TWO-SIDED ESTIMATES OF HEAT KERNELS OF JUMP TYPE DIRICHLET FORMS

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Abstract. We prove necessary and sufficient conditions for stable-like estimates of the heat kernel for jump type Dirichlet forms on metric measure spaces. The conditions are given in terms of the volume growth function, jump kernel and a generalized capacity.

Contents

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1. Introduction

1.1. **Historical background and motivation.** The heat kernel $p_t(x,y)$ in $\mathbb{R}^n$ is the fundamental solution of the classical heat equation

$$\partial_t u - \Delta u = 0$$

that is given by the Gauss-Weierstrass formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{1.1}$$

The heat kernel of a similar heat equation with non-local operator

$$\partial_t u + (-\Delta)^{1/2} u = 0$$

is also known and coincides with the Cauchy-Poisson kernel in $\mathbb{R}^n$:

$$p_t(x,y) = \frac{C_n}{t^n}\left(1+\frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}}, \tag{1.2}$$

where $C_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$.

We are interested in estimates of heat kernels in rather general abstract setting. Let (M,d) be a locally compact separable metric space and let μ be a Radon measure on M with full support. Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2(M,\mu)$. The generator $\mathcal{L}$ of $(\mathcal{E},\mathcal{F})$ is a self-adjoint, unbounded, non-negative definite operator in $L^2(M,\mu)$ that gives rise to the heat semigroup $P_t = e^{-t\mathcal{L}}$, $t \geq 0$. It is known that the operator P_t is Markovian, that is, $P_t f \geq 0$ if $f \geq 0$ and $P_t f \leq 1$ if $f \leq 1$. These properties allow to extend P_t to a bounded linear operator in all spaces $L^q(M,\mu)$, $q \in [1,\infty]$.

If P_t as an operator in $L^2(M,\mu)$ has for any $t>0$ an integral kernel $p_t(x,y)$ then the latter is called the *heat kernel* of $(\mathcal{E},\mathcal{F})$. The heat kernel coincides with the transition density of the Hunt process associated with $(\mathcal{E},\mathcal{F})$.

For example, the Gauss-Weierstrass function (1.1) is the heat kernel of the Dirichlet form

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n} |\nabla f|^2 \, dx,$$

where $f \in \mathcal{F} := W^{1,2}(\mathbb{R}^n)$. The generator of this form is $\mathcal{L} = -\Delta$. A more general Dirichlet form

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n} a_{ij}\partial_{x_i} f \partial_{x_j} f \, dx$$

with a uniformly elliptic symmetric matrix $(a_{ij}(x))$ has the generator $\mathcal{L} = -\partial_{x_i}\left(a_{ij}\partial_{x_j}\right)$. By Aronson's theorem [2], its heat kernel (equivalently, the transition density of the diffusion process generated by $\mathcal{L}$) satisfies the Gaussian estimate

$$p_t(x,y) \asymp \frac{C}{t^{n/2}} \exp\left(-c\frac{|x-y|^2}{t}\right),$$

where C, c are positive constants, and the sign $\asymp$ means that both $\leq$ and $\geq$ are true but with different values of C, c.

Another well-known example of the Dirichlet form is

$$\mathcal{E}(f,f) = \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x)-f(y))^2}{|x-y|^{n+\beta}} dxdy,$$

where $f \in \mathcal{F} := B_{2,2}^{\beta/2}(\mathbb{R}^n)$, where $\beta \in (0,2)$ is the *index* of this form. The generator of this Dirichlet form is $\mathcal{L} = (-\Delta)^{\beta/2}$, and its heat kernel (that is, the transition density of the symmetric stable process of index β) satisfies the estimate

$$p_t(x,y) \asymp \frac{C}{t^{n/\beta}}\left(1+\frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}.$$

Note that (1.2) matches this estimate with $\beta = 1$.

In the general setting, assume that the heat kernel exists and satisfies the following estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

with some function Φ and two positive parameters α, β. Then, by a result of [25], we have either $\Phi(s) \asymp C\exp\left(-cs^{\frac{\beta}{\beta-1}}\right)$ or $\Phi(s) \asymp C(1+s)^{-(\alpha+\beta)}$. In other words, either the Dirichlet form is local and the heat kernel satisfies *sub-Gaussian* bounds

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right), \tag{1.3}$$

or the Dirichlet form is of jump type and the heat kernel satisfies *stable-like* bounds

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{1.4}$$

The sub-Gaussian estimate (1.3) was proved for many fractal spaces like Sierpinski gaskets and carpets, see for example, [3] [4] [5], [8], [13], [27], [29], [30]. The stable-like estimate (1.4) follows from (1.3) by subordination, see [16], [34].

In the both cases α has to be the Hausdorff dimension of (M,d). Moreover, both (1.3) and (1.4) imply the α-regularity of the volume of balls in M, that is, for any metric ball $B(x,r)$ in M,

$$\mu(B(x,r)) \simeq r^{\alpha}, \tag{1.5}$$

where the sign $\simeq$ means that the ratio of the both sides is bounded from above and below by positive constants.

In the case of (1.3), the parameter β is called the *walk dimension*. In fact, the walk dimension happens to be an invariant of (M,d) as well. It is known that in this case necessarily $\beta \geq 2$. In fact, for most interesting examples, like Sierpinski gaskets and carpets, we have $\beta > 2$. In the case of (1.4), the parameter β is called the *index* of the associated jump process, and it can take in general arbitrary positive values.

The major question that arises in this area is to find some practical conditions on (M,d,μ) and $(\mathcal{E},\mathcal{F})$ that should be equivalent to (1.3) resp. (1.4). Certain equivalent conditions for (1.3) were obtained in [26] and [19], but they contain an elliptic Harnack inequality that is difficult to verify. Some equivalent conditions for the upper bound in (1.4) were obtained in [18] and [22].

If (M,d,μ) is a complete Riemannian manifold and $(\mathcal{E},\mathcal{F})$ is the standard Riemannian Dirichlet form given by

$$\mathcal{E}(f,f) = \int_M |\nabla f|^2 \, d\mu,$$

then it is known (cf. [15], [33]) that the Gaussian heat kernel estimate that corresponds to $\beta = 2$ in (1.3), is equivalent to the conjunction of the following two properties:

- the volume regularity (1.5);
- the Poincaré inequality

$$\int_{B(x,r)} |\nabla f|^2 \, d\mu \geq \frac{c}{r^2}\int_{B(x,r)} \left(f-\overline{f}\right)^2 d\mu, \tag{1.6}$$

where $\overline{f}$ is the arithmetic mean of f in $B(x,r)$.

In the case $\beta > 2$ one replaces (1.6) by the *β-Poincaré inequality*

$$\int_{B(x,r)} d\Gamma\langle f,f\rangle \geq \frac{c}{r^\beta}\int_{B(x,r)}\left(f-\overline{f}\right)^2 d\mu, \tag{1.7}$$

where $\Gamma\langle f,f\rangle$ is the energy measure of f. In general, (1.5) and (1.7) are necessary for (1.3), but not sufficient, so that one needs one more condition. The third condition was introduced for the first time by Barlow, Bass and Kumagai in [6]. They named that condition a *cutoff Sobolev inequality* (shortly, (CS)) and proved that (1.5), (1.7) and (CS) are equivalent to (1.3).

The meaning of (CS) is that it postulates the existence of test functions with certain properties. However, (CS) is quite difficult both to state and to verify, the search for another third condition continues.

Andres and Barlow introduced in [1] a much simpler *cutoff Sobolev inequality in annuli* (shortly, (CSA)) and used it to obtain equivalent conditions for upper bound of sub-Gaussian type. Grigor'yan, Hu and Lau proved in [23] that (1.5), (1.7) and (CSA) are equivalent to (1.3). Note that in [23] the authors used a slightly different version of (CSA) that was named a *generalized capacity estimate* (shortly, $(Gcap)$). It was conjectured in [23] that $(Gcap)$ can be replaced by the following much simpler *capacity condition:* for any ball B of radius r,

$$\operatorname{cap}(\tfrac{1}{2}B,B) \leq C\frac{\mu(B)}{r^\beta}, \tag{1.8}$$

where cap is the capacity associated with $(\mathcal{E},\mathcal{F})$ (see (1.12) for the definition). This conjecture is still open.

A similar question is in place for the stable-like estimate (1.4). In this case, we assume that $(\mathcal{E},\mathcal{F})$ is a jump-type Dirichlet form with a symmetric jump kernel $J(x,y)$. Chen and Kumagai proved in [10] that, in the case $\beta < 2$, the stable-like estimate (1.4) is equivalent to the volume regularity (1.5) and the following estimate of the jump kernel J

$$J(x,y) \simeq \frac{1}{d(x,y)^{\alpha+\beta}} \tag{1.9}$$

that replaces in this case the Poincaré inequality.

The main question that we address in the present paper is obtaining equivalent conditions for (1.4) for arbitrary values of the index β, in particular, for $\beta \geq 2$. In this case, apart from (1.5) and (1.9) one needs a third condition. Ideally, the third condition should be again the capacity condition (1.8), but like in the diffusion case we can state this only as a conjecture.

Our main result here is that (1.4) is equivalent to the conjunction of (1.5), (1.9) and a certain *generalized capacity condition* $(Gcap)$ that is stated below in Definition 1.11.

We are aware of a preprint of Chen, Kumagai and Wang [11] where they obtained a similar result using as a third condition a non-local version of (CSA). We should emphasize one significant advantage of our condition $(Gcap)$ – it can be stated in the same form both for local and non-local Dirichlet forms, whereas the conditions like (CSA) have to use a specific shape of $\mathcal{E}$.

We should also mention that Chen, Kumagai and Wang [11] use a more general volume doubling property instead of the volume regularity (1.5) and a more general gauge function instead of r^β. However, they have to assume also a reverse volume doubling property which implies that the underlying space must be non-compact. In contrast to that, our result is stated and proved in a localized form, that is, when all assumptions are made for a restricted range of radius and the heat kernel bound (1.4) is obtained for a restricted range of time. Consequently, our results apply also to compact spaces. Yet one more difference is that our proof is completely analytic whereas that of [11] uses quite strongly the jump process associated with $(\mathcal{E},\mathcal{F})$ and corresponding probabilistic tools.

We expect that our method should work also for non-regular Dirichlet forms but this would require a revision of a number of the previous works that we cite here. Let us emphasize that we assume neither conservativeness of $(\mathcal{E},\mathcal{F})$ nor compactness of metric balls, although these assumptions are commonly used in many papers on this subject.

1.2. **Statement of the main result.** Let (M, d) be a locally compact separable metric space. We denote by $B(x, r)$ the open metric ball in (M, d) of radius r centered at $x \in M$. For any ball $B = B(x, r)$ and for any $\lambda > 0$, we denote by λB the ball $B(x, \lambda r)$.

Let μ be a Radon measure on M with full support and let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. We assume that $(\mathcal{E}, \mathcal{F})$ is of *jump type*, that is, for all $u, v \in \mathcal{F} \cap C_0(M)$

$$\mathcal{E}(u, v) = \iint_{M \times M \setminus \mathrm{diag}} (u(x) - u(y)) (v(x) - v(y))\, dj(x, y),$$

where j is a jump measure defined on $M \times M \setminus \mathrm{diag}$ (see [14]). Assume in addition that the jump measure j has a density with respect to $\mu \times \mu$, which will be denoted by $J(x, y)$. Hence, by [14, Lemma 4.5.4, p.184], for all $u, v \in \mathcal{F}$,

$$\mathcal{E}(u, v) = \iint_{M \times M} (u(x) - u(y))(v(x) - v(y)) J(x, y) d\mu(x) d\mu(y). \tag{1.10}$$

Let us fix two positive parameters α, β as well as $\overline{R} \in (0, \mathrm{diam}\, M]$ that will be used throughout the paper.

Definition 1.1 (Condition (V))**.** We say that condition $(V_\leq)$ is satisfied if, for all $x \in M$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq C r^\alpha,$$

for some constant $C > 0$. We say that condition $(V_\geq)$ is satisfied if, for all $x \in M$ and all $r \in (0, \overline{R})$,

$$\mu(B(x, r)) \geq C^{-1} r^\alpha.$$

We say that (V) is satisfied if both $(V_\leq)$ and $(V_\geq)$ are satisfied.

Definition 1.2 (Condition (J))**.** We say that condition $(J_\leq)$ is satisfied if, for all distinct $x, y \in M$,

$$J(x, y) \leq C d(x, y)^{-(\alpha+\beta)}.$$

Similarly, condition $(J_\geq)$ means that

$$J(x, y) \geq C^{-1} d(x, y)^{-(\alpha+\beta)}.$$

We say that (J) is satisfied if both $(J_\leq)$ and $(J_\geq)$ are satisfied.

By the general theory of Dirichlet forms (cf. [14]), $(\mathcal{E}, \mathcal{F})$ has the *generator* $\mathcal{L}$ that is a non-negative definite, self-adjoint, symmetric operator on $L^2(M, \mu)$. The generator gives rise to the heat semigroup $\{P_t\}_{t \geq 0}$, where $P_t := e^{-t\mathcal{L}}$ is a bounded self-adjoint operator in $L^2(M, \mu)$. If, for any $t > 0$, P_t is an integral operator, that is, given by

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y),$$

where $p_t(x, y)$ is the integral kernel, then $p_t(x, y)$ is called the *heat kernel* of $(\mathcal{E}, \mathcal{F})$. If it exists then, for any $t > 0$, $p_t(x, y)$ is a non-negative measurable function of (x, y).

In this paper, we are concerned with the following stable-like estimates of the heat kernel.

Definition 1.3 (Condition (UE))**.** We say that the upper estimate (UE) is satisfied if the heat kernel $p_t(x, y)$ exists and satisfies the following estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)},$$

for all $t \in (0, \overline{R}^\beta)$ and μ-almost all $x, y \in M$.

Note that

$$\frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \asymp t^{-\alpha/\beta} \wedge \frac{t}{d(x, y)^{\alpha+\beta}}.$$

Definition 1.4 (Condition (LE))**.** We say that the lower estimate (LE) is satisfied if the heat kernel $p_t(x,y)$ exists and satisfies the following estimate

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)},$$

for all $t \in (0, \overline{R}^\beta)$ and μ-almost all $x, y \in M$.

To state the main result, we need some more definitions.

Definition 1.5. Let $U \subset M$ be an open set, A be any Borel subset of U and $\kappa \geq 1$ be a real number. A κ-*cutoff function* of the pair (A,U) is any function $\phi \in \mathcal{F}$ such that

- $0 \leq \phi \leq \kappa$ μ-a.e. in M;
- $\phi \geq 1$ μ-a.e. in A;
- $\phi = 0$ μ-a.e. in U^c.

We denote by κ-cutoff(A,U) the collection of all κ-cutoff functions of the pair (A,U).

Any 1-cutoff function will be simply referred to as a *cutoff function*. Clearly, $\phi \in \mathcal{F}$ is a cutoff function of (A,U) if $0 \leq \phi \leq 1$, $\phi|_A = 1$ and $\phi|_{U^c} = 0$. Also, we write

$$\text{cutoff}(A,U) := 1\text{-}\,\text{cutoff}(A,U).$$

Note that, for any $\kappa \geq 1$,

$$\text{cutoff}(A,U) \subset \kappa\text{-cutoff}(A,U),$$

and, for any $\phi \in \kappa$-cutoff(A,U), we have $1 \wedge \phi \in$ cutoff(A,U).

Remark 1.6. Let us emphasize that we do not require a cutoff function ϕ to have a compact support nor to be continuous, unlike some other papers where this notion was used.

Consider the following function space

$$\mathcal{F}' := \{v + a : v \in \mathcal{F},\ a \in \mathbb{R}\}.$$

The motivation for introducing this space is to include constant functions that are not necessarily in $\mathcal{F}$.

Definition 1.7. Let U be an open subset of M and A be any Borel subset of U. For any function $u \in \mathcal{F}' \cap L^\infty$ and a real number $\kappa \geq 1$, define the *generalized capacity* $\text{cap}_u^{(\kappa)}(A,U)$ of the pair (A,U) by

$$\text{cap}_u^{(\kappa)}(A,U) = \inf\left\{\mathcal{E}\left(u^2\phi, \phi\right) :\ \phi \in \kappa\text{-cutoff}(A,U)\right\}. \tag{1.11}$$

In the case $\kappa = 1$ and $u \equiv 1$ we obtain the usual capacity:

$$\text{cap}(A,U) := \text{cap}_1^{(1)}(A,U) = \inf\left\{\mathcal{E}(\phi,\phi) : \phi \in \text{cutoff}(A,U)\right\}. \tag{1.12}$$

Remark 1.8. Observe that the quantity $\mathcal{E}(u^2\phi, \phi)$ in the definition of the generalized capacity is well defined. Indeed, if $u = v + a$ where $v \in \mathcal{F} \cap L^\infty$ and $a \in \mathbb{R}$ then by [14, Theorem 4.2(ii), p.28] we have

$$u^2\phi = v^2\phi + 2av\phi + a^2\phi^2 \in \mathcal{F}.$$

Remark 1.9. Note that if u is not a constant then $\mathcal{E}\left(u^2\phi, \phi\right)$ can take negative values so that the generalized capacity can be negative (unlike the usual capacity that is always non-negative). Since we will be using only upper bounds for $\text{cap}_u^{(\kappa)}(A,U)$, one could have avoided negative values by using in the definition (1.11) $\mathcal{E}\left(u^2\phi, \phi\right)_+$ instead of $\mathcal{E}\left(u^2\phi, \phi\right)$. This would make the generalized capacity non-negative, while all the results and proofs of this paper remain unchanged.

Remark 1.10. The notion of a generalized capacity for local Dirichlet forms was defined in [23] in a somewhat different way. However, the main result in [23] can be also reformulated for the generalized capacity defined by (1.11). The advantage of the definition (1.11) is that it works equally well for local and non-local Dirichlet forms.

Definition 1.11. We say that the *generalized capacity condition* (*Gcap*) is satisfied if there exist two constants $\kappa \geq 1, C > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ and for all concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $x_0 \in M$ and $0 < R < R + r < \overline{R}$,

$$\mathrm{cap}_u^{(\kappa)}(B_0, B) \leq \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{1.13}$$

Clearly, (*Gcap*) is equivalent to the existence of a number $\kappa \geq 1$ (not depending on u) and a function $\phi \in \kappa\text{-cutoff}(B_0, B)$ (depending on u) such that

$$\mathcal{E}\left(u^2\phi, \phi\right) \leq \frac{C}{r^\beta} \int_B u^2 d\mu.$$

Our main result is the following theorem (it is a consequence of a more general Theorem 2.10 to be stated below).

Theorem 1.12. *Let $(\mathcal{E}, \mathcal{F})$ be a regular jump type Dirichlet form on $L^2(M, \mu)$ with a jump kernel J. Assume that (M, d, μ) satisfies (V). Then the following equivalence holds:*

$$(J) + (Gcap) \Leftrightarrow (UE) + (LE). \tag{1.14}$$

Moreover, under these hypotheses, the heat kernel is Hölder continuous jointly in x, y and continuous jointly in x, y, t.

If $\overline{R} = \infty$ and if $(\mathcal{E}, \mathcal{F})$ is conservative then it is known that $(UE) + (LE) \Rightarrow (V)$ (see [20]). Hence, in this case the statement of Theorem 1.12 can be reformulated as follows:

$$(V) + (J) + (Gcap) \Leftrightarrow (UE) + (LE). \tag{1.15}$$

In the case $\beta < 2$ the condition (*Gcap*) can be derived from (V) and (J) (see Section 2.6), so that in this case we obtain

$$(V) + (J) \Leftrightarrow (UE) + (LE). \tag{1.16}$$

This equivalence (in a somewhat more restricted setting) was first proved by Z.-Q.Chen and T.Kumagai [10].

Definition 1.13 (Condition (*cap*))**.** We say that the *capacity condition* (*cap*) is satisfied if there exists a constant $C > 0$ such that, for any $B := B(x_0, R)$ with $R < \overline{R}$,

$$\mathrm{cap}(\frac{1}{2}B, B) \leq C\frac{\mu(B)}{R^\beta}. \tag{1.17}$$

It is easy to see that

$$(Gcap) \Rightarrow (cap).$$

Indeed, applying (*Gcap*) with $u \equiv 1$, we obtain a function $\phi \in \kappa\text{-cutoff}(\frac{1}{2}B, B)$ such that

$$\mathcal{E}\left(\phi, \phi\right) \leq \frac{C}{R^\beta} \int_B u^2 d\mu = C\frac{\mu(B)}{R^\beta}.$$

Replacing ϕ by $\widetilde{\phi} := 1 \wedge \phi \in \mathrm{cutoff}(\frac{1}{2}B, B)$, we obtain that $\mathcal{E}(\widetilde{\phi}, \widetilde{\phi})$ satisfies the same estimate, which implies (1.17).

Theorem 1.12 and the condition (*Gcap*) are motivated by the following conjecture.

Conjecture 1.14. $(V) + (J) + (cap) \Rightarrow (UE) + (LE)$.

At the time being we lack necessary technical tools to approach to this problem.

1.3. **Structure of the paper.** Let us describe the main structural elements of the paper.

Section 2. In Subsection 2.1 we obtain a consequence of $(Gcap)$ – the Andres–Barlow condition (AB). This is a non-local version of the condition (CSA) introduced by Andres and Barlow [1] for local Dirichlet forms. A similar condition in [11] is called (CSJ).

In Subsection 2.2 we discuss some properties of capacity. In Subsection 2.3 we show that $(Gcap)$ follows from a *survival condition* (S). In Subsection 2.4 we obtain a self-improved version of the condition (AB). The latter is used later in the proof of Lemma 3.10 that in turn is one of the ingredients of the proof of the crucial Lemma of Growth (Lemma 4.1).

In Subsection 2.5 we state an extended version of Theorem 1.12 – Theorem 2.10, and explain a general scheme of its proof.

In Subsection 2.6 we treat a special case $\beta < 2$. We prove that in this case the hypothesis $(Gcap)$ can be dropped from (1.14), cf. Corollary 2.12.

Section 3. We prove here some auxiliary technical results, mostly related to the fact that $(\mathcal{E}, \mathcal{F})$ is of jump type. The main results of this section are Lemmas 3.9 and 3.10.

Section 4. This section is central for the proof of Theorems 1.12 and 2.10. In a sequence of lemmas, we prove estimates of the Hölder norm of harmonic functions. The condition $(Gcap)$ in the form (AB) is used only in the proof of Lemma 4.3, which itself constitutes the main part of the proof of Lemma of Growth 4.1. The latter implies the Weak Harnack Inequality of Lemma 4.5.

In the case of a local Dirichlet form, the Weak Harnack Inequality implies immediately the Hölder continuity estimate for harmonic functions (cf. [23]). In the present non-local case, the Weak Harnack Inequality implies a weaker statement that requires further self-improvement. This is a quite elaborate argument that we have borrowed from the paper of Di Castro, Kuusi, Palatucci [12] and that is implemented in Lemma 4.7. The latter implies immediately Oscillation Lemma 4.8 containing the required estimate of the Hölder norm.

Section 5 is devoted to the proof of Theorem 2.10. In Subsection 5.1 we show that $(V) + (J) + (AB)$ imply (S) (Corollary 5.7). In Subsection 5.2 we prove the oscillation inequality for a weak solution u of the equation $\mathcal{L}u = f$ (Lemma 5.9), based on the Oscillation Lemma 4.8.

In Subsection 5.3 we prove ultracontractive estimates for the heat semigroup P_t^Ω and for its time derivative (Lemma 5.10), by means of the Faber-Krahn and Nash inequalities that follow from (V) and (J) (Lemma 3.5).

In Subsection 5.4 we prove the oscillation inequality and the Hölder continuity for the heat semigroup considering a function $u = P_t^\Omega f$ as a weak solution to $\mathcal{L}u = -\partial_t u$ and using Lemmas 5.9 and 5.10.

In Subsection 5.5 we obtain the existence of the heat kernel via the ultracontractivity of the heat semigroup, and prove the Hölder continuity of the heat kernel (Lemma 5.13).

In Subsection 5.6 we conclude the proof of Theorem 2.10. We first obtain from (S) the on-diagonal lower bound of the heat kernel. Then, using the Hölder norm estimate of the heat kernel of Lemma 5.13, we obtain the *near-diagonal lower* estimate (NLE) of the heat kernel (see definition in Section 2.5). Finally, we apply the following result of the companion paper of the authors [17, Theorem 2.9]: under the standing assumption (V),

$$(J) + (S) + (NLE) \Leftrightarrow (UE) + (LE), \tag{1.18}$$

which finishes the proof of (UE) and (LE).

Let us also mention that the techniques for obtaining (LE) in (1.18) was developed in [17], while the method for derivation of (UE) came from [22, Corollary 2.7].

Section 6. In this Section we obtain some consequences of our main result: Corollary 6.2 about the equivalent conditions for (UE) and (LE) in terms of the Green function instead of $(Gcap)$, and Corollary 6.3 about asymptotic behavior of the heat semigroup as $t \to \infty$. Finally, *Appendix* contains some technical results.

NOTATION. To shorten the formulas, we use everywhere measure j defined on $M \times M$ by

$$dj = J(x, y)d\mu(x)d\mu(y).$$

In expressions of the form

$$\int_{E_1\times E_2} F\left(x,y\right) dj$$

we always follow the convention that the variable x belongs to E_1 and y belongs to E_2.

We use the expression "μ-almost all $x, y \in M$ " as a short hand for "$\mu \times \mu$-almost all $(x,y) \in M \times M$".

Sometimes we use abbreviation $\mathcal{E}\left(u\right) := \mathcal{E}\left(u,u\right)$.

Letters C, c, C', c', C_1, c_1 etc are used to denote positive constants whose values are unimportant and can change at any occurrence. However, our results are quantitative in the sense that the value of such constants depends only on the parameters in the hypotheses in question.

The letters α, β and $\overline{R}$ denote the global parameters that have the same meaning all over the paper. The usage of other letters depends on the context.

2. Around condition ($Gcap$)

Let us extend $\mathcal{E}$ from $\mathcal{F}$ to $\mathcal{F}' := \mathcal{F} + \{\text{const}\}$ as follows:

$$\mathcal{E}(u+a, v+b) := \mathcal{E}(u,v), \quad \forall u,v \in \mathcal{F},\ a,b \in \mathbb{R}.$$

Then $\mathcal{E}$ has on $\mathcal{F}'$ the same expression as in (1.10). Some properties of $(\mathcal{E}, \mathcal{F}')$ are proved in Appendix.

