Surgery of the Faber-Krahn inequality and applications to heat kernel bounds

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1 Introduction

Let M be a non-compact connected Riemannian manifold and let μ be the Riemannian measure on M. For each non-empty open subset $\Omega \subset M$, denote by $\lambda(\Omega)$ the first eigenvalue of the Dirichlet problem in Ω for the Laplace-Beltrami operator Δ. The *Faber-Krahn*

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inequality is a lower bound on $\lambda(\Omega)$ in terms of the volume $\mu(\Omega)$ as follows:

$$\lambda(\Omega) \geq \Lambda(\mu(\Omega)), \tag{1.1}$$

where Λ is a non-negative function on $(0, +\infty)$. The function Λ is called the *Faber-Krahn function* of an open set $U \subset M$ if (1.1) holds for all $\Omega \subset U$. Since $\lambda(\Omega)$ decreases on expansion of Ω, we will always assume that the Faber-Krahn function is monotone decreasing.

Recall that the classical Faber-Krahn theorem states that, for any open set $\Omega \subset \mathbb{R}^N$,

$$\lambda(\Omega) \geq \lambda(B),$$

where B is the Euclidean ball with volume $\mu(B) = \mu(\Omega)$. It is easy to see that $\lambda(B) = c_N \mu(B)^{-2/N}$. Hence, according to the definition given above, $\mathbb{R}^N$ has the Faber-Krahn function

$$\Lambda(v) = c_N\, v^{-2/N}.$$

This paper describes how Faber-Krahn inequalities and Faber-Krahn functions behave under removal of a compact set with smooth boundary (Section 2.2, Proposition 2.1 and Theorem 2.4) and under gluing of several non-compact manifolds (Section 3, Theorem 3.3). This is somewhat a technical goal but these results should prove useful in various situations. In particular, they extend (in a sense) those of [2] where a Sobolev inequality for the exterior of certain compact domains in $\mathbb{R}^N$ was proved.

One specific application of these cutting and gluing results is presented in detail in Section 4. It concerns with the problem of estimating of the heat kernel on a manifold with ends. To describe more precisely this application, let us assume that M is geodesically complete and let $K \subset M$ be a compact set with smooth boundary such that $M \setminus K$ has k connected components $E_1, \ldots, E_k$. The sets E_i are called the *ends* of M with respect to K.

Furthermore, in many cases each end E_i can be considered as the exterior of a compact set with smooth boundary in another complete manifold M_i. In this case we say that M is a *connected sum* of $M_1, ..., M_k$ and write

$$M = \bigsqcup_{i=1}^{k} M_i$$

(see Section 3.1 for a careful definition).

Now, suppose that each M_i is a non-compact complete manifold for which we have a good heat kernel upper bound. What information can we obtain for the heat kernel on the connected sum M?

The study of the relationships between heat kernel bounds and functional inequalities (such as Faber-Krahn inequalities and others) has been an active area of research during the past decades (see, e.g., [4], [21], [8], [11]). In view of the previous experience is natural to attack the above question about heat kernel bounds on connected sums of manifolds by using the Faber-Krahn inequalities, which is done in this paper.

We obtain fairly satisfactory heat kernel bounds that are easy to apply in some cases. For example, let us consider the special case when each end E_i is the exterior of a compact with smooth boundary in a non-compact complete manifold M_i with non-negative Ricci curvature. Let $V_i(x,r)$ be the volume of the geodesic ball in M_i of radius r and center $x \in M_i$. For any $r>0$, set

$$V_{\min}(r) = \min_{1\le i\le k} V_i(o_i,r),$$

where $o_i \in \partial E_i$ is a fixed reference point. In this situation we prove that, for all $t>0$,

$$\sup_{x,y\in K} p(t,x,y) \le \frac{C}{V_{\min}(\sqrt{t})}. \tag{1.2}$$

(see Theorem 4.5). The estimate (1.2) is used in our paper [13] as a key ingredient for obtaining two-sided estimates of $p(t,x,y)$ for the full range $x,y\in M$ and $t>0$ in the above setting. In particular, it follows from [13] that (1.2) is sharp, that is, has a matching lower bound, provided each manifold M_i is non-parabolic.

We denote by the letters c, C, c', C' etc positive constants whose values can change at each occurrence.

2 Cutting Faber-Krahn inequalities

In this section we show that the Faber-Krahn inequality is roughly preserved under the removal of a compact set with smooth boundary.

2.1 FK-functions

Let (M,g) be a non-compact Riemannian manifold possibly with boundary[1] δM. Fix a positive smooth function σ on M and consider a Radon measure μ on M defined by

$$d\mu = \sigma^2 d\mu_0,$$

where μ_0 is the Riemannian measure on M. The couple (M,μ) is called a *weighted manifold*. The operator

$$\mathcal{L}u = \sigma^{-2}\operatorname{div}(\sigma^2\nabla u)$$

is defined on functions $u\in C^2(M)$ is called the Laplace operator of (M,μ). In particular, in the case $\sigma \equiv 1$ it coincides with the Laplace-Beltrami operator of the Riemannian manifold M. The operator $\mathcal{L}$ obviously satisfies the Green formula for all $u,v\in C_c^\infty(M)$

$$\int_M u\,\mathcal{L}v\,d\mu = -\int_M (\nabla u,\nabla v)d\mu = \int_M \mathcal{L}u\,v\,d\mu,$$

although in the case of non-empty boundary δM we have to assume in addition that u and v satisfy the Neumann boundary condition on δM.

It follows that $\mathcal{L}$ is symmetric with respect to measure μ and admits the Friedrichs extension that is a self-adjoint operator in $L^2(M,\mu)$ that will be denoted also by $\mathcal{L}$ (cf. [11]).

[1] We use the symbol δ to denote the boundary of a "manifold with boundary" M as opposed to the symbol ∂ that denotes the topological boundary of a subset. Note that the topological boundary ∂M of a "manifold with boundary" M is always empty. It would be preferable to use another term for δM, for example, the *border*, and to call M a "manifold with border" but, unfortunately, the confusing term "manifold with boundary" is commonly used and cannot be easily changed.

Let $d(x,y)$ be the geodesic distance between x and y. Let $B(x,r)$ be the open geodesic ball of radius r around x. We say that the Riemannian manifold (M,g) is complete if the metric space (M,d) is complete. It is known that the completeness of M is equivalent to the fact that all geodesic balls $B\left(x,r\right)$ are precompact.

For any region $\Omega\subset M$, we denote by $\lambda(\Omega)$ the first Dirichlet eigenvalue for the operator $\mathcal{L}$ in Ω. More precisely,

$$\lambda(\Omega):=\inf_{\phi\in\mathcal{C}_0^\infty(\Omega)}\frac{\int_\Omega|\nabla\phi|^2\,d\mu}{\int_\Omega\phi^2d\mu}.$$

Note that if M has a boundary then, topologically, Ω can contain points of δM as *interior* points. At those points, the test function ϕ does not necessarily vanish. Therefore, in this case, $\lambda(\Omega)$ is the smallest eigenvalue of $\mathcal{L}$ in Ω satisfying the Dirichlet condition on $\partial\Omega$ and the Neumann condition on $\delta M\cap\Omega$. Nonetheless, we will always refer to $\lambda(\Omega)$ as the Dirichlet eigenvalue, in order not to overload the terminology.

Definition. Let U be an open subset of M. We say that a non-negative function $\Lambda(v)$ on $(0,\infty)$ is a *Faber-Krahn function* (FK-function) for U (or we say that U admits a FK-function Λ) if Λ is non-increasing on $(0,\infty)$ and, for any precompact open subset Ω of U,

$$\lambda(\Omega)\geq\Lambda(\mu(\Omega)). \tag{2.1}$$

Clearly, $\Lambda\equiv 0$ is always a FK-function but, of course, only positive FK-functions are of interest. It can happen that M itself has a positive FK-function, that is, (2.1) holds for all precompact open subsets Ω of M. For example, in the case $M=\mathbb{R}^N$ the classical Faber-Krahn theorem implies that $\mathbb{R}^N$ has the FK-function

$$\Lambda\left(v\right)=cv^{-2/N} \tag{2.2}$$

where $c=c\left(N\right)>0$. It follows by a compactness argument that, for any weighted manifold M of dimension N, any precompact open set $U\subset M$ admits the FK-function

$$\Lambda\left(v\right)=c_Uv^{-2/N} \tag{2.3}$$

with some constant $c_U>0$ that depends on U.

It is known that any Cartan-Hadamard manifold of dimension N admits the FK-function (2.2) but with a different value of c (see [15]).

The fact that M has the FK-function (2.2) is equivalent to the Nash inequality:

$$\forall\, f\in\mathcal{C}_0^\infty(M),\quad \|f\|_2^{2(1+2/N)}\leq C\left(\int_M|\nabla f|^2d\mu\right)\|f\|_1^{4/N}$$

and, in the case $N>2$, to the Sobolev inequality

$$\forall\, f\in\mathcal{C}_0^\infty(M),\quad \|f\|_{2N/(N-2)}^2\leq C\int_M|\nabla f|^2d\mu$$

(see [1], [14]).

Let M be complete non-compact manifold that covers a compact manifold. Fix some reference point x_0 and set $V(r) = \mu(B(x_0, r))$. It follows from the isoperimetric inequality of [3, Theorem 4], that M has the FK-function

$$\Lambda(v) = cr(v)^{-2},$$

where the function $r(v)$ is determined by the equation $Cv = V(r)$ and c, C are positive constants depending only on M. The same result holds for any non-compact connected real unimodular Lie group M of dimension N equipped with an invariant Riemannian metric.

Let us consider two explicit examples of the volume growth function on a covering manifold M.

Example. If, for large r, $V(r) \geq Cr^{\nu}$, where ν is a positive constant, then $r(v) \leq Cv^{1/\nu}$ and

$$\Lambda(v) = cv^{-2/\nu} \tag{2.4}$$

for large v.

If there exists $0 < \alpha \leq 1$ such that, for large r,

$$V(r) \geq \exp(cr^{\alpha}), \tag{2.5}$$

then $r(v) \leq C(\log v)^{1/\alpha}$ and

$$\Lambda(v) = \frac{c}{(\log v)^{2/\alpha}}, \tag{2.6}$$

for large v. In all cases for small v we have

$$\Lambda(v) = cv^{-2/N}.$$

Example. In the case of Lie group M as above there are two possibilities for the volume growth function: either $V(r) \asymp r^{\nu}$ for a positive integer ν or $\log V(r) \asymp r$. In the first case we obtain that M has the FK-function (2.4). In the second case, either M is amenable and then, for large v,

$$\Lambda(v) = \frac{c}{\log^2 v}$$

or M is non-amenable and then, for large v, $\Lambda(v) = c > 0$ (see [17]).

Example. Let us give one more example of different type. Let $M = \mathbb{R}^n \times K$ where K is a compact Riemannian manifold of dimension $N - n$. Then M admits the FK-function

$$\Lambda(v) = c \begin{cases} v^{-2/N}, & v < 1, \\ v^{-2/n}, & v \geq 1 \end{cases}$$

(see [5]).

On the other hand, for a general manifold one cannot expect to have a positive FK-function. The following notion is more flexible.

