ is increasing, the limit does exist but we need still to show that it is finite. For that write

$$\begin{aligned} a_n &= \sum_{k=1}^{n-1} \int_k^{k+1} (\ln x - f(x))\, dx \\ &\le \sum_{k=1}^{\infty} \int_k^{k+1} (\ln x - f(x))\, dx \\ &= \sum_{k=1}^{\infty} \left((k+1) \ln \frac{k+1}{e} - k \ln \frac{k}{e} - \frac{\ln k + \ln(k+1)}{2} \right) \\ &= \sum_{k=1}^{\infty} \left((k+1) \ln(k+1) - k \ln k - \frac{1}{2} \ln k - \frac{1}{2} \ln(k+1) - 1 \right) \\ &= \sum_{k=1}^{\infty} \left(\left(k + \frac{1}{2} \right) (\ln(k+1) - \ln k) - 1 \right) \\ &= \sum_{k=1}^{\infty} \left(\left(k + \frac{1}{2} \right) \ln \left(1 + \frac{1}{k} \right) - 1 \right). \end{aligned}$$

Expanding $\ln\left(1 + \frac{1}{k}\right)$ in Taylor series, we have

$$\ln\left(1 + \frac{1}{k}\right) = \frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} + o\left(\frac{1}{k^3}\right) \quad \text{as } k \to \infty,$$

whence

$$\begin{aligned}\left(k+\frac{1}{2}\right)\ln\left(1+\frac{1}{k}\right)-1 &= \left(k+\frac{1}{2}\right)\left(\frac{1}{k}-\frac{1}{2k^2}+\frac{1}{3k^3}+o\left(\frac{1}{k^3}\right)\right)-1\\ &= \left(1-\frac{1}{2k}+\frac{1}{3k^2}\right)+\left(\frac{1}{2k}-\frac{1}{4k^2}\right)-1+o\left(\frac{1}{k^2}\right)\\ &= \frac{1}{12k^2}+o\left(\frac{1}{k^2}\right)\\ &\sim \frac{1}{12k^2} \text{ as } k\to\infty.\end{aligned}$$

Hence, the series

$$\sum_{k=1}^{\infty}\left(\left(k+\frac{1}{2}\right)\ln\left(1+\frac{1}{k}\right)-1\right)$$

is convergent as it is equivalent to the convergent series $\sum\frac{1}{k^2}$. It follows that the sequence $\{a_n\}$ is bounded and, hence, $\lim a_n$ exists and is finite.

Consequently, the limit (5) exists. It follows that also the following limit exists

$$\lim_{n\to\infty}\frac{\sqrt{n}\left(\frac{n}{e}\right)^n}{n!}$$

and is a positive number. That is, for some constant $c>0$,

$$n!\sim c\sqrt{n}\left(\frac{n}{e}\right)^n. \tag{6}$$

To determine c, compute

$$\frac{(2^n n!)^2}{(2n)!}\sim\frac{\left(2^n c\sqrt{n}\left(\frac{n}{e}\right)^n\right)^2}{c\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}=\frac{c^2 2^{2n} n\, n^{2n}e^{-2n}}{c\sqrt{2n}\,2^{2n}n^{2n}e^{-2n}}=c\sqrt{\frac{n}{2}}.$$

By Theorem 1 we conclude

$$c\sqrt{\frac{n}{2}}\sim\sqrt{\pi n},$$

whence $c=\sqrt{2\pi}$. Substituting c into (6), we obtain (4). ■

3 Gauss integral

Theorem 3 *We have*

$$\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}.$$

Proof. Consider for any $n\in\mathbb{N}$ the function

$$f_n(x)=\frac{1}{\left(1+\frac{x^2}{n}\right)^n}.$$

It is known that the sequence $\left\{\left(1+\frac{x^2}{n}\right)^n\right\}_{n\in\mathbb{N}}$ is monotone increasing and converges to e^{x^2}. Hence, the sequence $\{f_n(x)\}$ is monotone decreasing and converges to e^{-x^2}. Since

$$f_n(x) \leq f_1(x) = \frac{1}{1+x^2}$$

and f_1 is integrable on $\mathbb{R}$, we obtain by the dominated convergence theorem that

$$\int_{-\infty}^{\infty} f_n(x)\, dx \to \int_{-\infty}^{\infty} e^{-x^2} dx. \tag{7}$$

On the other hand,

$$\int_{-\infty}^{\infty} f_n(x)\, dx = \int_{-\infty}^{\infty} \frac{dx}{\left(1+\frac{x^2}{n}\right)^n} = \sqrt{n}\int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^n}.$$

The change $y = \cot t$, $t \in (0,\pi)$ yields

$$\int_{-\infty}^{\infty} f_n(x)\, dx = \sqrt{n}\int_0^{\pi} \frac{dt}{\sin^2 t\left(1+\frac{\cos^2 t}{\sin^2 t}\right)^n} = \sqrt{n}\int_0^{\pi} \sin^{2n-2} t dt.$$

By (3), we have

$$\int_0^{\pi} \sin^{2n-2} t dt \sim \sqrt{\frac{2\pi}{2n-2}} \sim \sqrt{\frac{\pi}{n}} \quad \text{as } n \to \infty,$$

which implies

$$\int_{-\infty}^{\infty} f_n(x)\, dx \to \sqrt{\pi},$$

which together with (7) finishes the proof. ■