2.1. Condition (AB).

Definition 2.1. Given $\zeta \geq 0$, we say that condition (AB_ζ) is satisfied if there is $C > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ and for any three concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R+r)$ and $\Omega := B(x_0, R')$ with $0 < R < R + r < R' < \overline{R}$, there exists $\phi \in \text{cutoff}(B_0, B)$ such that

$$\int_{\Omega\times\Omega} u^2(x) \left(\phi(x) - \phi(y)\right)^2 dj \leq \ \zeta \int_{B\times B} \phi^2\left(x\right)\left(u(x)-u(y)\right)^2 dj + \frac{C}{r^\beta}\int_\Omega u^2 d\mu. \tag{2.1}$$

We say that (AB) holds if (AB_ζ) holds for some $\zeta \geq 0$.

The condition (AB) is named after Andres and Barlow, who first introduced in [1] a similar condition for local Dirichlet forms (although in [1] function ϕ had to be the same for all u). They called their condition by (CSA) – a cutoff Sobolev inequality in annuli. The condition (CSA) was a significantly simplified version of a cutoff Sobolev inequality introduced earlier by Barlow, Bass, Kumagai [6]. Since none of all these conditions is actually related to the classical Sobolev inequality, we have decided to give to it a more appropriate name.

To confuse the reader even more, let us also mention that a version of the condition (CSA) for local Dirichlet forms was used in [23] under the name ($Gcap$). Here we use ($Gcap$) for a different condition as stated above.

It follows from ($Gcap$) or (AB) that, for any couple of concentric balls B_1, B_2 with radii $0 < R_1 < R_2 < \overline{R}$, the set cutoff$(B_1, B_2)$ is non-empty. This observation will be frequently used. In this section we establish a relation between ($Gcap$) and (AB) that is needed for the proof of Theorem 1.12.

Lemma 2.2. *For any measurable set $E \subset M$ and for all measurable functions f, g on E, the following inequality holds:*

$$\begin{aligned}\int_{E\times E} f^2\left(x\right)\left(g\left(x\right)-g\left(y\right)\right)^2 dj \ \ &\leq\ \ 2\int_{E\times E}\left(g\left(x\right)-g\left(y\right)\right)\left(f^2\left(x\right)g\left(x\right)-f^2\left(y\right)g\left(y\right)\right) dj \\ &\quad +4\int_{E\times E} g^2\left(x\right)\left(f\left(x\right)-f\left(y\right)\right)^2 dj,\end{aligned} \tag{2.2}$$

provided the middle integral is greater than $-\infty$.

Remark 2.3. In general, the integrals in (2.2) can take value $+\infty$. However, all the integrals in (2.2) are finite provided $f, g \in \mathcal{F}' \cap L^\infty$. Indeed, by the expression (1.10) of $\mathcal{E}(g,g)$,

$$\int_{E\times E} f^2(x)\,(g(x)-g(y))^2\,dj \leq \ \|f\|^2_{L^\infty}\int_{M\times M}(g(x)-g(y))^2\,dj = \ \|f\|^2_{L^\infty}\,\mathcal{E}(g,g)<\infty.$$

Similarly, the third integral in (2.2) is finite. It follows from Proposition 6.5(ii) that $f^2g \in \mathcal{F}'\cap L^\infty$. Then, using the Cauchy-Schwarz inequality and the expressions of $\mathcal{E}(g,g)$ and $\mathcal{E}(f^2g, f^2g)$, we obtain

$$\int_{E\times E}|g(x)-g(y)|\left|f^2(x)g(x)-f^2(y)g(y)\right|dj \leq \sqrt{\mathcal{E}(g,g)}\sqrt{\mathcal{E}(f^2g,f^2g)}<\infty.$$

Proof of Lemma 2.2. By a direct computation, we have the following identity

$$\frac{1}{2}\left(X^2+Y^2\right)(a-b)^2=(a-b)(X^2a-Y^2b)-\frac{1}{2}(a^2-b^2)(X^2-Y^2)$$

for all numbers a, b, X, Y. Let us estimate the last term here as follows:

$$\begin{aligned}\left|(a^2-b^2)(X^2-Y^2)\right| &= \ |(X+Y)(a-b)|\cdot|(a+b)(X-Y)| \\ &\leq \frac{1}{4}(X+Y)^2(a-b)^2+(a+b)^2(X-Y)^2 \\ &\leq \frac{1}{2}(X^2+Y^2)(a-b)^2+2(a^2+b^2)(X-Y)^2.\end{aligned}$$

Substitution into the above identity yields

$$\frac{1}{2}\left(X^2+Y^2\right)(a-b)^2 \leq 2(a-b)(X^2a-Y^2b)+2(a^2+b^2)(X-Y)^2.$$

In particular, for arbitrary $x, y \in E$, setting here $X=f(x)$, $Y=f(y)$, $a=g(x)$ and $b=g(y)$, we obtain

$$\begin{aligned}&\frac{1}{2}\left(f^2(x)+f^2\,(y)\right)(g(x)-g(y))^2 \\ \leq \ &2(g(x)-g(y))(f^2(x)g(x)-f^2(y)g(y))+2(g^2(x)+g^2(y))(f(x)-f(y))^2.\end{aligned}$$

Integrating this inequality over $E\times E$ against dj and symmetrizing in x, y we obtain (2.2). □

Lemma 2.4. $(V_\leq)+(J_\leq)+(Gcap)\Rightarrow(AB)$.

Proof. Let B_0, B, Ω and u be as stated in condition (AB). Consider also the intermediate ball $B_1 := B(x_0, R + r/2)$. Applying ($Gcap$) to the pair (B_0, B_1), we obtain that there is a function $g \in \kappa\text{-cutoff}(B_0, B_1)$ such that

$$\mathcal{E}(u^2g,g)\leq\frac{C}{r^\beta}\int_{B_1}u^2d\mu. \tag{2.3}$$

Let us apply (2.2) with this g and with $f=u$, $E=B$. Since $g|_{B^c}=0$, we have

$$\begin{aligned}&\int_{B\times B}(g\,(x)-g\,(y))\left(u^2\,(x)\,g\,(x)-u^2\,(y)\,g\,(y)\right)dj \\ = \ &\mathcal{E}(u^2g,g)-\left(\int_{B^c\times B}+\int_{B\times B^c}\right)(g(x)-g(y))\left(u^2(x)g(x)-u^2(y)g(y)\right)dj \\ = \ &\mathcal{E}(u^2g,g)-\int_{B^c\times B}u^2(y)g^2(y)dj-\int_{B\times B^c}u^2(x)g^2(x)dj \\ \leq \ &\mathcal{E}(u^2g,g).\end{aligned}$$

Substituting this into (2.2) and using (2.3), we obtain

$$\begin{aligned}&\int_{B\times B}u^2(x)\,(g(x)-g(y))^2\,dj \leq \ 2\mathcal{E}(u^2g,g)+4\int_{B\times B}g^2(x)\,(u(x)-u(y))^2\,dj \\ &\leq 4\int_{B\times B}g^2(x)\,(u(x)-u(y))^2\,dj+\frac{2C}{r^\beta}\int_{B_1}u^2d\mu.\end{aligned} \tag{2.4}$$

Define the function

$$\phi := 1 \wedge g \in \text{cutoff}(B_0, B_1).$$

Since for all $x, y \in M$ we have $|\phi(x) - \phi(y)| \leq |g(x) - g(y)|$ and $g(x) \leq \kappa\phi(x)$, we obtain from (2.4)

$$\int_{B\times B} u^2(x)\left(\phi(x) - \phi(y)\right)^2 dj \leq 4\kappa^2 \int_{B\times B} \phi^2(x)\left(u(x) - u(y)\right)^2 dj + \frac{2C}{r^\beta}\int_{B_1} u^2 d\mu. \tag{2.5}$$

Since

$$\Omega \times \Omega = [B \times B] \sqcup [(\Omega \setminus B) \times (\Omega \setminus B)] \sqcup [(\Omega \setminus B) \times B] \sqcup [B \times (\Omega \setminus B)],$$

we have

$$\begin{aligned}
&\int_{\Omega\times\Omega} u^2(x)\left(\phi(x) - \phi(y)\right)^2 dj \\
&\qquad = \left(\int_{B\times B} + \int_{(\Omega\setminus B)\times(\Omega\setminus B)} + \int_{(\Omega\setminus B)\times B} + \int_{B\times(\Omega\setminus B)}\right)\cdots \\
&\qquad =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We estimate $I_1, \cdots, I_4$ separately. By (2.5), we have

$$I_1 \leq \zeta \int_{B\times B} \phi^2(x)\left(u(x) - u(y)\right)^2 dj + \frac{C}{r^\beta}\int_B u^2(x) d\mu(x),$$

where $\zeta = 4\kappa^2$. Since $\phi \equiv 0$ on B_1^c and, hence, $\phi \equiv 0$ on B^c, we have

$$I_2 = 0.$$

Using (6.8) from Appendix (which requires $(V_\leq)$ and $(J_\leq)$) and the fact that $\phi \leq 1$, we obtain

$$\begin{aligned}
I_3 &= \int_{\Omega\setminus B}\left(\int_B u^2(x)\phi^2(y)J(x,y)d\mu(y)\right)d\mu(x) \\
&= \int_{\Omega\setminus B}\left(\int_{B_1} u^2(x)\phi^2(y)J(x,y)d\mu(y)\right)d\mu(x) \\
&\leq \int_{\Omega\setminus B} u^2(x)\left(\int_{\{y:d(x,y)\geq\frac{r}{2}\}} J(x,y)d\mu(y)\right)d\mu(x) \\
&\leq \frac{C}{r^\beta}\int_{\Omega\setminus B} u^2(x)d\mu(x).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_4 &= \int_B\left(\int_{\Omega\setminus B} u^2(x)\phi^2(x)J(x,y)d\mu(y)\right)d\mu(x) \\
&= \int_B u^2(x)\phi^2(x)\left(\int_{\Omega\setminus B} J(x,y)d\mu(y)\right)d\mu(x) \\
&\leq \int_{B_1} u^2(x)\left(\int_{\{y:d(x,y)\geq\frac{r}{2}\}} J(x,y)d\mu(y)\right)d\mu(x) \\
&\leq \frac{C}{r^\beta}\int_{B_1} u^2(x)d\mu(x).
\end{aligned}$$

Adding up the estimates of $I_1, \cdots, I_4$, we obtain (2.1). □

2.2. **Condition** (cap)**.**

Lemma 2.5. *If (V) holds, then (cap) is equivalent to the following condition: for any $\lambda \in (0,1)$, there is a constant $C>0$ depending on λ so that for any ball $B := B(x_0, R)$ with $R \in (0, \overline{R})$,*

$$\mathrm{cap}(\lambda B, B) \le C \frac{\mu(B)}{R^\beta}. \tag{2.6}$$

Proof. Indeed, (cap) follows from (2.6) by setting $\lambda = 1/2$. Let us prove that (cap) implies (2.6). If $\lambda \le \frac{1}{2}$ then this trivially follows by the monotonicity of capacity. Now, assume that $\lambda \in (\frac{1}{2}, 1)$ and set $a = (1-\lambda)/2$. It follows from (V) by a standard covering argument that there exist an integer $N = N(\lambda) > 0$ and N balls $B_i = B(x_i, aR)$ with the centers $x_i \in \lambda B$, $i = 1, 2, ..., N$ such that

$$\lambda B \subset \bigcup_{i=1}^N B_i.$$

By the definition of a, we have $2B_i \subset B$. Using the subadditivity of capacity and its monotonicity properties, we obtain

$$\mathrm{cap}(\lambda B, B) \le \sum_{i=1}^N \mathrm{cap}(B_i, B) \le \sum_{i=1}^N \mathrm{cap}(B_i, 2B_i).$$

By (cap) we obtain

$$\mathrm{cap}(B_i, 2B_i) \le C \frac{\mu\left(2B_i\right)}{\left(2aR\right)^\beta} \le C' \frac{\mu\left(B\right)}{R^\beta},$$

whence (2.6) follows. □

Recall that $(Gcap) \Rightarrow (cap)$. In the next statement we show that also (AB) implies (cap).

Lemma 2.6. $(V_\le) + (J_\le) + (AB) \Longrightarrow (cap)$.

Proof. We need to prove that, for any ball $B := B(x_0, R) \subset M$ with $R \in (0, \overline{R})$,

$$\mathrm{cap}(\frac{1}{2}B, B) \le C \frac{\mu(B)}{R^\beta}, \tag{2.7}$$

with some constant $C > 0$ independent of R.

Applying the condition (AB) with function $u \equiv 1$ for the triple $\frac{1}{2}B, \frac{3}{4}B, B$, we obtain that there exists a function $\phi \in \mathrm{cutoff}(\frac{1}{2}B, \frac{3}{4}B)$ such that

$$\int_{B\times B} (\phi(x) - \phi(y))^2 dj \le C \frac{\mu(B)}{R^\beta}.$$

Using this inequality together with $(J_\le)$ and $(V_\le)$, we obtain

$$\begin{aligned}
\mathcal{E}(\phi, \phi) &= \int_{M\times M} \left(\phi(x) - \phi(y)\right)^2 dj \\
&= \int_{B\times B} \left(\phi(x) - \phi(y)\right)^2 dj + 2 \int_{\left(\frac{3}{4}B\right)\times B^c} \phi^2(x) dj \\
&\le C \frac{\mu(B)}{R^\beta} + 2 \left(\int_{\left(\frac{3}{4}B\right)} \phi^2(x) d\mu(x) \right) \left(\sup_{x \in \frac{3}{4}B} \int_{B^c} J(x,y) d\mu(y) \right) \\
&\le C \frac{\mu(B)}{R^\beta} + 2\mu(B) \left(\sup_{x\in B} \int_{\{d(x,y) \ge R/4\}} J(x,y) d\mu(y) \right) \\
&\le C \frac{\mu(B)}{R^\beta} + 2\mu(B) \frac{c}{(R/4)^\beta} \qquad \text{(by (6.8))} \\
&\le C' \frac{\mu(B)}{R^\beta},
\end{aligned}$$

which implies (2.7). □

2.3. **Condition** (S)**.** Given a non-empty open set $\Omega \subset M$, let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F} \cap C_0(\Omega)$ in $\mathcal{F}$ with respect to the norm $\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \|u\|_{L^2}^2$. It is well known (see [14]) that if $(\mathcal{E},\mathcal{F})$ is regular, then $(\mathcal{E},\mathcal{F}(\Omega))$ is also a regular Dirichlet form. In this case, we denote the corresponding generator, heat semigroup and heat kernel (if it exists) respectively by $\mathcal{L}^{\Omega}$, $\{P_t^{\Omega}\}$ and $p_t^{\Omega}(x,y)$.

Definition 2.7 (Condition (S))**.** We say that a *survival condition* (S) is satisfied if there exist constants $\varepsilon, \delta > 0$ such that, for any ball $B \subset M$ of radius $r \in (0, \overline{R})$ the following inequality holds:

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} P_t^B 1 \geq \varepsilon,$$

provided $t^{1/\beta} \leq \delta r$.

In this section, we will prove the following implication.

Lemma 2.8. *If every metric ball of radius* $< \overline{R}$ *has finite measure then* $(S) \Rightarrow (Gcap)$.

Proof. Assuming (S), we will prove that there exists a number $\kappa \geq 1$ such that, for any pair of balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $R + r < \overline{R}$, there is a function $\phi \in \kappa\text{-cutoff}(B_0, B)$ such that, for all $u \in \mathcal{F}' \cap L^{\infty}$,

$$\mathcal{E}(u^2\phi, \phi) \leq \frac{\kappa^2}{r^\beta} \int_B u^2 d\mu, \tag{2.8}$$

which yields $(Gcap)$. Construction of ϕ is motivated by the argument of [1, Lemma 5.4]. Set $\lambda = r^{-\beta}$ and consider the function

$$h = G_\lambda^B 1_B := \int_0^\infty e^{-\lambda t} P_t^B 1_B dt.$$

It follows from [14, Theorem 4.4.1] that $h \in \mathcal{F}(B)$. We first obtain two-sided bounds of h and then construct a κ-cutoff function ϕ using h. By the definition of h, we have $h = 0$ in B^c. Hence, for any $0 \leq f \in L^1 \cap L^2(B)$,

$$(h, f) = \int_0^\infty e^{-\lambda t} \left(P_t^B 1_B, f\right) dt \leq \int_0^\infty e^{-\lambda t} dt \cdot \|f\|_{L^1} = \lambda^{-1} \|f\|_{L^1} = r^\beta \|f\|_{L^1},$$

which implies that

$$h \leq r^\beta, \quad \mu\text{-a.e. on } B.$$

Let us now obtain a lower bound of h in B_0. Fix $x \in B_0$ and consider the ball $\widetilde{B} := B(x, r) \subset B$. By the definition of h and condition (S), we have, for any $0 \leq f \in L^1(\frac{1}{4}\widetilde{B})$,

$$\begin{aligned}
(h, f) = \int_0^\infty e^{-\lambda t} \left(P_t^B 1_B, f\right) dt &\geq \int_0^{(\delta r)^\beta} e^{-\lambda t} \left(P_t^{\widetilde{B}} 1_{\widetilde{B}}, f\right) dt \\
&\geq \int_0^{(\delta r)^\beta} e^{-\lambda t} dt \cdot \varepsilon \|f\|_{L^1} = \lambda^{-1} \left(1 - e^{-\lambda \delta^\beta r^\beta}\right) \varepsilon \|f\|_{L^1} \\
&= r^\beta (1 - e^{-\delta^\beta}) \varepsilon \|f\|_{L^1},
\end{aligned}$$

where the constants ε, δ are those from (S). Since B_0 can be covered by a family of countable balls like $\widetilde{B}$ and f is arbitrary, we obtain that

$$h \geq \kappa^{-1} r^\beta \quad \mu\text{-a.e. on } B_0.$$

where $\kappa := \varepsilon^{-1}(1 - e^{-\delta^\beta})^{-1}$.

Now consider the function

$$\phi := \frac{\kappa h}{r^\beta},$$

which satisfies the conditions $\phi \in \mathcal{F}(B)$, $0 \leq \phi \leq \kappa$, $\phi|_{B_0} \geq 1$ and $\phi|_{B^c} = 0$. It remains to prove that ϕ satisfied (2.8). Let us use the notation

$$\mathcal{E}_\lambda(w, v) = \mathcal{E}(w, v) + \lambda(w, v),$$

where $w, v \in \mathcal{F}$. If $u \in \mathcal{F}' \cap L^\infty$ then $u^2\phi \in \mathcal{F}$. By [14, Theorem 4.4.1], we obtain

$$\begin{aligned}\mathcal{E}(u^2\phi, \phi) &\leq \mathcal{E}_\lambda(u^2\phi, \phi) = \frac{\kappa}{r^\beta}\mathcal{E}_\lambda(u^2\phi, G_\lambda^B 1_B) = \frac{\kappa}{r^\beta}(u^2\phi, 1_B)\\ &= \frac{\kappa}{r^\beta}\int_B u^2\phi d\mu \leq \frac{\kappa^2}{r^\beta}\int_B u^2 d\mu,\end{aligned}$$

which finishes the proof of (2.8). □

2.4. Self-improvement of (AB).

Lemma 2.9. *If* $(V_\leq)$ *and* $(J_\leq)$ *hold then* $(AB) \Rightarrow \left(AB_{1/8}\right)$.

The difference between (AB) and $\left(AB_{1/8}\right)$ is that the constant ζ in (AB) may be large, whereas in $\left(AB_{1/8}\right)$ we have $\zeta = \frac{1}{8}$. In fact, the value $\frac{1}{8}$ is chosen for convenience of application, while in the statement and proof of Lemma 2.9 it can be replaced by arbitrarily small positive number.

The proof below follows essentially the argument of Andres and Barlow [1] that was done in the setting of local Dirichlet forms. We have to overcome two new difficulties, though: the non-locality and the fact that the test function ϕ in (AB) is allowed to depend on u, which makes the derivation of $\left(AB_{1/8}\right)$ much more involved.

Proof of Lemma 2.9. Let $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ and $\Omega := B(x_0, R')$ be as in the definition of (AB). Fix a function $u \in \mathcal{F}' \cap L^\infty$. We need to find $\phi \in \text{cutoff}(B_0, B)$ such that

$$\int_{\Omega\times\Omega} u^2(x)\left(\phi(x) - \phi(y)\right)^2 dj \leq \frac{1}{8}\int_{B\times B}\phi^2(x)\left(u(x) - u(y)\right)^2 dj + \frac{C}{r^\beta}\int_\Omega u^2 d\mu. \tag{2.9}$$

If $u \equiv 0$ in Ω then any ϕ will do. Assume in the rest of the proof that $\|u\|_{L^2(\Omega)} > 0$. Fix some $\varepsilon > 0$ to be specified below in (2.10), and set $u_\varepsilon := |u| + \varepsilon$. Note that $u_\varepsilon \in \mathcal{F}' \cap L^\infty$.

Let $q > 1$ be a parameter also to be determined later. Define the sequences $\{r_n\}_{n=0}^\infty$ and $\{s_n\}_{n=1}^\infty$ by

$$r_n = \left(1 - q^{-n}\right) r, \quad s_n = r_n - r_{n-1} = (q-1)\, q^{-n} r$$

and set

$$\begin{aligned}B_n &:= B(x_0, R + r_n),\\ U_n &:= B_{n+1} \setminus B_n.\end{aligned}$$

Obviously, $r_n \uparrow r$ and, hence, $B_n \uparrow B$ as $n \to +\infty$, and $\cup_{n=1}^\infty U_n = B \setminus B_1$ (see Fig. 1).

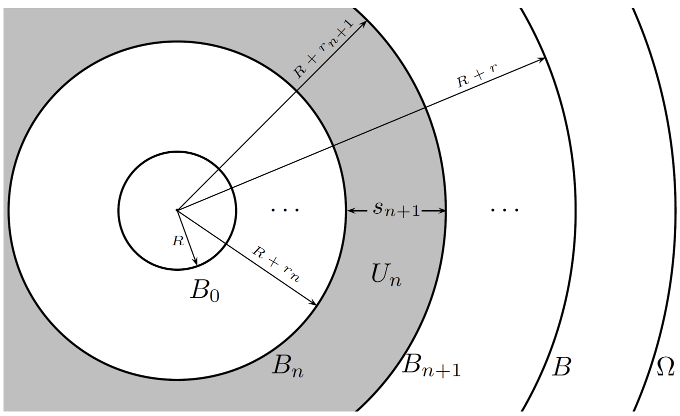

FIGURE 1. Sets B_n and U_n

Applying (AB) to the function u_ε and to each triple (B_n, B_{n+1}, Ω), we obtain that there exists $\phi_n \in \text{cutoff}(B_n, B_{n+1})$ such that,

$$\int_{\Omega\times\Omega} u_\varepsilon^2(x)\left(\phi_n(x)-\phi_n(y)\right)^2 dj \le \ \zeta \int_{B_{n+1}\times B_{n+1}} (u_\varepsilon(x)-u_\varepsilon(y))^2\, dj \ + \frac{C}{s_{n+1}^\beta}\int_\Omega u_\varepsilon^2 d\mu.$$

Note that

$$|u_\varepsilon(x)-u_\varepsilon(y)| \le |u(x)-u(y)|$$

and

$$u_\varepsilon^2 \le 2u^2+2\varepsilon^2.$$

In particular,

$$\int_\Omega u_\varepsilon^2 d\mu \le 2\int_\Omega u^2 d\mu + 2\varepsilon^2\mu(\Omega).$$

Choosing

$$\varepsilon := \left(\fint_\Omega u^2 d\mu\right)^{1/2} = \left(\frac{1}{\mu(\Omega)}\int_\Omega u^2 d\mu\right)^{1/2}, \tag{2.10}$$

we obtain that

$$\int_\Omega u_\varepsilon^2 d\mu \le 4\int_\Omega u^2 du.$$

It follows that

$$\int_{\Omega\times\Omega} u_\varepsilon^2(x)\left(\phi_n(x)-\phi_n(y)\right)^2 dj \le \ \zeta \int_{B_{n+1}\times B_{n+1}} (u(x)-u(y))^2\, dj \ + \frac{C}{s_{n+1}^\beta}\int_\Omega u^2 d\mu. \tag{2.11}$$

Consider the sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=0}^\infty$ defined by

$$b_n = q^{-\beta n}, \quad a_n = b_{n-1}-b_n = \left(q^\beta-1\right)q^{-\beta n},$$

so that

$$\sum_{n=1}^\infty a_n = 1,$$

and define the following function

$$\phi := \sum_{n=1}^\infty a_n\phi_n. \tag{2.12}$$

We will prove the following two claims:

- (i) $\phi \in \mathcal{F}$ (which will imply that $\phi \in \text{cutoff}(B_0, B)$ because by construction $\phi|_{B_0} = 1$ and $\phi|_{B^c} = 0$);
- (ii) if q is close enough to 1 then ϕ satisfies (2.9), which will finish the proof of $\left(AB_{1/8}\right)$.

To verify (i), observe first that $\|\phi_n\|_{L^2} \le \mu(B)^{1/2}$ and, hence,

$$\sum_{n=1}^\infty \|a_n\phi_n\|_{L^2} \le \mu(B)^{1/2}\sum_{n=1}^\infty a_n < \infty,$$

which implies that $\phi \in L^2(M)$. Since $\mathcal{F}$ is complete with respect to the norm $\|\cdot\|_{L^2} + \mathcal{E}(\cdot,\cdot)^{1/2}$, in order to prove that $\phi \in \mathcal{F}$, it suffices to verify that

$$\sum_{n=1}^\infty \mathcal{E}(a_n\phi_n, a_n\phi_n)^{1/2} < \infty.$$

Since $u_\varepsilon \ge \varepsilon$, we obtain from (2.11) that

$$\int_{\Omega\times\Omega} \left(\phi_n(x)-\phi_n(y)\right)^2 dj \le \ \zeta\varepsilon^{-2}\mathcal{E}(u,u) \ + \frac{C\varepsilon^{-2}}{s_{n+1}^\beta}\int_\Omega u^2 d\mu.$$

Since ϕ_n is supported in B, we obtain

$$\mathcal{E}(\phi_n,\phi_n)=\int_{M\times M}(\phi_n(x)-\phi_n(y))^2\,dj=\int_{\Omega\times\Omega}(\phi_n(x)-\phi_n(y))^2\,dj+2\int_{B\times\Omega^c}\phi_n^2(x)dj.$$

Since $d(B,\Omega^c)\geq R'-(R+r)>0$ and $\phi_n^2\leq 1$, the last integral here is bounded from above by a constant that is independent of n, which follows from (6.8) that in turn is based on ($V_{\leq}$) and ($J_{\leq}$). Absorbing also $\mathcal{E}(u,u)$ and $\int_\Omega u^2d\mu$ into constants, using $s_n=(q-1)\,q^{-n}r$ and combining the above two lines, we obtain

$$\mathcal{E}(\phi_n,\phi_n)\leq Cq^{\beta n},$$

where the constant C depends on all variables in question except for n. Since $a_n=\left(q^\beta-1\right)q^{-\beta n}$, we obtain that

$$\sum_{n=1}^{\infty}\mathcal{E}(a_n\phi_n,a_n\phi_n)^{1/2}=\sum_{n=1}^{\infty}a_n\mathcal{E}(\phi_n,\phi_n)^{1/2}\leq C\sum_{n=1}^{\infty}q^{-\beta n}q^{\frac{1}{2}\beta n}<\infty,$$

which finishes the proof of (i).

