# Pointwise estimates for transition probabilities of random walks on infinite graphs

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## 1. Preliminaries

Let $\Gamma$ be a countable infinite locally finite connected graph. If the vertices $x, y \in \Gamma$ are connected by an edge then we write $x \sim y$ and denote the edge connecting $x$ and $y$ by $\overline{xy}$. Suppose that each edge $\overline{xy}$ is assigned a positive weight $\mu_{xy} = \mu_{yx}$. Then define a function $\mu$ on vertices by

$$\mu(x) = \sum_{\{y: y \sim x\}} \mu_{xy}$$

and extend it to a measure on finite sets $A \subset \Gamma$ by

$$\mu(A) = \sum_{x \in A} \mu(x).$$

On every graph $\Gamma$ one can consider the *standard weight*: $\mu_{xy} = 1$ for all edges $\overline{xy}$. In this case, $\mu(x)$ is equal to the degree of $x$, that is to the number of adjacent edges.

A graph $\Gamma$ equipped by a weight $\mu$ as above is called a *weighted graph*. Any weighted graph $(\Gamma, \mu)$ admits a random walk on $\Gamma$ defined by the one-step transition probabilities

$$P(x,y) := \begin{cases} \frac{\mu_{xy}}{\mu(x)}, & \text{if } x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $P$ is a Markov kernel, that is

$$\sum_{y \in \Gamma} P(x,y) = 1.$$

It is symmetric with respect to the measure $\mu$, that is $P(x,y)\mu(x) = P(y,x)\mu(y)$. A random walk $X_n$ on $\Gamma$ is defined for all $n \in \mathbb{N}$ by the transition rule

$$\mathbb{P}\left(X_{n+1} = y \,|\, X_n = x\right) = P(x,y).$$

Denote $P_n(x,y) = \mathbb{P}\left(X_n = y \,|\, X_0 = x\right)$. The main subject of this survey is the *heat kernel* $p_n(x,y)$, defined as the density of $P_n(x,y)$ with respect to $\mu$, that is

$$p_n(x,y) = \frac{P_n(x,y)}{\mu(y)}.$$

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Clearly, $p_n(x,y) = p_n(y,x)$.

Denote by $d(x,y)$ the graph distance between points $x, y \in \Gamma$, that is the minimal number of edges required to connect $x$ to $y$ by an edge path.

We will consider here results of two kinds: about the on-diagonal behavior of the heat kernel, that is $p_n(x,x)$ as a function of $n \to +\infty$, and about the off-diagonal behavior, that is $p_n(x,y)$ as a function of the distance $d(x,y)$. On the connection between $\sup_{x\in\Gamma} p_n(x,x)$ and isoperimetric profiles see [9], [10] and [11]. For a general overview of random walks on infinite graphs see [32].

An important characteristic of a weighted graph is the volume growth function $V(x,r)$ defined as follows. Denote by $B(x,r)$ the ball associated with the distance $d(x,y)$, that is $B(x,r) = \{y \in \Gamma : d(x,y) < r\}$, and set $V(x,r) = \mu(B(x,r))$.

For example, for $\mathbb{Z}^D$ with a standard weight we have $V(x,r) \simeq r^D$. For this graph, a modification of the local central limit theorem implies that, for all positive $n \geq d(x,y)$ such that $n \equiv d(x,y) \pmod 2$,

$$p_n(x,y) \simeq n^{-D/2} \exp\left(-\frac{d^2(x,y)}{Cn}\right). \tag{1.1}$$

A similar estimate holds for Cayley graphs of finitely generated groups with polynomial volume growth (see [24]). More precise estimates for abelian groups can be found in [27].

Note that on any infinite weighted graph $(\Gamma, \mu)$, the following estimate holds:

$$p_n(x,y) \leq \frac{C}{\sqrt{\mu(x)\mu(y)}} \exp\left(-\frac{d^2(x,y)}{Cn}\right)$$

(see [31], [8]). However, this estimate does not reflect the decay of the heat kernel as $n \to \infty$.

Let us mention finally an estimate of a different kind, which holds on supercritical bond percolation clusters in $\mathbb{Z}^D$:

$$\sup_{y\in\Gamma} p_n(x,y) \leq C(x) n^{-D/2}$$

(see [29]).

## 2. On-diagonal decay and volume growth

The Markov property implies that for all $n \in \mathbb{N}$

$$\sum_{y\in\Gamma} P_n(x,y) = 1$$

that is

$$\sum_{y\in\Gamma} p_n(x,y)\mu(y) = 1.$$

Restricting the summation to $B(x,r)$, we obtain

$$\inf_{y\in B(x,r)} p_n(x,y) \leq \frac{1}{V(x,r)}.$$

This inequality suggests that the faster the function $V(x,r)$ increases the smaller the heat kernel should be, which agrees with (1.1). There are several results reflecting this heuristic idea. We restrict ourselves to the case of polynomial volume growth.

**Theorem 2.1.** [12, Theorem 4.6] *If for some $x_0 \in \Gamma$ and all $r > 0$*

$$V(x_0, 2r) \leq CV(x_0, r), \tag{2.1}$$

*then for all even* $n > 0$,

$$\sup_{x\in\Gamma} p_n(x,x) \geq \frac{c}{V(x_0,\sqrt{n})}. \tag{2.2}$$

In fact, the proof in [12] uses the additional hypotheses that $x \sim x$ for all $x \in \Gamma$ and that $\inf_{x\sim y} P(x,y) > 0$. One can get rid of these assumptions using [14, Proposition 4.2].