Definition. Let $(B, v) \mapsto \Lambda(B, v)$ be a non-negative function where $B = B(x, r)$ varies among all geodesic balls of M and $v \in (0, +\infty)$. We say that Λ is a *relative FK-function* for (M,μ) (RFK-function) if $v \mapsto \Lambda(B,v)$ is a FK-function of the ball B; that is, the function $v \mapsto \Lambda(B,v)$ is non-increasing in v and, for any ball B in M and for any open set $\Omega \subset B$,

$$\lambda(\Omega) \geq \Lambda(B,\mu(\Omega)). \tag{2.7}$$

A complete non-compact manifold M always admits a RFK-function of the from

$$\Lambda(B,v) = a(B)v^{-2/N} \tag{2.8}$$

where $N = \dim M$ and $a(B) > 0$ (cf. (2.3)), which follows from the fact that B is precompact. An important class of manifolds for which $a(B)$ can be estimated explicitly, is the class of complete manifolds with non-negative Ricci curvature, μ being the Riemannian measure. For such a manifold one has

$$a(B) \asymp \rho(B)^{-2}\mu(B)^{2/N}, \tag{2.9}$$

where $\rho(B)$ is the radius of B (cf. [7, Theorem 1.4 and Theorem 2.1], [16], [18]).

2.2 Cutting a manifold

Let (M,μ) be a non-compact connected weighted manifold of dimension N. Let us fix a compact set $K \subset M$ that is the closure of a non-empty open set with smooth boundary[2] such that $M \setminus K$ is connected. Consider the set

$$M^* := M \setminus \overset{o}{K}$$

as a manifold with boundary $\delta M^* = \delta M \sqcup \partial K$. We will equip with the superscript $*$ all the notation related to M^*; in particular, M^* is endowed with the measure $\mu^* = \mu|_{M^*}$.

By construction M^* is complete and connected. Denote by d^* the geodesic distance on M^* and by $B^*(x,r)$ geodesic balls in M^*. Obviously, we have $d^*(x,y) \geq d(x,y)$ for all $x,y \in M^*$, which implies the inclusion

$$B^*(x,r) \subset B(x,r) \tag{2.10}$$

of the balls, for all $x \in M^*$ and $r > 0$.

The connectedness of $M \setminus K$ implies that there is a precompact open subset U of M with smooth boundary such that $K \subset U$ and $U \setminus K$ is connected (for example, U can be taken as a ball of large enough radius centered at K). The set U is used in all statements in this section.

Our goal is to provide a lower bound for the Dirichlet eigenvalue $\lambda^*(\Omega^*)$ for open sets $\Omega^* \subset M^*$ in terms of such quantities on M. If Ω^* is disjoint with K then Ω^* is also an open subset of M and we have $\lambda^*(\Omega^*) = \lambda(\Omega^*)$ and there is nothing to do. However, if $\Omega^* \cap K = \Omega^* \cap \partial K$ is non-empty then $\lambda^*(\Omega^*)$ is the first eigenvalue of the operator $\mathcal{L}^* = \mathcal{L}$ in Ω^* with the Dirichlet condition on $\partial\Omega^*$ and the Neumann condition on

$$\delta M^* \cap \Omega^* = (\delta M \cap \Omega^*) \sqcup (\partial K \cap \Omega^*),$$

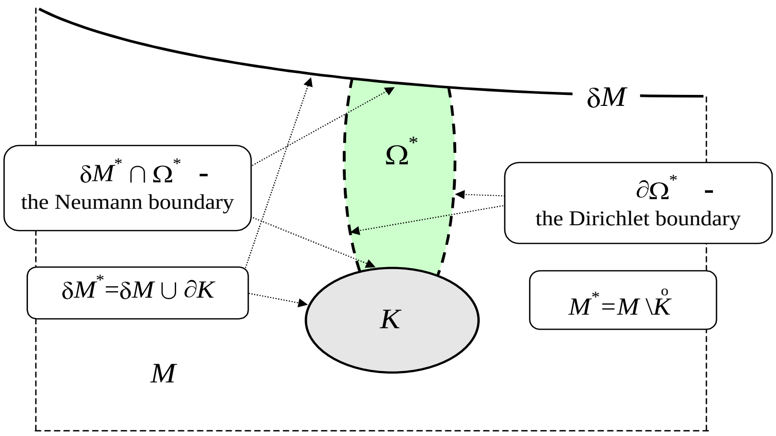

Figure 1: The eigenvalue problem for $\lambda^* (\Omega^*)$

as on Fig. 1.

Consider the set $\Omega = \Omega^* \setminus \partial K$ that is an open subset of M and observe that $\lambda(\Omega)$ is the first eigenvalue of $\mathcal{L}$ in Ω with the Dirichlet condition on $\partial\Omega = \partial\Omega^* \cup (\partial K \cap \Omega^*)$ and the Neumann condition on $\delta M \cap \Omega$. Hence, $\lambda(\Omega)$ in comparison with $\lambda^*(\Omega^*)$ has additional piece of the Dirichlet boundary which implies that

$$\lambda(\Omega) \geq \lambda^*(\Omega^*).$$

Therefore, obtaining a *lower* bound for $\lambda^*(\Omega^*)$ is a non-trivial task, that will be discussed in this section.

2.3 Non-parabolic case

To illustrate some technique that can be used for obtaining lower bounds for $\lambda^*(\Omega^*)$, we treat first the case when the weighted manifold (M, μ) is *non-parabolic* (see [10] for a detailed discussion of this notion). The crucial property that we will use is the following. A weighted manifold (M, μ) is non-parabolic if and only if for any open precompact set U there exists a constant $C_M(U)$ (which is called the non-parabolicity constant of U in M), such that

$$\forall\, f \in \mathcal{C}_c^\infty(M), \quad \int_U |f|^2 d\mu \leq C_M(U) \int_M |\nabla f|^2 d\mu. \tag{2.11}$$

Given two subsets $A \Subset B$ of M, we say that a function ϕ is a cutoff function of the pair A, B if $\phi \in \mathcal{C}_c^\infty(M)$, $0 \leq \phi \leq 1$, $\operatorname{supp}\phi \subset B$ and $\phi \equiv 1$ in an open neighborhood of $\overline{A}$.

Proposition 2.1 *Let (M, μ) be a non-parabolic. Under the above assumptions there exists a constant $c > 0$ such that, for any open subset $\Omega^* \subset M^*$,*

$$\lambda^*(\Omega^*) \geq c \min(\lambda(\Omega), 1), \tag{2.12}$$

[2]if M is a manifold with boundary then we assume in addition that ∂K and δM are disjoint

where $\Omega = \Omega^* \setminus \partial K$. The constant c depends only on $d(K, \partial U)$ and on the non-parabolicity constant $C_{M^*}(U \setminus K)$.

Proof. Observe that the manifold M^* is also non-parabolic (cf. [6]). Let ϕ a cutoff function of the pair K, U. Set

$$C_\phi = \sup_U |\nabla \phi|^2 .$$

Clearly, by choosing ϕ appropriately, C_ϕ can be bounded from above in terms of $d(K, \partial U)$. The restriction $\phi|_{M^*}$ also denote by ϕ. For any function $f \in \mathcal{C}_c^\infty(\Omega^*)$, we have $f = f_1 + f_2$ where

$$f_1 = f\phi, \quad f_2 = f(1-\phi)$$

(cf. Fig. 2).

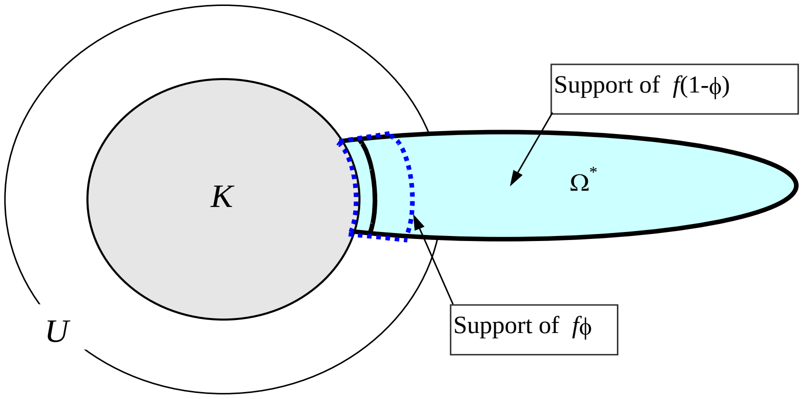

Figure 2: Functions $f\phi$ and $f(1-\phi)$

Since (M^*, μ^*) is non-parabolic, we have $C^* := C_{M^*}(U \setminus K) < \infty$ and, hence,

$$\int_{U \setminus K} |f|^2 d\mu \le C^* \int_{M^*} |\nabla f|^2 d\mu = C^* \int_\Omega |\nabla f|^2 d\mu. \tag{2.13}$$

Note that

$$\begin{aligned} \int_\Omega |\nabla f_1|^2 d\mu &\le 2 \int_\Omega |\nabla f|^2 \phi^2 d\mu + 2 \int_\Omega f^2 |\nabla \phi|^2 d\mu \\ &\le 2 \left(\int_\Omega |\nabla f|^2 d\mu + C_\phi \int_{U \setminus K} |f|^2 d\mu \right). \end{aligned}$$

Substituting here (2.13), we obtain

$$\int_\Omega |\nabla f_1|^2 d\mu \le 2(1 + C^* C_\phi) \int_\Omega |\nabla f|^2 d\mu.$$

The same estimate holds for f_2:

$$\begin{aligned} \int_\Omega |\nabla f_2|^2 d\mu &\le 2 \left(\int_\Omega |\nabla f|^2 d\mu + C_\phi \int_{U \setminus K} |f|^2 d\mu \right) \\ &\le 2(1 + C^* C_\phi) \int_\Omega |\nabla f|^2 d\mu. \end{aligned}$$

The function f_2 is compactly supported in the set $\Omega = \Omega^* \setminus K \subset M$ and thus

$$\int_\Omega f_2^2 d\mu \le \frac{1}{\lambda(\Omega)} \int_\Omega |\nabla f_2|^2 d\mu.$$

Using again the non-parabolicity of (M^*, μ^*), we obtain

$$\int_\Omega f_1^2 d\mu = \int_{U \setminus K} f_1^2 d\mu \le C^* \int_{M^*} |\nabla f_1|^2 d\mu = C^* \int_\Omega |\nabla f_1|^2 d\mu.$$

Combining all the above estimates, we obtain

$$\begin{aligned}
\int_\Omega f^2 d\mu &\le 2\left(\int_\Omega f_1^2 d\mu + \int f_2^2 d\mu\right) \\
&\le 2C^* \int_\Omega |\nabla f_1|^2 d\mu + \frac{2}{\lambda(\Omega)} \int_\Omega |\nabla f_2|^2 d\mu \\
&\le 4\left(C^* + \frac{1}{\lambda(\Omega)}\right)(1 + C^* C_\phi) \int_\Omega |\nabla f|^2 d\mu,
\end{aligned}$$

which implies (2.12). ■

2.4 FK-functions in balls

Now we pass to the case of a general (=possibly parabolic) manifold (M, μ). The main result of this section will be Theorem 2.4 below.

Instead of the non-parabolicity constant we will use the following three local versions of the Poincaré inequality. Let $W \subset M$ be a precompact open set.

1. For any $g \in W_0^1(W)$
$$\int_W g^2 d\mu \le C \int_W |\nabla g|^2 d\mu. \tag{2.14}$$

2. If W has smooth boundary and connected, then, for any $g \in W^1\left(\overline{W}\right)$,
$$\int_W (g - m)^2 d\mu \le C \int_W |\nabla g|^2 d\mu, \tag{2.15}$$
where $m = \frac{1}{\mu(W)} \int_M g d\mu$.

3. Moreover, if in the case 2
$$\mu(g \le 0) \ge \frac{1}{2}\mu(W)$$
then also
$$\int_W g_+^2 d\mu \le C \int_W |\nabla g_+|^2 d\mu \tag{2.16}$$
(cf. [7, Theorem 1.2]).

In all cases the constant C depends only on the intrinsic geometry of W.

Lemma 2.2 *Let U be a precompact open subset of M with smooth boundary such that $K \subset U$ and $U \setminus K$ is connected. There exists a constant $c > 0$ such that for any open subset $\Omega^* \subset M^* := M \setminus K$, we have*

$$\lambda^* (\Omega^*) \geq c \min (\lambda (\Omega^* \cup U), 1). \tag{2.17}$$

The constant c depends only of the local geometry of K and U.