For the proof of (ii), we consider the partial sums of the series (2.12):

$$\Phi_N:=\sum_{n=1}^{N}a_n\phi_n,$$

Clearly, $\Phi_N\uparrow\phi$ pointwise as $N\to\infty$. It suffices to prove the following inequality

$$\int_{\Omega\times\Omega}u^2(x)\,(\Phi_N(x)-\Phi_N(y))^2\,dj\leq\frac{1}{8}\int_{B\times B}\phi^2(x)\,(u(x)-u(y))^2\,dj+\frac{C}{r^\beta}\int_\Omega u^2d\mu,\tag{2.13}$$

because (2.9) will follow then from (2.13) as $N\to\infty$ by means of Fatou's lemma.

Set

$$S_N(u):=\int_{\Omega\times\Omega}u^2(x)\,(\Phi_N(x)-\Phi_N(y))^2\,dj=\int_{\Omega\times\Omega}u^2(x)\left(\sum_{n=1}^{N}a_n(\phi_n(x)-\phi_n(y))\right)^2dj.$$

Since $\phi_n=1$ on B_{m+1} for all $n\geq m+1$, and $\phi_m=0$ on B_{m+1}^c, we obtain, for all $x,y\in M$ and for all $n\geq m+2$,

$$\begin{aligned}(\phi_m(x)-\phi_m(y))\,(\phi_n(x)-\phi_n(y))&=\phi_m(x)-\phi_m(y)\phi_n(x)-\phi_m(x)\phi_n(y)+\phi_m(y)\\&=\phi_m(x)(1-\phi_n(y))+\phi_m(y)(1-\phi_n(x)).\end{aligned}$$

It follows that

$$\begin{aligned}&\left(\sum_{n=1}^{N}a_n(\phi_n(x)-\phi_n(y))\right)^2\\&=\sum_{n=1}^{N}a_n^2\,(\phi_n(x)-\phi_n(y))^2+2\sum_{m=1}^{N-1}\sum_{n=m+1}^{N}a_ma_n\,(\phi_m(x)-\phi_m(y))\,(\phi_n(x)-\phi_n(y))\\&=\sum_{n=1}^{N}a_n^2\,(\phi_n(x)-\phi_n(y))^2+2\sum_{m=1}^{N-1}a_ma_{m+1}\,(\phi_m(x)-\phi_m(y))\left(\phi_{m+1}(x)-\phi_{m+1}(y)\right)\\&\quad+2\sum_{m=1}^{N-2}\sum_{n=m+2}^{N}a_na_m\,(\phi_m(x)-\phi_m(y))\,(\phi_n(x)-\phi_n(y)),\\&\leq 3\sum_{n=1}^{N}a_n^2\,(\phi_n(x)-\phi_n(y))^2+2\sum_{m=1}^{N-2}\sum_{n=m+2}^{N}a_ma_n\Big(\phi_m(x)(1-\phi_n(y))+\phi_m(y)(1-\phi_n(x))\Big),\end{aligned}$$

whence

$$\begin{aligned} S_N(u) \le 3 & \underbrace{\sum_{n=1}^{N} a_n^2 \int_{\Omega\times\Omega} u^2(x)\left(\phi_n(x)-\phi_n(y)\right)^2 dj}_{I_1} \\ & + 2\underbrace{\sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \int_{\Omega\times\Omega} u^2(x)\phi_m(x)\left(1-\phi_n(y)\right) dj}_{I_2} \\ & + 2\underbrace{\sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \int_{\Omega\times\Omega} u^2(x)\phi_m(y)\left(1-\phi_n(x)\right) dj}_{I_3} \\ = {} & 3I_1 + 2I_2 + 2I_3. \end{aligned} \tag{2.14}$$

We estimate I_1, I_2, I_3 separately. By (2.11) we have

$$\begin{aligned} I_1 \le & \sum_{n=1}^{\infty} a_n^2 \int_{\Omega\times\Omega} u^2(x)\left(\phi_n(x)-\phi_n(y)\right)^2 dj \\ & \le \zeta\sum_{n=1}^{\infty} a_n^2 \int_{B_{n+1}\times B} \left(u(x)-u(y)\right)^2 dj \ + C\sum_{n=1}^{\infty}\frac{a_n^2}{s_{n+1}^{\beta}}\int_{\Omega} u^2(x)d\mu(x) \\ = & \ \zeta\underbrace{\sum_{n=1}^{\infty} a_n^2 \int_{B_1\times B}\left(u(x)-u(y)\right)^2 dj}_{I_{11}} \ + \zeta\underbrace{\sum_{n=1}^{\infty} a_n^2 \int_{(B_{n+1}\setminus B_1)\times B}\left(u(x)-u(y)\right)^2 dj}_{I_{12}} \\ & + C\underbrace{\sum_{n=1}^{\infty}\frac{a_n^2}{s_{n+1}^{\beta}}\int_{\Omega} u^2(x)d\mu(x)}_{I_{13}} = \ \zeta I_{11} + \zeta I_{12} + C I_{13}. \end{aligned} \tag{2.15}$$

Next, we estimate separately I_{11}, I_{12}, I_{13}. Since

$$\sum_{n=1}^{\infty} a_n^2 = (q^\beta-1)^2\sum_{n=1}^{\infty} q^{-2\beta n} = \frac{(q^\beta-1)^2}{q^{2\beta}-1} = \frac{q^\beta-1}{q^\beta+1}$$

and $\phi = 1$ on B_1, we obtain

$$I_{11} = \sum_{n=1}^{\infty} a_n^2 \int_{B_1\times B}\left(u(x)-u(y)\right)^2 dj \le \frac{q^\beta-1}{q^\beta+1}\int_{B\times B}\phi^2(x)\left(u(x)-u(y)\right)^2 dj. \tag{2.16}$$

Before we estimate I_{12}, which is the main term, observe that by (2.12) the function ϕ on each annulus U_m satisfies the estimate

$$\phi \ge \sum_{k=m+1}^{\infty} a_k\phi_k = \sum_{k=m+1}^{\infty} a_k = b_m,$$

which implies

$$a_m \le \frac{\phi}{b_m} a_m = (q^\beta-1)\phi \ \text{ on } U_m. \tag{2.17}$$

Using (2.17) and $a_n = q^{-(n-m)\beta} a_m$, we obtain

$$\begin{aligned} I_{12} &= \sum_{n=1}^{\infty} a_n^2 \int_{(B_{n+1}\setminus B_1)\times B} (u(x)-u(y))^2\, dj = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \int_{U_m\times B} a_n^2\,(u(x)-u(y))^2\, dj \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{n} q^{-2(n-m)\beta} \int_{U_m\times B} a_m^2\,(u(x)-u(y))^2\, dj \\ &= \sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty} q^{-2(n-m)\beta}\right) \int_{U_m\times B} a_m^2\,(u(x)-u(y))^2\, dj \\ &\le \sum_{m=1}^{\infty} \frac{q^{2\beta}}{q^{2\beta}-1} \int_{U_m\times B} (q^\beta-1)^2\phi^2(x)\,(u(x)-u(y))^2\, dj \\ &\le \frac{q^{2\beta}(q^\beta-1)}{q^\beta+1} \int_{B\times B} \phi^2(x)\,(u(x)-u(y))^2\, dj. \end{aligned} \tag{2.18}$$

In order to evaluate I_{13}, observe that, by the definitions of a_n and s_n,

$$\sum_{n=1}^{\infty} a_n^2 \frac{1}{s_{n+1}^\beta} = \sum_{n=1}^{\infty} \frac{\left(q^\beta-1\right)^2 q^{-2\beta n}}{(q-1)^\beta q^{-\beta(n+1)} r^\beta} = q^\beta \frac{\left(q^\beta-1\right)^2}{(q-1)^\beta r^\beta} \sum_{n=1}^{\infty} q^{-\beta n} = \frac{q^\beta\left(q^\beta-1\right)}{(q-1)^\beta r^\beta},$$

which implies that

$$I_{13} = \sum_{n=1}^{\infty} a_n^2 \frac{1}{s_{n+1}^\beta} \int_\Omega u^2 d\mu = \frac{q^\beta\left(q^\beta-1\right)}{(q-1)^\beta r^\beta} \int_\Omega u^2 d\mu. \tag{2.19}$$

Substitution of (2.16), (2.18), and (2.19) into (2.15) yields an upper bound of I_1.

Now let us estimate I_2. Using that $\phi_m = 0$ in B_{m+1}^c, $1-\phi_n = 0$ on B_n and

$$d(B_{m+1}, B_n^c) \ge r_n - r_{m+1} \ge s_{m+2}, \quad \text{provided } n \ge m+2,$$

we obtain

$$\begin{aligned} I_2 &= \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \int_{\Omega\times\Omega} u^2(x)\phi_m(x)\,(1-\phi_n(y))\, dj \\ &\le \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \int_{B_{m+1}\times(\Omega\setminus B_n)} u^2(x) dj \\ &\le \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \int_{B_{m+1}} u^2(x) \left(\int_{\{d(x,y)\ge s_{m+2}\}} J(x,y) d\mu(y)\right) d\mu(x) \\ &\le C \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \frac{1}{s_{m+2}^\beta} \int_\Omega u^2(x) d\mu(x), \end{aligned}$$

where we have used (6.8). Computing

$$\sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \frac{1}{s_{m+2}^\beta} = \sum_{m=1}^{\infty} \frac{a_m}{s_{m+2}^\beta} \sum_{n=m+2}^{\infty} a_n = \sum_{m=1}^{\infty} \frac{a_m}{s_{m+2}^\beta} b_{m+1} = \frac{q^\beta}{(q-1)^\beta r^\beta},$$

we obtain

$$I_2 \le C \frac{q^\beta}{(q-1)^\beta} \frac{1}{r^\beta} \int_\Omega u^2 d\mu. \tag{2.20}$$

We estimate I_3 similarly:

$$
\begin{aligned}
I_3 &= \sum_{m=1}^{\infty} \sum_{n=m+2}^{\infty} a_m a_n \int_{\Omega\times\Omega} u^2(x)\phi_m(y)\left(1-\phi_n(x)\right) dj \\
&\le \sum_{m=1}^{\infty} \sum_{n=m+2}^{\infty} a_m a_n \int_{(\Omega\setminus B_n)\times B_{m+1}} u^2(x) dj \\
&\le \sum_{m=1}^{\infty} \sum_{n=m+2}^{\infty} a_m a_n \int_{\Omega} u^2(x) \left(\int_{\{d(x,y)\ge s_{m+2}\}} J(x,y) d\mu(y) \right) d\mu(x) \\
&\le C \frac{q^\beta}{(q-1)^\beta} \frac{1}{r^\beta} \int_\Omega u^2(x) d\mu(x). \qquad (2.21)
\end{aligned}
$$

Combining (2.14), (2.15), (2.16), (2.18), (2.19), (2.20) and (2.21), we obtain

$$
S_N(u) \le \; 6\zeta \frac{q^{2\beta}(q^\beta-1)}{q^\beta+1} \int_{B\times B} \phi^2(x)\left(u(x)-u(y)\right)^2 dj + \frac{C(q)}{r^\beta} \int_\Omega u^2 d\mu. \qquad (2.22)
$$

Finally, by choosing q close enough to 1, we can make the coefficient in front of the first integral arbitrarily small, in particular $\le \frac{1}{8}$, which finishes the proof of (2.13). □

2.5. **Main theorem.** Now we formulate our main result that contains Theorem 1.12 from Introduction.

Theorem 2.10. *Let $(\mathcal{E},\mathcal{F})$ be a regular jump type Dirichlet form on $L^2(M,\mu)$ with a jump kernel J. If (M,d,μ) satisfies (V) then the following equivalences hold:*

$$
\begin{aligned}
(UE)+(LE) &\Leftrightarrow (J)+(S) \\
&\Leftrightarrow (J)+(Gcap) \\
&\Leftrightarrow (J)+(AB) \\
&\Leftrightarrow (J)+(AB_{1/8}).
\end{aligned}
$$

Under any of these conditions, the heat kernel $p_t(x,y)$ is Hölder continuous jointly in x, y and continuous jointly in x, y, t.

In the proof we use the following condition.

Definition 2.11 (Condition (NLE))**.** We say that a *near diagonal lower estimate* (NLE) is satisfied if the heat kernel $p_t(x,y)$ exists and satisfies the following estimate

$$
p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}},
$$

for all $t\in(0,\overline{R}^\beta)$ and μ-almost all $x,y\in M$ such that $d(x,y)\le \delta' t^{1/\beta}$, with some positive constants c, δ'.

In order to prove Theorem 2.10, we will use the following result of [17, Theorem 2.9]: under the standing assumption (V),

$$
(UE)+(LE) \Leftrightarrow (J)+(S)+(NLE). \qquad (2.23)
$$

Note that the main contribution of [17] was the proof of (LE) under $(J)+(S)+(NLE)$, while the other implications were based on [7, Theorem 1.2$(a)\Rightarrow(c)$], [21], [22, Theorem 2.1]. Combining (2.23) with the results obtained earlier in this section, we obtain

$$
\begin{aligned}
(UE)+(LE) &\Rightarrow (J)+(S) && \text{by (2.23)} \\
&\Rightarrow (J)+(Gcap) && \text{by Lemma 2.8} \\
&\Rightarrow (J)+(AB) && \text{by Lemma 2.4} \\
&\Rightarrow (J)+\left(AB_{1/8}\right) && \text{by Lemma 2.9.}
\end{aligned}
$$

Hence, in order to close the circle of implications in Theorem 2.10, it remains to verify that

$$(J)+(AB_{1/8}) \Rightarrow (S)+(NLE), \tag{2.24}$$

and then combine (2.24) with (2.23).

The proof of (2.24) will take the rest of the paper and will be concluded in Section 5.6. The existence and the continuity of the heat kernel are proved in Lemma 5.13.

2.6. **Case $\beta<2$.** In this section we make two mild additional assumptions:

(i) all the metric balls in (M,d) of radii $<\overline{R}$ are precompact;

(ii) $\mathcal{F}$ contains all functions $f\in C_0(M)$ such that $\mathcal{E}(f,f)<\infty$.

The main result of this Section is the following consequence of Theorem 2.10.

Corollary 2.12. *Let $(\mathcal{E},\mathcal{F})$ be a regular jump type Dirichlet form on $L^2(M,\mu)$ with a jump kernel J. Assume in addition that (i) and (ii) are satisfied, and that $\beta<2$. If (V) holds then*

$$(J) \Leftrightarrow (UE)+(LE). \tag{2.25}$$

If in addition $\overline{R}=\infty$ then we have the equivalence

$$(V)+(J) \Leftrightarrow (UE)+(LE). \tag{2.26}$$

This result was first proved by Chen and Kumagai [10], although in a more restricted setting.

Proof. We will prove that in the case $\beta<2$ we have

$$(V_{\leq})+(J_{\leq}) \Rightarrow (AB_0), \tag{2.27}$$

which will then imply (2.25) by Theorem 2.10.

Fix a point $x_0\in M$, numbers $0<R<R+r<\overline{R}$ and consider the function

$$\phi(x):=1\wedge\frac{(R+r-d(x_0,x))_+}{r}.$$

Clearly, the function ϕ is continuous, $\operatorname{supp}\phi\subset\overline{B}(x_0,R+r)$ and hence, $\operatorname{supp}\phi$ is compact. Observe also, that $0\leq\phi\leq1$ and $\phi\equiv1$ on $B(x_0,R)$. We will prove below that $\phi\in\mathcal{F}$, which will imply that ϕ is a cutoff function of the pair $(B(x_0,R),\,B(x_0,R+r))$. We will also prove that, for any open set $\Omega\supset B(x_0,R+r)$ and for any $u\in\mathcal{F}'\cap L^\infty$,

$$\int_{\Omega\times\Omega}u^2(x)\,(\phi(x)-\phi(y))^2\,J(x,y)\,d\mu(x)d\mu(y)\leq\frac{C}{r^\beta}\int_\Omega u^2d\mu \tag{2.28}$$

which is equivalent to (AB_0).

Let us start with the proof of the following inequality

$$\int_M(\phi(x)-\phi(y))^2J(x,y)\,d\mu(y)\leq Cr^{-\beta}, \tag{2.29}$$

for any $x\in M$. Because of $(J_{\leq})$, it suffices to prove (2.29) with $J(x,y)=d(x,y)^{-(\alpha+\beta)}$. Let us split the integral in (2.29) into the following two parts:

$$I_1(x):=\int_{\{d(x,y)<r\}}\frac{(\phi(x)-\phi(y))^2}{d(x,y)^{\alpha+\beta}}d\mu(y),\quad I_2(x):=\int_{\{d(x,y)\geq r\}}\frac{(\phi(x)-\phi(y))^2}{d(x,y)^{\alpha+\beta}}d\mu(y).$$

As follows from the definition of ϕ, we have

$$|\phi(x)-\phi(y)|\leq\frac{d(x,y)}{r},\quad\forall x,y\in M. \tag{2.30}$$

For any $k\geq0$, set $B_k:=B(x,2^{-k}r)$, so that

$$I_1(x)=\sum_{k=0}^\infty\int_{B_k\setminus B_{k+1}}\frac{(\phi(x)-\phi(y))^2}{d(x,y)^{\alpha+\beta}}d\mu(y).$$

Observe that, for any $y\in B_k\setminus B_{k+1}$,

$$2^{-(k+1)}r\leq d(x,y)<2^{-k}r$$

and, by (2.30)

$$\frac{(\phi(x)-\phi(y))^2}{d(x,y)^{\alpha+\beta}} \le \frac{d\,(x,y)^2}{r^2 d\,(x,y)^{\alpha+\beta}} \le c\frac{2^{-k(2-\alpha-\beta)}}{r^{\alpha+\beta}},$$

where $c=2^{\alpha+\beta}$. By $(V_{\le})$ we have

$$\mu\,(B_k\setminus B_{k+1}) \le \mu\,(B_k) \le C\left(2^{-k}r\right)^{\alpha},$$

whence it follows

$$I_1\,(x) \le \; c\sum_{k=0}^{\infty}\frac{2^{-k(2-\alpha-\beta)}}{r^{\alpha+\beta}}\mu(B_k\setminus B_{k+1}) \le \; \frac{cC}{r^{\beta}}\left(\sum_{k=0}^{\infty}2^{-k(2-\beta)}\right) = \; \frac{C'}{r^{\beta}},$$

where $C'<\infty$ because $2-\beta>0$.

Since $0\le\phi\le 1$, we obtain by $(V_{\le})$ and (6.7), that

$$I_2\,(x) \le \int_{\{d(x,y)\ge r\}}\frac{d\mu(y)}{d(x,y)^{\alpha+\beta}} \le cr^{-\beta}.$$

Combining the estimates of I_1 and I_2, we obtain (2.29).

Let us show that $\phi\in\mathcal{F}$. Since $\phi\in C_0\,(M)$, it suffices to verify that

$$\mathcal{E}\,(\phi,\phi) = \int_{M\times M}(\phi(x)-\phi(y))^2\,J\,(x,y)\,d\mu(x)d\mu(y) < \infty.$$

Set $B=B\,(x_0,R+r)$ and split the domain of integration in $\mathcal{E}\,(\phi,\phi)$ as follows:

$$\int_{M\times M} = \int_{B\times M} + \int_{B^c\times B} + \int_{B^c\times B^c}.$$

The third integral vanishes because $\phi=0$ in B^c. The first integral is estimated just by integrating (2.29) over B which yields that it is finite. The second integral is bounded by $\int_{M\times B}$ and the latter is equal to the first integral by the symmetry in x,y. Hence, $\mathcal{E}\,(\phi,\phi)<\infty$ and $\phi\in\mathcal{F}$.

Finally, multiplying (2.29) by u^2 and integrating over Ω, we obtain (2.28), which finishes the proof of (2.25).

Assume that $\overline{R}=\infty$. In the view of (2.25), in order to prove (2.26) we need only to ensure that $(UE)+(LE)\Rightarrow(V)$. This implication was proved in [20], although under the additional assumption that $(\mathcal{E},\mathcal{F})$ is conservative. However, the conservativeness of the Dirichlet form $(\mathcal{E},\mathcal{F})$ with the jump kernel $J\,(x,y)\simeq d\,(x,y)^{-(\alpha+\beta)}$ with $\beta<2$ follows from a result of [24], which finishes the proof. □

3. Auxiliary estimates

3.1. Subharmonic functions.

Definition 3.1. Let Ω be an open subset of M. We say that a function $u\in\mathcal{F}'$ is *subharmonic* (resp. *superharmonic*) in Ω if

$$\mathcal{E}\,(u,\varphi)\le 0 \quad (\text{resp. } \mathcal{E}\,(u,\varphi)\ge 0) \tag{3.1}$$

for any $0\le\varphi\in\mathcal{F}(\Omega)$. A function $u\in\mathcal{F}'$ is called *harmonic* in Ω if it is both subharmonic and superharmonic in Ω.

Lemma 3.2. *Let $u\in\mathcal{F}'$.*

(*i*) *Suppose that a function $f\in C^2(\mathbb{R})$ satisfies $f''\ge 0$ and $\sup_{\mathbb{R}}|f'|<\infty$, $\sup_{\mathbb{R}}f''<\infty$. Then, for any non-negative function $\phi\in\mathcal{F}\cap L^{\infty}$, we have $f\,(u)\in\mathcal{F}'$, $f'(u)\phi\in\mathcal{F}$ and*

$$\mathcal{E}(f(u),\phi)\le\mathcal{E}(u,f'(u)\phi). \tag{3.2}$$

(*ii*) *Let Ω be open subset of M. If $u\in\mathcal{F}'$ is subharmonic in Ω, then $u_+\in\mathcal{F}'$ and u_+ is also subharmonic in Ω.*

Proof. (i) By Proposition 6.5 (see Appendix), we conclude that $f(u) \in \mathcal{F}'$ and $f'(u)\phi \in \mathcal{F} \cap L^\infty$. In order to prove (3.2), we use the following elementary inequality: for all $X, Y \in \mathbb{R}$ and $a, b \in \mathbb{R}_+$

$$(f(X) - f(Y))(a-b) \le (X - Y)\left(f'(X)a - f'(Y)b\right). \tag{3.3}$$

Indeed, substituting here $X = u(x), Y = u(y), a = \phi(x), b = \phi(y)$, we obtain

$$\begin{aligned}\mathcal{E}(f(u),\phi) &= \int_{M\times M} (f(u(x)) - f(u(y)))(\phi(x) - \phi(y))\, dj \\ &\le \int_{M\times M} (u(x) - u(y))\left(f'(u(x))\phi(x) - f'(u(y))\phi(y)\right) dj = \mathcal{E}(u, f'(u)\phi),\end{aligned}$$

which proves (3.2).

To prove (3.3) we can assume without loss of generality that $a > b$ (otherwise switch a, b and X, Y). We have

$$f'(X)a - f'(Y)b = f'(X)(a-b) + \left(f'(X) - f'(Y)\right)b$$

whence

$$\begin{aligned}(X-Y)\left(af'(X) - bf'(Y)\right) &= (X-Y)f'(X)(a-b) + (X-Y)\left(f'(X) - f'(Y)\right)b \\ &\ge (X-Y)f'(X)(a-b),\end{aligned}$$

where we have used the monotonicity of f' and $b \ge 0$. Finally, it remains to observe that

$$(X-Y)f'(X) \ge f(X) - f(Y),$$

which follows from the monotonicity of f'.

(ii) Let $u = v + a$ with $v \in \mathcal{F}$ and $a \in \mathbb{R}$. Consider the function

$$g(t) = (t+a)_+ - a_+.$$

Since $g(v)$ is a normal contraction of v, we obtain $g(v) \in \mathcal{F}$ and, hence, $u_+ = g(v) + a_+ \in \mathcal{F}'$.

Since u is subharmonic in Ω, v is also subharmonic in Ω. In order to prove that u_+ is subharmonic in Ω, it suffices to verify that $g(v)$ is subharmonic in Ω. It is easy to see that there exists a sequence $\{g_k\}_{k=1}^\infty$ of C^2-functions on $\mathbb{R}$ such that

$$g_k \rightrightarrows g \quad \text{as} \quad k \to \infty.$$

and

$$g_k(0) = 0, \quad g_k' \ge 0, \quad g_k'' \ge 0, \ \sup_{\mathbb{R}} g_k'' < \infty, \quad \sup_k \sup_{\mathbb{R}} g_k' < \infty.$$

Fix a function $0 \le \phi \in \mathcal{F}(\Omega)$ and prove that $\mathcal{E}(g(v), \phi) \le 0$. By [14, Theorem 1.4.2(iii)], we can assume in addition that $\phi \in L^\infty$. Then $g_k'(v)\phi$ is non-negative and, by Proposition 6.5(ii)-(iii), $g_k'(v)\phi \in \mathcal{F}(\Omega)$. Applying (3.2) and using that v is subharmonic in Ω, we obtain, for any $k \ge 1$,

$$\mathcal{E}(g_k(v), \phi) \le \mathcal{E}(v, g_k'(v)\phi) \le 0.$$

It remains to verify that

$$\lim_{k\to\infty} \mathcal{E}(g_k(v), \phi) = \mathcal{E}(g(v), \phi), \tag{3.4}$$

which will imply that $g(v)$ is subharmonic in Ω.

