OFF-DIAGONAL UPPER ESTIMATES FOR THE HEAT KERNEL OF THE DIRICHLET FORMS ON METRIC SPACES

ALEXANDER GRIGOR'YAN AND JIAXIN HU

Abstract. We give equivalent characterizations for off-diagonal upper bounds of the heat kernel of a regular Dirichlet form on the metric measure space, in two settings: for the upper bounds with the polynomial tail (typical for jump processes) and for the upper bounds with the exponential tail (for diffusions). Our proofs are purely analytic and do not use the associated Hunt process.

Contents

1. Preliminaries and the main results

1.1. **General setup.** Let (M,d) be a locally compact, separable metric space, and let μ be a Radon measure on M with full support. Let $(\mathcal{E},\mathcal{F})$ be a *Dirichlet form* on $L^2(M) := L^2(M,\mu)$. That is, $\mathcal{E}$ is a closed, symmetric, non-negative definite, bilinear form on a dense subspace $\mathcal{F}$ of $L^2(M)$, which satisfies the Markov property. The *closedness* of the form $\mathcal{E}$ means that $\mathcal{F}$ is a Hilbert space with respect to the $\mathcal{E}_1$-inner product, where

$$\mathcal{E}_1(f,g) = \mathcal{E}(f,g) + (f,g).$$

Here $(\cdot,\cdot)$ stands for the inner product on $L^2(M)$. The *Markov property* means that if $f \in \mathcal{F}$ then the function $\widetilde{f} = \max(\min(f,1),0)$ is also in $\mathcal{F}$ and $\mathcal{E}(\widetilde{f}) \le \mathcal{E}(f)$ (here and in the sequel we use the abbreviation $\mathcal{E}(f) := \mathcal{E}(f,f)$).

Let Δ be the *generator* of $(\mathcal{E},\mathcal{F})$, that is, Δ is a non-positive definite self-adjoint operator in $L^2(M)$ with domain $\mathrm{dom}(\Delta) \subset \mathcal{F}$, and

$$-(\Delta f, g) = \mathcal{E}(f,g) \quad \text{for all } f \in \mathrm{dom}(\Delta), g \in \mathcal{F}.$$

Date: December 2007. Revised March 2008.

AG was supported by SFB 701 of the German Research Council (DFG). JH was supported by a Major Basic Research Grant of NSFC (Grant No. 10631040) and by a visiting grant of SFB 701.

The generator Δ gives rise to the *heat semigroup* $\{P_t\}_{t\geq 0}$, which is the family of bounded self-adjoint operators in $L^2(M)$, defined by

$$P_t = e^{t\Delta}. \tag{1.1}$$

Obviously, the heat semigroup is *contractive* in $L^2(M)$, that is, $\|P_t f\| \leq \|f\|$ for all $f \in L^2(M)$, where $\|\cdot\|$ is the L^2-norm, and *strongly continuous*, that is,

$$\|P_t f - f\| \to 0 \text{ as } t \to 0 \text{ for all } f \in L^2(M).$$

In addition, the semigroup $\{P_t\}$ is *Markovian*, that is, if $0 \leq f \leq 1$ a.e., then, for all $t > 0$,

$$0 \leq P_t f \leq 1 \quad \text{a.e.} \tag{1.2}$$

(see [10, Theorem 1.4.1, p. 23]). The Markovian property (1.2) allows to extend $P_t f$ to all $f \in L^\infty(M)$ so that P_t can be considered also as a contraction operator from $L^\infty(M)$ to $L^\infty(M)$.

Conversely, given a strongly continuous Markovian semigroup $\{P_t\}_{t\geq 0}$, one recovers the corresponding Dirichlet form by letting

$$\mathcal{E}(f) = \lim_{t\to 0}\left(\frac{f - P_t f}{t}, f\right), \tag{1.3}$$

and by letting $\mathcal{F}$ be the set of those $f \in L^2(M)$ for which $\mathcal{E}(f) < \infty$.

Let $C_0(M)$ be the space of all continuous functions on M with compact support. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ (in the $\mathcal{E}_1$-norm) and in $C_0(M)$ in the sup-norm. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *local* if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with disjoint compact supports[1] (cf. [10, p.6]).

The heat semigroup (or the Dirichlet form) is called *conservative* if $P_t 1 = 1$ a.e. for any $t > 0$.

1.2. **Upper bounds of the heat kernel.**

Definition 1.1. A family $\{p_t(x, y)\}_{t>0}$ of measurable functions on $M \times M$ is called the *heat kernel* of $(\mathcal{E}, \mathcal{F})$ if, for all $f \in L^2(M)$, $t > 0$, and μ-almost all $x \in M$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y). \tag{1.4}$$

The heat kernel does not have to exist but if it exists then it is unique (up to a set of measure zero) and satisfies the following properties, which follow immediately from the corresponding properties of the heat semigroup:

(i) For all $t > 0$ and almost all $x, y \in M$, $p_t(x, y) \geq 0$ and

$$\int_M p_t(x, y)\, d\mu(y) \leq 1. \tag{1.5}$$

(ii) For all $t, s > 0$ and almost all $x, y \in M$,

$$p_{t+s}(x, y) = \int_M p_t(x, z) p_s(z, y)\, d\mu(z). \tag{1.6}$$

(iii) For all $t > 0$ and almost all $x, y \in M$,

$$p_t(x, y) = p_t(y, x). \tag{1.7}$$

If the heat semigroup is conservative then (1.3) leads to the following relation between the heat kernel and the Dirichlet form: for all $f, g \in \mathcal{F}$,

$$\mathcal{E}(f, g) = \lim_{t\to 0} \mathcal{E}_t(f, g),$$

[1]By definition, the support $\operatorname{supp} f$ of a function $f \in L^2(M)$ is the set $M \setminus \Omega$ where Ω is the maximal open subset of M such that $f = 0$ a.e. in Ω.

where

$$\mathcal{E}_t(f,g)=\frac{1}{2t}\int_M\int_M (f(x)-f(y))(g(x)-g(y))\,p_t(x,y)\,d\mu(x)\,d\mu(y) \tag{1.8}$$

(see [12]).

Assuming that the heat kernel exists, we are interested in the following upper estimate:

$$p_t(x,y)\le C\min\left\{\frac{1}{V(\rho(t))},\quad \frac{1}{V(d(x,y))}h\left(\frac{d(x,y)}{\rho(t)}\right)\right\}, \tag{UE}$$

which is assumed to be true for all $t>0$ and almost all $x,y\in M$, with some constant $C>0$. Here and throughout the paper, assume that

- $\rho:[0,\infty]\to[0,\infty]$ is a strictly increasing continuous function such that $\rho(0)=0$ and $\rho(\infty)=\infty$;
- $V:[0,\infty)\to[0,\infty)$ is an increasing function such that $V(0)=0$ and $V(r)>0$ if $r>0$;
- $h:[0,\infty]\to[0,\infty]$ is a strictly decreasing continuous function such that $h(1)>0$ and $h(\infty)=0$.

Setting $r:=d(x,y)$, one can equivalently state (UE) as follows:

$$p_t(x,y)\le C\begin{cases}\dfrac{1}{V(\rho(t))}, & \text{if } r\le\rho(t),\\ \dfrac{1}{V(r)}h\left(\dfrac{r}{\rho(t)}\right), & \text{if } r>\rho(t).\end{cases} \tag{1.9}$$

Indeed, the implication (UE)$\Rightarrow$(1.9) is obvious. Now assume that (1.9) holds and we will deduce (UE). If $r\le\rho(t)$ then, using the monotonicity of V and h, we obtain

$$\frac{1}{V(\rho(t))}\le\frac{1}{V(r)}\le\frac{h(1)^{-1}}{V(r)}h\left(\frac{r}{\rho(t)}\right)$$

so that the first line in (1.9) implies (UE). If $r>\rho(t)$ then similarly

$$\frac{1}{V(r)}h\left(\frac{r}{\rho(t)}\right)\le h(1)\frac{1}{V(\rho(t))}$$

so that the second line in (1.9) implies (UE).

It is obvious that (UE) implies the *on-diagonal upper estimate* of $p_t(x,y)$:

$$p_t(x,y)\le\frac{C}{V(\rho(t))}, \tag{DUE}$$

for all $t>0$ and almost all $x,y\in M$.

Let us state two well-known particular cases of the estimate (UE). In the both cases, we assume that

$$V(r)=r^\alpha \ \text{ and }\ \rho(t)=t^{1/\beta}, \tag{1.10}$$

for some positive exponents α,β.

Example 1.2 (*Non-local Dirichlet form*)**.** Let

$$h(s)=s^{-\beta}. \tag{1.11}$$

We claim that (UE) is equivalent to

$$p_t(x,y)\le Ct^{-\alpha/\beta}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{1.12}$$

for all $t>0$ and almost all $x,y\in M$. Indeed, (UE) is equivalent to

$$p_t(x,y)\le\frac{C}{V(\rho(t))+\frac{V(r)}{h(r/\rho(t))}},$$

which for the selected functions V,ρ,h becomes[2]

$$\begin{aligned} p_t(x,y) &\le \frac{C}{t^{\alpha/\beta}+r^{\alpha}\left(r/t^{1/\beta}\right)^{\beta}} \\ &\simeq \frac{C}{t^{\alpha/\beta}+r^{\alpha+\beta}/t} \\ &\simeq \frac{C}{t^{\alpha/\beta}(1+r/t^{1/\beta})^{\alpha+\beta}}. \end{aligned}$$

The estimate (1.12) holds with $\alpha=n$ and with $0<\beta<2$ for the heat kernel of the operator $(-\Delta)^{\beta/2}$ in $\mathbb{R}^n$ (where Δ is the classical Laplace operator), which at the same time is the transition density of the symmetric β-stable process in $\mathbb{R}^n$.

The estimate (1.12) was shown in [7] to be true in the following setting: the metric space (M,d) is a subspace of some $\mathbb{R}^n$, the measure μ of a metric ball $B(x,r)$ (see (1.15)) satisfies the estimate

$$\mu(B(x,r))\simeq r^{\alpha},$$

for all $x\in M$ and $r>0$, and the Dirichlet form is defined by

$$\mathcal{E}(f,g)=\int_M\int_M (f(x)-f(y))(g(x)-g(y))\,J(x,y)\,d\mu(x)\,d\mu(y)$$

where

$$J(x,y)\simeq \frac{1}{d(x,y)^{\alpha+\beta}}.$$

Here α,β are arbitrary constants in the range $\alpha>0$ and $\beta\in(0,2)$.

Example 1.3 (*Local Dirichlet form*)**.** Assume that $\beta>1$ and set

$$h(s)=\exp\left(-c_0 s^{\beta/(\beta-1)}\right),$$

for some $c_0>0$. We claim that (UE) is equivalent to

$$p_t(x,y)\le Ct^{-\alpha/\beta}\exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right). \tag{1.13}$$

Indeed, if $r:=d(x,y)\le t^{1/\beta}$ then (1.13) is equivalent to

$$p_t(x,y)\le Ct^{-\alpha/\beta},$$

which is exactly the first case of (1.9). Assume now $r>t^{1/\beta}$. Then (1.9) becomes

$$p_t(x,y)\le Cr^{-\alpha}\exp\left(-c\left(\frac{r}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right), \tag{1.14}$$

which clearly implies (1.13). The converse implication (1.13)$\Rightarrow$(1.14) follows from the inequality

$$t^{-\alpha/\beta}\le C_\varepsilon r^{-\alpha}\exp\left(\varepsilon\left(\frac{r}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right),$$

which is true for any $\varepsilon>0$, with a big enough constant C_ε.

The purpose of this paper is to give some new equivalent characterizations of (UE). We emphasize that the argument in this paper is purely *analytical*, without recourse to the theory of Markov process. Our main results show how one can obtain the estimate (UE) from the diagonal upper bound (DUE) and some additional conditions. All known so far results have used some probabilistic conditions such as the first exit time from a ball, etc. – cf. [2, 13, 16]. For strongly recurrent graphs, the reader may refer to [3]. See [15] for an analytical approach on effective-resistance metric spaces.

[2]The relation $f\simeq g$ means that $C^{-1}f\le g\le Cf$ for some positive constant C, for the specified range of the arguments of functions f,g.

As for the diagonal upper bound, there are plenty of various equivalent characterization of (DUE) in terms of the Nash-type inequality [6], [8], [18], the Faber-Krahn inequality [11], the Sobolev inequality [19], the log-Sobolev inequality [9], etc.