Proof. Note that Ω is an open subset of M. If $\Omega^* \cap \partial K = \emptyset$ then the Neumann boundary condition on $\partial K \cap \Omega^*$ is void, and we obtain

$$\lambda^* (\Omega^*) = \lambda (\Omega^*) \geq \lambda (\Omega).$$

Consider a general case when $\Omega^* \cap \partial K$ is non-empty. Set

$$\Omega = \Omega^* \cup U.$$

We shall prove that, for any function $f \in \mathcal{C}_c^\infty (\Omega^*)$,

$$\int_{\Omega^*} f^2 d\mu \leq C \left(\frac{1}{\lambda(\Omega)} + 1 \right) \int_{\Omega^*} |\nabla f|^2 \, d\mu, \tag{2.18}$$

which then implies (2.17) with $c = (2C)^{-1}$. Let us extend f to a function in $\mathcal{C}_c^\infty (M^*)$ by setting $f = 0$ outside Ω^*. Set

$$m = \frac{1}{\mu (U \setminus K)} \int_{U \setminus K} f \, d\mu$$

and $\widetilde{f} = f - m$. Since the function f is smooth in $U \setminus K$, we obtain by the Poincaré inequality (2.15) in $U \setminus K$

$$\int_{U \setminus K} \widetilde{f}^2 d\mu \leq C_P \int_{U \setminus K} |\nabla f|^2 \, d\mu. \tag{2.19}$$

Choose an open subset V of M with smooth boundary such that $K \Subset V \Subset U$. Let h be the harmonic function in V such that $h = f$ on ∂V; set $\widetilde{h} = h - m$. Fix a function $\varphi \in \mathcal{C}_c^\infty (V)$ such that $\varphi \equiv 1$ in a neighborhood of K.

The function $(1 - \varphi) \widetilde{f}$ vanishes in a neighborhood of K and, hence, can be smoothly extended to the whole M by setting it to be 0 on K. Since the functions $\widetilde{h}$ and $(1 - \varphi) \widetilde{f}$ are defined in V and have the same boundary values on ∂V, the Dirichlet principle yields

$$\int_V \left| \nabla \widetilde{h} \right|^2 \, d\mu \leq \int_V \left| \nabla \left((1 - \varphi) \widetilde{f} \right) \right|^2 \, d\mu.$$

On the other hand, we have

$$\begin{aligned} \int_V \left| \nabla \left((1 - \varphi) \widetilde{f} \right) \right|^2 \, d\mu &= \int_{V \setminus K} \left| \nabla \left((1 - \varphi) \widetilde{f} \right) \right|^2 \, d\mu \\ &\leq C_\varphi \int_{U \setminus K} \left(\widetilde{f}^2 + \left| \nabla \widetilde{f} \right|^2 \right) d\mu, \end{aligned}$$

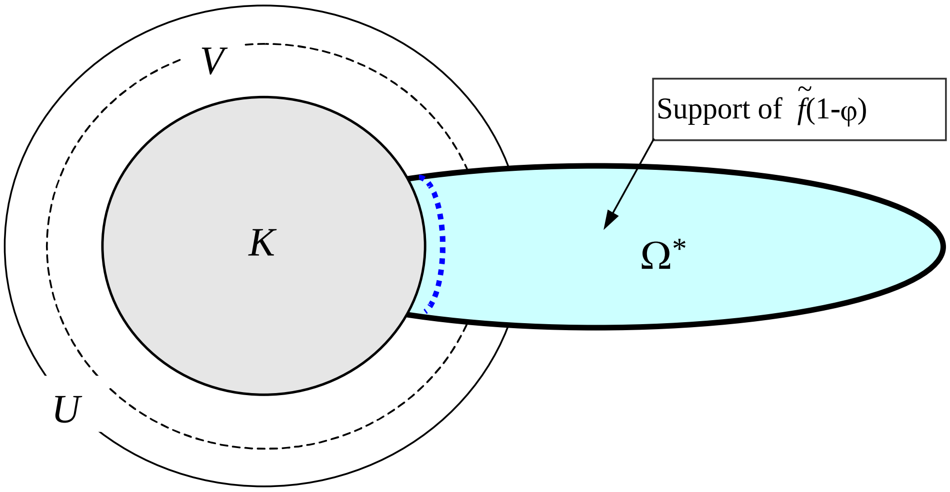

Figure 3: Set Ω^* and function $\widetilde{f}(1-\varphi)$

where C_φ depends only on the function φ. Combining with (2.19) and using $\nabla\widetilde{f} = \nabla f$, we obtain

$$\int_V \left|\nabla\left((1-\varphi)\widetilde{f}\right)\right|^2 d\mu \le C_\varphi\left(1+C_P\right)\int_{U\setminus K}|\nabla f|^2\, d\mu,$$

whence

$$\int_V |\nabla h|^2\, d\mu = \int_V \left|\nabla\widetilde{h}\right|^2 d\mu \le C_\varphi\left(1+C_P\right)\int_{U\setminus K}|\nabla f|^2\, d\mu. \tag{2.20}$$

Consider in Ω the function

$$g = \begin{cases} h & \text{in } V, \\ f & \text{in } \Omega\setminus V. \end{cases}$$

Since $g \in W_0^1(\Omega)$, we obtain by (2.20)

$$\|g\|^2_{L^2(\Omega)} \le \lambda(\Omega)^{-1}\|\nabla g\|^2_{L^2(\Omega)} \le \lambda(\Omega)^{-1}\left(1+C_\varphi\left(1+C_P\right)\right)\int_{\Omega\setminus K}|\nabla f|^2\, d\mu. \tag{2.21}$$

On the other hand, we have

$$\|f\|_{L^2(\Omega\setminus K)} \le \|g\|_{L^2(\Omega\setminus K)} + \|f-g\|_{L^2(V\setminus K)} \le \|g\|_{L^2(\Omega)} + \|f-h\|_{L^2(V\setminus K)}\,. \tag{2.22}$$

Consider the function $f-h$ in $V^* = V\setminus \overset{o}{K}$ on the manifold M^*. This function vanishes on $\partial V = \partial^* V^*$, so that $f-h \in W_0^1(V^*)$. Hence, using the Poincaré inequality (2.14) in V^* and (2.20), we obtain

$$\begin{aligned}\|f-h\|^2_{L^2(V\setminus K)} &\le C'_P\int_{V\setminus K}|\nabla(f-h)|^2\, d\mu \\ &\le 2C'_P\int_{V\setminus K}|\nabla f|^2 d\mu + 2C'_P\int_{V\setminus K}|\nabla h|^2 d\mu \\ &\le 2C'_P\left(1+C_\varphi\left(1+C_P\right)\right)\int_{U\setminus K}|\nabla f|^2\, d\mu,\end{aligned}$$

where the constant C_P' depends only on the intrinsic geometry of V^*.

Finally, combining (2.21)-(2.22) we obtain

$$\begin{aligned}\|f\|^2_{L^2(\Omega\setminus K)} &\leq 2\|g\|^2_{L^2(\Omega)} + 2\|f-h\|^2_{L^2(V\setminus K)} \\ &\leq \left(\frac{2}{\lambda(\Omega)} + 4C_P'\right)(1+C_\varphi(1+C_P))\int_{\Omega\setminus K}|\nabla f|^2\, d\mu,\end{aligned}$$

which is equivalent to (2.18). ■

Lemma 2.3 *Let U be a precompact open subset of M with smooth boundary such that $K \subset U$ and $U \setminus K$ is connected. There exists a constant $c > 0$ such that for any open subset $\Omega^* \subset M^* := M \setminus K$ with*

$$\mu(\Omega^*) \leq \frac{1}{2}\mu(U\setminus K) \tag{2.23}$$

the following inequality holds:

$$\lambda^*(\Omega^*) \geq c\,\min\left(\lambda(\Omega), \mu(\Omega^*)^{-2/N}\right), \tag{2.24}$$

where $\Omega = \Omega^* \setminus \partial K$. The constant $c > 0$ depends only of the local geometry of K and U.

Proof. Fix an open neighborhood V of K such that $\overline{V} \subset U$, and a cutoff function ψ of the pair V, U, that is, a function $\psi \in \mathcal{C}_c^\infty(U)$ such that $0 \leq \psi \leq 1$ and $\psi|_V \equiv 1$. For any $f \in \mathcal{C}_c^\infty(\Omega^*)$, we have

$$\int_{\Omega^*} f^2\, d\mu \leq 2\int_\Omega (f\psi)^2\, d\mu + 2\int_\Omega ((1-\psi)f)^2\, d\mu. \tag{2.25}$$

$(1-\psi)f$ is supported in $\Omega \subset M$, we have

$$\begin{aligned}\int_\Omega ((1-\psi)f)^2\, d\mu &\leq \lambda(\Omega)^{-1}\int_\Omega |\nabla((1-\psi)f)|^2\, d\mu \\ &\leq 2\lambda(\Omega)^{-1}\int\left(f^2|\nabla\psi|^2 + |\nabla f|^2(1-\psi)^2\right)d\mu \\ &\leq C_\psi\lambda(\Omega)^{-1}\left(\int\limits_{U\setminus K} f^2\, d\mu + \int_\Omega |\nabla f|^2\, d\mu\right),\end{aligned}$$

where we have used the boundedness of ψ and $|\nabla\psi|$.

Since ψf is supported in U, the first term on the right hand side of (2.25) can be bounded, by using the Faber-Krahn inequality in U, as follows:

$$\begin{aligned}\int_U |f\psi|^2\, d\mu &\leq C_{FK}\mu(\Omega^*)^{2/N}\int_U |\nabla(f\psi)|^2\, d\mu \\ &\leq C_{FK}C_\psi\mu(\Omega^*)^{2/N}\left(\int\limits_{U\setminus K} f^2\, d\mu + \int_\Omega |\nabla f|^2\, d\mu\right).\end{aligned}$$

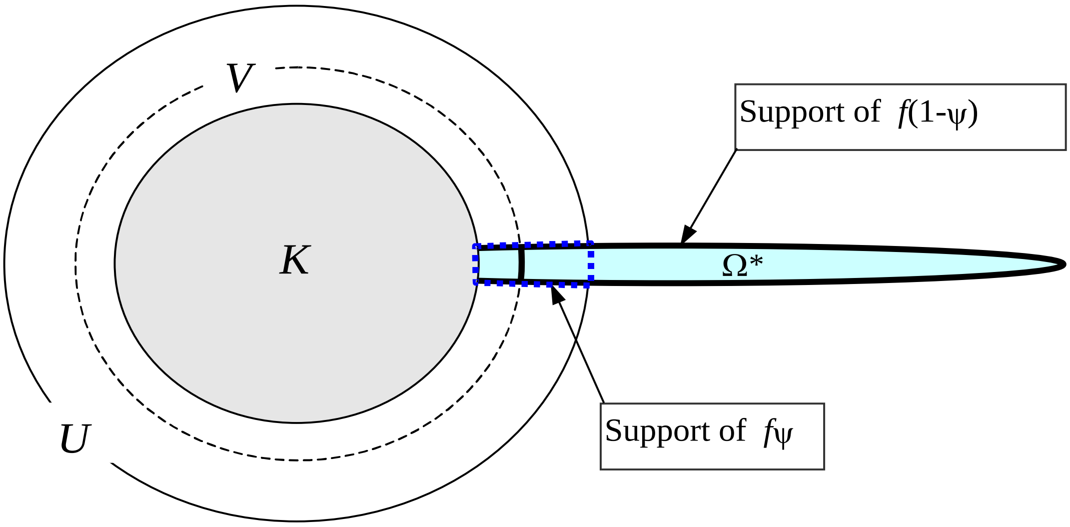

Figure 4: Set Ω^*

Hence, we obtain

$$\int_{\Omega^*} f^2 \, d\mu \leq C_\psi \left(\lambda(\Omega)^{-1} + C_{FK}\mu\left(\Omega^*\right)^{2/N} \right) \left(\int\limits_{U \setminus K} f^2 \, d\mu + \int_{\Omega} |\nabla f|^2 \, d\mu \right). \tag{2.26}$$

By hypothesis (2.23), we have

$$\mu\left(\{f = 0\} \cap U \setminus K\right) \geq \frac{1}{2}\mu\left(U \setminus K\right).$$

Applying the Poincaré inequality (2.15) to f_+ and f_-, we obtain

$$\int\limits_{U \setminus K} f^2 \, d\mu \leq C_P \int_{U \setminus K} |\nabla f|^2 \, d\mu. \tag{2.27}$$

Therefore, (2.26) yields

$$\int_{\Omega^*} f^2 \, d\mu \leq C_\psi \left(\lambda(\Omega)^{-1} + C_{FK}\mu\left(\Omega^*\right)^{2/N} \right) (1 + C_P) \int_{\Omega^*} |\nabla f|^2 \, d\mu\,,$$

whence (2.24) follows. ∎

Theorem 2.4 *Let K be a compact subset of M with smooth boundary such that $M^* := M \setminus K$ is connected. There exist constants $c \in (0,1)$ and $P, Q > 1$ such that any ball $B^*(x, R)$ in M^* admits the Faber-Krahn function*

$$v \mapsto \Lambda^*(v) := c\Lambda(Qv)\,, \tag{2.28}$$

where Λ is the Faber-Krahn function of the ball $B(x, PR)$ in M.