**Theorem 2.2.** *If for some* $x_0 \in \Gamma$ *and all* $r \geq 1$

$$V(x_0,r) \leq Cr^D,$$

*then for all even* $n > 0$

$$p_n(x_0,x_0) \geq \frac{c}{V(x_0,\sqrt{Cn\log n})}. \tag{2.3}$$

This Theorem follows from the proof in [28, Theorem 3] (see also [12, Remark 8 after Theorem 6.1]).

The lower bound (2.2) is sharp in $\mathbb{Z}^D$ (up to a constant factor). It is not known whether in Theorem 2.1 one can replace $\sup_x p_n(x,x)$ by $p_n(x_0,x_0)$. Theorem 2.2 provides a lower bound for $p_n(x_0,x_0)$ which in comparison with (2.2) is off by a factor $\sqrt{\log n}$. An example constructed in [7] indicates that one cannot get rid of the logarithm in (2.3). Note that the volume doubling condition (2.1) is not satisfied in this example.

**Theorem 2.3.** ([6, Theorem 2.1], [15, Section 5]) *Let the weighted graph* $(\Gamma,\mu)$ *satisfy the hypothesis*

$$\inf_{x\sim y} \mu_{xy} > 0. \tag{2.4}$$

*Suppose that, for all points* $x \in \Gamma$ *and all* $r \geq 1$,

$$V(x,r) \geq cr^D. \tag{2.5}$$

*Then, for all* $x \in \Gamma$ *and for all* $n \in \mathbb{N}$,

$$p_n(x,x) \leq Cn^{-\frac{D}{D+1}}. \tag{2.6}$$

Note that the hypothesis (2.4) is automatically satisfied for a standard weight. It also implies that (2.5) holds for $D = 1$. Hence, one can assume that $D \geq 1$.

In fact, [6, Theorem 2.1 and Remark 2.2] contains a more general result as follows. Suppose for all $x \in \Gamma$ and $r \geq 1$,

$$V(x,r) \geq v(r), \tag{2.7}$$

where $v$ is a continuous positive strictly increasing convex function on $[1,+\infty)$. Then, for all $x \in \Gamma$ and for all $n \in \mathbb{N}$,

$$p_n(x,x) \leq \frac{C}{\gamma(cn)}, \tag{2.8}$$

where $\gamma$ is determined by the relation

$$\gamma^{-1}(s) \simeq sv^{-1}(s). \tag{2.9}$$

For example, if $v(r) = cr^D$ then $\gamma^{-1}(s) \simeq s^{1+1/D}$ whence $\gamma(t) \simeq t^{\frac{D}{D+1}}$.

The upper bound (2.6) is by far not optimal for $\mathbb{Z}^D$. Of course, the value of Theorem 2.3 largely depends on whether the exponent $D/(D+1)$ is sharp, and it indeed is.

**Theorem 2.4.** [6, Theorem 4.1] *For arbitrarily large $D$ there exists a graph $\Gamma$ for which (with a standard weight)*

$$V(x,r) \simeq r^D \tag{2.10}$$

*and*

$$p_n(x,x) \simeq n^{-\frac{D}{D+1}} \tag{2.11}$$

*for all $x \in \Gamma$, $r \geq 1$ and even $n > 0$.*

Let us describe this example, motivated by fractal studies. Fix a positive integer $N$ and construct a graph $\Gamma$ called a *Vicsek tree* as a subset of $\mathbb{R}^N$. Let $Q_r$ denote the cube in $\mathbb{R}^N$

$$Q_r = \left\{x \in \mathbb{R}^N : 0 \leq x_i \leq r, \quad i = 1, 2, ..., N\right\}.$$

Construct an increasing sequence $\{\Gamma_k\}$ of finite graphs as subsets of $Q_{3^k}$. Let $\Gamma_1$ be the set of $2^N+1$ points containing all vertices of $Q_1$ and the center of $Q_1$. Define $2^N$ edges in $\Gamma_1$ as segments connecting the center with the corners. Assuming that $\Gamma_k$ is already constructed, define $\Gamma_{k+1}$ as follows. The cube $Q_{3^{k+1}}$ is naturally divided into $3^N$ congruent copies of $Q_{3^k}$; select $2^N+1$ of the copies of $Q_{3^k}$ by taking the corner cubes and the center one. In each of the selected copies of $Q_{3^k}$ construct a congruent copy of graph $\Gamma_k$, and define $\Gamma_{k+1}$ as the union of all $2^N+1$ copies of $\Gamma_k$ (merged at the corners). Then the Vicsek tree $\Gamma$ is the union of all $\Gamma_k$, $k \geq 1$ (see Fig. 1).

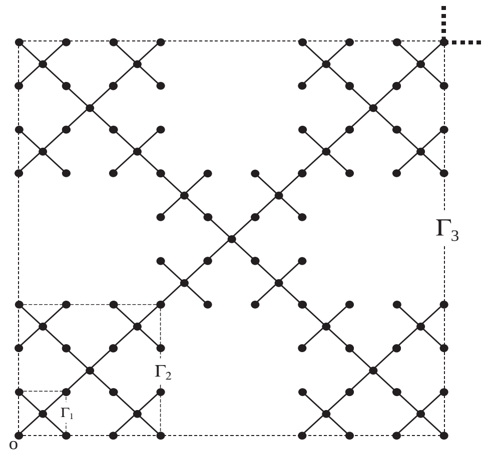


**Figure 1.** Vicsek tree in the plane

turns out that for the Vicsek tree both conditions (2.10) and (2.11) hold with

$$D = \log_3\left(2^N+1\right).$$

In [4], very interesting constructions of graphs are given, which yield the conclusion of Theorem 2.4 for every $D \geq 1$.

Concerning arbitrary types of volume growths, it is also shown in [6] that there are graphs satisfying (2.7) for any reasonable function $v$ and where the matching lower bound to (2.8) holds, up to a logarithmic factor.