Since $C := \sup_k \sup_{\mathbb{R}} g_k' < \infty$ and $g_k(0) = 0$, we have

$$|g_k(v)| \le C|v|.$$

Setting $w_k := g_k(v) \in \mathcal{F}$, and $w := g(v) \in \mathcal{F}$, we obtain by dominated convergence theorem, that

$$w_k \xrightarrow{L^2} w \quad \text{as} \quad k \to \infty. \tag{3.5}$$

On the other hand, since $C^{-1}w_k$ is a normal contraction of v, we have

$$\sup_k \mathcal{E}(w_k, w_k) \le c, \tag{3.6}$$

where $c = C^2\mathcal{E}(v,v) < \infty$. By [32, Lemma 2.12], (3.5) and (3.6) imply that $\{w_k\}$ converges to w weakly in $(\mathcal{E},\mathcal{F})$, that is,

$$\mathcal{E}(w_k,\phi) \to \mathcal{E}(w,\phi), \tag{3.7}$$

which is exactly (3.4). □

3.2. **Inequalities of Nash and Faber-Krahn.** Let us set

$$\nu = \frac{\beta}{\alpha}.$$

Definition 3.3. We say the *Nash's inequality* $(Nash)$ holds for $(\mathcal{E},\mathcal{F})$ if there exists a positive constant $C > 0$ such that

$$\|u\|_{L^2}^{2(1+\nu)} \le C\left(\mathcal{E}(u,u) + \overline{R}^{-\beta}\|u\|_{L^2}^2\right)\|u\|_{L^1}^{2\nu}$$

for all $u \in \mathcal{F}\cap L^1$.

Given a non-empty open set $\Omega \subset M$, let $\mathcal{L}^\Omega$ be the generator of the Dirichlet form $(\mathcal{E},\mathcal{F}(\Omega))$ (cf. Section 2.3). Denote by $\lambda_1(\Omega)$ the bottom of the spectrum of $\mathcal{L}^\Omega$ in $L^2(\Omega,\mu)$. It is known that

$$\lambda_1(\Omega) = \inf_{u\in\mathcal{F}(\Omega)\setminus\{0\}} \frac{\mathcal{E}(u,u)}{\|u\|_{L^2}^2}.$$

Definition 3.4. We say the *Faber-Krahn inequality* (FK) holds if there exist $\sigma \in (0,1)$ and $c > 0$ such that, for any ball $B := B(x_0,R) \subset M$ with $R \in (0,\sigma\overline{R})$ and for any non-empty open set $\Omega \subset B$,

$$\lambda_1(\Omega) \ge \frac{c}{R^\beta}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu.$$

Lemma 3.5. $(V) + (J_\ge) \Rightarrow (V_\le) + (Nash) \Rightarrow (FK)$.

Proof. The first implication $(V) + (J_\ge) \Rightarrow (Nash)$ follows from the argument in the proof of [28, Theorem 3.1]. Although the result of [28, Theorem 3.1] was stated and proved in the case $\overline{R} = \operatorname{diam} M$, this argument works also for any $\overline{R} \le \operatorname{diam}(M)$. Let us prove the second implication:

$$(V_\le) + (Nash) \Rightarrow (FK).$$

Let Ω be any open subset of a ball $B := B(x_0,R)$ with $R \in (0,\sigma\overline{R})$, where $\sigma > 0$ is a number to be determined later. We need to prove that, for any non-zero function $u \in \mathcal{F}(\Omega)$

$$\frac{\mathcal{E}(u,u)}{\|u\|_{L^2}^2} \ge \frac{c}{R^\beta}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu.$$

It suffices to prove this for any non-zero $u \in \mathcal{F}(\Omega)\cap L^1(\Omega)$. By the Cauchy-Schwarz inequality, we have

$$\|u\|_{L^1}^2 \le \|u\|_{L^2}^2\,\mu(\Omega).$$

Substituting this into $(Nash)$, we obtain

$$\begin{aligned}\|u\|_{L^2}^{2(1+\nu)} &\le C\left(\mathcal{E}(u,u) + \overline{R}^{-\beta}\|u\|_{L^2}^2\right)\|u\|_{L^1}^{2\nu}\\ &\le C\left(\mathcal{E}(u,u) + \left(\sigma^{-1}R\right)^{-\beta}\|u\|_{L^2}^2\right)\left(\|u\|_{L^2}^2\,\mu(\Omega)\right)^\nu,\end{aligned}$$

where we have also used that $\overline{R} > \sigma^{-1}R$. Dividing by $\|u\|_{L^2}^{2\nu}$, we obtain

$$\|u\|_{L^2}^2 \le C\mathcal{E}(u,u)\mu(\Omega)^\nu + C\sigma^\beta\frac{\mu(\Omega)^\nu}{R^\beta}\|u\|_{L^2}^2.$$

By $\Omega \subset B$ and $(V_\le)$ we have

$$\mu(\Omega)^\nu \le \mu(B)^\nu \le (CR^\alpha)^{\beta/\alpha} = C^\nu R^\beta,$$

whence

$$\|u\|_{L^2}^2 \le C\mathcal{E}(u,u)\mu(\Omega)^\nu + C^{1+\nu}\sigma^\beta\|u\|_{L^2}^2.$$

Choosing σ from $C^{1+\nu}\sigma^{\beta} = \frac{1}{2}$ we obtain

$$\|u\|_{L^2}^2 \le 2C\mathcal{E}(u,u)\mu(\Omega)^{\nu},$$

whence

$$\frac{\mathcal{E}(u,u)}{\|u\|_{L^2}^2} \ge \frac{(2C)^{-1}}{(\mu(\Omega))^{\nu}} = \frac{(2C)^{-1}}{(\mu(B))^{\beta/\alpha}}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^{\nu} \ge \frac{c}{R^{\beta}}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^{\nu},$$

which proves (FK). □

3.3. **Some energy estimates.** The main results of this section are Lemmas 3.9 and 3.10 that will be used in the next sections.

Lemma 3.6. *For all $u, v > 0$ and $a, b \in \mathbb{R}$, we have*

$$(u-v)\left(\frac{a^2}{u} - \frac{b^2}{v}\right) \le -\frac{1}{2}\left(\ln\frac{v}{u}\right)^2 \left(a^2 \wedge b^2\right) + 3\,(a-b)^2. \tag{3.8}$$

A similar estimate is contained implicitly in [12, p.1289].

Proof. We start with the following elementary inequality that is true for any $\varepsilon > 0$:

$$a^2 = \ (b + a - b)^2 \le \ (1+\varepsilon)b^2 + \left(1+\varepsilon^{-1}\right)(a-b)^2. \tag{3.9}$$

By the symmetry of (3.8) and the triviality of (3.8) when $u = v$, we can assume without loss of generality that $v < u$. Setting

$$t := \frac{v}{u} \in (0,1),$$

substituting $v = tu$ into (3.8) and using (3.9), we obtain

$$\begin{aligned}(u-v)\left(\frac{a^2}{u} - \frac{b^2}{v}\right) &= (1-t)\left(a^2 - t^{-1}b^2\right)\\ &\le (1-t)\left[(1+\varepsilon)b^2 + \left(1+\varepsilon^{-1}\right)(a-b)^2 - t^{-1}b^2\right]\\ &= (1-t)\left(1+\varepsilon - t^{-1}\right)b^2 + (1-t)\left(1+\varepsilon^{-1}\right)(a-b)^2.\end{aligned}$$

Set $\varepsilon = \frac{1}{2}(1-t)$ in the above inequality. Then $0 < \varepsilon < \frac{1}{2}$ and, hence,

$$(1-t)\left(1+\varepsilon^{-1}\right) = 2\varepsilon(1+\varepsilon^{-1}) = 2\varepsilon + 2 \le 3.$$

Since $b^2 \ge a^2 \wedge b^2$ and

$$(1-t)\left(1+\varepsilon - t^{-1}\right) = (1-t)\left(\frac{3}{2} - \frac{1}{2}t - t^{-1}\right) = \frac{1}{2}t^2 - \frac{1}{t} - 2t + \frac{5}{2},$$

the inequality (3.8) will be proved if we verify that

$$\frac{1}{2}t^2 - \frac{1}{t} - 2t + \frac{5}{2} \le -\frac{1}{2}(\ln t)^2. \tag{3.10}$$

Since the both sides of (3.10) vanish at the endpoint $t = 1$, it suffices to prove the following inequality between the derivatives of the both sides of (3.10):

$$t + \frac{1}{t^2} - 2 \ge -\frac{\ln t}{t},$$

which is equivalent to

$$t^2 + \frac{1}{t} - 2t \ge -\ln t. \tag{3.11}$$

Again, since the both sides of (3.11) vanish at $t = 1$, it suffices to prove the following inequality between their derivatives:

$$2t - \frac{1}{t^2} - 2 \le -\frac{1}{t},$$

which is equivalent to

$$2t^3 - 2t^2 + t - 1 \le 0$$

and which is true because $2t^3 - 2t^2 + t - 1 = (t-1)\left(2t^2+1\right)$. □

Lemma 3.7. *Let a function $u \in \mathcal{F}' \cap L^\infty$ be non-negative in an open set $B \subset M$ and $\phi \in \mathcal{F} \cap L^\infty$ be such that $\phi = 0$ in B^c. Fix any $\lambda > 0$ and set $u_\lambda := u + \lambda$. Then $\frac{\phi^2}{u_\lambda} \in \mathcal{F}$ and*

$$\begin{aligned} \mathcal{E}(u, \frac{\phi^2}{u_\lambda}) \leq & -\frac{1}{2}\int_{B\times B} \left(\phi^2(x) \wedge \phi^2(y)\right) \left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj \\ & + 3\mathcal{E}(\phi, \phi) - 2\int_{B\times B^c} u_\lambda(y) \frac{\phi^2(x)}{u_\lambda(x)} dj. \end{aligned} \tag{3.12}$$

Proof. We first prove that $\frac{\phi^2}{u_\lambda} \in \mathcal{F} \cap L^\infty$. Indeed, the function

$$F(t) := \frac{1}{|t| + \lambda}$$

is a bounded Lipschitz function on $\mathbb{R}$. Since u is non-negative in B and $\phi = 0$ in B^c, the function $\frac{\phi^2}{u_\lambda}$ is well defined on M and $\frac{\phi^2}{u_\lambda} = F(u)\phi^2$. Hence, by Proposition 6.5(ii),

$$\frac{\phi^2}{u_\lambda} = F(u)\phi^2 \in \mathcal{F} \cap L^\infty.$$

Now we prove (3.12). We split the integral in the definition of $\mathcal{E}(u, \frac{\phi^2}{u_\lambda})$ into four parts as follows:

$$\begin{aligned} \mathcal{E}(u, \frac{\phi^2}{u_\lambda}) &= \left(\int_{B\times B} + \int_{B\times B^c} + \int_{B^c\times B} + \int_{B^c\times B^c}\right) (u(x) - u(y)) \left(\frac{\phi^2(x)}{u_\lambda(x)} - \frac{\phi^2(y)}{u_\lambda(y)}\right) dj \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Since $\phi = 0$ in B^c, we have that $I_4 = 0$ and, by symmetry,

$$\begin{aligned} I_2 + I_3 &= 2\int_{B\times B^c} (u_\lambda(x) - u_\lambda(y)) \frac{\phi^2(x)}{u_\lambda(x)} dj \\ &= 2\int_{B\times B^c} \phi^2(x) dj - 2\int_{B\times B^c} u_\lambda(y) \frac{\phi^2(x)}{u_\lambda(x)} dj \\ &= 2\int_{B\times B^c} (\phi(x) - \phi(y))^2 dj - 2\int_{B\times B^c} u_\lambda(y) \frac{\phi^2(x)}{u_\lambda(x)} dj. \end{aligned}$$

In order to estimate I_1, we use Lemma 3.6 that yields

$$(u(x) - u(y)) \left(\frac{\phi^2(x)}{u_\lambda(x)} - \frac{\phi^2(y)}{u_\lambda(y)}\right) \leq -\frac{1}{2}\left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 \left(\phi^2(x) \wedge \phi^2(y)\right) + 3\left(\phi(x) - \phi(y)\right)^2.$$

Integrating this inequality over $B \times B$ against dj, we obtain

$$I_1 \leq -\frac{1}{2}\int_{B\times B} \left(\phi^2(x) \wedge \phi^2(y)\right) \left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj + 3\int_{B\times B} (\phi(x) - \phi(y))^2 dj.$$

Combining the estimates of $I_1, I_2 + I_3$ and I_4, we obtain

$$\begin{aligned} \mathcal{E}(u, \frac{\phi^2}{u_\lambda}) \leq & -\frac{1}{2}\int_{B\times B} \left(\phi^2(x) \wedge \phi^2(y)\right) \left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj \\ & + 3\left(\int_{B\times B} + \int_{B\times B^c}\right) (\phi(x) - \phi(y))^2 dj \\ & - 2\int_{B\times B^c} u_\lambda(y) \frac{\phi^2(x)}{u_\lambda(x)} dj, \end{aligned}$$

whence (3.12) follows. □

Lemma 3.8. *Let $u \in \mathcal{F}' \cap L^\infty$ be non-negative and superharmonic in a ball $2B$, where B is an arbitrary ball in M. Fix any $\lambda > 0$ and set $u_\lambda := u + \lambda$. Then the following inequality holds:*

$$\int_{B\times B} \left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj \le 6\,\mathrm{cap}(B, \frac{3}{2}B) + 4\int_{\frac{3}{2}B\times(2B)^c} \frac{(u_\lambda(y))_-}{u_\lambda(x)} dj. \tag{3.13}$$

Proof. If $\mathrm{cap}(B, \frac{3}{2}B) = \infty$ then (3.13) holds trivially. Hence, assume that $\mathrm{cap}(B, \frac{3}{2}B) < \infty$, and let ϕ be a cutoff function for the pair $(B, \frac{3}{2}B)$.

Observe that $\frac{\phi^2}{u_\lambda} \in \mathcal{F}(2B)$. Indeed, since ϕ vanishes outside $\frac{3}{2}B$, we have $\phi \in \mathcal{F}(2B)$. By the same argument as in the first part of the proof of Lemma 3.7, we conclude that $\frac{\phi^2}{u_\lambda} \in \mathcal{F}(2B)$. Since $\frac{\phi^2}{u_\lambda}$ is non-negative and u is superharmonic in $2B$, we obtain that

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) \ge 0. \tag{3.14}$$

Applying Lemma 3.7 with B replaced by $2B$ and using (3.14), we obtain

$$\frac{1}{2}\int_{2B\times 2B} \left(\phi^2(x)\wedge\phi^2(y)\right) \left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj \le 3\mathcal{E}(\phi,\phi) - 2\int_{(2B)\times(2B)^c} u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)} dj.$$

Since $\phi = 1$ in B, $\phi = 0$ in $(\frac{3}{2}B)^c$ and $\phi \le 1$ in $2B$, it follows that

$$\int_{B\times B} \left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj \le 6\mathcal{E}(\phi,\phi) + 4\int_{\frac{3}{2}B\times(2B)^c} \frac{(u_\lambda(y))_-}{u_\lambda(x)} dj.$$

Minimizing the last inequality over $\phi \in \mathrm{cutoff}(B, \frac{3}{2}B)$, we finish the proof. □

Fix a reference point $x_0 \in M$. For any measurable function v on M and for any ball $B = B(x_0, R)$ on M, define the *tail* of v outside B by

$$T_B(v) := \int_{B^c} |v(y)|\, J(x_0, y) d\mu(y). \tag{3.15}$$

Lemma 3.9. *Assume that (V), (J), and (cap) are satisfied. Let a function $u \in \mathcal{F}' \cap L^\infty$ be non-negative and superharmonic in the ball $2B$, where $B := B(x_0, R)$ and $R < \frac{1}{2}\overline{R}$. Fix three positive numbers a, b, λ, set $u_\lambda := u + \lambda$, and consider in $2B$ the function*

$$v := \left(\ln \frac{a}{u_\lambda}\right)_+ \wedge b.$$

Then

$$\fint_B \fint_B (v(x) - v(y))^2 d\mu(x) d\mu(y) \le C\left(1 + \frac{R^\beta T_{2B}((u_\lambda)_-)}{\lambda}\right), \tag{3.16}$$

where the constant C depends only on the constants in the conditions (cap), (J) and $(V_\le)$.

Proof. It follows from the definition of v that, for all $x, y \in 2B$,

$$|v(x) - v(y)| \le \left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|.$$

For all $x \in \frac{3}{2}B$, $y \in (2B)^c$, we have $d(x_0, x) < \frac{3}{2}R < 3d(x, y)$ and, hence,

$$\frac{d(x_0, y)}{d(x, y)} \le \frac{d(x_0, x) + d(x, y)}{d(x, y)} \le 3 + 1 = 4.$$

It follows from (J), that, for the above range of x, y,

$$J(x, y) \le CJ(x_0, y).$$

Using the above three inequalities, (V), (J), (cap) and Lemma 3.8, we obtain

$$\begin{aligned}
&\fint_B \fint_B (v(x)-v(y))^2 d\mu(x)d\mu(y)\\
&= \frac{1}{\mu(B)^2}\int_{B\times B}(v(x)-v(y))^2\frac{d(x,y)^{\alpha+\beta}}{d(x,y)^{\alpha+\beta}}d\mu(x)d\mu(y)\\
&\le C(2R)^{\alpha+\beta}\frac{1}{R^{2\alpha}}\int_{B\times B}\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 J(x,y)\,d\mu(x)d\mu(y)\\
&\le C2^{\alpha+\beta}R^{\beta-\alpha}\left(6\,\mathrm{cap}(B,\tfrac{3}{2}B)+4\int_{\frac{3}{2}B\times(2B)^c}\frac{(u_\lambda(y))_-}{u_\lambda(x)}J(x,y)\,d\mu(x)d\mu(y)\right)\\
&\le C'R^{\beta-\alpha}\left(R^{\alpha-\beta}+\int_{\frac{3}{2}B}d\mu(x)\int_{(2B)^c}\frac{(u_\lambda(y))_-}{u_\lambda(x)}J(x_0,y)d\mu(y)\right)\\
&\le C'+C'R^{\beta-\alpha}\mu(\tfrac{3}{2}B)\int_{(2B)^c}\frac{(u_\lambda(y))_-}{\lambda}J(x_0,y)d\mu(y)\\
&\le C'+C''\frac{R^\beta T_{2B}((u_\lambda)_-)}{\lambda},
\end{aligned}$$

which finishes the proof. □

Lemma 3.10. *Assume that $(AB_{1/8})$ is satisfied. Let $B_0 := B(x_0,R)$, $B := B(x_0,R+r)$ and $\Omega := B(x_0,R')$ be three balls so that $0<R<R+r<R'<\overline{R}$. Then, for any $u\in\mathcal{F}'\cap L^\infty$, there exists $\phi\in\mathrm{cutoff}(B_0,B)$ such that*

$$\mathcal{E}(u\phi)\le \frac{3}{2}\mathcal{E}(u,u\phi^2)+\frac{c}{r^\beta}\int_\Omega u^2d\mu+3\int_{\Omega\times\Omega^c}u(x)u(y)\phi^2(x)dj, \tag{3.17}$$

where the constant $c>0$ depends only on the constant in the condition $(AB_{1/8})$.

Proof. We first prove the following identity

$$\mathcal{E}(u\phi)=\mathcal{E}(u,u\phi^2)+\int_{M\times M}u(x)u(y)\left(\phi(x)-\phi(y)\right)^2dj, \tag{3.18}$$

for all $u,\phi\in\mathcal{F}'\cap L^\infty$. Note that, by Proposition 6.5(i)-(ii), both $u\phi$ and $u\phi^2$ belong to $\mathcal{F}'\cap L^\infty$. By a direct computation, we have the following identity for all numbers a,b,X,Y,

$$(Xa-Yb)^2=(X-Y)\left(Xa^2-Yb^2\right)+XY\left(a-b\right)^2.$$

Setting here $X=u(x),Y=u(y),a=\phi(x)$ and $b=\phi(y)$ and integrating this identity in $(x,y)\in M\times M$ with respect to dj, we obtain (3.18).

Assume further that $\phi\in\mathrm{cutoff}(B_0,B)$. Since

$$M\times M=(\Omega\times\Omega)\sqcup(\Omega^c\times M)\sqcup(\Omega\times\Omega^c), \tag{3.19}$$

and $\phi|_{\Omega^c=0}$, by (3.18), Cauchy-Schwarz inequality and symmetrization, we obtain

$$\begin{aligned}
\mathcal{E}(u\phi)&=\mathcal{E}(u,u\phi^2)+\left(\int_{\Omega\times\Omega}+\int_{\Omega^c\times\Omega}+\int_{\Omega\times\Omega^c}\right)u(x)u(y)\left(\phi(x)-\phi(y)\right)^2dj \quad \text{(by (3.19))}\\
&\le \mathcal{E}(u,u\phi^2)+\int_{\Omega\times\Omega}u^2(x)\left(\phi(x)-\phi(y)\right)^2dj \quad \text{(by Cauchy-Schwarz and symmetrization)}\\
&\quad +2\int_{\Omega\times\Omega^c}u(x)u(y)\phi^2(x)dj. \qquad \text{(by symmetrization)}
\end{aligned} \tag{3.20}$$

By condition $(AB_{1/8})$, there exists $\phi\in\mathrm{cutoff}(B_0,B)$ such that

$$\int_{\Omega\times\Omega}u^2(x)(\phi(x)-\phi(y))^2dj\le\frac{1}{8}\int_{\Omega\times\Omega}\left(u(x)-u(y)\right)^2\phi^2(x)dj+\frac{C}{r^\beta}\int_\Omega u^2d\mu. \tag{3.21}$$

Let us estimate the middle integral in (3.21). Applying (2.2) and observing that $\phi|_{\Omega^c} = 0$, we obtain

$$\int_{\Omega\times\Omega} (u(x)-u(y))^2 \phi^2(x)dj \leq 2\int_{\Omega\times\Omega} F dj + 4\int_{\Omega\times\Omega} u^2(x) (\phi(x)-\phi(y))^2 dj, \tag{3.22}$$

where

$$F(x,y) := (u(x)-u(y)) \left(u(x)\phi^2(x) - u(y)\phi^2(y)\right),$$

and all the integrals are finite by Remark 2.3. Note that

$$\int_{M\times M} F dj = \mathcal{E}\left(u, u\phi^2\right).$$

Using (3.19) again, we obtain

$$\int_{\Omega\times\Omega} F dj = \int_{M\times M} F dj - \int_{\Omega^c\times M} F dj - \int_{\Omega\times\Omega^c} F dj. \tag{3.23}$$

Since $\phi(x) = 0$ in Ω^c, we have

$$-\int_{\Omega^c\times M} F dj = \int_{\Omega^c\times\Omega} (u(x)-u(y)) u(y)\phi^2(y)dj \leq \int_{\Omega^c\times\Omega} u(x) u(y) \phi^2(y)dj.$$

Similarly, we obtain

$$-\int_{\Omega\times\Omega^c} F dj \leq \int_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj.$$

Symmetrizing the former integral and substituting into (3.23), we obtain

$$\int_{\Omega\times\Omega} F dj \leq \mathcal{E}\left(u, u\phi^2\right) + 2\int_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj.$$

Substitution into (3.22) yields

$$\begin{aligned}\int_{\Omega\times\Omega} (u(x)-u(y))^2 \phi^2(x)dj \quad &\leq \quad 2\mathcal{E}\left(u, u\phi^2\right) + 4\int_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj \\ &\quad + 4\int_{\Omega\times\Omega} u^2(x) (\phi(x)-\phi(y))^2 dj.\end{aligned}$$

Substituting this into (3.21), we obtain

$$\begin{aligned}&\int_{\Omega\times\Omega} u^2(x) (\phi(x)-\phi(y))^2 dj \\ &\qquad \leq \frac{1}{4}\mathcal{E}(u, u\phi^2) + \frac{1}{2}\int_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj \\ &\qquad\quad + \frac{1}{2}\int_{\Omega\times\Omega} u^2(x) (\phi(x)-\phi(y))^2 dj + \frac{C}{r^\beta}\int_\Omega u^2 d\mu,\end{aligned}$$

which implies that

$$\int_{\Omega\times\Omega} u^2(x) (\phi(x)-\phi(y))^2 dj \leq \frac{1}{2}\mathcal{E}(u, u\phi^2) + \int_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj + \frac{2C}{r^\beta}\int_\Omega u^2 d\mu.$$

Finally, substituting the above inequality into (3.20), we obtain (3.17). □

4. Superharmonic and harmonic functions

In this section we establish estimates of the Hölder norm of harmonic functions. The main result is stated in Lemma 4.8.

4.1. **Lemma of growth.** Recall that, by Lemma 3.5, the hypotheses (V) and (J) imply the Faber-Krahn inequality (FK). Let σ and $\nu = \frac{\beta}{\alpha}$ be the constants from (FK). Without loss of generality, we can always assume that $\sigma \in \left(0, \frac{1}{2}\right)$.

Lemma 4.1 (Lemma of growth)**.** *Assume that* (V)*,* (J) *and* (AB) *are satisfied. Then there exists* $\varepsilon_0 \in (0,1)$ *depending only on the constants in the above conditions, such that the following is true: if a function* $u \in \mathcal{F}' \cap L^\infty$ *is superharmonic and non-negative in a ball* $2B$*, where* $B = B(x_0, R)$ *has radius* $R < \sigma \overline{R}$*, and if, for some* $a > 0$*,*

$$\frac{\mu(B \cap \{u < a\})}{\mu(B)} \leq \varepsilon_0 \left(1 + \frac{R^\beta T_{2B}(u_-)}{a}\right)^{-\alpha/\beta}, \tag{4.1}$$

then

$$\operatorname*{ess\,inf}_{\frac{1}{2}B} u \geq \frac{a}{2} \tag{4.2}$$

(see Fig. 2).

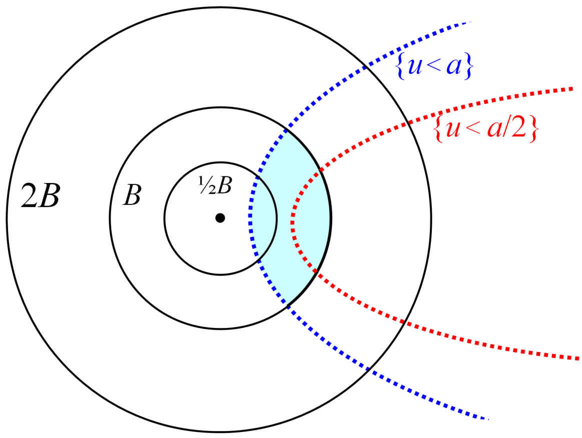

FIGURE 2. Level sets $\{u < a\}$ and $\{u < a/2\}$

Recall that the tail function $T_B(v)$ was defined by (3.15). Observe also that if $u \geq 0$ on M then $T_{2B}(u_-) = 0$ and the condition (4.1) simplifies.