To explain the results, let us introduce some notation and terminology. Let

$$B(x,r) := \{y \in M : d(y,x) < r\} \tag{1.15}$$

denote a metric ball in (M,d). Consider the following conditions:

- For all $r \geq 0$ and $x \in M$,

$$\mu\left(B(x,r)\right) \leq CV(r). \tag{V}$$

- For all $t, r > 0$ and almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,z)\, d\mu(z) \leq Ch\left(\frac{r}{\rho(t)}\right). \tag{T}$$

- For all $t, r, R > 0$, $y \in M$ and almost all $x \in M$,

$$\int_{B(x,r)^c \cap B(y,R)} p_t(x,z)\, d\mu(z) \leq C\frac{V(R)}{V(r)} h\left(\frac{r}{\rho(t)}\right). \tag{T'}$$

In all conditions, C is a positive constant that is independent on the variables in question and that can take different values on difference occurrences.

The integral in (T) should be understood as follows:

$$\int_{B(x,r)^c} p_t(x,z)\, d\mu(z) := \int_M p_t\left(x,z\right) \mathbf{1}_{B(x,r)^c}\left(z\right) d\mu\left(z\right).$$

Indeed, the function $F\left(x,z\right) = p_t\left(x,z\right) \mathbf{1}_{B(x,r)^c}\left(z\right)$ is obviously measurable jointly in x, z so that by Fubini's theorem the integral $\int_M F\left(x,z\right) d\mu\left(z\right)$ is well-defined as a measurable function of x. On the contrary, the integral in (T) cannot be understood as the value on the diagonal $\{x = y\}$ of the function

$$G\left(x,y\right) := \int_{B(y,r)^c} p_t\left(x,z\right) d\mu\left(z\right),$$

because a measurable function cannot be restricted to a set of measure zero.

It is easy to show that (T) can be equivalently stated as follows: for all $t, r > 0$ and all $x_0 \in M$,

$$\operatorname*{essup}_{x \in B(x_0,r/2)} \int_{B(x_0,r)^c} p_t(x,z)\, d\mu(z) \leq Ch\left(\frac{r}{\rho(t)}\right) \tag{1.16}$$

(cf. Remark 3.3 for the proof). The estimate (1.16) can be interpreted as an upper bound for the function $u\left(t,x\right) = P_t \mathbf{1}_{B(x_0,r)^c}$, which is illustrated on Fig. 1.

The estimate (T') can be reformulated similarly and is illustrated on Fig. 2.

Our first main result – Theorem 2.1, says that, if the function ρ satisfies the doubling property[3], functions V and h are polynomial-like (see (2.1)-(2.5)), and the measure of the balls satisfies (V), then

$$(UE) \Leftrightarrow (DUE) + (T) + (T').$$

For example, the setting of the Example 1.2 matches the hypotheses of Theorem 2.1. A similar equivalence for (UE) (under the assumptions (1.10) and (1.11)) was proved in [4] although instead of the conditions (T) and (T'), two alternative hypotheses were used, which were stated in terms of the exit time of the associated jump process.

[3] A function $f : [0,\infty) \to [0,\infty)$ is said to satisfy the *doubling property* if there is a constant $c > 0$ such that

$$f(2r) \leq cf(r) \quad \text{for all } r \geq 0. \tag{1.17}$$

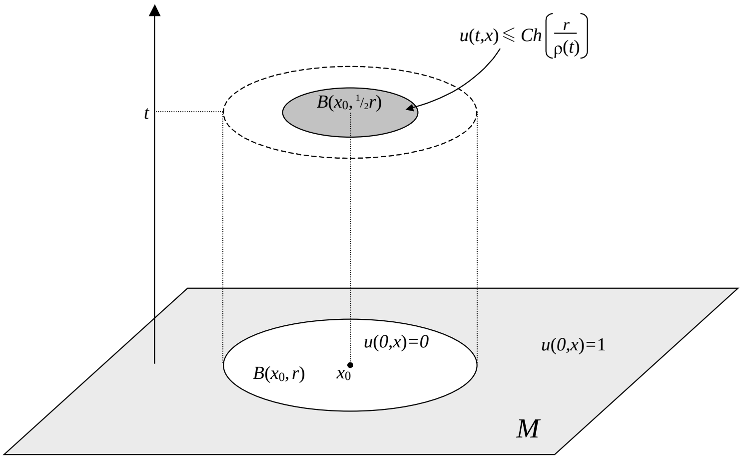

FIGURE 1. Function $u(t,\cdot) = P_t \mathbf{1}_{B(x_0,r)^c}$

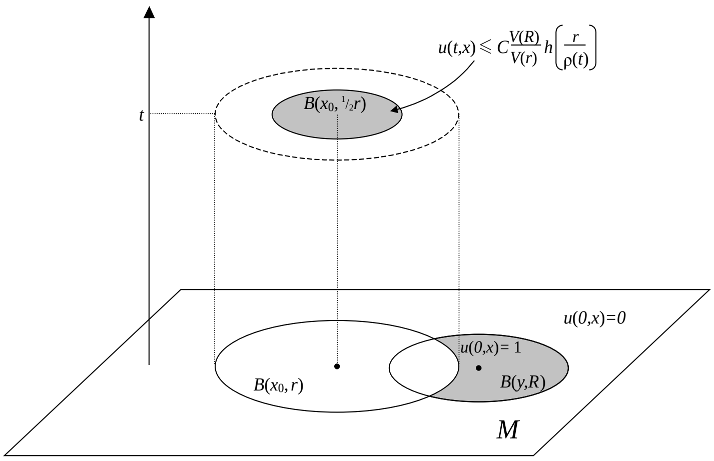

FIGURE 2. Function $u(t,\cdot) = P_t \, \mathbf{1}_{B(x_0,r)^c \cap B(y,R)}$

Our second main result – Theorem 3.10, treats the case when the Dirichlet form is local. In this case, one expects the upper bound of sub-Gaussian type as in Example 1.3. Consider the following modification of the condition (UE):

$$(UE_{\exp}) \qquad p_t(x,y) \leq \frac{C}{V(\rho(t))} \exp\left(-t\,\Phi\left(\frac{c\,d(x,y)}{t}\right)\right),$$

where

$$\Phi(s) = \sup_{\lambda>0}\left\{\frac{s}{\rho(1/\lambda)} - \lambda\right\}.$$

For example, under the conditions (1.10) with $\beta > 1$, we obtain $\Phi(s) = cs^{\frac{\beta}{\beta-1}}$, and ($UE_{\exp}$) becomes (1.13).

Let us introduce the following weak version of the condition (T): for any $\varepsilon > 0$ there exists $K > 0$ such that, for all r and t such that $r \geq K\rho(t)$ and for almost all $x \in M$,

$$(T_{weak}) \qquad \int_{B(x,r)^c} p_t(x,z)\,d\mu(z) \leq \varepsilon.$$

Equivalently, (T_{weak}) means that (T) holds with some (unspecified) function h such that $h(s) \to 0$ as $s \to \infty$. Theorem 3.10 says that if $(\mathcal{E},\mathcal{F})$ is a regular, conservative, local Dirichlet form in

$L^2(M)$, if all metric balls in M are precompact and satisfy (V), and if the functions ρ and V are doubling, then

$$(UE_{\exp}) \Leftrightarrow (DUE) + (T_{weak})$$

(in this result, we do not use the condition (T')). For the case $\rho(t) = t^{1/\beta}$, this equivalence was proved in [13], using the probabilistic approach.

The main ingredients in the proof of Theorem 3.10 are Theorems 3.1, 3.4 that are of independent interest. The main point of those theorems is that the locality of the Dirichlet form allows to self-improve the condition (T_{weak}) thus leading to the following inequality

$$\int_{B(x,r)^c} p_t(x,z)\, d\mu(z) \le C \exp\left(-t\,\Phi\left(c\frac{r}{t}\right)\right),$$

which is true for all $t, r > 0$ and for almost all $x \in M$. The proof of Theorems 3.1, 3.4 use the maximum principles for the resolvent equation (Proposition 4.6) and for the heat equation (Proposition 4.11), through their consequences – Corollary 4.15 and Lemma 4.18. To make the account self-contained, we present in Appendix the analytic proofs of the maximum principles, which may be of their own interest.

NOTATION. Letters $c, C, K > 0$ and $\varepsilon \in (0,1)$ denote the constants whose values may change at each occurrence.

If A is a subset of M then A^c is its complement, that is, $A^c = M \setminus A$.

If $B = B(x,r)$ is a ball in (M,d) then $\alpha B := B(x, \alpha r)$.

ACKNOWLEDGEMENTS. The authors are grateful to Takashi Kumagai for valuable discussions, which have led to improvements of Theorem 3.4.

2. Upper bound with a polynomial tail

In this section, we show that (UE) is equivalent to $(DUE)+(T)+(T')$ under some additional mild assumptions on V, h and ρ. Consider the following conditions:

$$\frac{V(r_2)}{V(r_1)} \le C\left(\frac{r_2}{r_1}\right)^{\alpha_1}, \tag{2.1}$$

$$\frac{V(r_2)}{V(r_1)} \ge c\left(\frac{r_2}{r_1}\right)^{\alpha_2}, \tag{2.2}$$

$$\frac{h(r_2)}{h(r_1)} \le C\left(\frac{r_2}{r_1}\right)^{-\beta_1}, \tag{2.3}$$

$$\frac{h(r_2)}{h(r_1)} \ge c\left(\frac{r_2}{r_1}\right)^{-\beta_2}, \tag{2.4}$$

where each inequality is assumed to be true for all $0 < r_1 < r_2 < \infty$ and for some positive constants $c, C, \alpha_1, \alpha_2, \beta_1, \beta_2$. Clearly, if (2.1) and (2.2) hold simultaneously then $\alpha_2 \le \alpha_1$, and if (2.3) and (2.4) hold simultaneously then $\beta_1 \le \beta_2$.

Theorem 2.1. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in $L^2(M)$ with the heat kernel $p_t(x,y)$. Assume that the function ρ is doubling, and that V and h satisfy (2.1)-(2.2) and (2.3)-(2.4), respectively, with the additional condition that*

$$\frac{\alpha_1}{\beta_1} < \left[\frac{\alpha_2}{\beta_2}\right] + 1. \tag{2.5}$$

Assume also that the volume of the metric balls in M satisfies (V). Then

$$(DUE) + (T) + (T') \Leftrightarrow (UE).$$

Here $[s]$ denotes by the integer part of the number s. The proof of Theorem 2.1 will be split into a series of Lemmas.

Lemma 2.2. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form with the heat kernel $p_t(x,y)$. Assume that the volume of the metric balls in M satisfies (V). Let function V be doubling and h satisfy (2.3)-(2.4). Then*

$$(UE) \Rightarrow (T) + (T') + (DUE).$$

Proof. The implication $(UE)\Rightarrow(DUE)$ is trivial. Let us show that $(UE)\Rightarrow(T)$. Fix $t>0$ and $r>0$. If $\rho(t)\geq r$ then the monotonicity of h implies, for almost all $x\in M$,

$$\int_{B(x,r)^c} p_t(x,y)\,d\mu(y) \leq 1 \leq Ch\left(\frac{r}{\rho(t)}\right),$$

where $C=\frac{1}{h(1)}$, which obviously matches (T). Assume now that $\rho(t)<r$. It follows from (UE) and the monotonicity of V and h that, for almost all $x,z\in M$ such that $d(z,x)\geq s$,

$$p_t(x,z) \leq \frac{C}{V(s)} h\left(\frac{s}{\rho(t)}\right). \tag{2.6}$$

Using (2.6), (V), the doubling property of V, and (2.3)-(2.4), we obtain, for almost all $x\in M$,

$$\begin{aligned}
\int_{B(x,r)^c} p_t(x,z)d\mu(z) &\leq \sum_{k=0}^{\infty}\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} p_t(x,z)\,d\mu(z)\\
&\leq \sum_{k=0}^{\infty}\frac{C}{V(2^k r)}h\left(\frac{2^k r}{\rho(t)}\right)\mu\left(B(x,2^{k+1}r)\setminus B(x,2^k r)\right)\\
&\leq \sum_{k=0}^{\infty} C\,\frac{V(2^{k+1}r)}{V(2^k r)}h\left(\frac{2^k r}{\rho(t)}\right)\\
&\leq \sum_{k=0}^{\infty} C\left(2^k\right)^{-\beta_1} h\left(\frac{r}{\rho(t)}\right)\\
&\leq Ch\left(\frac{r}{\rho(t)}\right).
\end{aligned}$$

Condition (T') is deduced from (2.6) and (V) as follows:

$$\begin{aligned}
\int_{B(x,r)^c\cap B(y,R)} p_t(x,z)\,d\mu(z) &\leq \frac{C}{V(r)}h\left(\frac{r}{\rho(t)}\right)\mu\left(B(y,R)\cap B(x,r)^c\right)\\
&\leq \frac{C}{V(r)}h\left(\frac{r}{\rho(t)}\right)V(R).
\end{aligned}$$

This finishes the proof. □

For any $q\geq 0$, consider the following condition

$$p_t(x,y) \leq \frac{C_q}{V(\rho(t))}h^q\left(\frac{d(x,y)}{\rho(t)}\right), \tag{H_q}$$

which should be true for some $C_q>0$, all $t>0$, and almost all $x,y\in M$. Clearly (DUE) is equivalent to (H_0), and

$$(H_0)+(H_{q_2}) \Rightarrow (H_{q_1}) \tag{2.7}$$

provided $0\leq q_1<q_2<\infty$.