As we see from the proof, the constants c, P, Q depend only on the intrinsic geometry of some precompact neighbourhood of K but, of course, they do not depend on x, R.

Proof. Since $M \setminus K$ is connected, there exists a precompact open neighborhood U of K such that $U \setminus K$ is connected. Let Ω^* be an open subset of a ball $B^*(x, R)$ in M^*; set $v := \mu(\Omega^*)$. To prove (2.28), we have to show that

$$\lambda^*(\Omega^*) \geq c\Lambda(Qv). \tag{2.29}$$

Set $\Omega = \Omega^* \setminus \partial K$ and so that Ω is a subset of M. Moreover, by (2.10) we have

$$\Omega \subset B(x, R).$$

Let us consider the following cases.

Case 1. Assume that $B(x, R)$ does not intersect K. Then $\lambda^*(\Omega^*)$ and $\lambda(\Omega)$ are the eigenvalues of the same boundary value problem, whence by (FK) in $B(x, PR)$ and by the monotonicity of $\Lambda(\cdot)$

$$\lambda^*(\Omega^*) = \lambda(\Omega) \geq \Lambda(v) \geq \Lambda(Qv). \tag{2.30}$$

Case 2. Assume that $B(x, R) \subset U$. By the Euclidean Faber-Krahn inequality in $U \setminus \overset{o}{K} \subset M^*$, it follows that

$$\lambda^*(\Omega^*) \geq cv^{-2/N}.$$

We are left to verify that in this case

$$\Lambda(v) \leq Cv^{-2/N}. \tag{2.31}$$

Indeed, since $v \leq \mu(B(x, R))$, then there is a ball $B(y, r) \subset B(x, R)$ with $\mu(B(y, r)) = v$. Its eigenvalue $\lambda(B(y, r))$ is comparable to that of the Euclidean ball of the Euclidean volume v, that is,

$$C^{-1}v^{-2/N} \leq \lambda(B(y, r)) \leq Cv^{-2/N}. \tag{2.32}$$

Note that $B(y, r)$ lies in U so that the constant C in (2.32) depends only on the intrinsic geometry of U. Therefore,

$$\Lambda(v) \leq \lambda(B(y, r)) \leq Cv^{-2/N}, \tag{2.33}$$

whence

$$\lambda^*(\Omega^*) \geq c\Lambda(v). \tag{2.34}$$

Case 3. Assume that $B(x, R)$ intersects K and $B(x, R)$ is not contained in U. Then

$$2R \geq d(K, U^c) \tag{2.35}$$

which implies that

$$U \subset B(x, PR) \tag{2.36}$$

where

$$P := 1 + 2\frac{\operatorname{diam} U}{d(K, U^c)} \tag{2.37}$$

Indeed, since $B(x,R)$ intersects both K and U^c, then

$$2R = \operatorname{diam} B(x,R) \geq d(K,U^c).$$

If $x' \in B(x,R) \cap K$ then

$$U \subset B\left(x', \operatorname{diam} U\right) \subset B(x, R + \operatorname{diam} U).$$

By (2.35) and (2.37) we have

$$R + \operatorname{diam} U \leq R + \frac{\operatorname{diam} U}{d(K,U^c)} 2R = PR,$$

whence (2.36) follows.

Consider two subcases.

Case 3a: let $v \leq \frac{1}{2}\mu(U \setminus K)$. Then we obtain by Lemma 2.3

$$\begin{aligned} \lambda^*(\Omega^*) &\geq c \min\left(\lambda(\Omega), v^{-2/N}\right) \\ &\geq c \min\left(\Lambda(v), v^{-2/N}\right). \end{aligned}$$

Note that in this case (2.31) is satisfied because, as in Case 2, we can choose a ball $B(y,r) \subset U \subset B(x,PR)$ such that

$$\mu(B(y,r)) = v,$$

whence (2.33) follows. Hence, we obtain

$$\lambda^*(\Omega^*) \geq c\Lambda(v) \geq c\Lambda(Qv).$$

Case 3b: let $v > \frac{1}{2}\mu(U \setminus K)$. Then Lemma 2.2 yields

$$\lambda^*(\Omega^*) \geq c \min(\lambda(\Omega^* \cup U), 1).$$

Since by (2.36)

$$\Omega^* \cup U \subset B(x, PR),$$

we have

$$\lambda(\Omega^* \cup U) \geq \Lambda(\mu(\Omega^* \cup U)).$$

Since

$$\mu(\Omega^* \cup U) \leq v + \mu(U) \leq v + \frac{\mu(U)}{\mu(U \setminus K)} 2v = Cv$$

where

$$C = 1 + 2\frac{\mu(U)}{\mu(U \setminus K)},$$

we obtain

$$\lambda(\Omega^* \cup U) \geq \Lambda(Cv)$$

and, hence,

$$\lambda^*(\Omega^*) \geq c \min(\Lambda(Cv), 1) \tag{2.38}$$

Note that

$$\Lambda\left(Cv\right) \leq \Lambda\left(\frac{C}{2}\mu\left(U\setminus K\right)\right) =: L_0.$$

If $L_0 \leq 1$ then we obtain from (2.38)

$$\lambda^*\left(\Omega^*\right) \geq c\Lambda\left(Cv\right).$$

If $L_0 > 1$ then it follows that

$$\lambda^*\left(\Omega^*\right) \geq \frac{c}{L_0}\min\left(\Lambda\left(Cv\right), L_0\right) \geq \frac{c}{L_0}\Lambda\left(Cv\right),$$

which finishes the proof. ■

2.5 Relative FK-function

Corollary 2.5 *Assume that we are in the setting of Theorem 2.4.*

(a) If M admits the FK-function $\Lambda(v)$ then M^* admits the FK-function

$$\Lambda^*\left(v\right) = c\Lambda\left(Qv\right),$$

where $c, Q > 0$ are the constants from Theorem 2.4.

(b) If M admits the RFK-function $\Lambda(B, v)$ then M^* admits the RFK-function

$$\Lambda^*(B^*\left(x, R\right), v) = c\Lambda(B(x, PR), Qv), \tag{2.39}$$

where c, P, Q are the constants from Theorem 2.4.

Proof. Clearly, it suffices to proof (b). By Theorem 2.4, any ball $B^*\left(x, R\right)$ in M^* admits a FK-function

$$v \mapsto c\Lambda\left(B\left(x, PR\right), Qv\right),$$

which means, that (2.39) is the RFK-function of M^*. ■

Example. Consider the case $M = \mathbb{R}^N$, μ being the Lebesgue measure. Then M has the FK-function $\Lambda(v) = cv^{-2/N}$ and by Corollary 2.5 M^* has the FK-function $\Lambda^*(v) = c'v^{-2/N}$. In particular, this implies the Nash inequality on M^*:

$$\forall\, f \in \mathcal{C}_0^\infty(M^*), \quad \|f\|_2^{2(1+2/N)} \leq C\left(\int_{M^*}|\nabla f|^2 d\mu\right)\|f\|_1^{4/N}$$

and, in the case $N > 2$, also the Sobolev inequality

$$\forall\, f \in \mathcal{C}_0^\infty(M^*), \quad \|f\|_{2N/(N-2)}^2 \leq C\int_{M^*}|\nabla f|^2 d\mu$$

The Sobolev inequality in $M^* = \mathbb{R}^N \setminus K$ (in fact, in a greater generality as far as the regularity of ∂K is concerned) was proved in [2]. The same conclusion holds, of course, for Cartan-Hadamard manifolds and, more generally, for any Riemannian manifold M with the FK-function $\Lambda(v) = cv^{-2/N}$.

Example. Let M be a complete non-compact manifold with non-negative Ricci curvature and μ its Riemannian measure. Then M admits the RFK-function

$$\Lambda(B,v)=\frac{c}{\rho\,(B)^2}\left(\frac{\mu\,(B)}{v}\right)^{2/N},$$

where $c>0$ (cf. [7], [19]). Theorem 2.4 yields that M^* admits the same RFK-function but with a different value of c. Some versions of the Nash and Sobolev inequalities hold in this case as well – see [18], [20]. Besides, this RFK-function implies a certain upper bound of the heat kernel in M^* with the Neumann boundary condition om ∂K (cf. Section 4.1). Two sides estimates of the heat kernel with the Dirichlet boundary condition on ∂K are also available – see [12].

3 Gluing FK-functions

The purpose of this section is to obtain the RFK-function on the connected sum of manifolds $M_1,...,M_k$ assuming that the RFK-functions are known for each M_i. The main result is Theorem 3.3.

3.1 Manifolds with ends

Let M be a Riemannian manifold. We say that an open set $E\subset M$ is an *end* if E is connected, E is not relatively compact and ∂E is compact (note that such an end may correspond to more than one asymptotic ends). Let K be a compact set with smooth boundary such that $M\setminus K$ has k connected components $E_1,\dots E_k$ and each E_i is an end. If M has a boundary δM, then we always assume that δM does not intersect K. We describe such a situation by writing

$$M=\bigsqcup_{i=1}^{k}\Big|_{K} E_i\,.$$

The closure $\overline{E}_i$ will be regarded as a manifold with boundary $\delta\overline{E}_i$ that consists of two disjoint pieces: the topological boundary ∂E_i that lies on ∂K and the rest that lies on δM. Clearly, K can also be regarded as a manifold with boundary $\delta K=\partial K$. We will always assume that all sets $\overline{E}_i$ are disjoint.

Since we are not interested in the effects of the specific geometry of K, we will sometimes omit K from the notation introduced above and write $M=\bigsqcup_{i=1}^{k}E_i$. Furthermore, in many cases, each E_i can be considered as the exterior of a compact in another manifold M_i. In this case we also write

$$M=\bigsqcup_{i=1}^{k}M_i$$

and refer to M as a *connected sum* of the manifolds M_i (see Fig. 5).

Here M_i may be a manifold with boundary and, in particular, $M_i=\overline{E}_i$ is allowed here. If M_i is a manifold with boundary, then we always assume that its boundary δM_i is the disjoint union of two pieces δ_i' and δ_i'' where $\delta_i'=\emptyset$ or $\delta_i'=\partial E_i$, and $\delta_i''=\delta M\cap E_i$.