## 3. Gaussian upper bound

For any function $f$ on $\Gamma$ define its energy by

$$\mathcal{E}(f) = \frac{1}{2} \sum_{\{x,y:x\sim y\}} (f(x) - f(y))^2 \mu_{xy}.$$

Define the Laplace operator on a function $f$ on $\Gamma$ by

$$\Delta f(x) = \sum_{y \sim x} P(x, y) f(y) - f(x).$$

One says that $f$ is *harmonic* on $\Omega \subset \Gamma$ if $f$ is defined on $\Omega$ and all neighbors of $\Omega$ and if

$$\Delta f(x) = 0 \; \forall x \in \Omega$$

(in other words, $f$ has the mean value property with respect to $P$). Similarly, $f$ is subharmonic if $\Delta f \geq 0$.

For any non-empty finite set $A \subset \Gamma$, let $c_0(A)$ be the set of all functions on $\Gamma$ which vanish outside $A$; then define

$$\lambda_1 (A) = \inf_{f \in c_0(A), f \not\equiv 0} \frac{\mathcal{E}(f)}{\sum_x f^2(x) \mu(x)}.$$

Alternatively, $\lambda_1(A)$ is the minimal eigenvalue of the Laplace operator $\Delta$ restricted to the space $c_0(A)$ of all functions on $\Gamma$ vanishing outside $A$.

For example, in $\mathbb{Z}^D$ the following lower bound for $\lambda_1$ is true:

$$\lambda_1 (A) \geq c \mu (A)^{-2/D}. \tag{3.1}$$

Consider the following conditions which in general may or may not hold:

- *Volume doubling property:* for some $C > 0$ and all $x \in \Gamma$, $r > 0$

$$V(x, 2r) \leq C V(x, r). \tag{$VD$}$$

- *On-diagonal upper estimate*: for all $x \in \Gamma$ and $n > 0$

$$p_n(x, x) \leq \frac{C}{V(x, \sqrt{n})}. \tag{$DUE$}$$

- *On-diagonal lower estimate*: for all $x \in \Gamma$ and even $n > 0$

$$p_n(x, x) \geq \frac{c}{V(x, \sqrt{n})}, \tag{$DLE$}$$

  for some $c > 0$.

- *Gaussian upper estimate*: for all $x, y \in \Gamma$ and $n > 0$

$$p_n(x, y) \leq \frac{C}{V(x, \sqrt{n})} \exp\left( -\frac{d^2(x, y)}{Cn} \right). \tag{$UE$}$$

- *Faber-Krahn type inequality:* for any ball $B(x, r)$ on $\Gamma$ and any non-empty set $A \subset B(x, r)$

$$\lambda_1 (A) \geq \frac{c}{r^2} \left( \frac{V(x, r)}{\mu(A)} \right)^{\alpha} \tag{$FK$}$$

  for some $\alpha, c > 0$.

Observe that $(FK)$ holds in $\mathbb{Z}^D$ with $\alpha = 2/D$, which follows from $V(x, r) \approx r^D$ and (3.1).

**Theorem 3.1.** [13, Theorems 1.1, 6.1] *For any weighted graph $(\Gamma, \mu)$, we have*

$$(FK) \Longleftrightarrow (UE) + (VD) \Longleftrightarrow (DUE) + (VD) \Longrightarrow (DLE).$$

*In particular, if $(\Gamma, \mu)$ satisfies $(VD)$ then*

$$(FK) \Longleftrightarrow (UE) \Longleftrightarrow (DUE) \Longrightarrow (DLE).$$

A continuous analogue of this theorem for the heat kernel on Riemannian manifolds was proved in [21].

Let $\nu$ be another weight on the graph $\Gamma$ such that $\mu(x) \simeq \nu(x)$ for all $x \in \Gamma$. Observe that the condition $(FK)$ is invariant under such a change of weight. More generally, it is also invariant under quasi-isometry of weighted graphs (see for instance [16]). By Theorem 3.1, the conditions $(UE)$ and $(DUE)$ are preserved by quasi-isometry, which is a priori not obvious.

## 4. Gaussian two-sided estimates

For any subset $A \subset \Gamma$ and any function $f$ on $\Gamma$, define the energy of $f$ on $A$ by

$$\mathcal{E}(f;A) := \frac{1}{2} \sum_{\{x,y: x\in A, y\sim x\}} (f(x)-f(y))^2 \mu_{xy}.$$

Alternatively,

$$\mathcal{E}(f;A) = \sum_{x\in A} |\nabla f|^2(x)\mu(x)$$

where

$$|\nabla f|^2(x) := \frac{1}{2}\sum_{y\sim x} (f(y)-f(x))^2 P(x,y).$$

Denote also

$$f_A = \frac{1}{\mu(A)} \sum_{y\in A} f(y)\mu(y).$$

Let us introduce the following hypotheses which in general may or may not be true.

- *Poincaré inequality*: for some $c>0$, $\delta \in (0,1]$ and for all $x\in\Gamma$, $R>0$, for any function $f$ on $\Gamma$,

$$(PI) \qquad \frac{c}{R^2} \sum_{y\in B(x,\delta R)} |f(y)-f_{B(x,R)}|^2 \mu(y) \le \mathcal{E}(f; B(x,R)).$$

- *Lower estimate* of the heat kernel: for all $x,y\in\Gamma$ and all positive $n \ge d(x,y)$,

$$(LE) \qquad p_n(x,y) \ge \frac{c}{V(x,\sqrt{n})} \exp\left(-\frac{d(x,y)^2}{cn}\right).$$

- *Parabolic Harnack* inequality: for all $x\in\Gamma$, $R\ge 1$ and for any non-negative function $u_n(y)$ defined for $n\in[0,4T]$, $y\in B(x,2R+1)$ and satisfying the heat equation $u_{n+1} = Pu_n$ in $[0,4T)\times B(x,2R)$, the following inequality holds

$$(PH) \qquad \max_{\substack{n\in[T,2T)\\ y\in B(x,R)}} u_n(y) \le C \min_{\substack{n\in[3T,4T)\\ y\in B(x,R)}} u_n(y),$$

provided $T$ is a positive integer such that $T \simeq R^2$ and $T\ge 2R$.