Lemma 2.3. *Assume that (T) and (T') hold where ρ is doubling, V satisfies (2.1)-(2.2), and h satisfies (2.4) If $p_t(x,y)$ satisfies (H_q) where*

$$0\leq q<\frac{\alpha_2}{\beta_2}, \tag{2.8}$$

then it also satisfies (H_{q+1}).

Proof. We need to prove that, for any $t>0$ and almost all $x,y\in M$,

$$p_{2t}(x,y)\leq \frac{C}{V(\rho(t))}h^{q+1}\left(\frac{r}{\rho(t)}\right), \tag{2.9}$$

where $r=\frac{1}{2}d(x,y)$, which implies (H_{q+1}) due to the doubling properties of functions V,ρ and (2.4). If $r\leq\rho(t)$ then (2.9) follows from (H_q) so that we can assume $r>\rho(t)$.

By the semigroup property, we have, for all $x,y\in M$,

$$\begin{aligned} p_{2t}(x,y) &= \int_M p_t(x,z)p_t(z,y)\,d\mu(z) \\ &\leq \int_{B(x,r)^c} p_t(x,z)p_t(z,y)\,d\mu(z)+\int_{B(y,r)^c} p_t(x,z)p_t(z,y)\,d\mu(z). \end{aligned} \tag{2.10}$$

On order to estimate the first term on the right-hand side of (2.10) (the second term can be treated similarly), split it into a "good" and a "bad" part:

$$\int_{B(x,r)^c} p_t(x,z)p_t(z,y)\,d\mu(z)=g(t,x,y)+b(t,x,y), \tag{2.11}$$

where

$$g(t,x,y):=\int_{B(x,r)^c\setminus B(y,r)} p_t(x,z)p_t(z,y)\,d\mu(z) \tag{2.12}$$

and

$$b(t,x,y):=\int_{B(y,r)} p_t(x,z)p_t(z,y)\,d\mu(z). \tag{2.13}$$

Let us first estimate the "good" part. It follows from (H_q) that, for almost all $y\in M$ and $z\notin B(y,r)$,

$$p_t(z,y)\leq \frac{C}{V(\rho(t))}h^{q}\left(\frac{r}{\rho(t)}\right). \tag{2.14}$$

Substituting this into (2.12) and using (T) we obtain, for almost all $x,y\in M$,

$$\begin{aligned} g(t,x,y) &\leq \frac{C}{V(\rho(t))}h^{q}\left(\frac{r}{\rho(t)}\right)\int_{B(x,r)^c} p_t(x,z)\,d\mu(z) \\ &\leq \frac{C}{V(\rho(t))}h^{q+1}\left(\frac{r}{\rho(t)}\right). \end{aligned} \tag{2.15}$$

In order to estimate the "bad" part, represent it in the form

$$b(t,x,y)=\sum_{k=0}^{\infty}\int_{B(y,2^{-k}r)\setminus B(y,2^{-(k+1)}r)} p_t(x,z)p_t(z,y)\,d\mu(z) \tag{2.16}$$

(see Fig. 3).

By condition (H_q) and (2.4), we see that, for almost all $y\in M$ and $z\in B(y,2^{-k}r)\setminus B(y,2^{-(k+1)}r)$,

$$p_t(z,y)\leq \frac{C}{V(\rho(t))}h^{q}\left(\frac{2^{-(k+1)}r}{\rho(t)}\right)\leq C\frac{2^{k\beta_2 q}}{V(\rho(t))}h^{q}\left(\frac{r}{\rho(t)}\right). \tag{2.17}$$

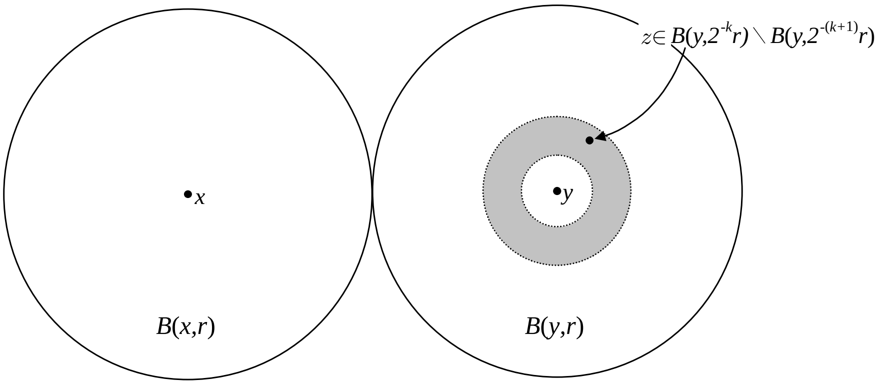

FIGURE 3. Estimating the "bad" part

Using (2.17), (T'), and (2.2), we obtain

$$\begin{aligned}
&\int_{B(y,2^{-k}r)\setminus B(y,2^{-(k+1)}r)} p_t(x,z)p_t(z,y)\,d\mu(z)\\
&\qquad\le C\frac{2^{k\beta_2 q}}{V(\rho(t))}h^q\left(\frac{r}{\rho(t)}\right)\int_{B(x,r)^c\cap B(y,2^{-k}r)} p_t(x,z)\,d\mu(z)\\
&\qquad\le C\frac{2^{k\beta_2 q}}{V(\rho(t))}h^q\left(\frac{r}{\rho(t)}\right)\frac{V(2^{-k}r)}{V(r)}h\left(\frac{r}{\rho(t)}\right)\\
&\qquad\le C\frac{2^{k\beta_2 q}2^{-k\alpha_2}}{V(\rho(t))}h^{q+1}\left(\frac{r}{\rho(t)}\right) \qquad (2.18)
\end{aligned}$$

(here we have used the obvious fact that the balls $B(x,r)$ and $B(y,2^{-k}r)$ are disjoint).
It follows from (2.16), (2.18), and (2.8) that

$$b(t,x,y)\le\frac{C}{V(\rho(t))}h^{q+1}\left(\frac{r}{\rho(t)}\right)\sum_{k=1}^{\infty}2^{k(\beta_2 q-\alpha_2)}\le\frac{C}{V(\rho(t))}h^{q+1}\left(\frac{r}{\rho(t)}\right). \qquad (2.19)$$

Combining (2.11), (2.15) and (2.19) we obtain (2.9), which finishes the proof. □

Corollary 2.4. *Under the hypotheses of Lemma 2.3, we have*

$$(DUE)\Rightarrow(H_q)$$

for any $0\le q<\left[\frac{\alpha_2}{\beta_2}\right]+1$.

Proof. Recall that (DUE) is equivalent to (H_0). Repeatedly applying Lemma 2.3, we obtain the conclusion. □

Lemma 2.5. *Assume that (T') holds where ρ is doubling, V satisfies (2.1)-(2.2), and h satisfies (2.3)-(2.4). Then (DUE) and (H_q) with some $q>\frac{\alpha_1}{\beta_1}$ imply that*

$$p_t(x,y)\le\frac{C}{V(d(x,y))}h\left(\frac{d(x,y)}{\rho(t)}\right) \qquad (2.20)$$

for all $t>0$ and almost all $x,y\in M$.

Proof. The argument is similar to that of Lemma 2.3. Let us prove that, for all $t>0$ and almost all $x,y\in M$,

$$p_{2t}(x,y)\le\frac{C}{V(r)}h\left(\frac{r}{\rho(t)}\right), \qquad (2.21)$$

where $r=\frac{1}{2}d\left(x,y\right)$, which will imply (2.20). If $r\leq\rho(t)$ then (2.21) immediately follows from (DUE). Assume in the sequel that $r>\rho\left(t\right)$. As in the proof of Lemma 2.3, it suffices to show that

$$\int_{B(x,r)^c} p_t(x,z)p_t(z,y)d\mu(z)\leq\frac{C}{V\left(r\right)}h\left(\frac{r}{\rho(t)}\right).$$

Writing for simplicity ρ instead of $\rho\left(t\right)$, we have

$$\begin{aligned}\int_{B(x,r)^c} p_t(x,z)p_t(z,y)d\mu(z) &\leq \int_{B(y,\rho)} p_t(x,z)p_t(z,y)d\mu(z)\\ &\quad+\sum_{k=0}^{\infty}\int_{B(x,r)^c\cap\left(B(y,2^{k+1}\rho)\setminus B(y,2^k\rho)\right)} p_t(x,z)p_t(z,y)\,d\mu(z)\end{aligned}\tag{2.22}$$

(see Fig. 4).

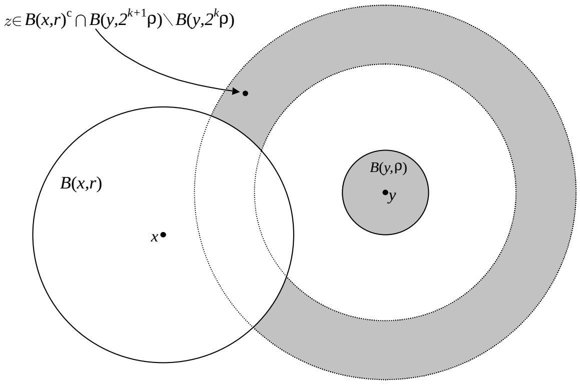

FIGURE 4. Illustration to the estimate (2.22)

It follows from $p_t(z,y)\leq\frac{C}{V(\rho)}$ a.e. and (T') that

$$\begin{aligned}\int_{B(y,\rho)} p_t(x,z)p_t(z,y)d\mu(z) &\leq \frac{C}{V(\rho)}\int_{B(y,\rho)} p_t(x,z)d\mu(z)\\ &= \frac{C}{V(\rho)}\int_{B(x,r)^c\cap B(y,\rho)} p_t(x,z)d\mu(z)\\ &\leq \frac{C}{V(\rho)}\frac{V(\rho)}{V(r)}h\left(\frac{r}{\rho}\right)\\ &= \frac{C}{V\left(r\right)}h\left(\frac{r}{\rho}\right)\end{aligned}\tag{2.23}$$

(we have used here the fact that the balls $B\left(x,r\right)$ and $B(y,\rho)$ are disjoint).

On the other hand, (H_q) and (2.3) imply that, for almost all $y\in M$ and $z\in B(y,2^{k+1}\rho)\setminus B(y,2^k\rho)$,

$$p_t(z,y)\leq\frac{C}{V(\rho)}h^q\left(\frac{2^k\rho}{\rho}\right)\leq\frac{C\,2^{-kq\beta_1}}{V(\rho)}.\tag{2.24}$$

Next, by (T') and (2.1),

$$\begin{aligned}\int_{B(x,r)^c\cap B(y,2^{k+1}\rho)} p_t(x,z)\,d\mu(z) &\leq C\frac{V(2^{k+1}\rho)}{V(r)}h\left(\frac{r}{\rho}\right)\\ &\leq C2^{k\alpha_1}\frac{V(\rho)}{V(r)}h\left(\frac{r}{\rho}\right).\end{aligned}\tag{2.25}$$

Therefore, we obtain from (2.24) and (2.25) that, for almost all $x, y \in M$,

$$\sum_{k=0}^{\infty} \int_{B(x,r)^c \cap \left(B(y,2^{k+1}\rho) \setminus B(y,2^k\rho)\right)} p_t(x,z) p_t(z,y)\, d\mu(z)$$

$$\leq \sum_{k=0}^{\infty} \frac{C 2^{-kq\beta_1}}{V(\rho)} \int_{B(x,r)^c \cap B(y,2^{k+1}\rho)} p_t(x,z)\, d\mu(z)$$

$$\leq \sum_{k=0}^{\infty} \frac{C 2^{-kq\beta_1}}{V(\rho)} \left(2^{k\alpha_1} \frac{V(\rho)}{V(r)} h\left(\frac{r}{\rho}\right)\right)$$

$$\leq \frac{C}{V(r)} h\left(\frac{r}{\rho}\right), \tag{2.26}$$

where the series converges due to $q\beta_1 > \alpha_1$. Combining (2.22), (2.23) and (2.26), we finish the proof. □

Proof of Theorem 2.1. The implication

$$(UE) \Rightarrow (DUE) + (T) + (T')$$

was obtained in Lemma 2.2. Let us prove the converse, that is,

$$(DUE) + (T) + (T') \Rightarrow (UE).$$

By Corollary 2.4, we have

$$(DUE) + (T) + (T') \Rightarrow (H_q),$$

for any $0 < q < \left[\frac{\alpha_2}{\beta_2}\right] + 1$. Since $\frac{\alpha_1}{\beta_1} < \left[\frac{\alpha_2}{\beta_2}\right] + 1$, we obtain that (H_q) holds for some $q > \frac{\alpha_1}{\beta_1}$. Hence, by Lemma 2.5, we obtain (2.20). Combining (2.20) with (DUE) we obtain (UE). □

3. Upper bound with an exponential tail

The main purpose of this section is to prove the upper bound for $p_t(x,y)$ with the exponential tail provided the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular, conservative, and local.