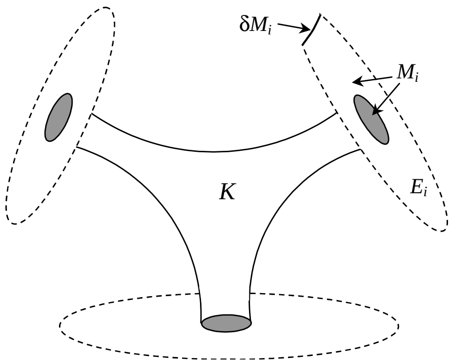

Figure 5: The manifold M that is a connected sum of M_i's

Example. A specific realization of $\mathbb{R}^n \bigsqcup \mathbb{R}^n$ is obtained as follows:

$$M = (\mathbb{R}^n \setminus \mathbb{B}^n) \bigsqcup\Big|_K (\mathbb{R}^n \setminus \mathbb{B}^n),$$

where $K = \mathbb{S}^{n-1} \times [-1, 1]$ is equipped with an appropriate metric. Here $\mathbb{B}^n$ is the unit closed ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere.

We assume that M is equipped with a measure μ on M with smooth density σ^2, and that each M_i is equipped with a measure μ_i with smooth densities σ_i^2 so that $\sigma_i = \sigma$ on E_i. We will equip with a subscript i the names of all the objects related to the manifold M_i. In particular, we will denote geodesic balls in M_i by $B_i(x, r)$. Note that a ball B in M that is contained in E_i is at the same time a ball in M_i. Moreover, for such a ball B we have $\mu(B) = \mu_i(B_i)$.

3.2 The main results

Let M be the connected sum of $M_1, ..., M_k$ as described in Section 3.1. The goal of this section is to obtain a RFK-function Λ for M in terms of given RFK-functions Λ_i on M_i, $i \geq 1$. Set $r_0 := \operatorname{diam} K$ and fix E_0 – an open set in M with smooth boundary such that

$$K_{2r_0} \subset E_0 \subset K_{3r_0},$$

where K_r is an open r-neighborhood of K. Since E_0 is a precompact open subset of M, it has the Faber-Krahn function

$$\Lambda_0(v) := c \begin{cases} v^{-2/N}, & \text{if} \quad v \leq v_0 \\ v_0^{-2/N}, & \text{if} \quad v > v_0 \end{cases}, \tag{3.1}$$

where $N = \dim M$ and $v_0 = \mu(E_0)$.

Given the RFK-function $\Lambda_i(B, v)$ on M_i, $i = 1, ..., k$, define the function

$$\overline{\Lambda}(B, v) := \min_{1 \leq i \leq k} \inf_{y \in \partial E_i} \Lambda_i(B_i(y, 3\rho(B)), v), \tag{3.2}$$

where $v > 0$, B is a ball in M, and $\rho(B)$ is the radius of B. Now, define the function $\Lambda(B, v)$ as follows:

$$\Lambda(B,v) := \begin{cases} \Lambda_0(v), & \text{if } B \subset E_0; \\ \Lambda_i(B,v), & \text{if } B \subset E_i,\ i \geq 1, \text{ but } B \not\subset E_0; \\ \overline{\Lambda}(B,v), & \text{otherwise.} \end{cases} \tag{3.3}$$

Proposition 3.1 *Assume that each of the manifolds $M_i, i = 1, ..., k$, is connected, non-compact, complete, and*

$$\overline{E}_i = M_i\,. \tag{3.4}$$

Assume also that, for each i, the weighted manifold (M_i, μ_i) admits the RFK-function Λ_i. Then the manifold $M = \bigsqcup_i M_i$ admits the RFK-function

$$(B, v) \mapsto c\Lambda(B, v)\,, \tag{3.5}$$

where Λ is defined by (3.3) and the constant $c > 0$ depends only on the intrinsic geometry of a precompact neighbourhood of K.

Proof. Fix a ball $B = B(x, R) \subset M$, an open set $\Omega \subset B$, a test function $f \in C_0^\infty(\Omega)$. We need to prove that

$$\int_\Omega |\nabla f|^2 \, d\mu \geq c\Lambda(B, v) \int_\Omega f^2 d\mu, \tag{3.6}$$

where $v := \mu(\Omega)$. Consider the following cases.

Case 1. Assume that $B \subset E_i$ for some $i \geq 0$ (if B is contained in both E_0 and E_j, $j \geq 1$, then take $i = 0$). If $i \geq 1$ then the Faber-Krahn inequality in $B = B_i(x, R) \subset E_i$ gives

$$\int_\Omega |\nabla f|^2 \, d\mu \geq \Lambda_i(B, v) \int_\Omega f^2 d\mu = \Lambda(B, v) \int_\Omega f^2 d\mu \tag{3.7}$$

If $i = 0$, then the ball $B = B_0(x, R)$ has the FK-function $\Lambda_0\,(v)$ and we obtain

$$\int_\Omega |\nabla f|^2 \, d\mu \geq \Lambda_0(v) \int_\Omega f^2 d\mu = \Lambda(B, v) \int_\Omega f^2 d\mu.$$

In particular, if $R \leq r_0$ then this case applies. Indeed, if $x \in K_{r_0}$, then $B(x, R) \subset K_{2r_0} \subset E_0$. If $x \notin K_{r_0}$, then $x \in E_i$, for some $i \geq 1$, and $B(x, R)$ is contained in the same E_i.

Cases 2,3: preliminary remarks. In the next cases 2 and 3, we assume that $B = B(x, R)$ is not contained in any $E_i, i \geq 0$. In particular, we have $R > r_0$. Let us set $\Omega_i = \Omega \cap \overline{E}_i$ for $i \geq 1$ and observe that the restriction of f to Ω_i belongs to $\mathcal{C}_0^\infty\,(\Omega_i)$ on the manifold M_i because non-zero values of f on $\partial\Omega_i$ lie on $\partial E_i = \delta M_i$ (here we use the assumption that $M_i = \overline{E}_i$). Set

$$I_i := \int_{\Omega_i} f^2 d\mu\,, i \geq 1, \quad \text{and } I_0 = \int_K f^2 d\mu.$$

Observe that

$$\sum_{i=0}^{k} I_i = \int_{\Omega} f^2 d\mu. \tag{3.8}$$

Case 2 ("main case"). Fix some positive $\varepsilon < \frac{1}{k}$ to be specified later on, and assume that, for some $i \geq 1$,

$$I_i \geq \varepsilon \int_{\Omega} f^2 d\mu. \tag{3.9}$$

Then the ball B has a non-empty intersection with E_i, but it is not contained in E_i by the aforementioned assumption. Thus, B intersects ∂E_i. Let y be a point in $B \cap \partial E_i$. We claim that

$$\Omega_i \subset B_i(y, 3R). \tag{3.10}$$

Indeed, $y \in B(x, R)$ implies that $B(y, 2R) \supset B(x, R) \supset \Omega$. Next,

$$B(y, 2R) \cap \overline{E}_i \subset B_i(y, 2R + \operatorname{diam} \partial E_i) \subset B_i(y, 3R),$$

because $\operatorname{diam} \partial E_i \leq r_0 < R$, whence

$$\Omega_i \subset B(y, 2R) \cap \overline{E}_i \subset B_i(y, 3R).$$

(see Fig. 6).

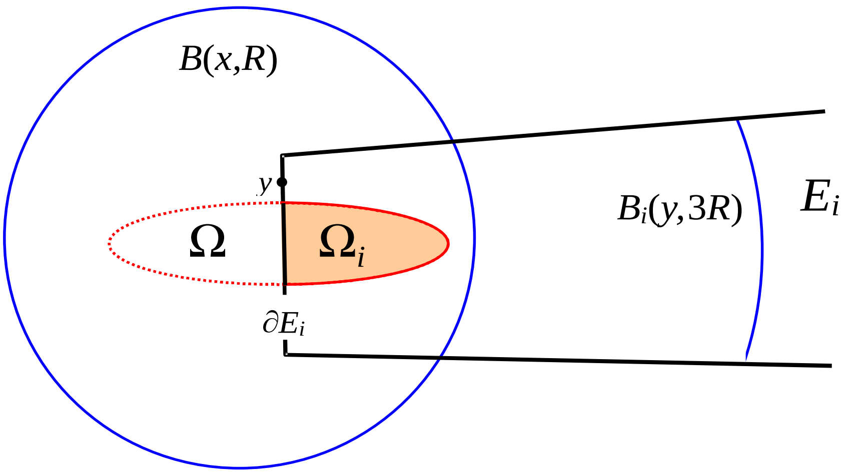

Figure 6: Illustration to the case 2

Now we can apply the Faber-Krahn inequality in Ω_i in the ball $B_i(y, 3R) \subset M_i$, which yields by (3.9)

$$\begin{aligned} \int\limits_{\Omega} |\nabla f|^2 \, d\mu &\geq \int\limits_{\Omega_i} |\nabla f|^2 \, d\mu \geq \Lambda_i \left(B_i(y, 3R), \mu\left(\Omega_i\right)\right) \int\limits_{\Omega_i} f^2 d\mu \\ &\geq \varepsilon \Lambda_i(B_i(y, 3R), v) \int\limits_{\Omega} f^2 \, . \end{aligned}$$

To conclude (3.6), we are left to verify that

$$\Lambda_i(B_i(y,3R),v) \geq \Lambda(B,v). \tag{3.11}$$

Indeed, since B is not contained in any $E_j, j \geq 0$, we have

$$\Lambda(B,v) \overset{(3.3)}{=} \overline{\Lambda}(B,v) \overset{(3.2)}{\leq} \Lambda_i(B_i(y,3R),v)\,,$$

whence (3.11) follows.

Case 3. Assume that (3.9) is not satisfied for any $i \geq 1$. Then we have

$$\int_{M\setminus K} f^2 d\mu = I_1 + ... + I_k \leq k\varepsilon \int_\Omega f^2 d\mu\,, \tag{3.12}$$

and by (3.8)

$$\int_K f^2 d\mu = I_0 \geq (1-k\varepsilon)\int_\Omega f^2 d\mu. \tag{3.13}$$

Since $1-k\varepsilon > 0$, we see that the intersection $K \cap B$ is non-empty. Fix a point $y \in K \cap B$; then we have $B \subset B(y,2R)$. Observe that $K \subset B(y,r_0)$ and $B(y,r_1) \subset E_0$ where $r_1 := 2r_0$. Choose a function $\phi \in \mathcal{C}_0^\infty(B(y,r_1))$ such that $\phi|_{B(y,r_0)} = 1$. Clearly, ϕ can be chosen in the way that

$$|\nabla\phi| \leq \frac{2}{r_1 - r_0} =: C_0 \tag{3.14}$$

(see Fig. 7).

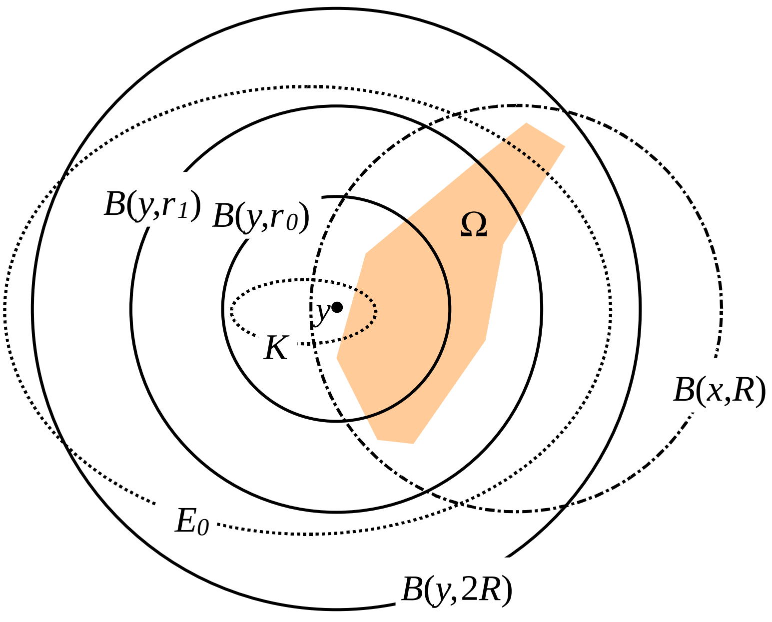

Figure 7: Illustration to the case 3.