The relation between these hypotheses is given by the following theorem.

**Theorem 4.1.** [18] *Assume that* $(\Gamma,\mu)$ *satisfies the following conditions:*

$(i)$ $x\sim x$ *for all* $x\in\Gamma$

$(ii)$ *there exists* $\varepsilon>0$ *such that* $P(x,y)\ge\varepsilon>0$, *for all* $x\sim y$.

*Then*

$$(UE)+(LE) \Longleftrightarrow (PH) \Longleftrightarrow (VD)+(PI).$$

The continuous version of this theorem was proved in [30] and [20]. It is possible to show that the conditions $(VD)$ and $(PI)$ are stable under quasi-isometry (see [16]); hence both $(PH)$ and $(UE)+(LE)$ are stable under quasi-isometry.

The role of the condition $(i)$ is to avoid the parity problem. Indeed, it excludes in particular bipartite graphs for which $p_n(x,y)=0$ whenever $n$ and $d(x,y)$ have different parities. Of course, for such graphs $(LE)$ cannot hold.

The hypothesis $(ii)$ implies that the degree of each vertex $x\in\Gamma$ is uniformly bounded from above. If $\mu$ is a standard weight then $(ii)$ is equivalent to this condition.

Note that $(VD)+(PI)\Longrightarrow(FK)$ (see [13]). Therefore, the implication $(VD)+(PI)\Longrightarrow (UE)$ can be deduced by Theorem 3.1. By the same theorem, one obtains the on-diagonal lower bound $(DLE)$ for the heat kernel. Hence, the main point of Theorem 4.1 is in obtaining the off-diagonal lower estimate, that is $(VD) + (PI) \Longrightarrow (LE)$.

A more direct approach than the one in [18] to off-diagonal lower bounds was given in [3]. The idea is to prove elliptic regularity estimates as a consequence of $(VD)$ and $(PI)$ by coming back to the ideas of De Giorgi, then deduce parabolic regularity from elliptic regularity by using the method introduced in [2] in a continuous context, and finally obtain the full off-diagonal lower bound.

Let us start with some definitions:

**Definition 4.2.** We say that $(\Gamma, \mu)$ satisfies the *De Giorgi property* if there exist $C > 0$ and $\alpha \in (0,1)$ such that for every $x \in \Gamma$, every $r, R$ such that $1 \le r \le R$, and every function $f$ which is harmonic on $B(x,R)$, one has

$$(DG) \qquad \mathcal{E}\left(f; B(x,r)\right) \le C \left(\frac{R}{r}\right)^{2(1-\alpha)} \frac{V(x,r)}{V(x,R)} \mathcal{E}\left(f; B(x,R)\right).$$

**Definition 4.3.** We say that a *parabolic oscillation estimate* holds on $(\Gamma, \mu)$ if there exist $\beta, \delta, C > 0$, such that

$$(PO) \qquad |p_n(x,y) - p_n(y,y)| \le C \left(\frac{d(x,y)}{\sqrt{n}}\right)^{\beta} \frac{1}{V(y,\sqrt{n})}$$

for all $x, y \in \Gamma$ and $n \in \mathbb{N}^*$ such that $d(x,y) \le \delta\sqrt{n}$.

It is easy to check that $(PO)$ together with the on-diagonal lower bound $(DLE)$ (which follows from $(UE)$ and $(VD)$) imply $(LE)$. Indeed, one can write

$$|p_n(x,y) - p_n(y,y)| \le C' \left(\frac{d(x,y)}{\sqrt{n}}\right)^{\beta} p_n(y,y),$$

and if one chooses $a \le \delta$ such that $C'a^\beta \le \frac{1}{2}$, then

$$p_n(x,y) \ge \frac{1}{2} p_n(y,y) \ge \frac{c'}{V(y,\sqrt{n})} \ge \frac{c''}{V(x,\sqrt{n})}$$

for all $n \in \mathbb{N}^*$ and all $x, y \in \Gamma$ such that $d(x,y) \le a\sqrt{n}$. Then, a classical iteration argument of Aronson [1] yields $(LE)$. Therefore, the main task is to obtain $(PO)$.

We recall here the discrete version of a classical result, known as Cacciopoli inequality.

**Proposition 4.4.** *There exists an absolute constant $C > 0$ such that, for all $x \in \Gamma$, $0 < r < R$, and for all non-negative subharmonic functions $f$ on $B(x,R)$,*

$$(C) \qquad \mathcal{E}\left(f; B(x,r)\right) \le \frac{C}{(R-r)^2} \sum_{y \in B(x,R)} |f(y)|^2 \mu(y).$$

It was proved in [17], using the Moser iteration argument, that $(VD)$ and $(PI)$ imply an elliptic regularity estimate. An alternative approach uses the ideas of De Giorgi (see [3]).