3.1. The tail of the heat semigroup.

Here we do not assume the existence of the heat kernel and work with the function $P_t \mathbf{1}_{B^c}$, where $B = B(x_0, r)$. We use αB as a shorthand for $B(x_0, \alpha r)$ as stated before. The reader may refer to the definition of P_t^B in Section 4.

Theorem 3.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular conservative Dirichlet form in $L^2(M)$ and let all metric balls in M be precompact. Let $\rho : [0, \infty) \to [0, \infty)$ be any continuous strictly increasing function with $\rho(0) = 0$, $\rho(\infty) = \infty$, and let ρ satisfy the doubling property. Then the following conditions are equivalent.*

(i) For any $\varepsilon \in (0,1)$ there exists $K > 0$ such that, for any $t > 0$ and any ball $B = B(x_0, r)$ with $r \geq K\rho(t)$,

$$P_t \mathbf{1}_{B^c} \leq \varepsilon \ \text{ a.e. } in\ \frac{1}{4}B. \tag{3.1}$$

(ii) For any $\varepsilon \in (0,1)$ there exists $K > 0$ such that, for any $t > 0$ and any ball $B = B(x_0, r)$ with $r \geq K\rho(t)$,

$$P_t^B \mathbf{1}_B \geq 1 - \varepsilon \ \text{ a.e. } in\ \frac{1}{4}B. \tag{3.2}$$

(iii) For any $\varepsilon \in (0,1)$ there exists $K > 0$ such that, for any $\lambda > 0$ and any ball $B = B(x_0, r)$ with $r \geq K\rho\left(\frac{1}{\lambda}\right)$,

$$\lambda R_\lambda^B \mathbf{1}_B \geq 1 - \varepsilon \ \text{ a.e. } in\ \frac{1}{4}B. \tag{3.3}$$

Remark 3.2. As one can see below, the doubling property of ρ is mildly used only in the proof of the implication $(ii) \Rightarrow (iii)$. In fact, the doubling property can be dropped from the hypotheses, but then conditions $r \ge K\rho(t)$ and $r \ge K\rho\left(\frac{1}{\lambda}\right)$ should be replaced respectively by $r \ge K\rho(ct)$ and $r \ge K\rho\left(\frac{c}{\lambda}\right)$, for a positive constant $c > 0$.

Remark 3.3. If the heat semigroup P_t possesses the heat kernel $p_t(x, y)$ then condition (i) can be equivalently stated as follows: for any $\varepsilon \in (0, 1)$ there exists $K > 0$ such that, for all $t > 0$, $r \ge K\rho(t)$, and almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x, y)\, d\mu(y) \le \varepsilon. \tag{3.4}$$

Indeed, for any ball $B(x_0, r)$ and for almost all $x \in B(x_0, r/4)$ (or even $x \in B(x_0, r/2)$), we have

$$P_t \mathbf{1}_{B(x_0,r)^c}(x) = \int_{B(x_0,r)^c} p_t(x, y)\, d\mu(y) \le \int_{B(x,r/2)^c} p_t(x, y)\, d\mu(y),$$

so that (3.4) implies (3.1) (with K being replaced by $2K$). Similarly, for almost all $x \in B(x_0, r/2)$,

$$\int_{B(x,r)^c} p_t(x, y)\, d\mu(y) \le \int_{B(x_0,r/2)^c} p_t(x, y)\, d\mu(y) = P_t \mathbf{1}_{B(x_0,r/2)^c}(x),$$

so that (3.1) implies (3.4), for almost all $x \in B(x_0, r/8)$. Covering M by a countable family of balls of radius $r/8$, we obtain that (3.4) holds for almost all $x \in M$.

Proof of Theorem 3.1. $(i) \Rightarrow (ii)$. Applying estimate (4.29) of Lemma 4.18 to function $f = \mathbf{1}_{\frac{1}{2}B}$, we obtain that

$$P_t^B \mathbf{1}_{\frac{1}{2}B}(x) \ge P_t \mathbf{1}_{\frac{1}{2}B}(x) - \sup_{s\in[0,t]} \operatorname*{essup}_{\left(\frac{3}{4}\overline{B}\right)^c} P_s \mathbf{1}_{\frac{1}{2}B}, \tag{3.5}$$

for $t > 0$ and a.e. $x \in M$. For any $x \in \frac{1}{4}B$, we have that $B(x, r/4) \subset \frac{1}{2}B$ (see Fig. 5). Using (3.1) and the identity $P_t 1 = 1$ a.e., we obtain, for any $x \in \frac{1}{4}B$,

$$P_t \mathbf{1}_{\frac{1}{2}B} = 1 - P_t \mathbf{1}_{\left(\frac{1}{2}B\right)^c} \ge 1 - P_t \mathbf{1}_{B(x,r/4)^c}.$$

Applying hypothesis (i) for the ball $B(x, r/4)$, we obtain that

$$P_t \mathbf{1}_{B(x,r/4)^c} \le \varepsilon \ \text{ a.e. in } B(x, r/16),$$

provided

$$\frac{r}{4} \ge K\rho(t) \tag{3.6}$$

with sufficiently large K. It follows that, for any $x \in \frac{1}{4}B$,

$$P_t \mathbf{1}_{\frac{1}{2}B} \ge 1 - \varepsilon \text{ a.e. in } B(x, r/16),$$

whence

$$P_t \mathbf{1}_{\frac{1}{2}B} \ge 1 - \varepsilon \text{ a.e. in } \frac{1}{4}B. \tag{3.7}$$

On the other hand, for any $y \in \left(\frac{3}{4}\overline{B}\right)^c$, we have $\frac{1}{2}B \subset B(y, r/4)^c$ (see Fig. 5), whence

$$P_s \mathbf{1}_{\frac{1}{2}B} \le P_s \mathbf{1}_{B(y,r/4)^c}.$$

Applying hypothesis (i) for the ball $B(y, r/4)$ at time s, we obtain that if (3.6) holds for sufficiently large K then, for all $0 < s \le t$,

$$P_s \mathbf{1}_{B(y,r/4)^c} \le \varepsilon \ \text{ a.e. in } B(y, r/16).$$

It follows that, for any $y \in \left(\frac{3}{4}\overline{B}\right)^c$,

$$P_s \mathbf{1}_{\frac{1}{2}B} \le \varepsilon \text{ a.e. in } B(y, r/16),$$

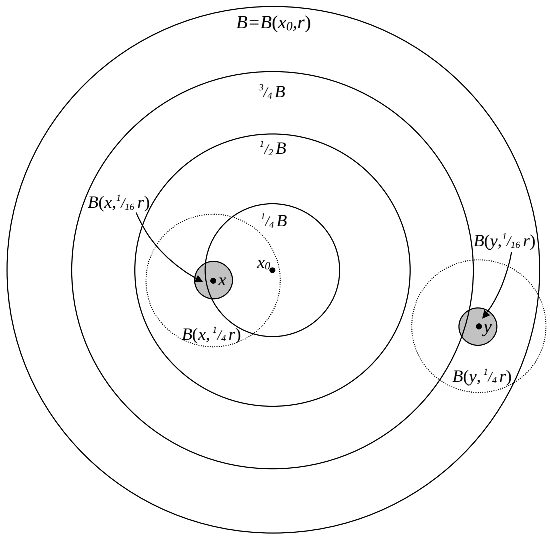

FIGURE 5. Illustration to the proof of $(i) \Rightarrow (ii)$

whence

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \varepsilon \text{ a.e. in } \left(\frac{3}{4}\overline{B}\right)^c . \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we obtain that, under condition (3.6),

$$P_t^B \mathbf{1}_B \geq P_t^B \mathbf{1}_{\frac{1}{2}B} \geq 1 - 2\varepsilon \; \text{ a.e. in } \frac{1}{4}B, \tag{3.9}$$

which is equivalent to (3.2).

$(ii) \Rightarrow (iii)$. By (ii), we have (3.2) provided $t \leq \rho^{-1}(t/K)$, whence

$$\lambda R_\lambda^B \mathbf{1}_B = \lambda \int_0^\infty e^{-\lambda t} P_t^B \mathbf{1}_B dt \geq \lambda \int_0^{\rho^{-1}(\frac{r}{K})} e^{-\lambda t} P_t^B \mathbf{1}_B dt \geq (1-\varepsilon)\left(1 - e^{-\lambda \rho^{-1}(\frac{r}{K})}\right),$$

which holds almost everywhere in $\frac{1}{4}B$. If

$$\lambda \rho^{-1}\left(\frac{r}{K}\right) \geq \log \frac{1}{\varepsilon} \tag{3.10}$$

then we obtain

$$\lambda R_\lambda^B \mathbf{1}_B \geq (1-\varepsilon)^2 \; \text{ a.e. in } \frac{1}{4}B,$$

which is equivalent to (3.3). Condition (3.10) is equivalent to

$$r \geq K\rho\left(\frac{\log \frac{1}{\varepsilon}}{\lambda}\right), \tag{3.11}$$

which, by the doubling property of ρ, is a consequence of $r \geq K_1 \rho\left(\frac{1}{\lambda}\right)$ for sufficiently large K_1.

$(iii) \Rightarrow (i)$. Let us first show that, for all $t, \lambda > 0$,

$$P_t^B \mathbf{1}_B \geq 1 - e^{\lambda t}\left(1 - \lambda R_\lambda^B \mathbf{1}_B\right). \tag{3.12}$$

Indeed, using the facts that $P_s^B \mathbf{1}_B \leq \mathbf{1}_B$ and

$$P_{s+t}^B \mathbf{1}_B = P_t^B \left(P_s^B \mathbf{1}_B\right) \leq P_t^B \mathbf{1}_B,$$

we obtain that

$$\begin{aligned}\lambda R_\lambda^B \mathbf{1}_B &= \lambda \int_0^\infty e^{-\lambda s} P_s^B \mathbf{1}_B \, ds \\ &= \lambda \int_0^t e^{-\lambda s} P_s^B \mathbf{1}_B \, ds + \lambda \int_t^\infty e^{-\lambda s} P_s^B \mathbf{1}_B \, ds \\ &\le \left(1 - e^{-\lambda t}\right) + \lambda \int_0^\infty e^{-\lambda(s+t)} P_{s+t}^B \mathbf{1}_B \, ds \\ &\le 1 - e^{-\lambda t} + e^{-\lambda t} P_t^B \mathbf{1}_B,\end{aligned}$$

thus giving (3.12).

Given $\varepsilon \in (0,1)$, $t > 0$ and $r \ge K\rho(t)$ (where K is defined by hypothesis (iii)), choose λ from condition $r = K\rho\left(\frac{1}{\lambda}\right)$. Then it follows from (3.3) and (3.12) that

$$P_t \mathbf{1}_B \ge P_t^B \mathbf{1}_B \ge 1 - \varepsilon e^{\lambda t} \text{ a.e. in } \frac{1}{4}B.$$

Using the identity $P_t 1 = 1$ and observing that

$$\lambda t \le \lambda \rho^{-1}\left(\frac{r}{K}\right) = 1,$$

we obtain

$$P_t \mathbf{1}_{B^c} = 1 - P_t \mathbf{1}_B \le \varepsilon e^{\lambda t} \le \varepsilon e \text{ a.e. in } \frac{1}{4}B,$$

which is equivalent to (3.1). □

The following statement is an extension of Theorem 3.1 in the case of a local Dirichlet form.

Theorem 3.4. *Assume that all the hypotheses of Theorem 3.1 hold, and in addition that the Dirichlet form $(\mathcal{E},\mathcal{F})$ is local. Then each of conditions (i), (ii), (iii) of Theorem 3.1 is equivalent to the following:*

(iv) *There are $c, C > 0$ such that, for any $t > 0$ and any ball $B = B(x_0, r)$,*

$$\operatorname*{essup}_{\frac{1}{2}B} P_t \mathbf{1}_{B^c} \le C \exp\left(-t\,\Phi\left(c\frac{r}{t}\right)\right), \tag{3.13}$$

where

$$\Phi(s) := \sup_{\lambda > 0}\left\{\frac{s}{\rho(1/\lambda)} - \lambda\right\}. \tag{3.14}$$

Remark 3.5. Obviously, estimate (3.13) with function Φ defined by (3.14) is equivalent to the following: for all $\lambda > 0$,

$$\operatorname*{essup}_{\frac{1}{2}B} P_t \mathbf{1}_{B^c} \le C \exp\left(\lambda t - \frac{cr}{\rho(1/\lambda)}\right). \tag{3.15}$$

Remark 3.6. If the heat semigroup P_t possesses the heat kernel $p_t(x,y)$ then condition (iv) can be equivalently stated as follows: for all $t, r > 0$ and for almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \le C \exp\left(-t\,\Phi\left(c\frac{r}{t}\right)\right), \tag{3.16}$$

which is proved by the argument of Remark 3.3. Estimate (3.15) can be reformulated similarly.