Since $f\phi \in \mathcal{C}_0^\infty(\Omega \cap B(y,r_1))$, we obtain by the Faber-Krahn inequality in E_0

$$\int |\nabla(f\phi)|^2 d\mu \geq \Lambda_0(\mu(\Omega \cap B(y,r_1))) \int (f\phi)^2 d\mu \geq \Lambda_0(v) \int (f\phi)^2 d\mu. \tag{3.15}$$

On the other hand,

$$\left|\nabla\left(f\phi\right)\right|^{2}\leq2\left|\nabla f\right|^{2}\phi^{2}+2f^{2}\left|\nabla\phi\right|^{2}$$

whence by (3.14) and (3.12)

$$\begin{aligned}\int\left|\nabla\left(f\phi\right)\right|^{2}d\mu &\leq 2\int\left|\nabla f\right|^{2}\phi^{2}d\mu+2\int f^{2}\left|\nabla\phi\right|^{2}d\mu\\ &\leq 2\int\left|\nabla f\right|^{2}d\mu+2C_{0}\int_{M\setminus K}f^{2}d\mu\\ &\leq 2\int\left|\nabla f\right|^{2}d\mu+2C_{0}k\varepsilon\int f^{2}d\mu. \qquad (3.16)\end{aligned}$$

Since by (3.13)

$$\int_{M}\left(f\phi\right)^{2}d\mu\geq\int_{K}f^{2}d\mu\geq\left(1-k\varepsilon\right)\int_{M}f^{2}d\mu\,, \qquad (3.17)$$

we obtain by putting together (3.15), (3.16), (3.17), that

$$\begin{aligned}\int\left|\nabla f\right|^{2}d\mu+C_{0}k\varepsilon\int f^{2}d\mu &\geq \frac{1}{2}\int\left|\nabla\left(f\phi\right)\right|^{2}d\mu \qquad (3.18)\\ &\geq \frac{1}{2}\Lambda_{0}\left(v\right)\int\left(f\phi\right)^{2}d\mu\\ &\geq \frac{1-k\varepsilon}{2}\Lambda_{0}\left(v\right)\int f^{2}d\mu. \qquad (3.19)\end{aligned}$$

Note that, by (3.1), $m:=\inf\Lambda_{0}>0$. Choose $\varepsilon>0$ so small that

$$C_{0}k\varepsilon<\frac{1-k\varepsilon}{4}m.$$

Then the second term on the left hand side of (3.18) is absorbed by the right hand side of (3.19), and we obtain

$$\int\left|\nabla f\right|^{2}d\mu\geq c'\Lambda_{0}\left(v\right)\int f^{2}d\mu.$$

We are left to show that

$$\Lambda_{0}(v)\geq c\Lambda(B,v). \qquad (3.20)$$

We have in this case, by (3.3),

$$\Lambda(B,v)=\overline{\Lambda}(B,v).$$

Therefore, (3.20) will follow from the following lemma, which will finish the proof of Proposition 3.1. ■

Lemma 3.2 *If $B=B\left(x,R\right)$ is not contained in any E_i, $i\geq0$, then*

$$\forall\, v>0,\quad\overline{\Lambda}(B,v)\leq C\Lambda_{0}(v), \qquad (3.21)$$

where the constant C depends on the intrinsic geometry of a precompact neighborhood of K.

Proof. Taking into account definition (3.2) of $\overline{\Lambda}$, it suffices to verify that, for some $i \geq 1$ and $y \in \partial E_i$,

$$\Lambda_i(B_i(y, 3R), v) \leq C\Lambda_0(v). \tag{3.22}$$

In fact, (3.22) holds for any $i \geq 1$ and $y \in \partial E_i$. So, fix some $i \geq 1$ and $y \in \partial E_i$.

Consider first the case when

$$v \leq v_1 := V_i(y, r_0).$$

Then, for some $r \leq r_0$, we have

$$v = V_i(y, r).$$

Set $\Omega = B_i(y, r)$ and observe that $\Omega \subset B_i(y, r_0) \subset B_i(y, 3R)$, because $R \geq r_0$. Therefore, by the Faber-Krahn inequality in $B_i(y, 3R)$,

$$\lambda(\Omega) \geq \Lambda_i(B_i(y, 3R), v).$$

On the other hand, the ball Ω may vary only within a compact region around $\partial E_i \subset \partial K$, which means that its first eigenvalue is comparable to that of the ball of the same volume in $\mathbb{R}^N$. In other words,

$$\lambda(\Omega) \asymp v^{-2/N},$$

whence we obtain

$$\Lambda_i(B_i(y, 3R), v) \leq Cv^{-2/N}.$$

Assume now $v > v_1$. Since Λ_i is non-increasing in v, we conclude

$$\Lambda_i(B_i(y, 3R), v) \leq \Lambda_i(B_i(y, 3R), v_1) \leq Cv_1^{-2/N}.$$

Combining the two cases, we obtain

$$\begin{aligned}
\Lambda_i(B_i(y, 3R), v) &\leq C \begin{cases} v^{-2/N}, & \text{if} \quad v \leq v_1 \\ v_1^{-2/N}, & \text{if} \quad v > v_1 \end{cases} \\
&\leq C' \left(1 + \frac{v_0}{v_1}\right)^{2/N} \Lambda_0(v) \\
&= C''\Lambda_0(v),
\end{aligned}$$

which was to be proved. ■

Our next goal is to state and prove the main result: an extension of Proposition 3.1 where we do not assume any more that $\overline{E}_i = M_i$. The statement will be very similar to that of Proposition 3.1, but we must modify the definition (3.3) of the function Λ. Fix some constants $P, Q > 1$ and set, for any ball $B \subset M$ and $v > 0$,

$$\overline{\Lambda}^*(B, v) := \min_{1 \leq i \leq k} \inf_{y \in \partial E_i} \Lambda_i(B_i(y, 3P\rho(B)), Qv), \tag{3.23}$$

where $\rho(B)$ is the radius of B, and B_i denotes geodesic balls in M_i. Fix also a constant $c > 0$ and set

$$\Lambda(B, v) := \begin{cases} \Lambda_0(v), & \text{if} \ \ B \subset E_0; \\ \Lambda_i(B, v), & \text{if} \ \ B \subset E_i, \ i \geq 1, \ \text{but} \ B \not\subset E_0; \\ c\overline{\Lambda}^*(B, v), & \text{otherwise}, \end{cases} \tag{3.24}$$

We can now state and prove our main result.

Theorem 3.3 *Assume that each of the manifolds $M_i, i = 1, ..., k$, is connected, non-compact, and complete. Assume also that, for each i, the weighted manifold (M_i, μ_i) admits the RFK-function Λ_i. Then the connected sum $M = \bigsqcup_i M_i$ admits the RFK-function Λ defined by (3.23)–(3.24) for some $P, Q > 1$ and $c > 0$.*

Proof. We denote by $B_i(x,r)$ the geodesic balls in M_i and by $B_i^*(x,R)$ – the geodesic balls in $\overline{E}_i$ considered as a manifold. By Corollary 2.5 (cf. (2.39)), the manifold $\overline{E}_i$ admits the RFK-function

$$\Lambda_i^*(B_i^*(x,R), v) = c\Lambda_i(B_i(x, PR), Qv) \tag{3.25}$$

Let us choose c so small and P, Q so large that they serve all E_i, $i \geq 1$.

Obviously, M is a connected sum of the manifolds $\overline{E}_i$, which allows us to use Proposition 3.1 to compute a RFK-function of M. Let $B = B(x,R)$ be a geodesic ball in M. If $B \subset E_i$, $i \geq 1$, then B is also a ball in M_i and, hence, B admits the FK-function

$$v \mapsto \Lambda_i(B, v) = \Lambda(B, v).$$

If $B \subset E_0$ then B admits the FK-function

$$\Lambda_0(v) = \Lambda(B, v).$$

Assume now that B does not lie in any E_i, $i \geq 0$. Using Proposition 3.1 (cf. (3.2)) and (3.25), we obtain that B admits a FK-function

$$\begin{aligned} v \quad &\mapsto \quad \min_{1\leq i\leq k} \inf_{y\in\partial E_i} \Lambda_i^*(B_i^*(y, 3R), v) \\ &= \quad \min_{1\leq i\leq k} \inf_{y\in\partial E_i} c\Lambda_i(B_i(y, 3PR)), Qv) \\ &= \quad c\overline{\Lambda}^*(B, v) = \Lambda(B, v), \end{aligned}$$

which finishes the proof. ■

3.3 Specific RFK-functions

We derive here consequences of Theorem 3.3 in two special cases as in Theorems 3.4 and 3.5 below. We keep the notation and hypotheses introduced in Section 3.1. The hypotheses of Theorem 3.3 are also assumed to hold.

Theorem 3.4 *Assume that each manifold M_i admits the FK-function $\Lambda_i(v)$. Then there exist constants $c > 0, Q > 1$ such that $M = \bigsqcup_{i=1}^k M_i$ admits the FK-function*

$$\Lambda(v) = c \min_{1\leq i\leq k} \Lambda_i(Qv). \tag{3.26}$$

Proof. Indeed, each M_i has the RFK-function

$$(B, v) \mapsto \Lambda_i(v),$$

and by Theorem 3.3 M has the RFK-function $\Lambda(B, v)$ given by (3.23)-(3.24). Given a precompact open set $\Omega \subset M$, choose a ball B containing Ω so large that B not contained in any E_i. Setting $v = \mu(\Omega)$, we obtain by Theorem 3.3

$$\lambda(\Omega) \geq \Lambda(B, v) = c\overline{\Lambda}^*(B, v) = c \min_{1\leq i\leq k} \Lambda_i(Qv) = \Lambda(v),$$

which was to be proved. $\blacksquare$

Example. Assume that each M_i has the FK-function

$$\Lambda_i(v) = c_i \begin{cases} v^{-2/N}, & v \le 1, \\ v^{-2/n_i}, & v > 1, \end{cases} \tag{3.27}$$

where N is the common topological dimension of all M_i and n_i can be called the *dimension at infinity* of M_i. Then $M = \bigsqcup_{i=1}^k M_i$ admits the FK-function

$$\Lambda(v) = c \begin{cases} v^{-2/N}, & v \le 1, \\ v^{-2/n}, & v > 1, \end{cases}$$

where $n = \min_{1 \le i \le k} n_i$, which follows immediately from (3.26).

Example. Assume that each M_i satisfies the Nash inequality

$$\forall f \in \mathcal{C}_0^\infty(M_i), \quad \|f\|_2^{2(1+2/n_i)} \le C_i \left(\int_{M_i} |\nabla f|^2 d\mu \right) \|f\|_1^{4/n_i} \tag{3.28}$$

for some $n_i > 0$. Then $M = \bigsqcup_{i=1}^k M_i$ satisfies the Nash inequality

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f\|_2^{2(1+2/n)} \le C \left(\int_{M_i} |\nabla f|^2 d\mu \right) \|f\|_1^{4/n}$$

with $n = \min_{1 \le i \le k} n_i$. This follows from Theorem 3.4 and from the equivalence between the Faber-Krahn inequality and the Nash inequality that was mentioned in Section 2.1.