**Proposition 4.5.** *Assume that $(\Gamma, \mu)$ satisfies $(VD)$ and $(PI)$. Then there exist $\alpha, C > 0$ such that for every $x_0 \in \Gamma$, $R \ge 1$, $f \in \mathbb{R}^\Gamma$ harmonic in $B(x_0,R)$ and $x, y \in B(x_0, R/4)$, one has*

$$(ER) \qquad |f(x) - f(y)| \le C \left(\frac{d(x,y)}{R}\right)^{\alpha} \left(\frac{1}{V(x_0,R)} \sum_{z \in B(x_0,R)} |f(z) - f_R(x_0)|^2 \mu(z)\right)^{\frac{1}{2}}.$$

We now show that

$$(ER) \implies (DG).$$

Indeed, if $f$ is harmonic in $B(x_0,R)$ and $1 \le r \le R/8$, write, for $x \in B(x_0,2r)$,

$$\begin{aligned}
|f(x) - f_{2r}(x_0)| &\le \frac{1}{V(x_0,2r)} \sum_{y\in B(x_0,2r)} |f(x)-f(y)|\mu(y) \\
&\le \frac{C}{V(x_0,2r)} \sum_{y\in B(x_0,2r)} \left(\frac{d(x,y)}{R}\right)^{\alpha} \left(\frac{1}{V(x_0,R)} \sum_{z\in B(x_0,R)} |f(z)-f_R(x_0)|^2\mu(z)\right)^{\frac12} \mu(y) \\
&\le C\left(\frac{r}{R}\right)^{\alpha} \left(\frac{1}{V(x_0,R)} \sum_{z\in B(x_0,R)} |f(z)-f_R(x_0)|^2\mu(z)\right)^{\frac12},
\end{aligned}$$

thus

$$\left(\sum_{x\in B(x_0,2r)} |f(x)-f_{2r}(x_0)|^2\mu(x)\right)^{\frac12} \le C\left(\frac{r}{R}\right)^{\alpha} \left(\frac{V(x_0,2r)}{V(x_0,R)} \sum_{z\in B(x_0,R)} |f(z)-f_R(x_0)|^2\mu(z)\right)^{\frac12}.$$

By applying $(VD)$, $(PI)$ and $(C)$, we obtain

$$\mathcal{E}\,(f;B(x_0,r))^{\frac12} \le C\left(\frac{r}{R}\right)^{\alpha-1} \left(\frac{V(x_0,r)}{V(x_0,R)}\mathcal{E}\,(f,B(x_0,R))\right)^{\frac12}$$

whenever $1 \le r \le R/8$. On the other hand, if $R/8 < r \le R$ then the result is trivial, hence $(DG)$ is obtained.

Finally, from the De Giorgi property $(DG)$ one can deduce $(PO)$ by proving first the following "inhomogeneous" $(DG)$

$$\mathcal{E}\,(f;B(x_0,r)) \le 8C\left(\frac{r}{R}\right)^{2(\alpha-1)} \frac{V(x_0,r)}{V(x_0,R)}\mathcal{E}\,(f;B(x_0,R)) + 2C'R^2 \sum_{y\in B(x_0,R)} |f(y)|^2\mu(y),$$

where $\Delta u = f$ on $\Gamma$ and $C,\alpha$ are the constants in De Giorgi's property. This inequality is similar to Morrey's fundamental estimate on inhomogeneous elliptic equations and can be used to derive parabolic estimates by following the ideas of [2].

## 5. Sub-Gaussian estimates

Recent development of analysis on fractals has brought into attention a new class of graphs which exhibit in the large scale the same phenomena as fractal sets in the small. One of the examples of such graphs is the Vicsek tree discussed in Section 2. Another example is shown on Fig. 2.

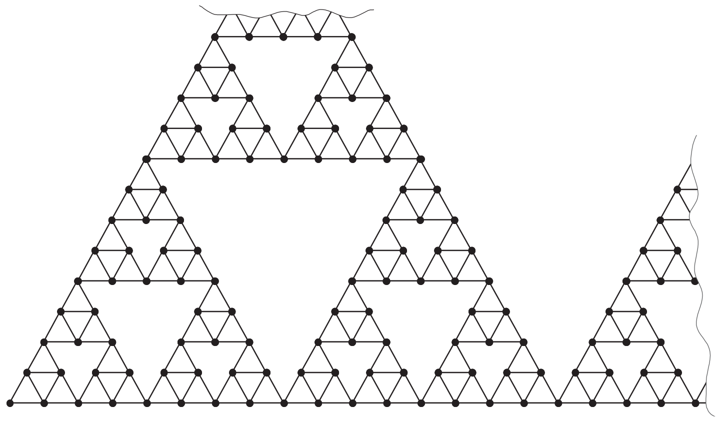

**Figure 2.** Graphical Sierpinski gasket

A common feature of fractal-like graphs is that a standard random walk on them exhibits a subdiffusive behavior. To be precise, let us introduce the following hypotheses, for an arbitrary weighted graph $(\Gamma, \mu)$:

- Upper estimate with parameter $\beta$: for all $x, y \in \Gamma$ and $n > 0$

$$(UE_\beta) \qquad p_n(x,y) \leq \frac{C}{V(x, n^{1/\beta})} \exp\left(-\left(\frac{d^\beta(x,y)}{Cn}\right)^{\frac{1}{\beta-1}}\right).$$

- Lower estimate with parameter $\beta$: for all $x, y \in \Gamma$ and positive $n \geq d(x,y)$

$$(LE_\beta) \qquad (p_n + p_{n+1})(x,y) \geq \frac{c}{V(x, n^{1/\beta})} \exp\left(-\left(\frac{d^\beta(x,y)}{cn}\right)^{\frac{1}{\beta-1}}\right).$$

Clearly, the hypothesis $(UE)$ introduced in the previous section coincides with $(UE_2)$. The condition $(LE)$ seems slightly stronger than $(LE_2)$ as it claims the lower bound for $p_n$ rather than for $p_n + p_{n+1}$. However, this has to do with the fact that $(LE)$ was obtained in the previous section under the assumption $x \sim x$ that will not be assumed here. In particular, now we do not exclude bipartite graphs.