Remark 4.2. The term "Lemma of growth" was introduced by E.M. Landis [31] in the context of second order elliptic PDEs in $\mathbb{R}^n$. In order to understand this terminology, let us reformulate the statement assuming that $\inf_{2B} u = 0$ and $a = \frac{1}{2}\sup_{2B} u$. Then, for the function $v := 2a - u$, we have $\inf_{2B} v = 0$, $\sup_{2B} v = 2a$, and the smallness of $\frac{\mu(B \cap \{v > a\})}{\mu(B)}$ implies that

$$\sup_{\frac{1}{2}B} v \leq \frac{3}{2} a,$$

which can be rewritten in the form

$$\sup_{2B} v \geq \frac{4}{3} \sup_{\frac{1}{2}B} v.$$

The latter means that $\sup v$ exhibits a *growth* by a factor $\geq \frac{4}{3} > 1$ when passing from $\frac{1}{2}B$ to $2B$, which gives the name to this type of statements. In the context of local Dirichlet forms, a similar Lemma of growth was proved in [23, Lemmas 7.2, 7.6].

The most essential part of the proof of Lemma 4.1 is contained in the following lemma. We use the notation $B_r := B(x_0, r)$.

Lemma 4.3. *Assume that* (V), (J) *and* (AB) *are satisfied. Let a function* $u \in \mathcal{F}' \cap L^\infty$ *be superharmonic and non-negative in a ball* B_{2R}, *where* $R < \sigma \overline{R}$. *Fix some* $0 < a < b$, $r_1 < r_2 < R$ *and set*

$$m_1 = \frac{\mu(B_{r_1} \cap \{u < a\})}{\mu(B_{r_1})} \quad \textit{and} \quad m_2 = \frac{\mu(B_{r_2} \cap \{u < b\})}{\mu(B_{r_2})}.$$

Then

$$m_1 \le \; CA \left(\frac{b}{b-a}\right)^2 \left(\frac{r_2}{r_1}\right)^\alpha \left(\frac{r_2}{r_2 - r_1}\right)^{\alpha+\beta} m_2^{1+\beta/\alpha}, \tag{4.3}$$

where

$$A := 1 + \frac{r_2^\beta T_{B_{r_2}}(u_-)}{b},$$

and the constant $C > 0$ *depends only on the constants in* (V), (J) *and* (AB).

Proof. We use in the proof the following facts from [14]:

(1) any function $u \in \mathcal{F}$ admits a *quasi-continuous* version $\widetilde{u}$ [14, Theorem 2.1.3, p.71];
(2) for any $u \in \mathcal{F}$ and any open subset Ω of M, we have $u \in \mathcal{F}(\Omega)$ if and only if $\widetilde{u} = 0$ q.e. in Ω^c, where q.e. means *quasi-everywhere* [14, Corollary 2.3.1 p.98].

Let us fix a quasi-continuous modification of a given superharmonic function u and denote it also by the same letter u. Set $v := (b-u)_+$ and

$$\widetilde{m}_1 := \mu(B_{r_1} \cap \{u < a\}), \quad \widetilde{m}_2 := \mu(B_{r_2} \cap \{u < b\}).$$

Let ϕ be any cutoff function of the pair $(B_{r_1}, B_{\frac{1}{2}(r_1+r_2)})$; without loss of generality, we can assume that ϕ is quasi-continuous. Then we have

$$\widetilde{m}_1 = \int_{B_{r_1} \cap \{u<a\}} \phi^2 d\mu \le \int_{B_{r_1}} \phi^2 \underbrace{\left(\frac{(b-u)_+}{b-a}\right)^2}_{\ge 1 \text{ on } \{u<a\}} d\mu = \frac{1}{(b-a)^2} \int_{B_{r_1}} (\phi v)^2 d\mu. \tag{4.4}$$

Consider the set

$$E := B_{\frac{1}{2}(r_1+r_2)} \cap \{u < b\}.$$

By the outer regularity of μ, for any $\varepsilon > 0$, there is an open set Ω such that $E \subset \Omega \subset B_{r_2}$ and

$$\mu(\Omega) \le \mu(E) + \varepsilon \le \widetilde{m}_2 + \varepsilon \tag{4.5}$$

(see Fig. 3).

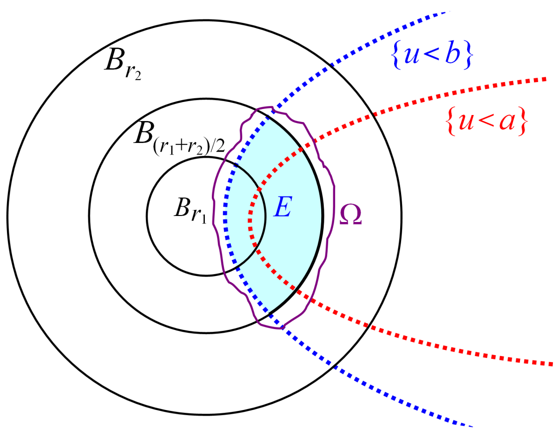

FIGURE 3. Sets E and Ω

On the other hand, since $\phi = 0$ q.e. outside $B_{\frac{1}{2}(r_1+r_2)}$ and $v = 0$ outside $\{u < b\}$, we obtain that $\phi v = 0$ q.e. in E^c. Hence, since $\phi v \in \mathcal{F}$ and $\phi v = 0$ q.e. in $\Omega^c \subset E^c$, we conclude that

$$\phi v \in \mathcal{F}(\Omega). \tag{4.6}$$

By the definition of $\lambda_1(\Omega)$, we obtain

$$\int_\Omega (\phi v)^2 d\mu \leq \frac{\mathcal{E}(\phi v)}{\lambda_1(\Omega)}.$$

Using again that ϕv vanishes outside Ω and combining this inequality with (4.4), we obtain

$$\widetilde{m}_1 \leq \frac{1}{(b-a)^2} \int_\Omega (\phi v)^2 d\mu \leq \frac{\mathcal{E}(\phi v)}{(b-a)^2 \lambda_1(\Omega)}. \tag{4.7}$$

By (FK) (Lemma 3.5) and (4.5), we have

$$\lambda_1(\Omega) \geq \frac{c}{r_2^\beta} \left(\frac{\mu(B_{r_2})}{\mu(\Omega)} \right)^\nu \geq \frac{c_1}{r_2^\beta} \left(\frac{\mu(B_{r_2})}{\widetilde{m}_2 + \varepsilon} \right)^\nu, \tag{4.8}$$

where $\nu = \beta/\alpha$.

Let us now estimate $\mathcal{E}(\phi v)$ from above. Since u is superharmonic in B_{2R}, the function $b - u$ is subharmonic in B_{2R} and, by Lemma 3.2(ii), the function $v = (b-u)_+$ is also subharmonic in B_{2R}. Furthermore, by Proposition 6.5(iii) and (4.6), we have $v\phi^2 = v\phi \cdot \phi \in \mathcal{F}(\Omega) \subset \mathcal{F}(B_{2R})$. Hence, by the definition of subharmonic functions, we obtain

$$\mathcal{E}(v, v\phi^2) \leq 0. \tag{4.9}$$

By Lemma 2.9, we have $(AB_{1/8})$. Applying Lemma 3.10 to the triple B_{r_1}, $B_{(r_1+r_2)/2}$, B_{r_2} and the function v, we see that there exists $\phi \in \text{cutoff}(B_{r_1}, B_{(r_1+r_2)/2})$ such that

$$\mathcal{E}(v\phi) \leq \; 2\mathcal{E}(v, v\phi^2) + \frac{c}{r^\beta} \int_{B_{r_2}} v^2 d\mu + 3 \int_{B_{r_2} \times B_{r_2}^c} v(x)v(y)\phi^2(x) dj$$

where $r = r_2 - r_1$, $x \in B_{r_2}$ and $y \in B_{r_2}^c$. Applying here (4.9) and using that $\phi = 0$ outside $B_{(r_1+r_2)/2}$, we obtain

$$\begin{aligned}
\mathcal{E}(v\phi) &\leq \frac{c}{r^\beta} \int_{B_{r_2}} v^2 d\mu + 3 \int_{B_{(r_1+r_2)/2}} v(x)\, d\mu(x) \cdot \operatorname*{ess\,sup}_{x \in B_{(r_1+r_2)/2}} \int_{B_{r_2}^c} v(y) J(x,y) d\mu(y) \\
&\leq \frac{c}{r^\beta} \int_{B_{r_2}} v^2 d\mu + 3 \int_{B_{r_2}} v d\mu \cdot C^2 \left(\frac{3r_2}{r} \right)^{\alpha+\beta} \int_{B_{r_2}^c} v(y) J(x_0, y) d\mu(y) \\
&\leq \frac{cb^2}{r^\beta} \mu(B_{r_2} \cap \{u < b\}) + 3C^2 b \mu(B_{r_2} \cap \{u < b\}) \left(\frac{3r_2}{r} \right)^{\alpha+\beta} T_{B_{r_2}}(v) \quad (\text{using } v \leq b 1_{\{u<b\}}) \\
&\leq c\widetilde{m}_2 \frac{b^2}{r_2^\beta} \left(\left(\frac{r_2}{r} \right)^\beta + \frac{r_2^\beta}{b} \left(\frac{r_2}{r} \right)^{\alpha+\beta} (T_{B_{r_2}}(b) + T_{B_{r_2}}(u_-)) \right) \qquad (\text{by definition of } \widetilde{m}_2 \text{ and } v) \\
&\leq c\widetilde{m}_2 \frac{b^2}{r_2^\beta} \left(\left(\frac{r_2}{r} \right)^\beta + \left(\frac{r_2}{r} \right)^{\alpha+\beta} + \frac{r_2^\beta}{b} \left(\frac{r_2}{r} \right)^{\alpha+\beta} T_{B_{r_2}}(u_-) \right) \qquad (\text{by } (6.8)) \\
&\leq c\widetilde{m}_2 \frac{b^2}{r_2^\beta} \left(\frac{r_2}{r} \right)^{\alpha+\beta} \left(1 + \frac{r_2^\beta T_{B_{r_2}}(u_-)}{b} \right) \qquad (\text{using } r_2 \geq r) \\
&= c\widetilde{m}_2 \frac{b^2}{r_2^\beta} \left(\frac{r_2}{r} \right)^{\alpha+\beta} A,
\end{aligned} \tag{4.10}$$

where in the second line we used that, for all $x \in B_{(r_1+r_2)/2}$ and all $y \in B_{r_2}^c$,

$$\frac{d(x_0,y)}{d(x,y)} \leq \frac{d(x_0,x) + d(x,y)}{d(x,y)} = \frac{d(x_0,x)}{d(x,y)} + 1 \leq \frac{2r_2}{r} + 1 \leq \frac{3r_2}{r},$$

which implies by (J) that

$$J(x,y) \le C^2 \left(\frac{3r_2}{r}\right)^{\alpha+\beta} J(x_0,y).$$

Combining (4.7), (4.8), (4.10) and letting $\varepsilon \to 0$, we obtain

$$\widetilde{m}_1 \le c\left(\frac{b}{b-a}\right)^2 \frac{\widetilde{m}_2^{\nu}\widetilde{m}_2}{\mu(B_{r_2})^{\nu}}\left(\frac{r_2}{r}\right)^{\alpha+\beta} A.$$

Dividing this inequality by $\mu(B_{r_1})$ and observing that

$$m_1 = \frac{\widetilde{m}_1}{\mu(B_{r_1})} \quad \text{and} \quad m_2 = \frac{\widetilde{m}_2}{\mu(B_{r_2})},$$

we obtain

$$\begin{aligned} m_1 &\le c\left(\frac{b}{b-a}\right)^2 m_2^{1+\nu}\frac{\mu(B_{r_2})}{\mu(B_{r_1})}\left(\frac{r_2}{r}\right)^{\alpha+\beta} A \\ &\le C\left(\frac{b}{b-a}\right)^2\left(\frac{r_2}{r_1}\right)^{\alpha}\left(\frac{r_2}{r}\right)^{\alpha+\beta} A\cdot m_2^{1+\beta/\alpha}, \end{aligned}$$

which finishes the proof. □

Proof of Lemma 4.1. Let $u \in \mathcal{F}' \cap L^\infty$ be superharmonic and non-negative in B_{2R} with $R < \sigma\overline{R}$ and let $a > 0$. Consider the following sequences

$$R_k := \frac{1}{2}(1+2^{-k})R, \quad \text{and} \quad a_k := \frac{1}{2}(1+2^{-k})a,$$

where k is a non-negative integer. Clearly, $R_0 = R$, $a_0 = a$, $R_k \searrow \frac{1}{2}R$, and $a_k \searrow \frac{1}{2}a$ as $k \to \infty$. Set also

$$m_k := \frac{\mu(B_{R_k} \cap \{u < a_k\})}{\mu(B_{R_k})}$$

(see Fig. 4).

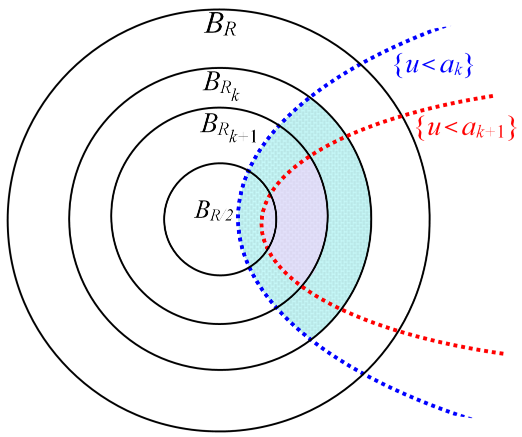

FIGURE 4. Sets $B_{R_k} \cap \{u < a_k\}$ and $B_{R_{k+1}} \cap \{u < a_{k+1}\}$

Applying the inequality (4.3) of Lemma 4.3 with $a = a_k$, $b = a_{k-1}$, $r_1 = R_k$ and $r_2 = R_{k-1}$, we obtain, for any $k \ge 1$,

$$m_k \le CA_k\left(\frac{a_{k-1}}{a_{k-1}-a_k}\right)^2\left(\frac{R_{k-1}}{R_k}\right)^{\alpha}\left(\frac{R_{k-1}}{R_{k-1}-R_k}\right)^{\alpha+\beta} m_{k-1}^{1+\beta/\alpha},$$

where

$$A_k := 1 + \frac{R_{k-1}^{\beta} T_{B_{R_{k-1}}}(u_-)}{a_{k-1}}.$$

Since u is non-negative in B_{2R} and $B_{R_{k-1}} \subset B_{2R}$ we have

$$T_{B_{R_{k-1}}}(u_-) = T_{B_{2R}}(u_-).$$

Since $R_{k-1} \le R$ and $a_{k-1} \ge \frac{1}{2}a$, we obtain

$$A_k \le 2A,$$

where

$$A = 1 + \frac{R^{\beta} T_{B_{2R}}(u_-)}{a}.$$

Using that

$$\frac{R_{k-1}}{R_k} \le 2, \quad \frac{a_{k-1}}{a_{k-1}-a_k} = \frac{1+2^{-(k-1)}}{2^{-(k-1)}-2^{-k}} \le 2^{k+1} \text{ and } \frac{R_{k-1}}{R_{k-1}-R_k} \le 2^{k+1},$$

we obtain that

$$m_k \le C \cdot 2A \cdot 2^{2(k+1)} \cdot 2^{\alpha} \cdot 2^{(k+1)(\alpha+\beta)} \cdot m_{k-1}^{1+\beta/\alpha} = C' A \cdot 2^{ck} \cdot m_{k-1}^{q},$$

where $C' := 2^{2\alpha+\beta+3} C$, $c = \alpha+\beta+2$, and

$$q = 1 + \beta/\alpha.$$

Applying the above inequality inductively, we obtain,

$$\begin{aligned} m_k &\le (C'A) \cdot 2^{ck} \cdot m_{k-1}^{q} \\ &\le (C'A)^{1+q} \cdot 2^{ck+cq(k-1)} \cdot m_{k-2}^{q^2} \\ &\quad \cdots \\ &\le (C'A)^{1+q+\cdots+q^{k-1}} \cdot 2^{c(k+q(k-1)+\cdots+q^{k-1})} \cdot m_0^{q^k}. \end{aligned}$$

Note that

$$k + q(k-1) + \cdots + q^{k-1} = \frac{q^{k+1}-(k+1)q+k}{(q-1)^2} \le \frac{q}{(q-1)^2} q^k,$$

$$1 + q + \cdots + q^{k-1} = \frac{q^k - 1}{q-1} = \frac{q^k}{q-1} - \frac{1}{q-1}.$$

Hence, we obtain that

$$m_k \le \left(2^{\frac{cq}{(q-1)^2}} \cdot (C'A)^{\frac{1}{q-1}} \cdot m_0\right)^{q^k} (C'A)^{-\frac{1}{q-1}}.$$

It follows from the last inequality and from $q > 1$ that if

$$2^{\frac{cq}{(q-1)^2}} \cdot (C'A)^{\frac{1}{q-1}} \cdot m_0 \le \frac{1}{2}, \tag{4.11}$$

then

$$\lim_{k\to\infty} m_k = 0. \tag{4.12}$$

Note that (4.11) is equivalent to

$$m_0 \le 2^{-\frac{cq}{(q-1)^2}-1} \cdot (C'A)^{-\frac{1}{q-1}},$$

that is, to

$$\frac{\mu(B_R \cap \{u < a\})}{\mu(B_R)} \le 2^{-\frac{cq}{(q-1)^2}-1} (C')^{-\frac{1}{q-1}} \left(1 + \frac{R^{\beta} T_{B_{2R}}(u_-)}{a}\right)^{-\frac{\alpha}{\beta}},$$

which is equivalent to the hypothesis (4.1) with

$$\varepsilon_0 := 2^{-\frac{cq}{(q-1)^2}-1} (C')^{-\frac{1}{q-1}}. \tag{4.13}$$

Assuming that ε_0 is defined by (4.13), we see that (4.11) is satisfied and, hence, we have (4.12). It follows that

$$\frac{\mu(B_{R/2}\cap\{u\le\frac{a}{2}\})}{\mu(B_{R/2})}=0,$$

which implies (4.2). □

Corollary 4.4. *Assume that (V), (J) and (AB) are satisfied. There is a constant $\varepsilon>0$ depending only on α,β and such that, for any ball $B:=B(x_0,R)$ with $R\in(0,\sigma\overline{R})$, and for any function $u\in\mathcal{F}'\cap L^\infty$ that is superharmonic and non-negative in $2B$ and satisfies*

$$R^\beta T_{2B}(u_-)\le\varepsilon\left(\fint_B\frac{1}{u}d\mu\right)^{-1}, \tag{4.14}$$

the following is true:

$$\operatorname*{ess\,inf}_{\frac{1}{2}B}u\ge\frac{\varepsilon}{2}\left(\fint_B\frac{1}{u}d\mu\right)^{-1}.$$

Proof. We will apply Lemma 4.1 with a suitable constant $a>0$. Indeed, for any $a>0$, we have

$$\mu(B\cap\{u<a\})=\mu(B\cap\{\frac{1}{u}>\frac{1}{a}\})\le a\int_B\frac{1}{u}d\mu=a\mu(B)\fint_B\frac{1}{u}d\mu.$$

In order to fulfill the condition (4.1) of Lemma 4.1, the constant a should satisfy the inequality:

$$a\fint_B\frac{1}{u}d\mu\le\varepsilon_0\left(1+\frac{R^\beta T_{2B}(u_-)}{a}\right)^{-\alpha/\beta}. \tag{4.15}$$

Let us set

$$\varepsilon:=2^{-\alpha/\beta}\varepsilon_0.$$

Assuming that (4.14) holds with this ε, we claim that (4.15) holds with the following value of a:

$$a:=\varepsilon\left(\fint_B\frac{1}{u}d\mu\right)^{-1}.$$

Indeed, for this a we have by (4.14)

$$R^\beta T_{2B}(u_-)\le a$$

and, hence,

$$a\fint_B\frac{1}{u}d\mu=\varepsilon=2^{-\frac{\alpha}{\beta}}\varepsilon_0\le\varepsilon_0\left(1+\frac{R^\beta T_{2B}(u_-)}{a}\right)^{-\frac{\alpha}{\beta}}.$$

Therefore, by Lemma 4.1, we conclude that

$$\operatorname*{ess\,inf}_{\frac{1}{2}B}u\ge\frac{a}{2}=\frac{\varepsilon}{2}\left(\fint_B\frac{1}{u}d\mu\right)^{-1},$$

which finishes the proof. □

4.2. Weak Harnack inequality.

Lemma 4.5 (Weak Harnack inequality)**.** *Assume that (V), (J) and (AB) are satisfied. Then there exists $\varepsilon\in(0,1)$ depending only on the constants in the above hypotheses, such that the following is true: if a function $u\in\mathcal{F}'\cap L^\infty$ is superharmonic and non-negative in a ball $2B$, where $B=B(x_0,R)$ has radius $R<\sigma\overline{R}$, and if, for some $a>0$,*

$$\frac{\mu(B\cap\{u\ge a\})}{\mu(B)}\ge\frac{1}{2} \tag{4.16}$$

and

$$R^\beta T_{2B}(u_-)\le\varepsilon a, \tag{4.17}$$

then

$$\operatorname*{ess\,inf}_{\frac{1}{2}B}u\ge\varepsilon a.$$

Proof. Let λ, b be two positive parameters to be determined later. Consider the functions $u_\lambda := u+\lambda$ and

$$v := \left(\ln \frac{a+\lambda}{u_\lambda}\right)_+ \wedge b.$$

Note that $0 \le v \le b$ and

$$\begin{aligned} v = 0 \quad &\Leftrightarrow \quad \frac{a+\lambda}{u_\lambda} \le 1 \quad \Leftrightarrow \quad u \ge a, \\ v = b \quad &\Leftrightarrow \quad \frac{a+\lambda}{u_\lambda} \ge e^b \quad \Leftrightarrow \quad u_\lambda \le (a+\lambda)e^{-b} =: q. \end{aligned}$$

We will apply Lemma 4.1 to u_λ instead of u. Set

$$\omega := \frac{\mu(B \cap \{u \ge a\})}{\mu(B)} = \frac{\mu(B \cap \{v = 0\})}{\mu(B)} \tag{4.18}$$

and

$$m_0 := \frac{\mu(B \cap \{u_\lambda \le q\})}{\mu(B)} = \frac{\mu(B \cap \{v = b\})}{\mu(B)} \tag{4.19}$$

(cf. Fig. 5).

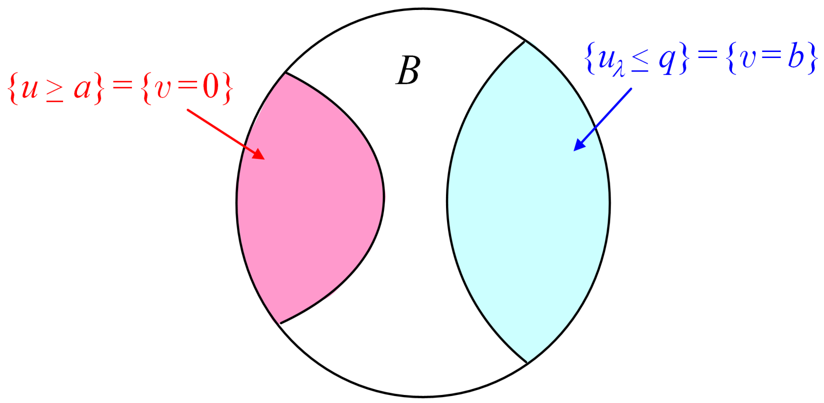

FIGURE 5. Sets $\{u \ge a\} = \{u_\lambda \ge a+\lambda\}$ and $\{u_\lambda \le q\}$

By Lemma 4.1 we know that if

$$m_0 \le \varepsilon_0 \left(1 + \frac{R^\beta T_{2B}((u_\lambda)_-)}{q}\right)^{-\frac{\alpha}{\beta}}, \tag{4.20}$$

then

$$\operatorname*{ess\,inf}_{\frac{1}{2}B} u_\lambda \ge \frac{q}{2}. \tag{4.21}$$

Clearly, we have

$$A := R^\beta T_{2B}(u_-) \ge R^\beta T_{2B}\left((u_\lambda)_-\right).$$

Hence, in order to have (4.20), it suffices to ensure that

$$m_0 \le \varepsilon_0 \left(1 + \frac{A}{q}\right)^{-\frac{\alpha}{\beta}}. \tag{4.22}$$

Using (4.19), (4.18), and Lemma 3.9, we obtain

$$\begin{aligned} b^2 m_0 \omega &= \frac{1}{\mu(B)^2} \int_{B\cap\{v=0\}} \int_{B\cap\{v=b\}} b^2 d\mu(x) d\mu(y) \\ &= \frac{1}{\mu(B)^2} \int_{B\cap\{v=0\}} \int_{B\cap\{v=b\}} (v(x)-v(y))^2 d\mu(x) d\mu(y) \\ &\le \fint_B \fint_B (v(x)-v(y))^2 d\mu(x) d\mu(y) \\ &\le c\left(1+\frac{R^\beta T_{2B}((u_\lambda)_-)}{\lambda}\right) \qquad \text{(by (3.16))} \\ &\le c\left(1+\frac{A}{\lambda}\right). \end{aligned}$$

It follows that

$$m_0 \le \frac{c}{b^2\omega}\left(1+\frac{A}{\lambda}\right) \le \frac{2c}{b^2}\left(1+\frac{A}{\lambda}\right),$$

where we have used that $\omega \ge 1/2$, which is true by (4.16). Hence, the condition (4.22) will be satisfied provided

$$\frac{2c}{b^2}\left(1+\frac{A}{\lambda}\right) \le \varepsilon_0 \left(1+\frac{A}{q}\right)^{-\frac{\alpha}{\beta}},$$

which is equivalent to

$$b^2 \ge \frac{2c}{\varepsilon_0}\left(1+\frac{A}{\lambda}\right)\left(1+\frac{A}{q}\right)^{\frac{\alpha}{\beta}}. \tag{4.23}$$

Fix $\varepsilon > 0$ to be determined later, and specify the parameters λ, b as follows:

$$\lambda := \varepsilon a, \quad b := \ln \frac{1+\varepsilon}{4\varepsilon}.$$

Then we have

$$q = (a+\lambda)e^{-b} = 4\varepsilon a,$$

and the inequality (4.23) is equivalent to

$$\left(\ln\frac{1+\varepsilon}{4\varepsilon}\right)^2 \ge \frac{2c}{\varepsilon_0}\left(1+\frac{A}{\varepsilon a}\right)\left(1+\frac{A}{4\varepsilon a}\right)^{\frac{\alpha}{\beta}}. \tag{4.24}$$

Since by (4.17) we have $A \le \varepsilon a$, the inequality (4.24) will follow from

$$\left(\ln\frac{1+\varepsilon}{4\varepsilon}\right)^2 \ge \frac{4c}{\varepsilon_0}\left(\frac{5}{4}\right)^{\alpha/\beta},$$

and the latter can be achieved by choosing ε small enough. With this choice of ε we conclude that (4.21) holds, which implies

$$\operatorname*{ess\,inf}_{\frac{1}{2}B} u \ge \frac{q}{2} - \lambda = 2\varepsilon a - \varepsilon a = \varepsilon a,$$

thus finishing the proof. □

4.3. **Oscillation inequalities.** In this section we frequently use the notation $B_r := B(x_0, r)$ assuming that x_0 is a fixed point on M.