Note also that a sufficient condition for (3.28) is the Faber-Krahn inequality with the FK-function (3.27) with $n_i \ge N$.

Our next result concerns the case of certain "nice" RFK-functions. Fix $\alpha > 0$. We say that a weighted manifold (M, μ) satisfies condition (RFKα) if there exists $c > 0$ such that M has the RFK-function

$$\Lambda(B, v) = c\rho(B)^{-2} \left(\frac{\mu(B)}{v} \right)^\alpha, \tag{3.29}$$

where $\rho(B)$ is the radius of the ball B. As we have already mentioned in Section 2.1, a complete non-compact Riemannian manifold with non-negative Ricci curvature satisfies (RFKα) with $\alpha = 2/N$ where $N = \dim M$.

Note that if M satisfies (RFKα) for some $\alpha > 0$ then M satisfies also (RFKβ) for any $0 < \beta < \alpha$. Indeed, replacing in (3.29) the value of α by a smaller value β reduces the right hand side in the case $v \le \mu(B)$. Since for any open set $\Omega \subset B$ we have always $v := \mu(\Omega) \le \mu(B)$, we obtain that the inequality $\lambda(\Omega) \ge \Lambda(B, v)$ will continue to hold after replacing α by β (the values of $\Lambda(B, v)$ for $v > \mu(B)$ do not matter).

It is known [8, Proposition 5.2] that if M is a complete manifold satisfying (RFKα) then M satisfies the following *volume regularity property*: there exists a constant C such that

$$\forall B' \subset B, \ \frac{\mu(B)}{\mu(B')} \le C \left(\frac{\rho(B)}{\rho(B')} \right)^{2/\alpha}. \tag{VR}$$

In particular, fixing B, letting $r(B') \to 0$ and using $\mu(B') \asymp \rho(B')^N$, we obtain from (VR) that $2/\alpha \geq N$ and, hence,

$$\alpha \leq 2/N.$$

Now we assume that, for each $i = 1, ..., k$, M_i is a complete non-compact manifold that satisfies (RFKα_i) for some $\alpha_i > 0$. For any ball $B_i(x,r)$ in M_i set

$$V_i(x,r) = \mu_i(B_i(x,r)).$$

For any i fix a reference point $o_i \in \partial E_i$ and set for any $r > 0$

$$V_{\min}(r) = \min_{1\leq i\leq k} V_i(o_i, r). \tag{3.30}$$

For any ball $B = B(x,r)$ in M set

$$F(B) = F(x,r) := \begin{cases} \mu(B), & \text{if } B \subset E_i, i \geq 1, \\ V_{\min}(r), & \text{otherwise.} \end{cases} \tag{3.31}$$

Theorem 3.5 *Assume that each M_i satisfies* (RFKα_i) *for some $\alpha_i > 0$. Then $M = \bigsqcup_{i=1}^k M_i$ admits the RFK-function*

$$\Lambda(B,v) = c\rho(B)^{-2}\left(\frac{F(B)}{v}\right)^{\alpha},$$

where $\alpha = \min \alpha_i$, $c > 0$ and F is defined in by (3.30)-(3.31).

Proof. Since $\alpha_i \leq \alpha$, we see that each M_i satisfies (RFKα), that is, M_i has the RFK-function

$$\Lambda_i(B_i(y,r), v) = \frac{c}{r^2}\left(\frac{V_i(y,r)}{v}\right)^{\alpha}.$$

By Theorem 3.3, M admits the RFK-function Λ given by (3.23)-(3.24). It follows from (3.23) that, for any ball $B(x,r)$ in M,

$$\begin{aligned} \overline{\Lambda}^*(B,v) &= \min_{1\leq i\leq k}\inf_{y\in\partial E_i} \Lambda_i(B_i(y,3Pr), Qv) \\ &= \frac{c}{r^2}\left[\min_{1\leq i\leq k}\inf_{y\in\partial E_i} V_i(y,3Pr)\right]^{\alpha} v^{-\alpha}. \end{aligned} \tag{3.32}$$

Let us show that for any $y \in \partial E_i$

$$V_i(y,3Pr) \geq cV_i(o_i,r).$$

Indeed, if r is large enough (compared to r_0) then $B(y,3Pr) \supset B(o_i,r)$, whereas for a bounded range of r we have

$$V_i(y,3Pr) \asymp r^N \asymp V_i(o_i,r).$$

Hence, we obtain from (3.32)

$$\overline{\Lambda}^*(B,v) \geq \frac{c}{r^2} V_{\min}(r)^{\alpha} v^{-\alpha}.$$

Substituting this estimate into (3.24) yields

$$\Lambda(B,v) \geq c \begin{cases} v^{-2/N}, & \text{if } B \subset E_0; \\ r^{-2} \left(\mu(B)/v\right)^{\alpha}, & \text{if } B \subset E_i,\ i \geq 1, \text{ but } B \not\subset E_0; \\ r^{-2} \left(V_{\min}(r)/v\right)^{\alpha}, & \text{otherwise.} \end{cases} \tag{3.33}$$

If $B \subset E_0$ and $v \leq \mu(B)$, then we obtain by $\alpha \leq 2/N$ and $\mu(B) \leq Cr^N$ that

$$v^{-2/N} \geq cr^{-2}(\mu(B)/v)^{2/N} \geq cr^{-2}(\mu(B)/v)^{\alpha}.$$

Therefore, in the right hand side of (3.33) the first and second line can be combined as follows:

$$\begin{aligned} \Lambda(B,v) &\geq c \begin{cases} r^{-2} \left(\mu(B)/v\right)^{\alpha}, & \text{if } B \subset E_i,\ i \geq 0, \\ r^{-2} \left(V_{\min}(r)/v\right)^{\alpha}, & \text{otherwise,} \end{cases} \\ &= cr^{-2} \left(\frac{F(B)}{v}\right)^{\alpha}, \end{aligned}$$

which finishes the proof. ■

4 Upper bound of the heat kernel

In this section we obtain upper bounds of the heat kernel on the connected sum $M = \bigsqcup_{i=1}^{k} M$. Each M_i is a complete, connected, non-compact weighted manifold, that satisfies a certain Faber-Krahn type inequality. By using the results of Section 3, we will derive a Faber-Krahn inequality on M. By [8], this Faber-Krahn inequality on M implies certain heat kernel upper bounds. We will also show how to obtain upper bounds for the heat kernel on M starting from upper bounds on the heat kernel of each M_i.

There are two main types of assumptions on M_i, under which the above scheme works:

(A) Each M_i satisfies (RFKα), that is, admits a relative Faber-Krahn function of the form (3.29).

(B) Each M_i admits a uniform Faber-Krahn function $\Lambda_i(v)$.

4.1 Faber-Krahn inequalities and heat kernel upper bounds

This section contains some preliminary material borrowed mainly from [8]. Let (M,μ) be a complete connected weighted manifold. For any ball $B(x,r)$ in M set $V(x,r) = \mu(B(x,r))$. Denote by $p_t(x,y)$ the heat kernel on M, that is, the minimal positive fundamental solution to the heat equation $\frac{\partial u}{\partial t} = \mathcal{L}u$ on M. If M has the boundary δM then the heat kernel satisfies the Neumann boundary condition on δM.

Proposition 4.1 ([8, Theorem 5.2]) *Assume that M has the RFK-function*

$$\Lambda(B(x,R),v) = a(x,R)v^{-\alpha}, \tag{4.1}$$

where $\alpha > 0$ and $a(x,R)$ is an arbitrary positive function of x and R. Then, for all $x,y \in M$ and $t,R>0$,

$$p(t,x,y) \le \frac{C}{\min(t,R^2)^{\frac{1}{\alpha}}\,(a(x,R)a(y,R))^{\frac{1}{2\alpha}}}\exp\left(-\frac{d^2(x,y)}{Dt}\right) \tag{4.2}$$

for some $C,D>0$ (in fact, D is any constant >4 and $C=C(\alpha,D)$).

In particular, if M satisfies (RFKα), that is, has the RFK-function

$$\Lambda(B(x,R),v) = \frac{c}{R^2}\left(\frac{V(x,R)}{v}\right)^{\alpha},$$

then (4.1) is satisfied with

$$a(x,R) = \frac{c}{R^2}V(x,R)^{\alpha}.$$

Substituting this into (4.2) and setting $R=\sqrt{t}$, we obtain

$$p(t,x,y) \le \frac{C}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}\exp\left(-\frac{d^2(x,y)}{Dt}\right). \tag{4.3}$$

Recall that (RFKα) implies the volume regularity condition (VR). Using the latter to estimate the ratio $V\left(x,\sqrt{t}\right)/V\left(y,\sqrt{t}\right)$, one obtains from (4.3)

$$p(t,x,y) \le \frac{C}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{Dt}\right). \tag{4.4}$$

For $x=y$ we obtain the diagonal upper estimate

$$p(t,x,x) \le \frac{C}{V(x,\sqrt{t})}. \tag{DUE}$$

The volume regularity condition (VR) implies trivially the volume doubling condition: for all $x \in M$ and $R>0$,

$$V(x,2R) \le CV(x,R). \tag{VD}$$

In particular, we obtain the implication

$$(\mathrm{RFK}\alpha) \Rightarrow (VD) + (DUE).$$

It turns out that the converse is also true.

Proposition 4.2 *(*[8, Proposition 5.2]*) For any complete manifold M, the following equivalence holds:*

$$(VD) + (DUE) \Leftrightarrow (\mathrm{RFK}\alpha) \text{ for some } \alpha>0. \tag{4.5}$$

For the rest of this section, we consider the case when the manifold (M, μ) admits a uniform FK-function $\Lambda(v)$. We assume that $\Lambda(v)$ is non-increasing positive continuous function on $(0, \infty)$, such that

$$\int_0 \frac{dv}{v\Lambda(v)} < \infty. \tag{4.6}$$

For example, the hypothesis (4.6) holds if $\Lambda(v) \asymp v^{-\varepsilon}$ for small v (where $\varepsilon > 0$).

Given such a function Λ, we associate with it a function $\digamma(t)$ defined for any $t > 0$ by means of the following identity:

$$t = \int_0^{\digamma(t)} \frac{dv}{v\Lambda(v)}. \tag{4.7}$$

The integral in (4.7) converges by (4.6). Due to the fact that Λ is non-increasing, the integral (4.7) takes arbitrarily large values so that $\digamma(t)$ is defined for all $t > 0$. Clearly, $\digamma$ is positive, continuous, increasing and $\lim_{t\to\infty} \digamma(t) = \infty$.

Proposition 4.3 ([8, Theorem 2.1]) *If (M, μ) admits a uniform FK-function Λ satisfying (4.6), then the following upper bound of the heat kernel holds, for all $x, y \in M$, $t > 0$,*

$$p(t, x, y) \leq \frac{C}{\digamma(ct)}. \tag{4.8}$$

Proposition 4.4 ([9, Theorem 1.1]) *Let us assume that the heat kernel on (M, μ) satisfies the upper estimate (4.8), where $\digamma(t)$ is a positive increasing function on $(0, \infty)$* (not necessarily given by (4.7))*. Assume in addition that the function $\digamma(t)$ satisfies the following regularity property:*

$$\text{for some } \gamma > 1 \text{ and all } t_2 > t_1 > 0, \quad \frac{\digamma(\gamma t_1)}{\digamma(t_1)} \leq C\frac{\digamma(\gamma t_2)}{\digamma(t_2)}. \tag{4.9}$$

Then, for all $x, y \in M$ and $t > 0$, we have

$$p(t, x, y) \leq \frac{C}{\digamma(ct)} \exp\left(-\frac{d^2(x, y)}{Dt}\right). \tag{4.10}$$

Remark. Condition (4.9) does not restrict the growth of the function $\digamma$. If $\digamma$ is of *at most polynomial* volume growth in the sense that $\digamma(\gamma t) \leq C\digamma(t)$, for some $\gamma > 1$ and for all $t > 0$, then (4.9) holds, due to the hypothesis that $\digamma$ is increasing. If $\digamma$ is of *at least polynomial* volume growth in the sense that $\digamma(\gamma t)/\digamma(t)$ is increasing, for some $\gamma > 1$, then (4.9) holds with $C = 1$.