It is possible to prove that if $(UE_\beta)$ and $(LE_\beta)$ hold on $(\Gamma, \mu)$ then necessarily $\beta \geq 2$. The case $\beta = 2$ is referred to as Gaussian; a more general case $\beta \geq 2$ is referred to as sub-Gaussian. If $\beta \geq 2$ and $n \geq d(x,y)$ then clearly

$$\left(\frac{d^\beta(x,y)}{n}\right)^{\frac{1}{\beta-1}} \geq \frac{d^2(x,y)}{n},$$

which means that $p_n(x,y)$ decays with $d(x,y)$ faster in the sub-Gaussian case rather than in Gaussian. At the same time, $V(x, n^{1/\beta}) \leq V(x, n^{1/2})$ that is $p_n(x,x)$ decays in $n$ slower in the sub-Gaussian case. Therefore, the propagation of the random walks in the sub-Gaussian case is in general slower than in the Gaussian; hence the name.

One may expect that the results discussed in Sections 3, 4 should be extended (with proper modification) to the sub-Gaussian case. However, such extensions are not straightforward and are not available at the present time. Instead, we present here alternative characterizations of sub-Gaussian estimates obtained in [22], [23].

For any non-empty finite set $A \subset \Gamma$, let $\tau_A$ be the *first exit time* from $A$, that is

$$\tau_A = \min\{n \geq 0 : X_n \notin A\}.$$

Set

$$E(x,r) = \mathbb{E}_x \tau_{B(x,r)},$$

which is the *mean exit time* from the ball $B(x,r)$ starting from the center.

For any couple of finite sets $A \subset B \subset \Gamma$, define the *capacity* $\operatorname{cap}(A,B)$ by

$$\operatorname{cap}(A,B) = \inf_{f|_A=1, f|_{\complement B}=0} \mathcal{E}(f). \tag{5.1}$$

Consider the following hypotheses, where $\beta$ is a positive parameter.

- *Mean exit time estimate*: for all $x \in \Gamma$ and $r \geq 1$,

$$E(x,r) \simeq r^\beta. \tag{$E_\beta$}$$

- *Capacity estimate:* for all $x \in \Gamma$ and $r \geq 1$,

$$\operatorname{cap}(B(x,r), B(x,2r)) \simeq \frac{V(x,r)}{r^\beta}. \tag{$Cap_\beta$}$$

- *Elliptic Harnack inequality:* for any non-negative harmonic function $u(x)$ in a ball $B(x,2r)$, the following inequality holds:

$$\max_{B(x,r)} u \leq C \min_{B(x,r)} u, \tag{$H$}$$

  with a constant $C$ that is the same for all $x \in \Gamma$ and $r > 0$.
- *Parabolic Harnack inequality* with parameter $\beta$: for all $x \in \Gamma$, $R \geq 1$ and for any non-negative function $u_n(y)$ defined for $n \in [0, 4T]$, $y \in B(x, 2R+1)$ and satisfying the heat equation $u_{n+1} = Pu_n$ in $[0,4T) \times B(x,2R)$, the following inequality holds

$$\max_{\substack{n \in [T,2T) \\ y \in B(x,R)}} u_n(y) \leq C \min_{\substack{n \in [3T,4T) \\ y \in B(x,R)}} \left(u_n(y) + u_{n+1}(y)\right), \tag{$PH_\beta$}$$

  provided $T$ is a positive integer such that $T \simeq R^\beta$ and $T \geq 2R$.

The hypothesis $(PH)$ introduced in the previous section differs from $(PH_2)$ only by using in the right hand side $u_n$ rather than $u_n + u_{n+1}$. Let us also observe that $(PH_\beta) \Longrightarrow (H)$ for any $\beta$.

**Theorem 5.1.** [22, Theorem 3.1] *Assume that* $(\Gamma, \mu)$ *satisfies the following condition:*

$$P(x,y) \geq \varepsilon > 0, \quad \text{for all } x \sim y. \tag{5.2}$$

*Then*

$$(UE_\beta) + (LE_\beta) \Longleftrightarrow (PH_\beta) \Longleftrightarrow (VD) + (H) + (E_\beta) \Longleftrightarrow (VD) + (H) + (Cap_\beta). \tag{5.3}$$

One of the points of Theorem 5.1 is that it shows the difference between the elliptic Harnack inequality $(H)$ and the parabolic one $(PH_\beta)$. Indeed, as one sees from (5.3) the difference is $(VD)$ and one of the conditions $(E_\beta)$ or $(Cap_\beta)$ that determine the parameter $\beta$. An example of Delmotte [19] shows that $(H)$ does not imply $(VD)$. It is easy to see $(H)$ does not imply $(E_\beta)$ either, because there are examples of graphs satisfying $(H)$ and $(E_\beta)$ with *arbitrary* $\beta \geq 2$ (see [4]).

For the case $\beta = 2$, this theorem admits the following extensions. Let us introduce the *radial Faber-Krahn inequality:*

$$\lambda_1(B(x,r)) \geq \frac{c}{r^2}, \tag{$RFK$}$$

for some $c > 0$ and all $x \in \Gamma$ and $r \geq 1$. Then

$$(UE_2) + (LE_2) \Longleftrightarrow (PH_2) \Longleftrightarrow (VD) + (H) + (RFK) \Longleftrightarrow (FK) + (H). \tag{5.4}$$

The first equivalence is a particular case of (5.3), and the second equivalence was proved in [22, Corollary 3.2]. To include $(FK) + (H)$ in the circle, note that the implication

$$(UE_2) + (LE_2) \Longrightarrow (FK) + (H)$$

follows from Theorem 3.1 and (5.3), whereas the implication

$$(FK) + (H) \Longrightarrow (VD) + (H) + (RFK)$$

follows from $(FK) \Longrightarrow (VD)$ (cf. Theorem 3.1) and the trivial observation that $(FK) \Longrightarrow (RFK)$. Note also that the equivalence $(PH_2) \Longleftrightarrow (FK) + (H)$ was proved in [25] in the setting of manifolds.