Example 3.7. If $\rho(t) = t^{1/\beta}$ for some $\beta > 1$ then

$$\Phi(s) = \sup_{\lambda > 0}\left\{s\lambda^{1/\beta} - \lambda\right\} = C_\beta s^{\frac{\beta}{\beta-1}}$$

so that (3.13) becomes

$$\operatorname*{essup}_{\frac{1}{2}B} P_t \mathbf{1}_{B^c} \le C \exp\left(-c\left(\frac{r^\beta}{t}\right)^{\frac{1}{\beta-1}}\right).$$

Proof of Theorem 3.4. Observe first that the function Φ is non-negative on $[0,+\infty)$ (just let $\lambda \to 0$ in (3.14)), monotone increasing, and satisfies the following inequality: for all $s \geq 0$ and $A \geq 1$,

$$\Phi(As) \geq A\Phi(s), \tag{3.17}$$

which is proved as follows:

$$\begin{aligned} \Phi(As) &= \sup_{\lambda>0}\left\{\frac{As}{\rho(1/\lambda)} - \lambda\right\} \\ &\geq A\sup_{\lambda>0}\left\{\frac{s}{\rho(1/\lambda)} - \lambda\right\} \\ &= A\Phi(s). \end{aligned}$$

Let us prove that $(iv) \Rightarrow (i)$. Assuming that $r \geq 2c^{-1}K\rho(t)$ (where $K > 1$ is to be specified below) and using (3.17), (3.14), we obtain

$$\begin{aligned} \Phi\left(c\frac{r}{t}\right) &\geq \Phi\left(2K\frac{\rho(t)}{t}\right) \geq K\Phi\left(\frac{2\rho(t)}{t}\right) \\ &= K\sup_{\lambda>0}\left\{\frac{2\rho(t)}{t\rho(1/\lambda)} - \lambda\right\} \geq \frac{K}{t}, \end{aligned}$$

where the last inequality follows by setting $\lambda = 1/t$. Hence, (3.13) implies that

$$P_t\mathbf{1}_{B^c} \leq C\exp(-K) \text{ a.e. in } \frac{1}{2}B.$$

Choosing K big enough, we obtain (3.1).

Now we prove the main implication $(iii) \Rightarrow (iv)$. This proof is rather long and will be split into five steps.

Step 1. We claim that, for any $\varepsilon > 0$, there exists $K > 0$ such that if a function $w \in \mathcal{F} \cap L^\infty(M)$ is such that $0 \leq w \leq 1$ in a ball $B = B(x_0, r)$ and w satisfies weakly in B the equation

$$-\Delta w + \lambda w = 0,$$

where $\lambda > 0$ and r are related by

$$r \geq K\rho\left(\frac{1}{\lambda}\right),$$

then

$$w \leq \varepsilon \text{ a.e. in } \frac{1}{4}B.$$

Indeed, since the Dirichlet form is local and the ball is precompact, we have by Corollary 4.15 (see Appendix) that

$$w \leq 1 - \lambda R_\lambda^B\mathbf{1}_B \text{ a.e. in } B.$$

By (iii), we have

$$\lambda R_\lambda^B\mathbf{1}_B \geq 1 - \varepsilon \text{ a.e. in } \frac{1}{4}B,$$

provided $r \geq K\rho\left(\frac{1}{\lambda}\right)$, where K is now defined by condition (iii). Combining the above two lines, we finish the proof of the claim.

Step 2. Let us show that there exists $c > 0$ such that, for any ball $B = B(x_0, r)$ and any $\lambda > 0$,

$$\operatorname*{essup}_{B(x_0,\delta)}(\lambda R_\lambda 1_{B^c}) \leq \exp\left(-\frac{cr}{\rho(1/\lambda)} + 1\right), \tag{3.18}$$

where $\delta = \delta(\lambda) > 0$. Choose some $R > 4r$ and consider the functions

$$\phi = \mathbf{1}_{B(x_0,R)\setminus B(x_0,r)}$$

and

$$u = \lambda R_\lambda \phi. \tag{3.19}$$

It suffices to prove that

$$\operatorname*{essup}_{B(x_0,\delta)} u \le \exp\left(-\frac{cr}{\rho\,(1/\lambda)} + 1\right), \tag{3.20}$$

and then let $R \to \infty$. Since $0 \le \phi \le 1$ and $\phi \in L^2(M)$, we have $0 \le u \le 1$ on M, $u \in \mathrm{dom}(\Delta) \subset \mathcal{F}$, and u satisfies in M the equation

$$-\Delta u + \lambda u = \lambda \phi. \tag{3.21}$$

It suffices to assume that

$$cr \ge \rho\left(\frac{1}{\lambda}\right), \tag{3.22}$$

(where $c > 0$ is to be specified later) because otherwise (3.20) is trivially satisfied due to $u \le 1$.

Let $n \ge 2$ be an integer to be determined later on. For any $1 \le i \le n$, set $r_i = \frac{ir}{n}$,

$$b_i = \operatorname*{essup}_{B(x_0,r_i)} u,$$

and, for $1 \le i < n$,

$$w_i(x) = \frac{u(x)}{b_{i+1}}.$$

Clearly, $w_i \in \mathcal{F} \cap L^\infty(M)$. Since $\phi = 0$ in $B(x_0, r)$, it follows from (3.21) that

$$-\Delta w_i + \lambda w_i = 0 \text{ in } B(x_0, r).$$

By definition of b_{i+1}, we have $0 \le w_i \le 1$ in $B(x_0, r_{i+1})$. In particular, the same inequality holds in any ball $B(x, r_1)$ for any $x \in B(x_0, r_i)$ (see Fig. 6). Therefore, by Step 1 with $\varepsilon = e^{-1}$,

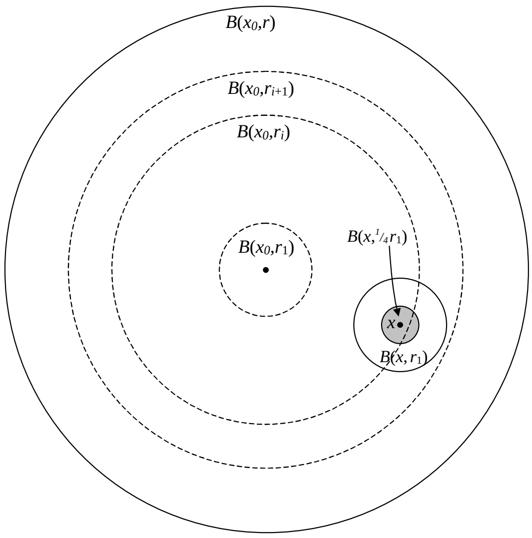

FIGURE 6. Balls $B(x_0, r_i)$ and $B(x_0, r_{i+1})$

we have that

$$w_i \le e^{-1} \text{ a.e. in } B\left(x, \frac{1}{4} r_1\right),$$

provided

$$r_1 \ge K\rho\left(\frac{1}{\lambda}\right), \tag{3.23}$$

for an appropriate constant K. It follows that

$$\operatorname*{essup}_{B(x_0,r_i)} w_i \le e^{-1},$$

that is,

$$b_i \le e^{-1} b_{i+1}. \tag{3.24}$$

Before we proceed further, let us make sure that condition (3.23) is satisfied. Since $r_1 = r/n$, it is equivalent to

$$n \le \frac{r}{K\rho\left(\frac{1}{\lambda}\right)},$$

so that we can choose

$$n = \left[\frac{r}{K\rho\left(\frac{1}{\lambda}\right)}\right].$$

Choosing in (3.22) $c = \frac{1}{2K}$, we obtain that $n \ge 2$. Note also that

$$n \ge \frac{2cr}{\rho\left(\frac{1}{\lambda}\right)} - 1.$$

Now, iterating (3.24) and using the fact that $b_n \le 1$, we obtain

$$b_1 \le e^{-(n-1)} b_n \le e^{-n/2} \le \exp\left(-\frac{cr}{\rho\left(\frac{1}{\lambda}\right)} + 1\right).$$

Clearly, this implies (3.20), where δ can be anything $\le r_1 = \frac{r}{n}$; for example, set $\delta = K\rho\left(\frac{1}{\lambda}\right)$.

Let us note that the iteration argument in this part of the proof is motivated by that in [14] for the setting of infinite graphs.

Step 3. Let us show that there is $K \ge 1$ such that for any ball $B = B(x_0, r)$ with

$$r \ge K\rho\left(\frac{1}{\lambda}\right), \tag{3.25}$$

we have

$$\operatorname*{essinf}_{(2B)^c} \left(\lambda R_\lambda \mathbf{1}_{B^c}\right) \ge \frac{1}{2}. \tag{3.26}$$

Indeed, for any $x \in (2B)^c$, we have $B(x, r) \subset B^c$, whence by condition (iii),

$$\lambda R_\lambda \mathbf{1}_{B^c} \ge \lambda R_\lambda \mathbf{1}_{B(x,r)} \ge \frac{1}{2} \text{ a.e. in } B\left(x, \frac{1}{4}r\right).$$

provided (3.25) is satisfied with an appropriate K. Hence, (3.26) follows.

Step 4. Let us show that, for any non-negative function $f \in L^\infty(M)$, the function $u = \lambda R_\lambda f$ satisfies the inequality

$$P_t u \le e^{\lambda t} u \text{ in } M. \tag{3.27}$$

for arbitrary $t, \lambda > 0$. Indeed, we have

$$\begin{aligned} P_t u &= \lambda \int_0^\infty e^{-\lambda s} P_{t+s} f\, ds \\ &= \lambda \int_t^\infty e^{-\lambda(s-t)} P_s f\, ds \\ &= e^{\lambda t} \lambda \int_t^\infty e^{-\lambda s} P_s f\, ds \le e^{\lambda t} u. \end{aligned}$$

Step 5. Finally, let us prove (3.13). Let c be the same as in (3.18) (Step 2), so that for any $\lambda > 0$ and for $u = \lambda R_\lambda 1_{B^c}$,

$$\operatorname*{essup}_{B(x_0,\delta)} u \le \exp\left(-\frac{cr}{\rho\,(1/\lambda)} + 1\right). \tag{3.28}$$

Let $\lambda > 0$ be such that (3.25) is satisfied. Then it follows from (3.26) that

$$u \ge \frac{1}{2}\mathbf{1}_{(2B)^c} \text{ in } M.$$

Applying P_t to the both sides of this inequality and using (3.27), we obtain

$$\frac{1}{2}P_t\mathbf{1}_{(2B)^c} \le P_t u \le e^{\lambda t}u, \tag{3.29}$$

which together with (3.28) yields

$$\operatorname*{essup}_{B(x_0,\delta)} P_t\mathbf{1}_{(2B)^c} \le C\exp\left(\lambda t - \frac{cr}{\rho\,(1/\lambda)}\right), \tag{3.30}$$

where $C = 2e$.

If λ is such that (3.25) fails, that is, $r < K\rho\,(1/\lambda)$ then (3.30) holds trivially with $C = e^{cK}$. Hence, (3.30) holds for all $\lambda > 0$, which is equivalent to (3.15). □

In applications it is frequently convenient to replace function Φ in (3.13) by a more explicit function as in the following statement.

Lemma 3.8. *Define a function $\Psi(\lambda)$ on $[0, +\infty)$ by*

$$\Psi\,(\lambda) = \begin{cases} \lambda\rho\left(\frac{1}{\lambda}\right), & \lambda > 0, \\ 0, & \lambda = 0 \end{cases} \tag{3.31}$$

and assume that $\Psi\,(\lambda)$ is a continuous, monotone increasing bijection from $[0, +\infty)$ onto $[0, +\infty)$, so that the inverse function Ψ^{-1} is defined on $[0, \infty)$. Then

$$\Phi\,(2s) \ge \Psi^{-1}\,(s) \ge \Phi\,(s) \ \textit{for all}\ s \ge 0. \tag{3.32}$$

Proof. Set $\lambda = \Psi^{-1}\,(s)$ so that

$$s = \lambda\rho\,(1/\lambda)\,.$$

It follows from (3.14) that

$$\Phi\,(2s) \ge \frac{2s}{\rho\,(1/\lambda)} - \lambda = \lambda,$$

which proves the left inequality in (3.32).