Lemma 4.6 (Oscillation Inequality)**.** *Assume that* (V), (J) *and* (AB) *are satisfied. Let* $u \in \mathcal{F}' \cap L^\infty$ *be harmonic in a ball* $B_R = B(x_0, R)$ *with* $R < \sigma\overline{R}$. *Then, there is a constant* $\varepsilon \in (0,1)$ *depending only on the constants from the hypotheses and such that, either*

$$\operatorname*{osc}_{B_{R/4}} u \le (1-\varepsilon) \operatorname*{osc}_{B_R} u, \tag{4.25}$$

or

$$\operatorname*{osc}_{B_R} u \le \frac{1}{\varepsilon} A, \tag{4.26}$$

where

$$A = R^\beta T_{B_R}((u-m)_- + (M-u)_-),$$

$m = \operatorname{ess\,inf}_{B_R} u$ *and* $M = \operatorname{ess\,sup}_{B_R} u$.

Proof. Set $a = \frac{M-m}{2}$. Consider the function $u - m \in \mathcal{F}' \cap L^\infty$, which is non-negative and harmonic in B_R. By the weak Harnack inequality of Lemma 4.5, there is a constant $\varepsilon' \in (0,1)$ such that if

$$\frac{\mu(B_{R/2} \cap \{u - m \geq a\})}{\mu(B_{R/2})} \geq \frac{1}{2}, \tag{4.27}$$

and

$$A_1 := R^\beta T_{B_R}((u-m)_-) \leq \varepsilon' a, \tag{4.28}$$

then,

$$\operatorname*{ess\,inf}_{B_{R/4}}(u-m) \geq \varepsilon' a.$$

The latter implies that

$$\operatorname*{osc}_{B_{R/4}} u = \operatorname*{osc}_{B_{R/4}} (u-m) \leq (M-m) - \varepsilon' a = \left(1 - \frac{\varepsilon'}{2}\right)(M-m) = \left(1 - \frac{\varepsilon'}{2}\right) \operatorname*{osc}_{B_R} u,$$

that is (4.25) with $\varepsilon = \varepsilon'/2$. Similarly, if

$$\frac{\mu(B_{R/2} \cap \{M - u \geq a\})}{\mu(B_{R/2})} \geq \frac{1}{2}, \tag{4.29}$$

and

$$A_2 := R^\beta T_{B_R}((M-u)_-) \leq \varepsilon' a, \tag{4.30}$$

then

$$\operatorname*{ess\,inf}_{B_{R/4}}(M-u) \geq \varepsilon' a,$$

and, hence,

$$\operatorname*{osc}_{B_{R/4}} u \leq M - m - \varepsilon' a = \left(1 - \frac{\varepsilon'}{2}\right)(M-m) = \left(1 - \frac{\varepsilon'}{2}\right) \operatorname*{osc}_{B_R} u,$$

that is (4.25) with $\varepsilon = \varepsilon'/2$. Since

$$u - m \geq a \quad \Leftrightarrow \quad u \geq \frac{M+m}{2},$$
$$M - u \geq a \quad \Leftrightarrow \quad u \leq \frac{M+m}{2},$$

we see that either (4.27) or (4.29) is always satisfied. Hence, if both (4.28) and (4.30) are satisfied, then we obtain (4.25). On the other hand, if one of (4.28) and (4.30) is not satisfied, then

$$A = A_1 + A_2 \geq \varepsilon' a = \varepsilon' \frac{M-m}{2} = \frac{\varepsilon'}{2} \operatorname*{osc}_{B_R} u,$$

which is equivalent to (4.26) with $\varepsilon = \varepsilon'/2$. □

Lemma 4.7 (Iterated Oscillation Inequality)**.** *Assume that* (V), (J) *and* (AB) *are satisfied. Let* $u \in \mathcal{F}' \cap L^\infty$ *be harmonic in a ball* $B_R = B(x_0, R)$ *with* $R < \sigma \overline{R}$. *Set* $R_k := q^{-k} R$, *where* $q \geq 4$, $k \geq 0$ *and*

$$Q_k := \operatorname*{osc}_{B_{R_k}} u.$$

If q *is large enough then, for all* $k \geq 0$,

$$Q_k \leq C_0 q^{-\gamma k} A, \tag{4.31}$$

where

$$A := R^\beta T_{B_R}(|u|) + \|u\|_{L^\infty(B_R)},$$

and the constants C_0, γ, q *depend only on the constants in the hypotheses.*

Proof. We prove (4.31) by induction in k. For $k=0$ and $k=1$ it is trivial, because

$$Q_1 \le Q_0 = \operatorname*{osc}_{B_R} u \le 2\left\|u\right\|_{L^\infty(B_R)} \le 2A = 2q^\gamma\left(q^{-\gamma}A\right),$$

so that (4.31) holds provided $C_0 \ge 2q^\gamma$.

Assuming that $k\ge 1$, let us make the induction step from $\le k$ to $k+1$. Since $q\ge 4$, we can apply Lemma 4.6 and obtain the following:

$$Q_{k+1} \le (1-\varepsilon)Q_k, \quad \text{or} \quad Q_k \le \varepsilon^{-1}A_k,$$

where $\varepsilon \in (0,1)$ is the constant from Lemma 4.6,

$$A_k := R_k^\beta T_{B_{R_k}}((u-m_k)_- + (M_k-u)_-),$$

and

$$m_k := \operatorname*{ess\,inf}_{B_{R_k}} u, \quad M_k := \operatorname*{ess\,sup}_{B_{R_k}} u.$$

In the first case, that is, when

$$Q_{k+1} \le (1-\varepsilon)Q_k,$$

we obtain by induction hypothesis

$$Q_{k+1} \le (1-\varepsilon)C_0 q^{-\gamma k}A = (1-\varepsilon)q^\gamma C_0 q^{-\gamma(k+1)}A \le C_0 q^{-\gamma(k+1)}A,$$

provided

$$(1-\varepsilon)q^\gamma \le 1. \tag{4.32}$$

Below we will make sure that (4.32) is satisfied as follows: we will first determine (a large) q and then specify γ to be small enough (and then choose C_0 large enough).

Consider now the second case when

$$Q_k \le \varepsilon^{-1}A_k. \tag{4.33}$$

We will show that, by choosing suitable values of $q,\ C_0,\ \gamma$ we can ensure that

$$Q_k \le C_0 q^{-(k+1)\gamma}A, \tag{4.34}$$

which will imply the same estimate for $Q_{k+1}\le Q_k$, thus finishing the induction step.

For that, set $v := (u-m_k)_- + (M_k-u)_-$ and estimate the quantity $A_k = R_k^\beta T_{B_{R_k}}(v)$ from above by using the induction hypothesis

$$Q_j \le C_0 q^{-\gamma j}A, \quad j=0,1,\cdots,k. \tag{4.35}$$

Decompose $T_{B_{R_k}}(v)$ as follows:

$$\begin{aligned} T_{B_{R_k}}(v) &= \int_{B_{R_k}^c} v(y)J(x_0,y)d\mu(y) \\ &= \sum_{i=0}^{k-1}\int_{B_{R_i}\setminus B_{R_{i+1}}} v(y)J(x_0,y)d\mu(y) + \int_{B_R^c} v(y)J(x_0,y)d\mu(y) \end{aligned} \tag{4.36}$$

(see Fig. 6).

Observe that the following inequality holds in B_{R_i} with $i\le k$:

$$v = (u-m_k)_- + (M_k-u)_- \le Q_i - Q_k. \tag{4.37}$$

Indeed, if $m_k \le u \le M_k$ then $v=0$ and (4.37) is trivial. If $u<m_k$ then we have in B_{R_i}

$$v = m_k - u \le m_k - m_i \le m_k - m_i + M_i - M_k = Q_i - Q_k,$$

and the same argument works if $u>M_k$:

$$v = u - M_k \le M_i - M_k \le Q_i - Q_k.$$

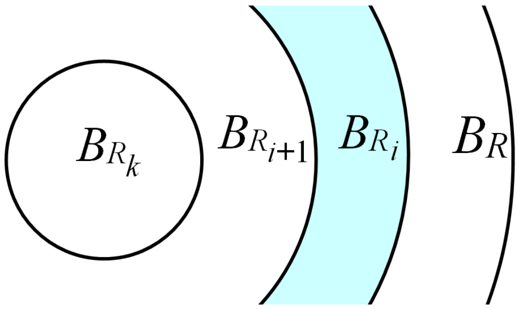

FIGURE 6. Balls B_{R_i}

Using $(J_{\leq})$, $(V_{\leq})$ and (6.8), we obtain, for any $i \leq k$,

$$\int_{B_{R_i}\setminus B_{R_{i+1}}} v(y)J(x_0,y)d\mu(y) \leq (Q_i - Q_k)\int_{B^c_{R_{i+1}}} J(x_0,y)d\mu(y) \leq \frac{c(Q_i-Q_k)}{R^\beta_{i+1}},$$

where c is the constant from (6.8); we take $c > 1$.

On the other hand, since everywhere

$$v \leq |u| + \max\left(|m_k|, |M_k|\right) \leq |u| + \|u\|_{L^\infty(B_R)},$$

we obtain

$$\begin{aligned}\int_{B^c_R} v\left(y\right)J(x_0,y)d\mu(y) &\leq \int_{B^c_R}(|u(y)| + \|u\|_{L^\infty(B_R)})J(x_0,y)d\mu(y)\\ &\leq T_{B_R}(|u|) + \frac{c\,\|u\|_{L^\infty(B_R)}}{R^\beta} \leq \frac{cA}{R^\beta}.\end{aligned}$$

Hence, we obtain from (4.36), that

$$T_{B_{R_k}}(v) \leq c\sum_{i=0}^{k-1}\frac{Q_i-Q_k}{R^\beta_{i+1}} + \frac{cA}{R^\beta},$$

which implies that

$$\begin{aligned}A_k = R^\beta_k T_{B_{R_k}}(v) &\leq c\sum_{i=0}^{k-1}\frac{R^\beta_k}{R^\beta_{i+1}}(Q_i-Q_k) + c\left(\frac{R_k}{R}\right)^\beta A\\ &= c\sum_{i=0}^{k-1} q^{\beta(i+1-k)}(Q_i-Q_k) + cq^{-k\beta}A.\end{aligned}$$

Assuming that $\gamma < \beta$ and using the induction hypothesis (4.35), we obtain

$$\begin{aligned}\sum_{i=0}^{k-1} q^{\beta(i+1-k)}Q_i &\leq \sum_{i=0}^{k-1} q^{\beta(i+1-k)}\cdot C_0 q^{-\gamma i}A\\ &= C_0 A q^{-(k-1)\gamma}\sum_{i=0}^{k-1} q^{(\beta-\gamma)(i+1-k)}\\ &= C_0 A q^{-(k-1)\gamma}\sum_{j=0}^{k-1} q^{-(\beta-\gamma)j}\\ &\leq C_0 A\frac{q^{-(k-1)\gamma}}{1-q^{-(\beta-\gamma)}}.\end{aligned}$$

On the other hand, since $k \geq 1$, we have

$$\sum_{i=0}^{k-1} q^{\beta(i+1-k)} \geq 1.$$

It follows that

$$A_k \leq cC_0 A \frac{q^{-(k-1)\gamma}}{1-q^{-(\beta-\gamma)}} - cQ_k + cq^{-k\beta} A.$$

Substituting into (4.33), we obtain

$$Q_k \leq \frac{A_k}{\varepsilon} \leq \frac{cC_0 A}{\varepsilon} \frac{q^{-(k-1)\gamma}}{1-q^{-(\beta-\gamma)}} - \frac{c}{\varepsilon} Q_k + \frac{c}{\varepsilon} q^{-k\beta} A,$$

which is equivalent to

$$Q_k \leq \frac{c}{c+\varepsilon} \left(C_0 \frac{q^{-(k-1)\gamma}}{1-q^{-(\beta-\gamma)}} + q^{-k\beta} \right) A.$$

To ensure (4.34), it suffices to have

$$\frac{c}{c+\varepsilon} \left(C_0 \frac{q^{-(k-1)\gamma}}{1-q^{-(\beta-\gamma)}} + q^{-k\beta} \right) A \leq C_0 q^{-(k+1)\gamma} A,$$

which is equivalent to

$$\frac{q^{2\gamma}}{1-q^{-(\beta-\gamma)}} + \frac{1}{C_0} q^{\gamma - k(\beta-\gamma)} \leq 1 + \frac{\varepsilon}{c}. \tag{4.38}$$

Let us now choose $q \geq 4$ big enough so that

$$\frac{1}{1-q^{-\frac{\beta}{2}}} < 1 + \frac{\varepsilon}{c}.$$

Then we choose $\gamma \in (0, \beta/2)$ small enough such that (4.32) is true and

$$\frac{q^{2\gamma}}{1-q^{-\frac{\beta}{2}}} < 1 + \frac{\varepsilon}{c}.$$

Since $\beta - \gamma > \beta/2$, it follows that also

$$\frac{q^{2\gamma}}{1-q^{-(\beta-\gamma)}} < 1 + \frac{\varepsilon}{c}.$$

Finally, we choose C_0 so big that (4.38) is satisfied, which finishes the proof. □

Lemma 4.8 (Oscillation Lemma)**.** *Assume that (V), (J) and (AB) are satisfied. Let $u \in \mathcal{F}' \cap L^\infty$ be harmonic in a ball $B_R = B(x_0, R)$ with $R < \sigma \overline{R}$. Then, for any $\rho \in (0, R]$,*

$$\operatorname*{osc}_{B_\rho} u \leq C \left(\frac{\rho}{R}\right)^\gamma \left(R^\beta T_{B_R}(|u|) + \|u\|_{L^\infty(B_R)} \right), \tag{4.39}$$

where $\gamma > 0$ is the constant from Lemma 4.7 and C depends only on the constants from the hypotheses.

Proof. We use the notation from Lemma 4.7. Since $\rho \in (0, R]$, there exists an integer $k \geq 0$ such that

$$q^{-(k+1)} < \frac{\rho}{R} \leq q^{-k}.$$

Hence, by Lemma 4.7,

$$\operatorname*{osc}_{B_\rho} u \leq \operatorname*{osc}_{B_{q^{-k}R}} u \leq C_0 \left(q^{-k}\right)^\gamma A = C_0 q^\gamma \left(q^{-(k+1)}\right)^\gamma A \leq C_0 q^\gamma \left(\frac{\rho}{R}\right)^\gamma A,$$

which is exactly (4.39) with $C = C_0 q^\gamma$. □

5. Heat semigroup and heat kernel

In this section we develop techniques for the proof of the implication (2.24), that is,

$$(V)+(J)+(AB)\Rightarrow(S)+(NLE)\,,$$

which will conclude the proof of Theorem 2.10.

5.1. Green operator and conditions (E) and (S).

The main result of this Section is Corollary 5.7 containing the implication

$$(V)+(J)+(AB)\Rightarrow(S)\,.$$

The proof uses condition (E) stated below in terms of the *Green operator*.

For any open set $\Omega\subset M$, the heat semigroup $\left\{P_t^\Omega\right\}$ was defined in Section 2.3. Note that, for any $f\in L^2(\Omega)$, the function $t\mapsto P_t^\Omega f$ is continuous as a mapping from $[0,\infty)$ to $L^2(\Omega)$, which allows to integrate $P_t^\Omega f$ in t as an L^2-valued function. Define the *Green operator* G^Ω by

$$G^\Omega f:=\int_0^\infty P_t^\Omega f\,dt,$$

where f so far is any non-negative function from $L^2(\Omega)$. The function $G^\Omega f$ takes values in $[0,\infty]$. The monotonicity of $G^\Omega f$ in f allows to extend this operator to all non-negative $f\in L^2_{loc}(\Omega)$, in particular, to $f\equiv1$.

By [26, Lemma 3.2, p.1232][1], if $G^\Omega 1\in L^\infty(\Omega)$ then G^Ω can be extended to a bounded operator on $L^2(\Omega)$ that satisfies the identity $G^\Omega=\left(\mathcal{L}^\Omega\right)^{-1}$.

Lemma 5.1. *If $G^\Omega 1\in L^\infty(\Omega)$ then, for any $f\in L^2(\Omega)$, the function $u=G^\Omega f$ belongs to $\mathcal{F}(\Omega)$ and satisfies the identity*

$$\mathcal{E}(u,\varphi)=(f,\varphi)\quad\forall\varphi\in\mathcal{F}(\Omega)\,.$$

If in addition $f\geq0$ then u is superharmonic in Ω.

Proof. Indeed, for any non-negative $\varphi\in\mathcal{F}(\Omega)$, we have

$$\mathcal{E}(u,\varphi)=\mathcal{E}\left(G^\Omega f,\varphi\right)=\left(\mathcal{L}^\Omega G^\Omega f,\varphi\right)=(f,\varphi)\,,$$

where we have used that $\mathcal{L}^\Omega G^\Omega=Id$. Consequently, if $f\geq0$ then $\mathcal{E}(u,\varphi)\geq0$ for any non-negative $\varphi\in\mathcal{F}(\Omega)$ which means that u is superharmonic in Ω. □

Definition 5.2 (Condition (E))**.** We say that condition $(E_\leq)$ holds if there exist $\epsilon\in(0,1)$ and $C>0$ such that, for any ball $B=B(x_0,R)$ of radius $R\in(0,\epsilon\overline{R})$,

$$\operatorname*{ess\,sup}_B G^B 1\leq CR^\beta. \tag{5.1}$$

We say that condition $(E_\geq)$ holds if, for any ball B of radius $R\in(0,\overline{R})$,

$$\operatorname*{ess\,inf}_{\frac14 B} G^B 1\geq C^{-1}R^\beta. \tag{5.2}$$

We say that condition (E) holds if both $(E_\leq)$ and $(E_\geq)$ are satisfied.

Remark 5.3. Using the monotonicity of G^B1 in B, it is easy to prove that the condition $(E_\geq)$ is equivalent to the following: there exist $\epsilon,\delta\in(0,1)$ such that, for any ball of radius $R\in\left(0,\epsilon\overline{R}\right)$,

$$\operatorname*{ess\,inf}_{\delta B} G^B 1\geq C^{-1}R^\beta.$$

Lemma 5.4. Let $(\mathcal{E},\mathcal{F})$ be a regular non-local Dirichlet form with a jump kernel J. Then

$$(V)+(J_\geq)\Rightarrow(E_\leq)\,.$$

[1] Although this lemma was stated for local Dirichlet forms, its proof goes through also for general Dirichlet forms.

Proof. Fix a ball $B(x_0, R)$ of radius $R < \overline{R}$ and first assume that there exists another ball $B(y_0, R)$ of the same radius such that $\overline{B(x_0, R)}$ and $\overline{B(y_0, R)}$ are disjoint. By [17, Lemma 4.4], for any $t > 0$ and for any non-negative function $f \in L^1 \cap L^2(M)$, we have the following inequality:

$$\left(1 - P_t^V 1, f\right) \geq 2\mu(B(y_0, R)) \inf_{\substack{x \in B(x_0,R), \\ y \in B(y_0,R)}} J(x, y) \int_0^t \left(f, P_s^V 1_B\right) ds \tag{5.3}$$

where $B = B(x_0, R)$ and $V = M \setminus \overline{B(y_0, R)}$. Observing that

$$\left(1 - P_t^V 1, f\right) \leq \|f\|_{L^1},$$

and letting in (5.3) $t \to \infty$, we obtain

$$\|f\|_{L^1} \geq 2\mu(B(y_0, R)) \inf_{\substack{x \in B(x_0,R), \\ y \in B(y_0,R)}} J(x, y) \left(f, G^B 1_B\right).$$

Since f is arbitrary, it follows that

$$\operatorname*{ess\,sup}_B G^B 1_B \leq \left[2\mu(B(y_0, R)) \inf_{\substack{x \in B(x_0,R), \\ y \in B(y_0,R)}} J(x, y)\right]^{-1}.$$

Assume in addition that

$$d(x_0, y_0) \leq CR. \tag{5.4}$$

Then by $(J_\geq)$

$$\inf_{\substack{x \in B(x_0,R), \\ y \in B(y_0,R)}} J(x, y) \geq cR^{-(\alpha+\beta)},$$

and by $(V_\geq)$

$$\mu(B(y_0, R)) \geq cR^\alpha,$$

whence it follows that

$$\operatorname*{ess\,sup}_B G^B 1_B \leq CR^\beta.$$

Hence, in order to prove $(E_\leq)$, we should, for any ball $B = B(x_0, R)$ of radius $R < \epsilon\overline{R}$, find a ball $B(y_0, R)$ such that the balls $\overline{B(x_0, R)}$ and $\overline{B(y_0, R)}$ are disjoint and (5.4) is satisfied.

To that end, observe the following consequence of the condition (V). Let C be the constant from (V). Then, for any $r \in \left(0, \overline{R}\right)$, $\lambda \in (0, 1)$ and $x \in M$, we have

$$\mu(B(x, \lambda r)) \leq C\lambda^\alpha r^\alpha \quad \text{and} \quad \mu(B(x, r)) \geq C^{-1} r^\alpha,$$

which implies

$$\mu(B(x, \lambda r)) < \mu(B(x, r))$$

provided λ is small enough, for example, $\lambda = (2C)^{-2/\alpha}$. For this λ, the annulus $B(x, r) \setminus B(x, \lambda r)$ is non-empty.

Set $\epsilon = \lambda/3$. Then $R < \epsilon\overline{R}$ implies that $r := 3\lambda^{-1}R < \overline{R}$, and we obtain that the annulus $B\left(x_0, 3\lambda^{-1}R\right) \setminus B(x_0, 3R)$ is non-empty. Let y_0 be any point from this annulus. Then the balls $\overline{B(x_0, R)}$ and $\overline{B(y_0, R)}$ are disjoint and (5.4) holds, which finishes the proof of $(E_\leq)$. □

Lemma 5.5. *Let $(\mathcal{E}, \mathcal{F})$ be a regular non-local Dirichlet form with jump kernel J. Then,*

$$(V) + (J) + (AB) \Rightarrow (E_\geq).$$

Proof. By Lemma 5.4, under the present hypotheses we have $(E_{\leq})$ with some $\epsilon > 0$. Without loss of generality, we can assume that $\epsilon \leq \sigma/2$ where σ is the parameter from the Faber-Krahn inequality that was used in Corollary 4.4.

Fix a ball $B = B(x_0, R)$ with $R \in (0, \epsilon\overline{R})$ and set $u = G^B 1$. It suffices to prove that

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} u \geq cR^{\beta}. \tag{5.5}$$

By $(E_{\leq})$ we have $G^B 1 \in L^{\infty}$. Hence, by Lemma 5.1, the function $u = G^B 1$ is superharmonic in B. Applying Corollary 4.4 to function u in B (instead of $2B$) and observing that $T_B(u_-) = 0$, we obtain that

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} u \geq c \left(\fint_{\frac{1}{2}B} \frac{1}{u} d\mu \right)^{-1}. \tag{5.6}$$

On the other hand, by Corollary 2.6, the condition (cap) holds under the present hypotheses. By (cap) and $(V_{\leq})$, there exists $\phi \in \operatorname{cutoff}(B_{R/2}, B_{3R/4})$ such that $\phi \in \mathcal{F}(B)$ and

$$\mathcal{E}(\phi, \phi) \leq CR^{\alpha-\beta}.$$

For any $\lambda > 0$, we have by Proposition 6.5(iii) that $\frac{\phi^2}{u+\lambda} \in \mathcal{F}(B)$, whence by Lemma 5.1

$$\int_{B_{R/2}} \frac{1}{u+\lambda} d\mu \leq (1_B, \frac{\phi^2}{u+\lambda}) = \mathcal{E}(G^B 1_B, \frac{\phi^2}{u+\lambda}) = \mathcal{E}(u, \frac{\phi^2}{u+\lambda}).$$

By Lemma 3.7, we obtain

$$\mathcal{E}(u, \frac{\phi^2}{u+\lambda}) \leq 3\mathcal{E}(\phi, \phi) \leq C' R^{\alpha-\beta}.$$

Combining the two previous lines, passing to the limit as $\lambda \to 0$ and using $(V_{\geq})$, we obtain

$$\fint_{B_{R/2}} \frac{1}{u} d\mu \leq CR^{-\beta}.$$

Finally, substituting this into (5.6), we obtain (5.5). □

Lemma 5.6. Let $(\mathcal{E}, \mathcal{F})$ be a regular non-local Dirichlet form with the jump kernel J. Then $(E) \Rightarrow (S)$.

Proof. Recall that (S) means the following: for any ball $B = B(x_0, R)$ with $R \in \left(0, \overline{R}\right)$,

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} P_t^B 1 \geq \varepsilon, \tag{5.7}$$

provided $t^{1/\beta} \leq \delta R$.

We first prove (5.7) assuming $R < \epsilon\overline{R}$ where ϵ is the parameter from $(E_{\leq})$. The ball B can be exhausted by an increasing family of precompact open sets $\{\Omega_n\}_{n=1}^{\infty}$ such that $\overline{\Omega}_n \subset \Omega_{n+1}$ for each $n \geq 1$. Since $(\mathcal{E}, \mathcal{F})$ is regular, for each $n \geq 1$, there is a cutoff function of the pair (Ω_n, M). On the other hand, the function $G^B 1$ is bounded by $(E_{\leq})$. Hence, it follows from the arguments in the proof of [22, (6.34) p.6429] that, for all $t > 0$ and for μ-a.a. $x \in \Omega_n$,

$$P_t^B 1(x) \geq \frac{G^B 1(x) - t}{\|G^B 1\|_{L^{\infty}}}. \tag{5.8}$$

Since $n \geq 1$ is arbitrary, this inequality holds also for μ-a.a. $x \in B$. By (E) we have

$$\left\| G^B 1 \right\|_{L^{\infty}} \leq CR^{\beta}$$

and

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} G^B 1 \geq C^{-1} R^{\beta}.$$

Substituting into (5.8) and assuming that $t \le R^\beta/(2C)$, we obtain

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} P_t^B 1 \ge \frac{C^{-1}R^\beta - t}{CR^\beta} \ge \frac{C^{-1}R^\beta/2}{CR^\beta} = \frac{1}{2C^2}, \tag{5.9}$$

which proves (5.7) in the case $R < \epsilon\overline{R}$.