A deep question related to the above considerations is to provide a geometric understanding of the elliptic Harnack inequality $(H)$. This question is still open, also in the continuous setting of manifolds.

If $(VD)$ and $(Cap_\beta)$ are satisfied then necessarily $\beta \geq 2$. Indeed, the following upper estimate of capacity is always true

$$\operatorname{cap}(B(x,r), B(x,2r)) \leq \frac{V(x,2r)}{r^2}$$

(it follows by taking a standard cut-off function in definition (5.1) of capacity). Combining with $(VD)$ and $(Cap_\beta)$, we conclude $\beta \geq 2$.

The following theorem extends further the line of equivalences (5.3).

**Theorem 5.2.** *Assume that* $(\Gamma, \mu)$ *satisfies the condition (5.2). Then*

$$(VD) + (H) + (UE_\beta) \Longrightarrow (LE_\beta)\,. \tag{5.5}$$

*In particular,*

$$(VD) + (H) + (UE_\beta) \Longleftrightarrow (UE_\beta) + (LE_\beta) \Longleftrightarrow (PH_\beta)\,.$$

A continuous version of this result was proved in [25]. A similar approach can be used in the graph case (see [23, Remark 15.1]).

Examples of graphs satisfying sub-Gaussian estimates $(UE_\beta) + (LE_\beta)$ can be found in [5], [26], [22]. In most of these examples, the volume growth function satisfies a uniform estimate

$$V(x,r) \simeq r^\alpha, \tag{$V_\alpha$}$$

for all $x \in \Gamma$, $r \geq 1$, for some parameter $\alpha$. In this case, the sub-Gaussian estimates become as follows:

$$p_n(x,y) \leq \frac{C}{n^{\alpha/\beta}} \exp\left(-\left(\frac{d^\beta(x,y)}{Cn}\right)^{\frac{1}{\beta-1}}\right) \tag{$UE_{\alpha,\beta}$}$$

and

$$(p_n + p_{n+1})(x,y) \geq \frac{c}{n^{\alpha/\beta}} \exp\left(-\left(\frac{d^\beta(x,y)}{cn}\right)^{\frac{1}{\beta-1}}\right). \tag{$LE_{\alpha,\beta}$}$$

Then Theorem 5.1 yields the equivalence

$$(UE_{\alpha,\beta}) + (LE_{\alpha,\beta}) \Longleftrightarrow (V_\alpha) + (H) + (E_\beta) \Longleftrightarrow (V_\alpha) + (H) + (Cap_\beta)\,. \tag{5.6}$$

In general, the parameters $\alpha, \beta$ must satisfy the following relations

$$2 \leq \beta \leq \alpha + 1 \tag{5.7}$$

(see [23, Section 2]). Indeed, $\beta \geq 2$ was proved above. To show that $\beta \leq \alpha + 1$ let us recall that $(V_\alpha)$ implies by Theorem 2.3

$$p_n(x,x) \leq Cn^{-\frac{\alpha}{\alpha+1}}.$$

Comparing with the on-diagonal lower bound which follows from $(LE_{\alpha,\beta})$ we obtain $\beta \leq \alpha + 1$.

It was shown in [4], that for *any* pair of $\alpha$ and $\beta$ satisfying (5.7) there exists a graph where $(V_\alpha)$, $(E_\beta)$, and $(H)$ hold. The Vicsek tree with parameter $D$ described in Section 2 (see Fig. 1) satisfies all the conditions (5.6) with $\alpha = D$ and $\beta = D+1$ (see [22]). Hence, in this case $\beta = \alpha+1$

which is the borderline case for $\alpha$ and $\beta$ in (5.7). The Sierpinski gasket on Fig. 2 satisfies (5.6) with $\alpha = \log_2 3$ and $\beta = \log_2 5$ (see [26]).

In these two examples we have $\alpha < \beta$. If on the contrary $\alpha > \beta$ then the sub-Gaussian estimates admit another characterization in terms of the *Green kernel* $g(x, y)$ that is defined by

$$g(x, y) = \sum_{n=0}^{\infty} p_n(x, y)$$

(alternatively, $g(x, y)$ is the kernel of the operator $(-\Delta)^{-1}$). It is straightforward that $(UE_{\alpha,\beta})$ and $(LE_{\alpha,\beta})$ imply

$$(G_{\alpha-\beta}) \qquad g(x, y) \simeq d(x, y)^{-(\alpha-\beta)}, \quad \text{for all } x \neq y,$$

provided $\alpha > \beta$.

**Theorem 5.3.** [23] *If $(\Gamma, \mu)$ satisfies (5.2) with $\alpha > \beta \geq 2$, then*

$$(UE_{\alpha,\beta}) + (LE_{\alpha,\beta}) \Longleftrightarrow (V_\alpha) + (G_{\alpha-\beta}).$$

## References


[1] **Aronson D.G.**, Non-negative solutions of linear parabolic equations, *Ann. Scuola Norm. Sup. Pisa. Cl. Sci. (4)*, **22** (1968) 607-694. Addendum **25** (1971) 221-228.

[2] **Auscher P.**, Regularity theorems and heat kernel for elliptic operators, *J. London Math. Soc.*, **2(54)** (1996) 284-296.