Since

$$\Phi\,(s) = \sup_{\nu>0}\left\{\frac{\lambda\rho\,(1/\lambda)}{\rho\,(1/\nu)} - \nu\right\},$$

the right inequality in (3.32) is equivalent to the inequality

$$\frac{\lambda\rho\,(1/\lambda)}{\rho\,(1/\nu)} - \nu \le \lambda, \text{ for all } \nu > 0,$$

which after division by ν becomes

$$\frac{\Psi\,(\lambda)}{\Psi\,(\nu)} \le 1 + \frac{\lambda}{\nu}. \tag{3.33}$$

Indeed, if $\nu \ge \lambda$ then by the monotonicity of Ψ, $\Psi\,(\lambda) \le \Psi\,(\nu)$, which obviously implies (3.33). If $\nu < \lambda$ then $\rho\,(1/\nu) \ge \rho\,(1/\lambda)$ and

$$\frac{\Psi\,(\lambda)}{\Psi\,(\nu)} = \frac{\lambda\rho\,(1/\lambda)}{\nu\rho\,(1/\nu)} \le \frac{\lambda}{\nu},$$

which implies (3.33) as well. □

Corollary 3.9. *Theorem 3.4 remains true if the function Φ in condition (iv) is replaced by Ψ^{-1}, provided Ψ^{-1} exists.*

3.2. **Pointwise estimates of the heat kernel.** Now we can state the main result about the relation between (DUE) and (UE) in the case of a local Dirichlet form. Let us first state and label all the required conditions in terms of the functions V and ρ:

- The upper bounds for the volume of balls: for all $x \in M$ and $r > 0$

$$\mu(B(x,r)) \leq CV(r). \tag{V}$$

- The version of condition (T): for any $\varepsilon > 0$ there is $K > 0$ such that whenever $r \geq K\rho(t)$ then, for almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,z)\, d\mu(z) \leq \varepsilon. \tag{T_{weak}}$$

Obviously, (T_{weak}) is equivalent to the fact that (T) holds with *some* function h such that $h(s) \to 0$ as $s \to \infty$.

- The on-diagonal upper bound: for all $t > 0$ and almost all $x, y \in M$,

$$p_t(x,y) \leq \frac{C}{V(\rho(t))}, \tag{DUE}$$

- The upper bound with the exponential tail: for all $t > 0$ and almost all $x, y \in M$,

$$p_t(x,y) \leq \frac{C}{V(\rho(t))} \exp\left(-t\,\Phi\left(c\frac{d(x,y)}{t}\right)\right), \tag{$UE_{\exp}$}$$

where Φ is defined by (3.14).

Theorem 3.10. *Let $(\mathcal{E}, \mathcal{F})$ be a regular, conservative, local Dirichlet form in $L^2(M)$ and let $p_t(x,y)$ be its heat kernel. Assume that all metric balls in M are precompact and satisfy (V), and let functions ρ and V be doubling. Then*

$$(DUE) + (T_{weak}) \Leftrightarrow (UE_{\exp}).$$

Proof. Let us prove the implication

$$(DUE) + (T_{weak}) \Rightarrow (UE_{\exp}).$$

Observe first that condition (T_{weak}) is equivalent to the condition (i) of Theorem 3.1 (cf. Remark 3.3). Hence, by Theorem 3.4, we obtain

$$\int_{B(x,r)^c} p_t(x,z)d\mu(z) \leq C\exp\left(-t\Phi\left(c\frac{r}{t}\right)\right), \tag{3.34}$$

for all $t, r > 0$ and almost all $x \in M$ (cf. Remark 3.6). Using the semigroup property, (DUE), and (3.34), we obtain, for almost all $x, y \in M$ and setting $r = \frac{1}{2}d(x,y)$, that

$$\begin{aligned}
p_{2t}(x,y) &= \int_M p_t(x,z)p_t(y,z)d\mu(z) \\
&\leq \int_{B(x,r)^c} p_t(x,z)p_t(y,z)\, d\mu(z) + \int_{B(y,r)^c} p_t(x,z)p_t(y,z)\, d\mu(z) \\
&\leq \operatorname*{essup}_{M\times M} p_t(\cdot,\cdot) \int_{B(x,r)^c} p_t(x,z)\, d\mu(z) + \operatorname*{essup}_{M\times M} p_t(\cdot,\cdot) \int_{B(y,r)^c} p_t(y,z)\, d\mu(z) \\
&\leq \frac{C}{V(\rho(t))} \exp\left(-t\,\Phi\left(c\frac{r}{t}\right)\right).
\end{aligned}$$

Renaming $2t$ by t and applying (3.17) and the doubling property of V and ρ, we obtain $(UE_{\exp})$. Note that in this part of the proof we have not used (V).

Let us now prove the converse, that is,

$$(UE_{\exp}) \Rightarrow (DUE) + (T_{weak}).$$

The on-diagonal bound (DUE) follows from ($UE_{\exp}$) trivially. To prove (T_{weak}), observe that, by ($UE_{\exp}$),

$$\int_{B(x,r)^c} p_t(x,z)\,d\mu(z) \le \frac{C}{V(\rho(t))}\int_{B(x,r)^c} \exp\left(-t\,\Phi\left(\frac{c\,d(x,z)}{t}\right)\right) d\mu(z).$$

Hence, we are left to prove that, for any $\varepsilon > 0$ there is $K = K(\varepsilon)$ such that

$$\frac{1}{V(\rho(t))}\int_{B(x,r)^c} \exp\left(-t\,\Phi\left(\frac{cd(x,z)}{t}\right)\right) d\mu(z) \le \varepsilon, \tag{3.35}$$

provided

$$r \ge K\rho(t). \tag{3.36}$$

For any non-negative integer k, set

$$\xi_k = t\,\Phi\left(c\frac{2^k r}{t}\right) \tag{3.37}$$

and observe that, by (3.17),

$$\xi_k \ge 2^k \xi_0. \tag{3.38}$$

Next, consider the following part of the integral (3.35):

$$\begin{aligned} I_k &= \frac{1}{V(\rho(t))}\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \exp\left(-t\,\Phi\left(\frac{cd(x,z)}{t}\right)\right) d\mu(z) \\ &\le C\frac{V\left(2^{k+1}r\right)}{V(\rho(t))}\exp\left(-t\,\Phi\left(\frac{c2^k r}{t}\right)\right) \\ &\le C\left(\frac{2^{k+1}r}{\rho(t)}\right)^{\alpha_1}\exp(-\xi_k) \\ &\le C\left(\frac{r}{\rho(t)\,\xi_0}\right)^{\alpha_1}\xi_k^{\alpha_1}\exp(-\xi_k), \end{aligned} \tag{3.39}$$

where we have used (V), (2.1) (which is a consequence of the doubling property of V), (3.37), and (3.38). Observe that by (3.37) and (3.14)

$$\xi_0 = t\Phi\left(c\frac{r}{t}\right) = \sup_{\lambda>0}\left\{\frac{cr}{\rho(1/\lambda)} - \lambda t\right\} \ge \frac{cr}{\rho(t)} - 1,$$

which follows by setting $\lambda = 1/t$. Assuming that r and t are related by (3.36) and K is so large that

$$cK \ge 2, \tag{3.40}$$

we obtain

$$\xi_0 \ge \frac{c}{2}\frac{r}{\rho(t)}. \tag{3.41}$$

Substituting into (3.39), we obtain

$$I_k \le C\xi_k^{\alpha_1}\exp(-\xi_k). \tag{3.42}$$

On the other hand, using $\xi_k/\xi_{k-1} \geq 2$, which is true by (3.17), we obtain

$$\begin{aligned}\int_{\xi_0}^{\infty} s^{\alpha_1-1}\exp(-s)\,ds &= \sum_{k=1}^{\infty}\int_{\xi_{k-1}}^{\xi_k} s^{\alpha_1-1}\exp(-s)\,ds\\ &\geq \sum_{k=1}^{\infty}\exp(-\xi_k)\int_{\xi_{k-1}}^{\xi_k} s^{\alpha_1-1}\,ds\\ &\geq c'\sum_{k=1}^{\infty}\xi_k^{\alpha_1}\exp(-\xi_k),\end{aligned}$$

where $c' = \frac{1}{\alpha_1}(1-2^{-\alpha_1})$. It follows that

$$\begin{aligned}\sum_{k=0}^{\infty}\xi_k^{\alpha_1}\exp(-\xi_k) &= \xi_0^{\alpha_1}\exp(-\xi_0) + \sum_{k=1}^{\infty}\xi_k^{\alpha_1}\exp(-\xi_k)\\ &\leq \xi_0^{\alpha_1}\exp(-\xi_0) + C\int_{\xi_0}^{\infty} s^{\alpha_1-1}\exp(-s)\,ds\\ &\leq C\exp\left(-\frac{\xi_0}{2}\right).\end{aligned}$$

Therefore, we obtain from (3.42), (3.41), (3.36)

$$\sum_{k=0}^{\infty} I_k \leq C\exp\left(-\frac{\xi_0}{2}\right) \leq C\exp\left(-\frac{c}{4}\frac{r}{\rho(t)}\right) \leq C\exp\left(-\frac{cK}{4}\right). \tag{3.43}$$

Since the left hand side of (3.35) is equal to $\sum_{k=0}^{\infty} I_k$ and K can be chosen arbitrarily large, we obtain that (3.35) can be satisfied with any $\varepsilon > 0$, which finishes the proof. □

Using Lemma 3.8, we obtain immediately the following consequence.

Corollary 3.11. *Theorem 3.10 remains true if the function Φ in ($UE_{\exp}$) is replaced by Ψ^{-1}, provided Ψ^{-1} exists.*

It follows from (3.43) that, for all $t, r > 0$ and almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,z)\,d\mu(z) \leq C\exp\left(-\frac{c}{4}\frac{r}{\rho(t)}\right). \tag{3.44}$$

Indeed, (3.43) was proved above under the assumption that $r/\rho(t) \geq 2/c$ (cf. (3.36) and (3.40)). If $r/\rho(t) < 2/c$ then (3.44) is trivially satisfied with $C = e^{1/2}$. Hence, under the hypotheses of Theorem 3.10, we obtain (T) with the function

$$h(s) = \exp(-cs). \tag{3.45}$$

Let us introduce a stronger version of condition (T), with an additional requirement on h: for all $t, r > 0$ and almost all $x \in M$,

$$\begin{aligned}&\int_{B(x,r)^c} p_t(x,z)\,d\mu(z) \leq Ch\left(\frac{r}{\rho(t)}\right),\\ &\text{and } \lim_{t\to 0}\frac{1}{t}h\left(\frac{r}{\rho(t)}\right) = 0.\end{aligned} \tag{T_{strong}}$$

For example, function (3.45) satisfies the second condition in (T_{strong}) if, for some $c, \eta > 0$,

$$\rho(t) \leq ct^{\eta},\ 0 < t < 1. \tag{3.46}$$

The above observation means that, under the hypotheses of Theorem 3.10 and assuming in addition (3.46), the following is true:

$$(UE_{\exp}) \Rightarrow (DUE) + (T_{strong}).$$

The converse implication

$$(DUE) + (T_{strong}) \Rightarrow (UE_{\exp})$$

is true as well, just because (T_{strong}) implies (T_{weak}). Hence, the hypothesis (T_{weak}) in the statement of Theorem 3.10 can be replaced by (T_{strong}), provided one assumes in addition (3.46). Besides, the condition (T_{strong}) allows to drop the hypothesis of the locality of the Dirichlet form in Theorem 3.10 as it is clear from the following statement.

Lemma 3.12. *Let $(\mathcal{E}, \mathcal{F})$ be a conservative Dirichlet form with the heat kernel $p_t(x,y)$. If (T_{strong}) holds then the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is local.*

Proof. We need to prove that if f and g are two functions from $\mathcal{F}$ whose supports are disjoint compact sets then $\mathcal{E}(f,g) = 0$. Since $(\mathcal{E}, \mathcal{F})$ is conservative, we have

$$\mathcal{E}(f,g) = \lim_{t \to 0} \mathcal{E}_t(f,g),$$

where $\mathcal{E}_t$ is defined by (1.8). Hence, it suffices to show that

$$\mathcal{E}_t(f,g) \to 0 \text{ as } t \to 0. \tag{3.47}$$

Let $A = \operatorname{supp} f$ and $C = \operatorname{supp} g$. Since A and C are disjoint, it follows from (1.8) that

$$\mathcal{E}_t(f,g) = -\frac{1}{t}\int_A \int_C f(x)\, g(y)\, p_t(x,y)\, d\mu(y)\, d\mu(x),$$

whence by the Cauchy-Schwarz inequality

$$\begin{aligned}
|\mathcal{E}_t(f,g)| &\le \left(\frac{1}{t}\int_A\int_C f^2(x)\, p_t(x,y)\, d\mu(y)\, d\mu(x)\right)^{1/2} \\
&\quad \times \left(\frac{1}{t}\int_A\int_C g^2(y)\, p_t(x,y)\, d\mu(y)\, d\mu(x)\right)^{1/2} \\
&\le \|f\|\,\|g\| \left(\operatorname*{essup}_{x\in A} \frac{1}{t}\int_C p_t(x,y)\, d\mu(y)\right)^{1/2} \\
&\quad \times \left(\operatorname*{essup}_{y\in C} \frac{1}{t}\int_A p_t(x,y)\, d\mu(x)\right)^{1/2}.
\end{aligned}$$

Let $r = \operatorname{dist}(A, C)$. Choose a finite covering $\{B_i\}_{i=1}^N$ of A by metric balls $B_i = B(x_i, r/2)$ ($x_i \in A$). Then $C \subset (2B_i)^c$ for any i, and we obtain by (T_{strong})

$$\begin{aligned}
\frac{1}{t}\operatorname*{essup}_{x\in A}\int_C p_t(x,y)\, d\mu(y) &\le \frac{1}{t}\sup_i \operatorname*{essup}_{x\in B_i}\int_{(2B_i)^c} p_t(x,y)\, d\mu(y) \\
&\le \frac{1}{t} h\left(\frac{2r}{\rho(t)}\right) \to 0 \text{ as } t \to 0.
\end{aligned}$$

Similarly, we have that

$$\frac{1}{t}\operatorname*{essup}_{y\in C}\int_A p_t(x,y)\, d\mu(x) \to 0 \text{ as } t \to 0,$$

whence (3.47) follows. $\square$

Combining Lemma 3.12 with the previous remarks, we obtain the following result.