Now let us prove (5.7) for any $R \in (0, \overline{R})$. Assume without loss of generality that $\epsilon < \frac{1}{2}$. Since M is separable, the ball $\frac{1}{4}B$ can be covered by at most countable family of balls $B(x_i, \frac{\epsilon}{4}R)$ where $x_i \in \frac{1}{4}B$. Applying (5.9) to each ball $B_i = B(x_i, \epsilon R)$, we obtain that if $t \le (\epsilon R)^\beta / 2C$ then

$$\operatorname*{ess\,inf}_{\frac{1}{4}B_i} P_t^{B_i} 1 \ge \frac{1}{2C^2}.$$

Since $P_t^B 1 \ge P_t^{B_i} 1$ and the union of $\frac{1}{4}B_i$ covers $\frac{1}{4}B$, we obtain

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} P_t^B 1 \ge \frac{1}{2C^2},$$

which proves (5.7) with $\varepsilon = \left(2C^2\right)^{-1}$ and $\delta = \epsilon \left(2C\right)^{-1/\beta}$. □

Corollary 5.7. *Let $(\mathcal{E}, \mathcal{F})$ be a regular non-local Dirichlet form with the jump kernel J. Then*

$$(V) + (J) + (AB) \Rightarrow (S).$$

Proof. Indeed, this implication follows from the three previous Lemmas 5.4, 5.5 and 5.6. □

5.2. **Oscillation inequality for $\mathcal{L}u = f$.** We use here the results of Section 4.3 in order to prove the existence of a Hölder continuous heat kernel of $(\mathcal{E}, \mathcal{F})$, under appropriate hypotheses. For a non-empty open set $\Omega \subset M$ and $f \in L^2(\Omega)$, we say that a function $u \in \mathcal{F}$ solves weakly the equation

$$\mathcal{L}u = f \text{ in } \Omega,$$

if, for any $\phi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u, \phi) = (f, \phi).$$

Proposition 5.8. *Let $u \in \mathcal{F}$ solves the equation $\mathcal{L}u = f$ weakly in Ω for some $f \in L^2(\Omega)$. Let $B \subset \Omega$ be an open subset.*

(*a*) *If $v \in \mathcal{F}$ solves the equation $\mathcal{L}v = f$ weakly in B, then $u - v$ is harmonic in B.*

(*b*) *If $\left\|G^B 1\right\|_{L^\infty} < \infty$ then $u - G^B f$ is harmonic in B.*

Proof. (*a*) By definition of a weak solution, we have, for any $\phi \in \mathcal{F}(B) \subset \mathcal{F}(\Omega)$,

$$\mathcal{E}(u, \phi) = (f, \phi) \quad \text{and} \quad \mathcal{E}(v, \phi) = (f, \phi),$$

which implies that

$$\mathcal{E}(u - v, \phi) = 0.$$

Hence, $u - v$ is harmonic in B.

(*b*) If $\left\|G^B 1\right\|_{L^\infty} < \infty$ then, by Lemma 5.1, the function $v = G^B f$ belongs to $\mathcal{F}(B)$ and satisfies $\mathcal{E}(v, \phi) = (f, \phi)$ for any $\phi \in \mathcal{F}(B)$, that is, v solves the equation $\mathcal{L}v = f$ weakly in B. Hence, we conclude by (*a*) that $u - G^B f$ is harmonic in B. □

Lemma 5.9. *Assume that (V), (J) and (AB) are satisfied. Let Ω be any open subset of M containing a ball $B := B(x_0, r) \subset \Omega$ of radius $r \in (0, \sigma\overline{R})$, where $\sigma \in (0, 1)$ depends on the constants from the present hypotheses. If $f \in L^2 \cap L^\infty(\Omega)$ and if $u \in \mathcal{F}(\Omega) \cap L^\infty$ solves the equation $\mathcal{L}u = f$ weakly in Ω, then, for any $0 < \rho \le r$,*

$$\operatorname*{osc}_{B(x_0,\rho)} u \le C\left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(\Omega)} + Cr^\beta \|f\|_{L^\infty(B)}, \tag{5.10}$$

where γ is the constant from Lemma 4.8 and C depends only on the constants in the conditions (V), (J), (AB).

Proof. So far we have denoted by σ the parameter from the Faber-Krahn inequality (cf. Lemmas 3.5 and 4.8). Let us reduce the value of σ to ensure that $\sigma \le \epsilon$ where ϵ is the parameter from $(E_\le)$ (cf. Lemma 5.4), and use in what follows the reduced value of σ.

By $(E_\le)$ and $r < \sigma\overline{R}$ we have $\left\|G^B 1\right\|_{L^\infty} < \infty$. Consider the function

$$v := u - G^B f,$$

that by Proposition 5.8 is harmonic in B. Besides, $v \in \mathcal{F}(\Omega) \cap L^\infty$. Hence, by Lemma 4.8,

$$\operatorname*{osc}_{B(x_0,\rho)} v \le C\left(\frac{\rho}{r}\right)^\gamma \left(r^\beta T_B(v) + \|v\|_{L^\infty(B)}\right). \tag{5.11}$$

Since $u = 0$ in Ω^c, by $(J_\le)$, $(V_\le)$ and (6.8), we obtain

$$\begin{aligned} T_B(v) &\le \int_{B^c} J(x_0,y)(|u(y)| + G^B f)d\mu(y) \\ &\le \left(\|u\|_{L^\infty(\Omega)} + \left\|G^B f\right\|_{L^\infty(\Omega)}\right)\int_{B^c} J(x_0,y)d\mu(y) \\ &\le C\left(\|u\|_{L^\infty(\Omega)} + \left\|G^B f\right\|_{L^\infty(B)}\right) r^{-\beta}. \end{aligned}$$

Substituting this estimate into (5.11) and using that

$$\|v\|_{L^\infty(B)} \le \|u\|_{L^\infty(B)} + \left\|G^B f\right\|_{L^\infty(B)},$$

we obtain

$$\operatorname*{osc}_{B(x_0,\rho)} v \le C\left(\frac{\rho}{r}\right)^\gamma \left(\|u\|_{L^\infty(\Omega)} + \left\|G^B f\right\|_{L^\infty(B)}\right). \tag{5.12}$$

By $(E_\le)$ we have

$$\left\|G^B f\right\|_{L^\infty(B)} \le \left\|G^B 1\right\|_{L^\infty} \|f\|_{L^\infty(B)} \le C r^\beta \|f\|_{L^\infty(B)}.$$

Combining this with (5.12), we obtain

$$\begin{aligned} \operatorname*{osc}_{B(x_0,\rho)} u &\le \operatorname*{osc}_{B(x_0,\rho)} v + \operatorname*{osc}_{B(x_0,\rho)} G^B f \\ &\le C\left(\frac{\rho}{r}\right)^\gamma \left(\|u\|_{L^\infty(\Omega)} + \left\|G^B f\right\|_{L^\infty(B)}\right) + 2\left\|G^B f\right\|_{L^\infty(B)} \\ &\le C\left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(\Omega)} + C r^\beta \|f\|_{L^\infty(B)}, \end{aligned}$$

which proves (5.10). □

5.3. Estimates for the heat semigroup.

Lemma 5.10. *Suppose (V) and $(J_\ge)$ are satisfied. Let Ω be an open subset of M. Fix $f \in L^1 \cap L^2(\Omega)$ and set $u = P_t^\Omega f$. Then u satisfies the following inequalities, for any $t > 0$:*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C\frac{e^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\|f\|_{L^1(\Omega)}, \tag{5.13}$$

and

$$\|\partial_t u(\cdot,t)\|_{L^\infty(\Omega)} \le C\frac{e^{\overline{R}^{-\beta}t}}{t^{1+\alpha/\beta}}\|f\|_{L^1(\Omega)}, \tag{5.14}$$

where $\partial_t u(\cdot,t)$ is the Fréchet derivative of the $L^2(\Omega)$-valued function $t \mapsto u(\cdot,t)$. Besides, for all $t > s > \tau > 0$,

$$\|u(\cdot,t) - u(\cdot,s)\|_{L^\infty(\Omega)} \le C(t-s)\frac{e^{\overline{R}^{-\beta}\tau}}{\tau^{1+\alpha/\beta}}\|f\|_{L^1(\Omega)}. \tag{5.15}$$

The constant C depends only on the constants in the hypotheses (V) and $(J_\le)$.

Proof. By (V), $(J_{\geq})$ and Lemma 3.5, we have the Nash inequality $(Nash)$ for $(\mathcal{E},\mathcal{F})$. Therefore, we have the Nash inequality also for $(\mathcal{E},\mathcal{F}(\Omega))$. Since in what follows we will use only the Nash inequality, we can assume without loss of generality and for the sake of simplicity of notations that $\Omega = M$, so that $u = P_t f$. Applying [9, Theorem 2.1], we obtain

$$\|P_t\|_{L^1\to L^\infty} \leq \frac{Ce^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}, \tag{5.16}$$

whence

$$\|P_t f\|_{L^\infty} \leq \frac{Ce^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\|f\|_{L^1},$$

which proves (5.13).

To prove (5.14), let us first obtain an upper bound of $\|P_t\|_{L^1\to L^2}$. By Markov property, P_t is a contraction in L^1, that is,

$$\|P_t g\|_{L^1} \leq \|g\|_{L^1} \text{ for all } g\in L^1. \tag{5.17}$$

Using (5.16) and (5.17), we obtain, for any $h\in L^1\cap L^2$ and $t>0$,

$$\begin{aligned}\sup_{\|g\|_{L^1}=1}|(P_t h,g)| &= \sup_{\|g\|_{L^1}=1}|(h,P_t g)| \leq \|h\|_{L^2}\sup_{\|g\|_{L^1}=1}\|P_t g\|_{L^2}\\ &\leq \|h\|_{L^2}\sup_{\|g\|_{L^1}=1}\sqrt{\|P_t g\|_{L^\infty}\|P_t g\|_{L^1}}\\ &\leq \|h\|_{L^2}\sup_{\|g\|_{L^1}=1}\sqrt{\frac{Ce^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\|g\|_{L^1}\|g\|_{L^1}}\\ &= \sqrt{C}\frac{e^{\overline{R}^{-\beta}t/2}}{t^{\alpha/(2\beta)}}\|h\|_{L^2}.\end{aligned}$$

It follows that

$$\|P_t\|_{L^2\to L^\infty} \leq \sqrt{C}\frac{e^{\overline{R}^{-\beta}t/2}}{t^{\alpha/(2\beta)}}, \tag{5.18}$$

whence by the duality

$$\|P_t\|_{L^1\to L^2} = \|P_t\|_{L^2\to L^\infty} \leq \sqrt{C}\frac{e^{\overline{R}^{-\beta}t/2}}{t^{\alpha/(2\beta)}}. \tag{5.19}$$

Since, for any $s\in(0,t)$,

$$P_t f = P_s P_{t-s} f$$

and P_s is a bounded operator in L^2, we obtain

$$\partial_t(P_t f) = P_s(\partial_t P_{t-s} f).$$

Using this identity and the following inequality (see [26, Lemma 5.4])

$$\|\partial_s(P_s f)\|_{L^2} \leq \frac{2}{s}\|P_{s/2}f\|_{L^2}, \tag{5.20}$$

we obtain

$$\begin{aligned}\|\partial_t(P_t f)\|_{L^\infty} &= \|P_s\partial_t(P_{t-s}f)\|_{L^\infty}\\ &\leq \|P_s\|_{L^2\to L^\infty}\|\partial_t(P_{t-s}f)\|_{L^2}\\ &\leq \|P_s\|_{L^2\to L^\infty}\frac{2}{t-s}\|P_{(t-s)/2}f\|_{L^2}\\ &\leq \frac{2}{t-s}\|P_s\|_{L^2\to L^\infty}\|P_{(t-s)/2}\|_{L^1\to L^2}\|f\|_{L^1}.\end{aligned}$$

Setting here $s=t/2$ and using (5.18), (5.19), we obtain

$$\|\partial_t(P_t f)\|_{L^\infty} \leq C\frac{e^{\overline{R}^{-\beta}t}}{t^{1+\alpha/\beta}}\|f\|_{L^1},$$

which proves (5.14).

Finally, let us prove (5.15). For simplicity let us rename τ into 2τ, so that $t > s > 2\tau$. We have

$$\|P_t f - P_s f\|_{L^\infty} = \|P_\tau (P_{t-\tau} f - P_{s-\tau} f)\|_{L^\infty} \le \|P_\tau\|_{L^2 \to L^\infty} \|P_{t-\tau} f - P_{s-\tau} f\|_{L^2} .$$

Next, using (5.20), we obtain

$$\begin{aligned} \|P_{t-\tau} f - P_{s-\tau} f\|_{L^2} &= \left\| \int_{s-\tau}^{t-\tau} \partial_\xi (P_\xi f)\, d\xi \right\|_{L^2} \le \int_{s-\tau}^{t-\tau} \frac{2}{\xi} \left\|P_{\xi/2} f\right\|_{L^2} d\xi \\ &\le (t-s) \frac{2}{\tau} \left\|P_{\tau/2} f\right\|_{L^2} \le (t-s)\frac{2}{\tau} \left\|P_{\tau/2}\right\|_{L^1 \to L^2} \|f\|_{L^1} . \end{aligned}$$

Hence, it follows that

$$\|P_t f - P_s f\|_{L^\infty} \le (t-s) \frac{2}{\tau} \|P_\tau\|_{L^2 \to L^\infty} \left\|P_{\tau/2}\right\|_{L^1 \to L^2} \|f\|_{L^1} .$$

Substituting the estimates (5.18) and (5.19), we obtain (5.15). □

5.4. **Oscillation inequality and Hölder continuity for the heat semigroup.**

Lemma 5.11. *Assume that (V), (J) and (AB) are satisfied. Let Ω be a non-empty open subset of M. Fix a function $f \in L^1(\Omega) \cap L^2(\Omega)$ and set $u(x,t) = P_t^\Omega f(x)$. Then, for any ball $B(x_0, R) \subset \Omega$ of radius $R \in (0, \overline{R})$, for any $t > 0$ and $\rho \le \eta(t^{1/\beta} \wedge R)$, the following inequality holds:*

$$\operatorname*{osc}_{B(x_0,\rho)} u(\cdot, t) \le C \frac{e^{\overline{R}^{-\beta} t}}{t^{\alpha/\beta}} \left(\frac{\rho}{t^{1/\beta} \wedge R}\right)^\theta \|f\|_{L^1(\Omega)} , \tag{5.21}$$

where

$$\theta = \frac{\beta\gamma}{\beta+\gamma}, \quad \eta = \sigma^{(\beta+\gamma)/\gamma}$$

and the constant $C > 0$ depends only on the constants in the conditions (V), (J) and (AB). Here γ and σ are the constants from Lemma 5.9.

Proof. It is known that, for all $t > 0$, the function $u(\cdot, t)$ belongs to $\operatorname{dom}\left(\mathcal{L}^\Omega\right)$, is Fréchet differentiable in t as a path in $L^2(\Omega)$, and satisfies $\partial_t u(\cdot, t) = -\mathcal{L}^\Omega u(\cdot, t)$. It follows that, for any $\phi \in \mathcal{F}(\Omega)$ and $t > 0$,

$$\mathcal{E}(u(\cdot,t), \phi) = \left(\mathcal{L}^\Omega u(\cdot,t), \phi\right) = -\left(\partial_t u(\cdot,t), \phi\right),$$

which means that u is a weak solution in Ω of the equation

$$\mathcal{L} u(\cdot, t) = -\partial_t u(\cdot, t).$$

By Lemma 5.10, we have $u(\cdot, t) \in L^\infty(\Omega)$ and $\partial_t u(\cdot, t) \in L^\infty(\Omega)$ for all $t > 0$.

Note that $\rho \le \eta(t^{1/\beta} \wedge R) < \sigma R$. Choose some $r \in [\rho, \sigma R]$ to be specified below and set $B := B(x_0, r)$. By Lemmas 5.9 and 5.10 we obtain, for any $t > 0$,

$$\begin{aligned} \operatorname*{osc}_{B(x_0,\rho)} u(\cdot,t) &\le C \left(\left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(\Omega)} + r^\beta \|\partial_t u\|_{L^\infty(B)} \right) \\ &\le C \frac{e^{\overline{R}^{-\beta} t}}{t^{\alpha/\beta}} \left(\left(\frac{\rho}{r}\right)^\gamma + \frac{r^\beta}{t} \right) \|f\|_{L^1} \\ &\le C \frac{e^{\overline{R}^{-\beta} t}}{t^{\alpha/\beta}} \left(\left(\frac{\rho}{r}\right)^\gamma + \frac{r^\beta}{\tau} \right) \|f\|_{L^1} , \end{aligned} \tag{5.22}$$

where

$$\tau = t \wedge R^\beta.$$

Let us now specify r from the equation

$$\left(\frac{\rho}{r}\right)^\gamma = \frac{r^\beta}{\tau},$$

that is

$$r=\left(\rho^{\gamma}\tau\right)^{\frac{1}{\beta+\gamma}}.$$

Note that

$$\rho\leq\eta(t^{1/\beta}\wedge R)=\eta\tau^{1/\beta}$$

whence it follows that

$$r\geq\left(\rho^{\gamma}\left(\rho/\eta\right)^{\beta}\right)^{\frac{1}{\beta+\gamma}}>\rho$$

and

$$r\leq\left(\left(\eta\tau^{1/\beta}\right)^{\gamma}\tau\right)^{\frac{1}{\beta+\gamma}}=\eta^{\frac{\gamma}{\beta+\gamma}}\tau^{1/\beta}\leq\eta^{\frac{\gamma}{\beta+\gamma}}R=\sigma R.$$

Hence, $r\in[\rho,\sigma R]$ as required. For this choice of r, we have

$$\frac{r^{\beta}}{\tau}=\frac{1}{\tau}\left(\rho^{\gamma}\tau\right)^{\frac{\beta}{\beta+\gamma}}=\rho^{\frac{\beta\gamma}{\beta+\gamma}}\tau^{-\frac{\gamma}{\beta+\gamma}}=\left(\frac{\rho}{\tau^{1/\beta}}\right)^{\frac{\beta\gamma}{\beta+\gamma}}.$$

Therefore, inequality (5.21) follows from (5.22). □

For any set $U\subset M$, denote by U_r the open r-neighborhood of U, that is,

$$U_r=\bigcup_{x\in U}B\left(x,r\right).$$

Lemma 5.12. *Assume that (V), (J) and (AB) are satisfied. Let Ω be an open subset of M. Fix a function $f\in L^1\cap L^2(\Omega)$ and set $u\left(\cdot,t\right)=P_t^{\Omega}f$.*

(a) For any $t>0$, the function $u(\cdot,t)$ has a locally Hölder continuous version $\widetilde{u}(\cdot,t)$ in Ω with the Hölder exponent θ where θ is the same as in Lemma 5.11. Moreover, the function $\widetilde{u}\left(x,t\right)$ is jointly continuous in $(x,t)\in\Omega\times(0,\infty)$.

(b) For any open subset U of Ω, we have, for all $x,x'\in U$,

$$\left|\widetilde{u}(x,t)-\widetilde{u}(x',t)\right|\leq C\frac{e^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\left(\frac{d(x,x')}{t^{1/\beta}\wedge R}\right)^{\theta}\|f\|_{L^1(\Omega)}, \tag{5.23}$$

where

$$R=\sup\left\{r\in[0,\overline{R}):U_r\subset\Omega\right\} \tag{5.24}$$

and θ is the same constant as in Lemma 5.11.

(c) In the case $\Omega=M$, the function $\widetilde{u}\left(x,t\right)$ is Hölder continuous in $x\in M$, jointly continuous in $(x,t)\in M\times(0,\infty)$, and (5.23) holds for all $x,x'\in M$ and with $R=\overline{R}$.

Proof. (a) The fact that $u\left(\cdot,t\right)$ has a Hölder continuous version $\widetilde{u}(\cdot,t)$ follows from (5.21) by a standard argument.

By (5.15), we have, for all $t>s>\tau>0$,

$$\sup_{x\in\Omega}\left|\widetilde{u}\left(x,t\right)-\widetilde{u}\left(x,s\right)\right|\leq C\left(t-s\right)\frac{e^{\overline{R}^{-\beta}\tau}}{\tau^{1+\alpha/\beta}}\|f\|_{L^1(\Omega)},$$

which implies that the function $t\mapsto\widetilde{u}\left(x,t\right)$ is continuous in $t\in(0,\infty)$ uniformly in $x\in\Omega$. Since the function $x\mapsto\widetilde{u}\left(x,t\right)$ is continuous in $x\in\Omega$, we conclude that $\widetilde{u}\left(x,t\right)$ is jointly continuous in (x,t).

(b) It suffices to prove (5.23) for any $R<\overline{R}$ such that $U_R\subset\Omega$. Then we have $B(x,R)\subset\Omega$ for any $x\in U$. Set $\tau=t\wedge R^{\beta}$. By Lemma 5.11, we obtain

$$\operatorname*{osc}_{B(x,\rho)}\widetilde{u}(\cdot,t)\leq C\frac{e^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\left(\frac{\rho}{\tau^{1/\beta}}\right)^{\theta}\|f\|_{L^1(\Omega)}, \tag{5.25}$$

provided $\rho \le \eta\tau^{1/\beta}$. If $\rho > \eta\tau^{1/\beta}$, then we obtain by Lemma 5.10

$$\begin{aligned}\operatorname*{osc}_{B(x,\rho)} \widetilde{u}(\cdot,t) &\le 2\|u(\cdot,t)\|_{L^\infty} \le C\frac{e^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\|f\|_{L^1(\Omega)} \\ &= C\frac{e^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\left(\frac{\rho}{\tau^{1/\beta}}\right)^{-\theta}\left(\frac{\rho}{\tau^{1/\beta}}\right)^{\theta}\|f\|_{L^1(\Omega)} \\ &\le C\eta^{-\theta}\frac{e^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\left(\frac{\rho}{\tau^{1/\beta}}\right)^{\theta}\|f\|_{L^1(\Omega)}.\end{aligned}$$

Hence, by adjusting the constant C, we obtain that (5.21) holds for all $\rho > 0$. Choosing $\rho = d(x,x')$, we obtain (5.23).

(c) If $\Omega = M$, then applying the first statement with $U = M$ we obtain by (5.24) that $R = \overline{R}$. Hence, (5.23) holds with $R = \overline{R}$ for all $x, x' \in M$, which implies that $\widetilde{u}$ is Hölder continuous in M. □

5.5. **Existence and the Hölder continuity of the heat kernel.** Recall that the heat kernel $p_t(x,y)$ is the integral kernel of the heat semigroup $\{P_t\}$. In particular, for any $t > 0$, the function $p_t(x,y)$ is a measurable function of $(x,y) \in M \times M$. By Lemma 5.12, under the hypotheses $(V), (J), (AB)$, the function $P_t f(x)$ has a continuous version for any $f \in L^1 \cap L^2(M)$. From now on let us use the notation $P_t f(x)$ for this continuous version. In particular, the semigroup identity

$$P_{t+s} f(x) = P_t(P_s f)(x)$$

holds for *all* $x \in M$ and $t, s > 0$.

We say that a function $p_t(x,y)$ of $t > 0$, $x, y \in M$ is a *continuous heat kernel* if, for any $t > 0$, the function $(x,y) \mapsto p_t(x,y)$ is continuous in $(x,y) \in M \times M$ and, for any $f \in L^1 \cap L^2(M)$,

$$P_t f(x) = \int_M p_t(x,y) f(y)\, d\mu(y),$$

for all $t > 0$ and $x \in M$.

If p_t is a continuous heat kernel then the properties of the heat semigroup $\{P_t\}$ imply the following properties of p_t, for all $x, y \in M$ and $t, s > 0$:

(1) $p_t(x,y) \ge 0$ and $\|p_t(x,\cdot)\|_{L^1} \le 1$;
(2) $p_t(x,y) = p_t(y,x)$;
(3) $p_{t+s}(x,y) = \int_M p_t(x,z) p_s(z,y)\, d\mu(z)$.

All these apply to the heat semigroup $\{P_t^\Omega\}$, where $\Omega \subset M$ is any open set. We use the notation $P_t^\Omega f(x)$ for the continuous version, and define the notion of a continuous heat kernel $p_t^\Omega(x,y)$ in the same way.

Lemma 5.13. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form with jump kernel J. Assume that (V), (J) and (AB) are satisfied.*

For any non-empty open set $\Omega \subset M$, there exists a (locally Hölder) continuous heat kernel $p_t^\Omega(x,y)$. Moreover, the function $p_t^\Omega(x,y)$ is jointly continuous in $(x,y,t) \in \Omega \times \Omega \times (0,\infty)$ and satisfies the upper bound

$$\sup_{x,y\in\Omega} p_t^\Omega(x,y) \le C\frac{e^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}. \tag{5.26}$$

In the case $\Omega = M$, the function $p_t(x,y)$ satisfies the following estimate: for all $x, x', y, y' \in M$ and $t > 0$,

$$\left|p_t(x,y) - p_t(x',y')\right| \le C\frac{e^{\overline{R}^{-\beta}t}}{t^{\alpha/\beta}}\left(\left(\frac{d(x,x')}{t^{1/\beta}\wedge\overline{R}}\right)^{\theta} + \left(\frac{d(y,y')}{t^{1/\beta}\wedge\overline{R}}\right)^{\theta}\right), \tag{5.27}$$

where θ is the constant of Lemma 5.11 and the constant $C > 0$ depends only on the constants in the conditions (V), (J) and (AB).

Proof. For simplicity we restrict ourself to the heat kernel $p_t(x,y)$, while the claims related to $p_t^{\Omega}(x,y)$ are proved in the same way.