[3] **Auscher P., Coulhon T.**, Gaussian lower bounds for random walks from elliptic regularity, *Ann. Inst. Henri Poincaré - Probabilités et Statistiques*, **35** (1999) no.5, 605-630.

[4] **Barlow M.T.**, Which values of the volume growth and escape time exponents are possible for graphs?, preprint.

[5] **Barlow M.T., Bass R.F.**, Random walks on graphical Sierpinski carpets, *in:* "Random walks and discrete potential theory (Cortona, Italy, 1997)", ed. M. Picardello, W. Woess, Symposia Math. 39, Cambridge Univ. Press, Cambridge, 1999. 26-55.

[6] **Barlow M.T., Coulhon T., Grigor'yan A.**, Manifolds and graphs with slow heat kernel decay, *Invent. Math.*, **144** (2001) 609-649.

[7] **Barlow M.T., Perkins A.**, Symmetric Markov chains in $\mathbb{Z}^d$: how fast can they move?, *Probab. Th. Rel. Fields*, **82** (1989) 95–108.

[8] **Carne K.**, A transmutation formula for Markov chains, *Bull. Sc. Math., 2ème série*, **109** (1985) 399-403.

[9] **Coulhon T.**, Ultracontractivity and Nash type inequalities, *J. Funct. Anal.*, **141** (1996) no.2, 510-539.

[10] **Coulhon T.**, Analysis on infinite graphs with regular volume growth, *in:* "Random walks and discrete potential theory", ed. M. Picardello, W. Woess, Symposia Math. XXXIX, Cambridge University Press, 1999. 165-187.

[11] **Coulhon T.**, Random walks and geometry on infinite graphs, *in:* "Lecture notes on analysis on metric spaces", ed. Luigi Ambrosio, Francesco Serra Cassano, Scuola Normale Superiore di Pisa, 2000. 5-30.

[12] **Coulhon T., Grigor'yan A.**, On-diagonal lower bounds for heat kernels on non-compact manifolds and Markov chains, *Duke Math. J.*, **89** (1997) no.1, 133-199.

[13] **Coulhon T., Grigor'yan A.**, Random walks on graphs with regular volume growth, *Geom. and Funct. Analysis*, **8** (1998) 656-701.

[14] **Coulhon T., Grigor'yan A., Pittet Ch.**, A geometric approach to on-diagonal heat kernel lower bounds on groups, *Ann. Inst. Fourier, Grenoble*, **51** (2001) no.6, 1763-1827.

[15] **Coulhon T., Saloff-Coste L.**, Isopérimétrie pour les groupes et les variétés, *Revista Matemática Iberoamericana*, **9** (1993) no.2, 293-314.

[16] **Coulhon T., Saloff-Coste L.**, Variétés riemanniennes isométriques à l'infini, *Revista Matemática Iberoamericana*, **11** (1995) no.3, 687-726.

[17] **Delmotte T.**, Inégalité de Harnack elliptique sur les graphes, *Colloq. Math*, **72** (1997) no.1, 19-37.

[18] **Delmotte T.**, Parabolic Harnack inequality and estimates of Markov chains on graphs, *Revista Matemática Iberoamericana*, **15** (1999) no.1, 181-232.

[19] **Delmotte T.**, Graphs between elliptic and parabolic Harnack inequalities, to appear in *Potential Analysis.*

[20] **Grigor'yan A.**, The heat equation on non-compact Riemannian manifolds, (in Russian) *Matem. Sbornik*, **182** (1991) no.1, 55-87. Engl. transl. *Math. USSR Sb.*, **72** (1992) no.1, 47-77.

[21] **Grigor'yan A.**, Heat kernel upper bounds on a complete non-compact manifold, *Revista Matemática Iberoamericana*, **10** (1994) no.2, 395-452.

[22] **Grigor'yan A., Telcs A.**, Harnack inequalities and sub-Gaussian estimates for random walks, preprint.

[23] **Grigor'yan A., Telcs A.**, Sub-Gaussian estimates of heat kernels on infinite graphs, *Duke Math. J.*, **109** (2001) no.3, 452-510.

[24] **Hebisch W., Saloff-Coste, L.**, Gaussian estimates for Markov chains and random walks on groups, *Ann. Prob.*, **21** (1993) 673–709.

[25] **Hebisch W., Saloff-Coste, L.**, On the relation between elliptic and parabolic Harnack inequalities, *Ann. Inst. Fourier*, **51** (2001) no.5, 1437-1481.

[26] **Jones O.D.**, Transition probabilities for the simple random walk on the Sierpinski graph, *Stoch. Proc. Appl.*, **61** (1996) 45-69.

[27] **Kotani M., Shirai T., Sunada T.**, Asymptotic behavior of the transition probability of a random walk on an infinite graph, *J. Funct. Anal.*, **159** (1998) 664-689.

[28] **Lust-Piquard F.**, Lower bounds on $\|K^n\|_{1\to\infty}$ for some contractions $K$ of $L^2(\mu)$, with some applications to Markov operators, *Math.Ann.*, **303** (1995) 699-712.

[29] **Mathieu P., Remy E.**, Décroissance du noyau de la chaleur et isopérimétrie sur un amas de percolation, *C.R.A.S. Paris*, **332** (2001) no.10, 927-931.

[30] **Saloff-Coste L.**, A note on Poincaré, Sobolev, and Harnack inequalities, *Duke Math J.*, **65** no.3, *Internat. Math. Res. Notices*, **2** (1992) 27-38.

[31] **Varopoulos N.Th.**, Long range estimates for Markov chains, *Bull. Sc. Math., 2ème série*, **109** (1985) 113-119.

[32] **Woess W.**, "Random walks on infinite graphs and groups", Cambridge Tracts in Mathematics 138, Cambridge Univ. Press., 2000.



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