Corollary 3.13. *Under the hypotheses of Theorem 3.10, drop the assumption of the locality of $(\mathcal{E}, \mathcal{F})$ and add condition (3.46). Then the following is true:*

$$(DUE) + (T_{strong}) \Leftrightarrow (UE_{\exp}).$$

4. Appendix: Markovian properties

We prove here a number of the consequences of the Markov property of the Dirichlet forms, such as the maximum and comparison principles, the properties of the resolvents and the heat semigroups in subsets, etc., which are necessary for Section 3 (Corollary 4.15 and Lemma 4.18 are explicitly used in the proofs of Theorem 3.4 and 3.1, respectively). These results are "well-known", but it is hardly possible to give accurate references. Besides, the existing proofs would normally use the Hunt process associated with the Dirichlet form. We give self-contained, analytic proofs of all these results, some of which are new.

Let us state some frequently used basic facts about Dirichlet forms:

- (*Extension of the Markov property*). If $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with the Lipschitz constant ≤ 1 and $\varphi(0) = 0$ then, for any $u \in \mathcal{F}$, the function $\varphi(u)$ is also in $\mathcal{F}$ and $\mathcal{E}(\varphi(u)) \leq \mathcal{E}(u)$. For the function $\varphi(s) = \max(\min(s,1),0)$ this property holds by the definition of the Markov property. The proof for a general φ can be found in [10, Theorem 1.4.1, p.23].
- If $u, v \in \mathcal{F} \cap L^\infty(M)$ then $uv \in \mathcal{F}$ (see [10, Theorem 1.4.2(*ii*)]).

Definition 4.1. For any open subset Ω of M, let $\mathcal{F}_0(\Omega)$ be the set of functions from $\mathcal{F}$ whose support is compact and is contained in Ω. Then define $\mathcal{F}(\Omega)$ as the closure of $\mathcal{F}_0(\Omega)$ in $\mathcal{F}$ with respect to the $\mathcal{E}_1$-norm.

In particular, it follows that any function from $\mathcal{F}(\Omega)$ vanishes in Ω^c and, hence, can be identified as an element of $L^2(\Omega)$. If $\mathcal{F}(\Omega)$ is dense in $L^2(\Omega)$ then $(\mathcal{E}, \mathcal{F}(\Omega))$ is a Dirichlet form in $L^2(\Omega)$. In this case, denote by Δ_Ω and P_t^Ω respectively the generator and the semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$. If $f \in L^2(M)$ then set $P_t^\Omega f := P_t^\Omega (f|_\Omega)$.

In general $\mathcal{F}(\Omega)$ need not be dense in $L^2(\Omega)$. However, if the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular then $\mathcal{F}_0(\Omega)$ is obviously dense in $L^2(\Omega)$. In this case, $\mathcal{F}(\Omega)$ admits the following two equivalent definitions (see [10, Corollary 2.3.1, p.95 and Theorem 4.4.2, p.154]):

(1) $\mathcal{F}(\Omega)$ is the closure of $\mathcal{F} \cap C_0(\Omega)$ in $\mathcal{F}$.
(2) $\mathcal{F}(\Omega) = \left\{ f \in \mathcal{F} : \widetilde{f} = 0 \text{ q.e. on } \Omega^c \right\}$ where $\widetilde{f}$ is a quasi-continuous modification of f and "q.e." stands for "quasi everywhere".

Let us state the following useful properties of *regular* Dirichlet forms:

- If $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form then $(\mathcal{E}, \mathcal{F}(\Omega))$ is also a regular Dirichlet form.
- For any open set $\Omega \subset M$ and any set $S \Subset \Omega$, there is a function $\varphi \in \mathcal{F} \cap C_0(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in an open neighborhood of $\overline{S}$ (see [10, p.27]). Such a function φ is called a *cut-off function* of the pair (S, Ω).

4.1. Maximum principle for weak solutions.

In this subsection, $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form in $L^2(M)$, not necessarily regular, unless otherwise stated.

Lemma 4.2. ([1]) *Let u, v be two functions from $\mathcal{F}$ such that $0 \leq u \leq 1$ and $v \geq 0$. If*

$$u \equiv 1 \text{ on the set } \{v > 0\}$$

then $\mathcal{E}(u, v) \geq 0$.

Proof. It follows from the hypotheses that, for any $\lambda > 0$,

$$\min(u + \lambda v, 1) = u.$$

By the Markov property and the bilinearity of the Dirichlet form, we obtain

$$\mathcal{E}(u) \leq \mathcal{E}(u + \lambda v) = \mathcal{E}(u) + 2\lambda \mathcal{E}(u, v) + \lambda^2 \mathcal{E}(v),$$

whence

$$2\lambda \mathcal{E}(u, v) + \lambda^2 \mathcal{E}(v) \geq 0.$$

Dividing by λ and then letting $\lambda \to 0$, we obtain $\mathcal{E}(u, v) \geq 0$, which was to be proved. $\square$

Lemma 4.3. *Let $\varphi(s)$ be an increasing function on $\mathbb{R}$ such that $\varphi(0)=0$ and $|\varphi(s_1)-\varphi(s_2)| \le |s_1-s_2|$ for all $s_1,s_2\in\mathbb{R}$. Then, for any $u\in\mathcal{F}$, also $\varphi(u)\in\mathcal{F}$ and*

$$\mathcal{E}(u,\varphi(u)) \ge \mathcal{E}(\varphi(u)). \tag{4.1}$$

In particular, (4.1) implies

$$\mathcal{E}(u,\varphi(u)) \ge 0. \tag{4.2}$$

For example, applying this with the function $\varphi(s)=s_+ := \max(s,0)$, we obtain that

$$\mathcal{E}(u,u_+) \ge 0. \tag{4.3}$$

Proof. That $\varphi(u)$ belongs to $\mathcal{F}$ is true by the Markov property. Fix some $\lambda\in(0,1)$ and consider the function

$$\psi(s) = \lambda s + (1-\lambda)\varphi(s). \tag{4.4}$$

Obviously, ψ is Lipschitz and $\psi(0)=0$, whence it follows that $\psi(u)\in\mathcal{F}$. Using $0\le\varphi'\le 1$ (where all relations involving the derivatives of Lipschitz functions are understood almost everywhere), we obtain from (4.4)

$$\psi' \ge \max\left(\lambda,\varphi'\right). \tag{4.5}$$

In particular, $\psi'\ge\lambda$, which implies that the function ψ has the inverse ψ^{-1} on $\mathbb{R}$, which is also a Lipschitz function. Using the identity

$$\left(\varphi\circ\psi^{-1}\right)'(s) = \frac{\varphi'\left(\psi^{-1}(s)\right)}{\psi'\left(\psi^{-1}(s)\right)}$$

and $\psi'\ge\varphi'$ (cf. (4.5)), we obtain

$$0 \le \left(\varphi\circ\psi^{-1}\right)' \le 1,$$

which implies by the Markov property that

$$\mathcal{E}(\varphi(u)) = \mathcal{E}\left(\left(\varphi\circ\psi^{-1}\right)(\psi(u))\right) \le \mathcal{E}(\psi(u)). \tag{4.6}$$

On the other hand, by (4.4),

$$\mathcal{E}(\psi(u)) = \lambda^2\mathcal{E}(u) + (1-\lambda)^2\mathcal{E}(\varphi(u)) + 2\lambda(1-\lambda)\mathcal{E}(u,\varphi(u)).$$

Expanding in λ and comparing with (4.6), we obtain

$$2\lambda\left(\mathcal{E}(u,\varphi(u)) - \mathcal{E}(\varphi(u))\right) + \lambda^2\left(\mathcal{E}(u) - 2\mathcal{E}(u,\varphi(u)) + \mathcal{E}(\varphi,u)\right) \ge 0.$$

Dividing by λ and then letting $\lambda\to 0$, we obtain (4.1). □

The maximum principles that will be stated below in Propositions 4.6 and 4.11, use the boundary condition

$$u \le 0 \text{ on } \Omega^c, \tag{4.7}$$

that is to be understood in a weak sense. The precise meaning of (4.7) is that $u_+\in\mathcal{F}(\Omega)$. The next statement provides a convenient equivalent way of stating this condition.

Lemma 4.4. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form. Let $u\in\mathcal{F}$ and Ω be an open subset of M. Then the following are equivalent:*

- *(i)* $u_+\in\mathcal{F}(\Omega)$.
- *(ii)* $u\le v$ *in* M *for some function* $v\in\mathcal{F}(\Omega)$.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial since we can take $v = u_+$. Let us prove $(ii) \Rightarrow (i)$. Set $f = u - v$ so that $f \in \mathcal{F}$ and $f \le 0$ in M. The question amounts to proving that

$$(v+f)_+ \in \mathcal{F}(\Omega), \tag{4.8}$$

for any non-positive $f \in \mathcal{F}$ and any $v \in \mathcal{F}(\Omega)$. Assume first that $f \in \mathcal{F} \cap L^\infty(M)$ and $v \in \mathcal{F}_0(\Omega)$. Let φ be a cut-off function of $\operatorname{supp} v$ in Ω. Since $\varphi \in \mathcal{F} \cap L^\infty(M)$ and $\operatorname{supp}\varphi \subset \Omega$, it follows that $\varphi f \in \mathcal{F}(\Omega)$. Observe that

$$(v+f)_+ = (v+\varphi f)_+ . \tag{4.9}$$

Indeed, on $\operatorname{supp} v$ we have $\varphi \equiv 1$ so that the identity (4.9) is trivially satisfied, while on the set $\{v = 0\}$ the both sides of (4.9) vanish because $f \le 0$ (see Fig. 7). Since $v + \varphi f \in \mathcal{F}(\Omega)$, we

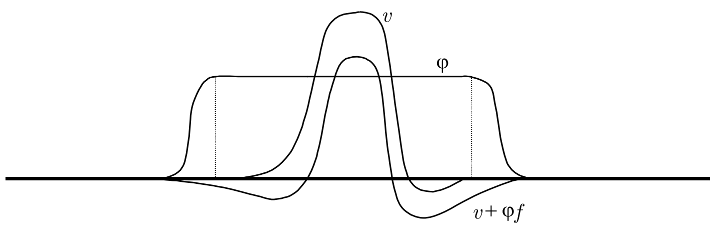

FIGURE 7. Function $v + \varphi f$

conclude that $(v+\varphi f)_+ \in \mathcal{F}(\Omega)$ whence (4.8) follows.

For an arbitrary non-positive function $f \in \mathcal{F}$, consider the sequence $f_k = \max(f, -k)$ so that $f_k \in \mathcal{F} \cap L^\infty(M)$, $f_k \le 0$, and $f_k \xrightarrow{\mathcal{E}_1} f$ (see [10, Theorem 1.4.2(iii), p.26]). Also, if $v \in \mathcal{F}(\Omega)$ then there is a sequence of functions $\{v_k\}_{k=1}^\infty \subset \mathcal{F} \cap C_0(\Omega)$ such that $v_k \xrightarrow{\mathcal{E}_1} v$. By the previous argument, we have

$$(v_k + f_k)_+ \in \mathcal{F}(\Omega). \tag{4.10}$$

Since

$$v_k + f \xrightarrow{\mathcal{E}_1} v + f,$$

it follows by [10, Theorem 1.4.2(v), p.26] that

$$(v_k + f)_+ \overset{\mathcal{E}_1}{\rightharpoonup} (v+f)_+ = u_+,$$

where the convergence is weak with respect to $\mathcal{E}_1$-norm. However, $\mathcal{F}(\Omega)$ being a closed subspace of the Hilbert space $\mathcal{F}$, is also weakly closed. Together with (4.10), this implies $u_+ \in \mathcal{F}(\Omega)$, which was to be proved. □

In the two propositions below, the regularity of $(\mathcal{E}, \mathcal{F})$ is not assumed.