By Lemma 5.12, for any $f \in L^1 \cap L^2(M)$ and $t>0$, the function $P_t f$ is Hölder continuous and satisfies the following estimate, for all $t>0$ and $x, x' \in M$,

$$\left|P_t f(x) - P_t f(x')\right| \le C \frac{e^{\overline{R}^{-\beta} t}}{t^{\alpha/\beta}} \left(\frac{d(x,x')}{t^{1/\beta} \wedge \overline{R}}\right)^{\theta} \|f\|_{L^1}. \tag{5.28}$$

Furthermore, by Lemma 5.10, we have, for all $t>0$ and $x \in M$,

$$|P_t f(x)| \le C \frac{e^{\overline{R}^{-\beta} t}}{t^{\alpha/\beta}} \|f\|_{L^1}. \tag{5.29}$$

It follows from (5.29) that the mapping $f \mapsto P_t f(x)$ (for any fixed $t>0$ and $x \in M$) extends to a bounded linear functional on $L^1(M)$. Hence, there exists a function $q_{t,x} \in L^{\infty}(M)$ such that, for any $f \in L^1(M)$,

$$P_t f(x) = \int_M q_{t,x} f \, d\mu = (q_{t,x}, f) \tag{5.30}$$

and

$$\|q_{t,x}\|_{L^\infty} \le C \frac{e^{\overline{R}^{-\beta} t}}{t^{\alpha/\beta}}. \tag{5.31}$$

By the Markov properties of P_t we have

$$q_{t,x} \ge 0 \quad \mu\text{-a.e. in } M \quad \text{and} \quad \|q_{t,x}\|_{L^1} \le 1. \tag{5.32}$$

In particular, $q_{t,x} \in L^\infty \cap L^1$ and, hence, $q_{t,x} \in L^2$.

For any $0<s<t$ and $x,y \in M$, let us define the function

$$p_{t,s}(x,y) := \int_M q_{t-s,x} q_{s,y} \, d\mu = P_{t-s} q_{s,y}(x) = P_s q_{t-s,x}(y). \tag{5.33}$$

Let us prove some properties of $p_{t,s}(x,y)$.

(i) For any $f \in L^1$ and $0<s<t$, we have

$$(p_{t,s}(x,\cdot), f) = (P_s q_{t-s,x}, f) = (q_{t-s,x}, P_s f) = P_{t-s}\left(P_s f\right)(x) = P_t f(x) = (q_{t,x}, f).$$

It follows that

$$p_{t,s}(x,\cdot) = q_{t,x} \quad \mu\text{-a.e. in } M. \tag{5.34}$$

(ii) Applying (5.28) with $f = q_{s,y}$ and using that $\|q_{s,y}\|_{L^1} \le 1$, we obtain, for all $x, x', y \in M$,

$$\begin{aligned} \left|p_{t,s}(x,y) - p_{t,s}(x',y)\right| &= \left|P_{t-s} q_{s,y}(x) - P_{t-s} q_{s,y}(x')\right| \\ &\le C \frac{e^{\overline{R}^{-\beta}(t-s)}}{(t-s)^{\alpha/\beta}} \left(\frac{d(x,x')}{(t-s)^{1/\beta} \wedge \overline{R}}\right)^{\theta}. \end{aligned} \tag{5.35}$$

Similarly, we have, for all $x', y, y' \in M$,

$$\left|p_{t,s}(x',y) - p_{t,s}(x',y')\right| \le C \frac{e^{\overline{R}^{-\beta} s}}{s^{\alpha/\beta}} \left(\frac{d(y,y')}{s^{1/\beta} \wedge \overline{R}}\right)^{\theta}. \tag{5.36}$$

Adding up the above two inequalities, we see that $p_{t,s}(x,y)$ is jointly Hölder continuous in (x,y) with the Hölder exponent θ.

(iii) It follows from (5.34) that, for all $x \in M$ and $s', s'' \in (0,t)$,

$$p_{t,s'}(x,y) = p_{t,s''}(x,y)$$

for μ-a.a. $y \in M$. By the continuity of $p_{t,s}(x,y)$ in (x,y), we conclude that this identity holds for all $y \in M$. In other words, $p_{t,s}(x,y)$ is independent of the choice of s. Hence, we set, for all $x,y \in M$ and $t>0$,

$$p_t(x,y) := p_{t,s}(x,y). \tag{5.37}$$

It follows from (5.30), (5.34) and (5.37) that $p_t(x,y)$ is a continuous heat kernel.

By (ii), the function $p_t(x,y)$ is Hölder continuous in (x,y) with the Hölder exponent θ. Moreover, letting $s\to 0$ in (5.35) and $s\to t$ in (5.36), and then adding up the two inequalities, we obtain (5.27).

The estimate (5.26) follows from (5.31), (5.34) and (5.37).

Finally, let us show that $p_t(x,y)$ is jointly continuous in (x,y,t). For that, it suffices to verify that the function $t\mapsto p_t(x,y)$ is continuous in t uniformly in (x,y). Indeed, by (5.33), (5.15) and (5.32), we obtain, for all $t>s>2\tau>0$

$$|p_t(x,y)-p_s(x,y)| = |P_{t-\tau}q_{\tau,y}(x)-P_{s-\tau}q_{\tau,y}(x)| \le C(t-s)\frac{e^{\overline{R}^{-\beta}2\tau}}{\tau^{1+\alpha/\beta}},$$

whence the claim follows. □

5.6. **Proof of Theorem 2.10.** As we have already mentioned in Section 2.5, in order to complete the proof of Theorem 2.10, it remains to prove (2.24), that is, to derive (S) and (NLE) under the standing assumptions (V), (J), (AB).

Condition (S) holds by Corollary 5.7. By Lemma 5.13, we obtain that there is a continuous heat kernel $p_t(x,y)$. Moreover, for each ball $B\subset M$, the heat kernel $p_t^B(x,y)$ also exists and is continuous.

Let us now prove (NLE), that is, for any $t\in(0,\overline{R}^\beta)$ and for all x,y such that $d(x,y)\le \eta t^{1/\beta}$, we have

$$p_t(x,y)\ge \frac{a}{t^{\alpha/\beta}}, \tag{5.38}$$

with some positive a,η. Let ε,δ be the constants in condition (S). We first prove (5.38) assuming that $t<\left(\delta\overline{R}\right)^\beta$. Fix $x\in M$, $t<\left(\delta\overline{R}\right)^\beta$ and consider the ball $B=B(x,r)$ with $r:=\delta^{-1}(t/2)^{1/\beta}<\overline{R}$. By conditions (S) and $(V_\le)$, we have

$$\begin{aligned} p_t(x,x) &= \int_M p_{t/2}(x,y)^2 d\mu(y) \ge \int_B p^B_{t/2}(x,y)^2 d\mu(y) \ge \frac{1}{\mu(B)}\left(\int_B p^B_{t/2}(x,y)d\mu(y)\right)^2 \\ &= \frac{\left(P^B_{t/2}1(x)\right)^2}{\mu(B)} \ge \frac{\varepsilon^2}{Cr^\alpha} = \frac{c_1}{t^{\alpha/\beta}}, \end{aligned}$$

with $c_1>0$. By the inequality (5.27) of Lemma 5.13, we have, for all $x,y\in M$ and $t\in(0,\overline{R}^\beta)$,

$$|p_t(x,x)-p_t(x,y)|\le \frac{c_2}{t^{\alpha/\beta}}\left(\frac{d(x,y)}{t^{1/\beta}}\right)^\theta.$$

It follows that

$$p_t(x,y)\ge p_t(x,x)-|p_t(x,x)-p_t(x,y)|\ge \frac{1}{t^{\alpha/\beta}}\left(c_1-c_2\left(\frac{d(x,y)}{t^{1/\beta}}\right)^\theta\right).$$

Therefore, if $d(x,y)\le \eta t^{1/\beta}$, where $\eta=\left(\frac{c_1}{2c_2}\right)^{1/\theta}$ then we have (5.38) with $a=c_1/2$.

Now let us extend this estimate to all $t<\overline{R}^\beta$. For that, it suffices to prove that if (5.38) holds for some $t<\overline{R}^\beta$ and all $d(x,y)\le \eta t^{1/\beta}$ then the same estimate holds also for $2t$ in place of t, but with different values of a and η. Without loss of generality, assume $\eta<1$. Fix $x\in M$, $t<\overline{R}^\beta$ and set $r=\frac{1}{2}\eta t^{1/\beta}$. By the semigroup property, we have, for any $y\in B(x,r)$,

$$p_{2t}(x,y)\ge \int_{B(x,r)} p_t(x,z)\,p_t(z,y)\,d\mu(z).$$

For any $z\in B(x,r)$, we have $d(z,y)<2r=\eta t^{1/\beta}$. Hence, by the previous step of the proof, we obtain that both $p_t(x,z)$ and $p_t(z,y)$ are bounded from below by $a/t^{\alpha/\beta}$. Since $r<\overline{R}$, we can use $(V_\ge)$, which implies

$$p_{2t}(x,y)\ge\left(\frac{a}{t^{\alpha/\beta}}\right)^2 cr^\alpha = \frac{a'}{t^{\alpha/\beta}},$$

for all y such that $d(x,y) \le r = \eta'(2t)^{1/\beta}$, where $\eta' = \eta 2^{-(1+\frac{1}{\beta})}$, which finishes the proof.

6. Some consequences

In this section we prove some consequences of Theorem 2.10.

6.1. Green function. Let Ω be an open subset of M. If the Green operator G^{Ω} (cf. Section 5.1) has an integral kernel then the latter is called the *Green function* in Ω and is denoted by $g^{\Omega}(x,y)$. In other words, $g^{\Omega}(x,y)$ is a μ-measurable function in x,y that satisfies the following identity

$$G^{\Omega}f(x) = \int_{\Omega} g^{\Omega}(x,y)f(y)d\mu(y)$$

for all non-negative $f \in L^2(\Omega)$ and μ-a.a. $x \in \Omega$.

Definition 6.1 (Condition $(g_{\geq})$)**.** There exist constants $\delta, \sigma \in (0,1)$ and $c > 0$ such that, for any ball $B := B(x_0, r)$ of radius $r \in (0, \sigma\overline{R})$, the Green function $g^B(x,y)$ exists and satisfies

$$g^B(x,y) \ge cr^{\beta-\alpha}, \text{ for } \mu\text{-a.a. } x,y \in B(x_0, \delta r).$$

Corollary 6.2. *Let $(\mathcal{E}, \mathcal{F})$ be a regular jump type Dirichlet form on $L^2(M,\mu)$ with a jump kernel J. If (M,d,μ) satisfies (V) then the following equivalence holds*

$$(J) + (g_{\geq}) \Leftrightarrow (UE) + (LE).$$

Proof. Let us first prove that

$$(J) + (g_{\geq}) \Rightarrow (UE) + (LE).$$

By Theorem 2.10, it suffices to prove the implication:

$$(V) + (J_{\geq}) + (g_{\geq}) \Rightarrow (S). \tag{6.1}$$

Indeed, for any ball $B := B(x_0, r)$ of radius $r \in (0, \sigma\overline{R})$, and for μ-a.a. $x \in \delta B$, we have, using $(g_{\geq})$ and $(V_{\geq})$,

$$G^B 1(x) = \int_B g^B(x,y)d\mu(y) \ge \int_{\delta B} g^B(x,y)d\mu(y) \ge C^{-1}r^{\beta-\alpha}\mu(\delta B) \ge cr^{\beta},$$

which together with Remark 5.3 implies the condition $(E_{\geq})$.

By Lemma 5.4, we have

$$(V) + (J_{\geq}) \Rightarrow (E_{\leq}),$$

and by Lemma 5.6 $(E) \Rightarrow (S)$, which finishes the proof of (6.1).

Let us prove the opposite implication

$$(UE) + (LE) \Rightarrow (J) + (g_{\geq}).$$

By Theorem 2.10, it suffices to prove that

$$(UE) + (LE) \Rightarrow (g_{\geq}).$$

Let us first verify the existence of the Green function g^B for any ball $B := B(x_0, r)$ of radius $r \in (0, \sigma\overline{R})$, where $\sigma \in (0,1)$ is to be specified. By Theorem 2.10, $(UE) + (LE)$ imply (J) whence by Lemma 3.5 we obtain (FK). Setting σ to be the constant from (FK), we obtain that $\lambda_1(B) > 0$. Recall that the heat kernel $p_t^B(x,y)$ is continuous jointly in t,x,y. Since $\left\|P_t^B\right\|_{L^2 \to L^2} \le e^{-\lambda_1(B)t}$, it follows that $p_t^B(x,y)$ decays exponentially as $t \to \infty$. Combining this with (UE), we see that the integral

$$\int_0^{\infty} p_t^B(x,y)\,dt$$

converges for all distinct $x,y \in B$ and, hence, determines the Green function $g^B(x,y)$.

Let us now verify the lower bound $(g_{\geq})$. Let $\delta \in (0,1/4)$ be a small number to be determined later. Let B be the ball as above and $f \in L^1 \cap L^\infty(\delta B)$ be an arbitrary non-negative function. By [21, (4.1), p.2626], we have, for all $t > 0$ and μ-a.a. $x \in B$,

$$P_t^B f(x) \geq P_t f(x) - \sup_{0<s\leq t} \|P_s f\|_{L^\infty(K^c)}, \tag{6.2}$$

where $K = \overline{(\frac{1}{2}B)}$. Since $P_t f$ and $P_t^B f$ are continuous by Theorem 2.10 and Lemma 5.12, this inequality holds for all $x \in B$. Since f is arbitrary and the heat kernels p_t^B and p_t are continuous functions (cf. Lemma 5.13), it follows from (6.2) that, for all $x, y \in \delta B$ and $t > 0$,

$$p_t^B(x,y) \geq p_t(x,y) - \sup_{0<s\leq t} \sup_{z\in K^c, w\in\delta B} p_s(z,w). \tag{6.3}$$

By (LE) we obtain, for all $t \leq (\delta r)^\beta < \overline{R}^\beta$ and $x, y \in \delta B$, that

$$p_t(x,y) \geq c\left(t^{-\alpha/\beta} \wedge \frac{t}{(2\delta r)^{\alpha+\beta}}\right) = \frac{ct}{(2\delta r)^{\alpha+\beta}}. \tag{6.4}$$

For all $z \in K^c$, $w \in \delta B$, we have

$$d(z,w) \geq d(x_0,z) - d(x_0,w) \geq \left(\frac{1}{2} - \delta\right) r \geq \frac{1}{4} r,$$

where we have used that $\delta < \frac{1}{4}$. By (UE), we obtain, for all such z, w and $0 < s \leq t$ that

$$p_s(z,w) \leq \frac{Cs}{d(z,w)^{\alpha+\beta}} \leq \frac{Ct}{(\frac{1}{4}r)^{\alpha+\beta}}. \tag{6.5}$$

Combining (6.3), (6.4) and (6.5), we obtain, for all $t \leq (\delta r)^\beta$ and $x, y \in \delta B$,

$$p_t^B(x,y) \geq \frac{ct}{(2\delta r)^{\alpha+\beta}} - \frac{Ct}{(\frac{1}{4}r)^{\alpha+\beta}} = c\frac{t}{r^{\alpha+\beta}},$$

assuming that $\delta = \delta(c,C) > 0$ is sufficiently small. It follows that

$$g^B(x,y) = \int_0^\infty p_t(x,y)\,dt \geq \int_0^{(\delta r)^\beta} c\frac{t}{r^{\alpha+\beta}} dt = \frac{c}{2}\frac{(\delta r)^{2\beta}}{r^{\alpha+\beta}} = c' r^{\beta-\alpha},$$

which finishes the proof of $(g_{\geq})$. □

6.2. **Asymptotic behavior of the heat semigroup.** If the heat kernel $p_t(x,y)$ exists then the operator P_t extends to all measurable functions f on M by

$$P_t f(x) = \int_M p_t(x,y) f(y)\,d\mu(y),$$

provided the integral converges.

Corollary 6.3. *Let $(\mathcal{E},\mathcal{F})$ be a regular, jump type Dirichlet form on $L^2(M,\mu)$ with a jump kernel J. Assume that (V), (J) and $(Gcap)$ are satisfied with $\overline{R} = \infty$. Fix some $x_0 \in M$. Then, for any measurable function f on M such that*

$$\int_M \left(1 + d(y,x_0)^\theta\right) |f(y)|\,d\mu(y) < \infty$$

we have

$$P_t f(x) = K p_t(x,x_0)\left(1 + O\left(t^{-\theta/\beta}\right)\right) \quad \text{as } t \to \infty, \tag{6.6}$$

where $K = \int_M f d\mu$ and $\theta > 0$ is the exponent from Lemma 5.11.

Proof. Using the definitions of $P_t f$ and K, we obtain

$$P_t f(x) - K p_t(x, x_0) = \int_M (p_t(x,y) - p_t(x,x_0)) f(y)\, d\mu(y).$$

By Lemma 5.13, we have

$$|p_t(x,y) - p_t(x,x_0)| \le \frac{C}{t^{\alpha/\beta}} \left(\frac{d(y,x_0)}{t^{1/\beta}} \right)^{\theta},$$

which implies

$$|P_t f(x) - K p_t(x,x_0)| \le \frac{C}{t^{\alpha/\beta+\theta/\beta}} \int_M d(y,x_0)^{\theta} |f(y)|\, d\mu(y) \le \frac{C}{t^{\alpha/\beta+\theta/\beta}}.$$

Finally, it remains to observe that, by Theorem 2.10 with $\overline{R} = \infty$,

$$p_t(x,x_0) \ge \frac{c}{t^{\alpha/\beta}}$$

for large enough t, whence (6.6) follows. □

Appendix

The inequalities in the following proposition are frequently used in this paper.

Proposition 6.4. *If $(V_\le)$ is satisfied then, for all $r > 0$ and $x \in M$,*

$$\int_{B(x,r)^c} \frac{d\mu(y)}{d(x,y)^{\alpha+\beta}} \le \frac{C}{r^{\beta}}. \tag{6.7}$$

Consequently, if $(V_\le)$ and $(J_\le)$ are satisfied, then, for all $r > 0$ and $x \in M$,

$$\int_{B(x,r)^c} J(x,y) d\mu(y) \le \frac{C}{r^{\beta}}. \tag{6.8}$$

The constant $C > 0$ depends only on α, β and the constants in the conditions $(V_\le)$ and $(J_\le)$.

Proof. Set $r_k = 2^k r$ for all non-negative integers k and $B_k = B(x, r_k)$. Using $(V_\le)$, we obtain

$$\begin{aligned} \int_{B(x,r)^c} \frac{d\mu(y)}{d(x,y)^{\alpha+\beta}} &= \sum_{k=0}^{\infty} \int_{B_{k+1}\setminus B_k} \frac{d\mu(y)}{d(x,y)^{\alpha+\beta}} \\ &\le \sum_{k=0}^{\infty} \int_{B_{k+1}\setminus B_k} \frac{d\mu(y)}{r_k^{\alpha+\beta}} \\ &\le \sum_{k=0}^{\infty} \frac{\mu(B_{k+1})}{r_k^{\alpha+\beta}} \\ &\le C \sum_{k=0}^{\infty} r_k^{-\beta} \le C' r^{-\beta}, \end{aligned}$$

which proves (6.7). Clearly, (6.8) follows from (6.7) and $(J_\le)$. □

Proposition 6.5. *Suppose that $u = w + a \in \mathcal{F}'$ with $w \in \mathcal{F}$ and $a \in \mathbb{R}$, $v \in \mathcal{F} \cap L^\infty$ and that $F : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz function. The following is true:*

(i) *$F(u) - F(a) \in \mathcal{F}$ and so, $F(u) \in \mathcal{F}'$.*
(ii) *If in addition $F(u) \in L^\infty$, then $F(u)v \in \mathcal{F} \cap L^\infty$.*
(iii) *Let Ω be an open subset of M. If in addition $v \in \mathcal{F}(\Omega)$, then, $F(u)v \in \mathcal{F}(\Omega)$.*

Proof. Denote the Lipschitz constant of F by L and define

$$h(t) := \frac{F(t+a) - F(a)}{L}, \quad t \in \mathbb{R}. \tag{6.9}$$

(i) Since $F(u) - F(a) = Lh(w)$, it suffices to prove $h(w) \in \mathcal{F}$. By (6.9), h satisfies

$$|h(t) - h(s)| \le |t - s| \quad \text{and} \quad h(0) = 0.$$

Hence, $h(w)$ is a normal contraction of w. By the Markov property of $(\mathcal{E}, \mathcal{F})$, $h(w) \in \mathcal{F}$ and so,

$$F(u) = Lh(u - a) + F(a) = Lh(w) + F(a) \in \mathcal{F}'.$$

(ii) By (i), the fact that $F(u) \in L^\infty$ implies $h(w) \in \mathcal{F} \cap L^\infty$. Hence by [14, Theorem 1.4.2(ii)], $h(w)v \in \mathcal{F} \cap L^\infty$, since $v \in \mathcal{F} \cap L^\infty$. Consequently,

$$F(u)v = (Lh(u - a) + F(a))v = Lh(w)v + F(a)v \in \mathcal{F} \cap L^\infty.$$

(iii) Let $\widetilde{h(w)}$ and $\widetilde{v}$ be the quasi-continuous modifications of $h(w)$ and v respectively. Then, $\widetilde{h(w)}\widetilde{v}$ is the quasi-continuous modification of $h(w)v$. Since $v \in \mathcal{F}(\Omega)$,

$$\widetilde{h(w)}\widetilde{v} = 0, \quad \text{q.e. in } \Omega^c.$$

Hence, $h(w)v \in \mathcal{F}(\Omega)$ and so, $F(u)v = Lh(w)v + F(a)v \in \mathcal{F}(\Omega)$. □

References

[1] S. Andres and M. T. Barlow. Energy inequalities for cutoff functions and some applications. *J. Reine Angew. Math.*, 699:183–215, 2015.

[2] D. Aronson. Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.*, 73:890–896, 1967.

[3] M. T. Barlow. *Diffusions on fractals*, volume 1690 of *Lect. Notes Math.*, pages 1–121. Springer, 1998.

[4] M. T. Barlow and R. Bass. Transition densities for Brownian motion on the the Sierpínski carpet. *Probab. Theory Related Fields*, 91:307–330, 1992.

[5] M. T. Barlow and R. Bass. Brownian motion and harmonic analysis on Sierpínski carpets. *Canad. J. Math.*, 51:673–744, 1999.

[6] M. T. Barlow, R. Bass, and T. Kumagai. Stability of parabolic Harnack inequalities on metric measure spaces. *J. Math. Soc. Japan*, 58:485–519, 2006.

[7] M. T. Barlow, A. Grigor'yan, and T. Kumagai. Heat kernel upper bounds for jump processes and the first exit time. *J. Reine Angew. Math.*, 626:135–157, 2009.

[8] M. T. Barlow and E. Perkins. Brownian motion on the Sierpínski gasket. *Probab. Theory Related Fields*, 79:543–623, 1988.

[9] E. A. Carlen, S. Kusuoka, and D. W. Stroock. Upper bounds for symmetric Markov transition functions. *Ann. Inst. H. Poincaré Probab. Statist.*, 23(2, suppl.):245–287, 1987.

[10] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for stable-like processes on d-sets. *Stochastic Process. Appl.*, 108(1):27–62, 2003.

[11] Z.-Q. Chen, T. Kumagai, and J. Wang. Stability of heat kernel estimates for symmetric jump processes on metric measure spaces. *ArXiv e-prints*, Apr. 2016.

[12] A. Di Castro, T. Kuusi, and G. Palatucci. Local behavior of fractional p-minimizers. *Annales de l'Institut Henri Poincaré Non Linear Analysis*, 33(5):1279–1299, 2015.

[13] P. Fitzsimmons, B. Hambly, and T. Kumagai. Transition density estimates for Brownian motion on affine nested fractals. *Comm. Math. Phys.*, 165:595–620, 1994.

[14] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2011.

[15] A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. *Mat. Sb.*, 182:55–87, 1991.

[16] A. Grigor'yan. *Heat kernels on metric measure spaces. In Handbook of Geometric Analysis 2*, volume 13 of *Advanced Lectures in Math.*, pages 1–60. International Press, 2010.

[17] A. Grigor'yan, E. Hu, and J. Hu. Lower estimates of heat kernels for non-local Dirichlet forms on metric measure spaces. *J. Funct. Anal.*, 272(8):3311–3346, 2017.

[18] A. Grigor'yan and J. Hu. Off-diagonal upper estimates for the heat kernel of the Dirichlet forms on metric spaces. *Invent. Math.*, 174(1):81–126, 2008.

[19] A. Grigor'yan and J. Hu. Heat kernels and Green functions on metric measure spaces. *Canad. J. Math.*, 66:641–699, 2014.

[20] A. Grigor'yan, J. Hu, and K.-S. Lau. Heat kernels on metric measure spaces and an application to semilinear elliptic equations. *Trans. Amer. Math. Soc.*, 355(5):2065–2095, 2003.

[21] A. Grigor'yan, J. Hu, and K.-S. Lau. Comparison inequalities for heat semigroups and heat kernels on metric measure spaces. *J. Funct. Anal.*, 259(10):2613–2641, 2010.
[22] A. Grigor'yan, J. Hu, and K.-S. Lau. Estimates of heat kernels for non-local regular Dirichlet forms. *Trans. Amer. Math. Soc.*, 366(12):6397–6441, 2014.
[23] A. Grigor'yan, J. Hu, and K.-S. Lau. Generalized capacity, harnack inequality and heat kernels of Dirichlet forms on metric measure spaces. *J. Math. Soc. Japan*, 67(4):1485–1549, 2015.
[24] A. Grigor'yan, X. Huang, and J. Masamune. On stochastic completeness of jump processes. *Math. Z.*, 271(3-4):1211–1239, 2012.
[25] A. Grigor'yan and T. Kumagai. On the dichotomy in the heat kernel two sided estimates. In P. E. et al., editor, *Analysis on Graphs and its Applications, Proc. of Symposia in Pure Math.*, volume 77, pages 199–210. Amer. Math. Soc., 2008.
[26] A. Grigor'yan and A. Telcs. Two-sided estimates of heat kernels on metric measure spaces. *Annals of Probability*, 40:1212–1284, 2012.
[27] B. Hambly and T. Kumagai. Transition density estimates for diffusion processes on post critically finite self-similar fractals. *Proc. London Math. Soc.*, 79:431–458, 1999.
[28] J. Hu and T. Kumagai. Nash-type inequalities and heat kernels for non-local Dirichlet forms. *Kyushu J. Math.*, 60(2):245–265, 2006.
[29] S. Kusuoka. A diffusion process on a fractal. In *Symposium on Probabilistic Methods in Mathematical Physics, Taniguchi, Katata*, pages 251–274. Academic Press, Amsterdam, 1987.
[30] S. Kusuoka. Lecture on diffusion processes on nested fractals. In *Lect. Notes Math.*, volume 1567, pages 39–98. Springer, 1993.
[31] E. Landis. *The second order equations of elliptic and parabolic type*, volume 171. Transl. of Mathematical Monographs, AMS publications, 1998.
[32] Z. M. Ma and M. Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms*. Universitext. Springer-Verlag, Berlin, 1992.
[33] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Internat. Math. Res. Notices*, (2):27–38, 1992.
[34] A. Stós. Symmetric α-stable processes on d-sets. *Bull. Pol. Acad. Sci. Math.*, 48:237–245, 2000.

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