4.2 Case of a relative Faber-Krahn inequality

Throughout this section, we will assume that a weighted manifold M is the connected sum of $M_1, ..., M_k$ as was explained in Section 3.1, where all M_i are connected, complete, non-compact.

Assume as in Section 3.3 that each M_i satisfies (RFKα), that is, admits the RFK-function

$$\Lambda_i(B, v) = \frac{c}{R(B)^2}\left(\frac{\mu_i(B)}{v}\right)^\alpha, \tag{4.11}$$

where c and α are positive constants (we can always take c and α to be so small that they serve M_i for all $i = 1, ..., k$). Recall the following notation from Section 3.3: $V_{\min}(r)$ is defined by (3.30), that is,

$$V_{\min}(r) := \min_{1 \le i \le k} V_i(o_i, r)$$

and $F(x, r)$ is defined by (3.31), that is,

$$F(x,r) = \begin{cases} V(x,r), & \text{if } B(x,r) \subset E_i, i \ge 1, \\ V_{\min}(r), & \text{otherwise.} \end{cases} \tag{4.12}$$

Note that if $B(x,r) \subset E_0$ then $V(x,r) \simeq r^N \simeq V_{\min}(r)$ so that the condition $i \ge 1$ in the first line of (4.12) can be replaced by $i \ge 0$.

Theorem 4.5 *Assume that each M_i satisfies (RFKα). Then the heat kernel on $M = \bigsqcup_{i=1}^k M_i$ satisfies for all $x, y \in M$ and $t > 0$ the following estimate:*

$$p(t,x,y) \le \frac{C}{\sqrt{F(x,\sqrt{t})F(y,\sqrt{t})}} \exp\left(-\frac{d^2(x,y)}{Dt}\right), \tag{4.13}$$

for some $C, D > 0$. Consequently, we have for all $t > 0$

$$\sup_{x,y \in K} p_t(x,y) \le \frac{C}{V_{\min}(\sqrt{t})}. \tag{4.14}$$

Proof. By Theorem 3.5, M admits the RFK-function

$$\Lambda(B(x,R), v) = b(x,R)v^{-\alpha},$$

where

$$b(x,R) = cR^{-2}F^{\alpha}(x,R). \tag{4.15}$$

Hence, Proposition 4.1 yields, for all $x, y \in M$ and all $t > 0$, $R > 0$,

$$p(t,x,y) \le \frac{C}{\min(t,R^2)^{\frac{1}{\alpha}} (b(x,R)b(y,R))^{\frac{1}{2\alpha}}} \exp\left(-\frac{d^2(x,y)}{Dt}\right).$$

By choosing $R = \sqrt{t}$ and by substituting b from (4.15), we obtain (4.13).

The estimate (4.14) is a trivial consequence of (4.13) as in the case $x \in K$ we have by (4.12) $F(x,r) = V_{\min}(r)$. ■

By using (3.31), the estimate (4.13) can be written in a more explicit form as follows. For $x \in E_i$, $y \in E_j$, where $i, j \ge 0$ and $d = d(x,y)$, we have

$$p(t,x,y) \le C \exp\left(-\frac{d^2}{Dt}\right) \begin{cases} \frac{1}{V_{\min}(\sqrt{t})}, & \text{if } B(x,\sqrt{t}) \not\subset E_i \text{ and } B(y,\sqrt{t}) \not\subset E_j, \\ \frac{1}{\sqrt{V_{\min}(\sqrt{t})V(x,\sqrt{t})}}, & \text{if } B(x,\sqrt{t}) \subset E_i \text{ and } B(y,\sqrt{t}) \not\subset E_j, \\ \frac{1}{\sqrt{V_{\min}(\sqrt{t})V(y,\sqrt{t})}}, & \text{if } B(x,\sqrt{t}) \not\subset E_i \text{ and } B(y,\sqrt{t}) \subset E_j, \\ \frac{1}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}, & \text{if } B(x,\sqrt{t}) \subset E_i \text{ and } B(y,\sqrt{t}) \subset E_j. \end{cases} \tag{4.16}$$

If we assume $0 < t \le r_0^2$, then $B(x, \sqrt{t})$ is necessarily contained in one of E_i, $i \ge 0$. Therefore, we obtain the following

Corollary 4.6 *Referring to Theorem 4.5, we have, for any $t \in (0, r_0^2]$ and all $x, y \in M$,*

$$p(t,x,y) \le \frac{C}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp\left(-\frac{d^2(x,y)}{Dt}\right). \tag{4.17}$$

Combining Theorem 4.5 with Proposition 4.2, we obtain the following result.

Corollary 4.7 *Assume that each M_i satisfies (VD) and (DUE). Then the heat kernel on $M = \bigsqcup_{i=1}^k M_i$ satisfies the upper bound (4.13).*

4.3 Case of a "flat" Faber-Krahn inequality

Let us assume that each M_i, $i = 1, ..., k$, admits a positive FK-function $\Lambda_i(v)$ satisfying (4.7). For each i define the function $\digamma_i(t)$ for $t > 0$ by

$$t = \int_0^{\digamma_i(t)} \frac{dv}{v\Lambda_i(v)}. \tag{4.18}$$

Theorem 4.8 *Under the above assumptions the heat kernel on $M = \bigsqcup_{i=1}^k M_i$ satisfies, for all $x, y \in M$ and all $t > 0$,*

$$p(t,x,y) \le \frac{C}{\digamma(ct)}, \tag{4.19}$$

for some positive constants C, c, where

$$\digamma(t) := \min_{1\le i\le k} \digamma_i(t). \tag{4.20}$$

Moreover, if $\digamma$ satisfies the regularity condition (4.9), then

$$p(t,x,y) \le \frac{C}{\digamma(ct)} \exp\left(-\frac{d^2(x,y)}{Dt}\right) \tag{4.21}$$

for all $x, y \in M$ and all $t > 0$.

Proof. By Theorem 3.4 M admits the FK-function

$$\Lambda(v) := c \min_{1\le i\le k} \Lambda_i(Qv)\,. \tag{4.22}$$

Let us define the function $\widetilde{\digamma}$ by

$$t = \int_0^{\widetilde{\digamma}(t)} \frac{dv}{v\Lambda(v)}. \tag{4.23}$$

By Proposition 4.3, the heat kernel $p(t,x,y)$ on M satisfies, for all $x, y \in M$, $t > 0$,

$$p(t,x,y) \le \frac{C}{\widetilde{\digamma}(ct)}. \tag{4.24}$$

Hence, the estimate (4.19) will follow if we prove that

$$\widetilde{F}(t) \geq cF(ct), \tag{4.25}$$

for some $c > 0$. Indeed, (4.22) and (4.23) yield

$$\frac{1}{\Lambda(v)} = \max_{1 \leq i \leq k} \frac{1}{c\Lambda_i(Qv)} \leq \sum_{i=1}^{k} \frac{1}{c\Lambda_i(Qv)}$$

and

$$ct = c\int_0^{\widetilde{F}(t)} \frac{dv}{v\Lambda(v)} \leq \sum_{i=1}^{k} \int_0^{\widetilde{F}(t)} \frac{dv}{v\Lambda_i(Qv)} = \sum_{i=1}^{k} \int_0^{Q\widetilde{F}(t)} \frac{dv}{v\Lambda_i(v)}.$$

Therefore, for some $i = 1, 2, ...k$,

$$\int_0^{Q\widetilde{F}(t)} \frac{dv}{v\Lambda_i(v)} \geq \frac{c}{k}t = c't,$$

which implies by (4.18) and (4.20),

$$Q\widetilde{F}(t) \geq F_i(c't) \geq F(c't),$$

and (4.25) follows.

Finally, if F satisfies the regularity property (4.9), then (4.21) follows from (4.19) by Proposition 4.4. ■

Example. Assume that each Λ_i is given by

$$\Lambda_i(v) = c\begin{cases} v^{-2/N}, & v \leq 1, \\ v^{-2/n_i}. & v > 1. \end{cases} \tag{4.26}$$

Then, by (4.18),

$$F_i(t) \asymp \begin{cases} t^{N/2}, & t \leq 1, \\ t^{n_i/2}, & t > 1, \end{cases},$$

which obviously satisfies (4.9). By (4.20), we obtain

$$F(t) \asymp \begin{cases} t^{N/2}, & t \leq 1, \\ t^{n/2}, & t > 1, \end{cases},$$

where $n = \min_{i \geq 1} n_i$. This function satisfies the regularity property (4.9).

Example. Assume that M consists of two ends, with the FK-functions, given for large v as follows:

$$\Lambda_1(v) = cv^{-1/\nu_1}$$

and

$$\Lambda_2(v) = c(\log v)^{1-\nu_2},$$

where $\nu_1, \nu_2 > 1$. Then we have, for large enough t,

$$\mathcal{F}_1(t) \asymp t^{\nu_1}$$

and

$$\mathcal{F}_2(t) = \exp\left((at+b)^{1/\nu_2}\right), \quad a > 0.$$

Obviously, we obtain $\mathcal{F}(t) = \mathcal{F}_1(t)$ for large t.

We say that a continuously differentiable function $f : (0,\infty) \to (0,\infty)$ satisfies (REG) if it has the following two properties.

- $f(0) = 0$, $f(\infty) = \infty$, $f' > 0$ and f'/f is monotone decreasing;
- there exist $\epsilon > 0$ such that the function $g = f'/f$ satisfies

$$g(t_2) \geq \epsilon g(t_1) \ \text{ for all } 0 < t_1 \leq t_2 \leq 2t_1.$$

By [8, Theorem 2.2] the following is true. If the heat kernel $p(t,x,y)$ on a complete non-compact manifold satisfies for all $x \in M$ and $t > 0$ the inequality

$$p(t,x,x) \leq \frac{1}{f(t)},$$

where f satisfies (REG) then M has the FK-function $\Lambda(v)$, where Λ is uniquely determined from f by the identity

$$t = \int_0^{f(\delta t)} \frac{dv}{v\Lambda(v)}$$

with some $\delta = \delta(\epsilon) > 0$.

Corollary 4.9 *Assume that heat kernel of each M_i satisfies for all $t > 0$ the estimate*

$$\sup_{x \in M_i} p_i(t,x,x) \leq \frac{1}{f_i(t)},$$

where f_i satisfies (REG). *Then the heat kernel on $M = \bigsqcup_{i=1}^k$ satisfies*

$$p(t,x,y) \leq \frac{C}{f(ct)}, \tag{4.27}$$

where $f(t) = \min_{1\leq i\leq k} f_i(t)$. Moreover, if f satisfies in addition the regularity condition (4.9)*, then*

$$p(t,x,y) \leq \frac{C}{f(ct)} \exp\left(-\frac{d^2(x,y)}{Dt}\right) \tag{4.28}$$

for all $x, y \in M$ and all $t > 0$.

Proof. By [8, Therem 2.2] each M_i has the FK-function Λ_i that satisfies

$$t = \int_0^{f_i(\delta t)} \frac{dv}{v\Lambda_i(v)}.$$

By Theorem 4.8 we obtain

$$p(t, x, y) \leq \frac{C}{\digamma(ct)},$$

where

$$\digamma(t) := \min_{1\leq i\leq k} f_i(\delta t),$$

which is equivalent to (4.27). ∎

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