Definition 4.5. Let Ω be an open subset of M, $f \in L^2(\Omega)$ and $\lambda \in \mathbb{R}$. We say that a function $u \in \mathcal{F}$ satisfies weakly the inequality

$$-\Delta u + \lambda u \le f \text{ in } \Omega,$$

if, for any non-negative function $\psi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u,\psi) + \lambda(u,\psi) \le (f,\psi). \tag{4.11}$$

Similarly one defines in the weak sense the inequality $-\Delta u + \lambda u \ge f$ and the identity $-\Delta u + \lambda u = f$.

Proposition 4.6 (Elliptic maximum principle)**.** *Let $u \in \mathcal{F}$ be a function such that, for some open set $\Omega \subset M$ and $\lambda > 0$,*

$$\begin{cases} -\Delta u + \lambda u \leq 0 \text{ weakly in } \Omega, \\ u_+ \in \mathcal{F}(\Omega). \end{cases}$$

Then $u \leq 0$ a.e. *in Ω.*

Proof. Since $u_+ \in \mathcal{F}(\Omega)$, we can take $\psi = u_+$ in (4.11) and obtain that

$$\mathcal{E}(u, u_+) + \lambda (u, u_+) \leq 0.$$

Since $\lambda > 0$ and by (4.3) $\mathcal{E}(u, u_+) \geq 0$, it follows that

$$\|u_+\|_{L^2(\Omega)} = (u, u_+) \leq 0,$$

whence $u_+ = 0$ in Ω. □

Remark 4.7. By Lemma 4.4, in the case when the Dirichlet form is regular, condition $u_+ \in \mathcal{F}(\Omega)$ can be replaced by $u \leq v$ for some $v \in \mathcal{F}(\Omega)$.

Remark 4.8. For the case $\lambda = 0$ some additional assumptions on the domain Ω are necessary for the validity of the maximum principle. Under different assumptions on Ω and for a local form $\mathcal{E}$, the following version of the elliptic maximum principle was proved in [5, p.140] and [17, Corollary 1.1]: if $u \in \mathcal{F}$, $-\Delta u \leq 0$ weakly in Ω and $\widetilde{u}$ is a quasi-continuous version of u then $\widetilde{u} \leq C$ q.e.in $M \setminus \Omega$ implies that $\widetilde{u} \leq C$ q.e. in Ω, where $C \geq 0$ is an arbitrary constant. Note that the condition $\widetilde{u} \leq 0$ q.e. in $M \setminus \Omega$ is equivalent to $u_+ \in \mathcal{F}(\Omega)$ (cf. the remark after Definition 4.1).

Definition 4.9. Let I be an open interval in $\mathbb{R}$, Ω be an open subset of M, and $f \in L^2(\Omega)$. We say that a function $u : I \to \mathcal{F}$ satisfies weakly the inequality

$$\frac{\partial u}{\partial t} - \Delta u \leq f \text{ in } I \times \Omega, \tag{4.12}$$

if the Fréchet derivative $\frac{\partial u}{\partial t}$ of u exists in $L^2(\Omega)$ for any $t \in I$ and, for any non-negative function $\psi \in \mathcal{F}(\Omega)$,

$$\left(\frac{\partial u}{\partial t}(t, \cdot), \psi\right) + \mathcal{E}(u(t, \cdot), \psi) \leq (f, \psi). \tag{4.13}$$

Similarly one defines in the weak sense the inequality $\frac{\partial u}{\partial t} - \Delta u \geq f$ and the identity $\frac{\partial u}{\partial t} - \Delta u = f$.

If $\frac{\partial u}{\partial t} - \Delta u = 0$ weakly in $I \times \Omega$ then the function u is called a *weak solution* to the heat equation in $I \times \Omega$. The weak inequality $\frac{\partial u}{\partial t} - \Delta u \leq 0$ (≥ 0) defines a *weak subsolution* (resp., *supersolution*) to the heat equation.

Example 4.10. For example, for any $f \in L^2(M)$, the function $u = P_t f$ is a weak solution to the heat equation in $(0, +\infty) \times M$, that is, for any $\psi \in \mathcal{F}(M)$,

$$\left(\frac{\partial}{\partial t} P_t f, \psi\right) + \mathcal{E}(P_t f, \psi) = 0. \tag{4.14}$$

Indeed, it follows from the spectral theory that the following equation is satisfied *strongly*

$$\frac{\partial}{\partial t} P_t f = \Delta (P_t f),$$

that is, Δ is understood here as an operator in $L^2(M)$. By the definition of the generator Δ, we have

$$(\Delta (P_t f), \psi) = -\mathcal{E}(P_t f, \psi),$$

whence (4.14) follows.

Proposition 4.11 (parabolic maximum principle). *Fix $T \in (0,+\infty]$ and an open subset $\Omega \subset M$, and assume that a function $u : (0,T) \to \mathcal{F}$ satisfies the following conditions:*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \leq 0 \text{ weakly in } (0,T)\times\Omega, \\ u_+(t,\cdot) \in \mathcal{F}(\Omega) \text{ for any } t \in (0,T) \\ u_+(t,\cdot) \overset{L^2(\Omega)}{\longrightarrow} 0 \text{ as } t \to 0. \end{cases}$$

Then $u \leq 0$ a.e. on $(0,T)\times\Omega$.

Proof. Fix a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi = 0$ on $(-\infty, 0]$, $\varphi > 0$ on $(0,\infty)$, and $0 \leq \varphi' \leq 1$ on $\mathbb{R}$. Choosing in (4.13) the function

$$\psi := \varphi(u(t,\cdot)) = \varphi(u_+(t,\cdot)) \in \mathcal{F}(\Omega),$$

we obtain

$$\left(\frac{\partial u}{\partial t}, \varphi(u)\right) + \mathcal{E}\left(u, \varphi(u)\right) \leq 0.$$

By (4.2), we have $\mathcal{E}(u,\varphi(u)) \geq 0$, whence

$$\left(\frac{\partial u}{\partial t}, \varphi(u)\right) \leq 0. \tag{4.15}$$

Now define the function Φ by

$$\Phi(s) = \left(\int_0^s \varphi(\xi)\,d\xi\right)^{1/2}, \quad s \in \mathbb{R}.$$

By choosing a suitable φ in a right neighborhood of 0, for example, letting $\varphi(s) = \frac{d}{ds}\exp\left(-s^{-2}\right)$, we can make $\Phi(s)$ and all its derivatives $\Phi^{(k)}(s)$ tend to 0 as $s \to 0$, so that $\Phi \in C^\infty(\mathbb{R})$. Note that $\Phi = 0$ on $(-\infty,0]$, $\Phi > 0$ on $(0,\infty)$, and $0 \leq \Phi' \leq 1$ on $\mathbb{R}$. All these properties are obvious except that $\Phi' \leq 1$ on $(0,\infty)$. The latter is proved as follows: since

$$\frac{d}{ds}\int_0^s \varphi(\xi)\,d\xi = \varphi(s) \geq \varphi'(s)\varphi(s) = \frac{1}{2}\frac{d}{ds}\varphi^2(s),$$

we have

$$\Phi(s)^2 = \int_0^s \varphi(\xi)\,d\xi \geq \frac{1}{2}\varphi^2(s), \quad s > 0,$$

whence

$$\Phi'(s) = \frac{\varphi(s)}{2\Phi(s)} \leq \frac{\sqrt{2}\Phi(s)}{2\Phi(s)} = \frac{\sqrt{2}}{2} < 1.$$

It follows that $\Phi(u) \in \mathcal{F}$. It is easy to show that the function $t \mapsto \Phi(u(t,\cdot))$ is Fréchet differentiable in $L^2(\Omega)$ and, by the chain rule,

$$\frac{\partial}{\partial t}\Phi(u) = \Phi'(u)\frac{\partial u}{\partial t}.$$

By the product rule for the Fréchet derivative , we obtain from (4.15) that

$$\begin{aligned} \frac{d}{dt}\left(\Phi(u),\Phi(u)\right) &= 2\left(\Phi'(u)\frac{\partial u}{\partial t}, \Phi(u)\right) \\ &= \left(\frac{\partial u}{\partial t}, 2\Phi'(u)\Phi(u)\right) = \left(\frac{\partial u}{\partial t}, \varphi(u)\right) \leq 0. \end{aligned}$$

Hence, the function $\|\Phi(u(t,\cdot))\|$ is non-increasing in t. As $\Phi(s) \leq s_+$, it follows that, for any $t \in (0,T)$,

$$\|\Phi(u(t,\cdot))\| \leq \lim_{s\to 0+}\|\Phi(u(s,\cdot))\| \leq \lim_{s\to 0+}\|u_+(s,\cdot)\| = 0,$$

which implies that $u \leq 0$ a.e. on $(0,T)\times\Omega$. ∎

4.2. **Comparison lemmas for the resolvent.** In this subsection, $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2(M)$. For any open subset $\Omega\subset M$ and any $\lambda>0$, define the resolvent operator $R_\lambda^\Omega : L^2(\Omega)\to L^2(\Omega)$ by

$$R_\lambda^\Omega f=(-\Delta_\Omega+\lambda)^{-1} f=\int_0^\infty e^{-\lambda t}P_t^\Omega f\, dt, \tag{4.16}$$

for any $f\in L^2(\Omega)$. It follows that R_λ^Ω is a bounded operator in $L^2(\Omega)$ and, for any $f\in L^2(\Omega)$,

$$R_\lambda^\Omega f\in \operatorname{dom}(\Delta_\Omega)\subset \mathcal{F}(\Omega).$$

It is clear from (4.16) that $f\geq 0$ implies $R_\lambda^\Omega f\geq 0$ and $f\leq 1$ implies $\lambda R_\lambda^\Omega f\leq 1$.

If $f\in L^2(M)$ then set $R_\lambda^\Omega f:=R_\lambda^\Omega\left(f|_\Omega\right)$.

Lemma 4.12. *Let $\Omega\subset M$ be an open set, $\lambda>0$, and let a non-negative function $u\in\mathcal{F}$ satisfy weakly in Ω the inequality*

$$-\Delta u+\lambda u\geq f, \tag{4.17}$$

where $0\leq f\in L^2(\Omega)$. Then

$$u\geq R_\lambda^\Omega f.$$

Proof. It follows from the definition (4.16) of the resolvent that the function $v=R_\lambda^\Omega f$ satisfies in Ω the equation

$$-\Delta_\Omega v+\lambda v=f. \tag{4.18}$$

Multiplying (4.18) by $\psi\in\mathcal{F}(\Omega)$ and integrating over Ω, we obtain

$$\mathcal{E}(v,\psi)+\lambda(v,\psi)=(f,\psi),$$

that is, v satisfies *weakly* the equation

$$-\Delta v+\lambda v=f \text{ in } \Omega.$$

It follows from (4.17) that the function $w=v-u$ belongs to $\mathcal{F}$ and satisfies weakly the inequality

$$-\Delta w+\lambda w\leq 0 \text{ in } \Omega.$$

Since $w\leq v$ and $v\in\mathcal{F}(\Omega)$, we conclude by Lemma 4.4 that $w_+\in\mathcal{F}(\Omega)$. Then by Proposition 4.6, we obtain $w\leq 0$ in Ω, that is, $u\geq v$, which was to be proved. □

It follows from Lemma 4.12 that the function $u=R_\lambda^\Omega f$ is the minimal non-negative solution from the class $\mathcal{F}$ to the equation

$$-\Delta u+\lambda u=f \text{ in } \Omega,$$

which is understood in the weak sense.

Lemma 4.13. *If $\{\Omega_i\}_{i=1}^\infty$ is an increasing sequence of open subsets of M and $\Omega=\bigcup_{i=1}^\infty\Omega_i$ then, for any $\lambda>0$ and any $0\leq f\in L^2(\Omega)$,*

$$R_\lambda^{\Omega_i}f\xrightarrow{\text{a.e.}}R_\lambda^\Omega f.$$

Proof. Set $u_i=R_\alpha^{\Omega_i}f$ and observe that, by Lemma 4.12, the sequence $\{u_i\}$ is increasing and

$$0\leq u_i\leq R_\alpha^\Omega f.$$

Therefore, u_i converges almost everywhere to a measurable function u on Ω such that

$$0\leq u\leq R_\alpha^\Omega f.$$

This implies that $u\in L^2(\Omega)$ and, by the dominated convergence theorem, $u_i\to u$ in $L^2(\Omega)$. We need to prove that $u=R_\lambda^\Omega f$.

Let us first show that $u\in\mathcal{F}(\Omega)$. The function u_i belongs $\mathcal{F}(\Omega_i)$ and, hence, $u_i\in\mathcal{F}(\Omega)$. Let us show that the sequence $\{u_i\}$ is Cauchy in $\mathcal{F}(\Omega)$ with respect to the norm $\mathcal{E}_1$. Each function u_i satisfies the equation

$$\mathcal{E}(u_i,\varphi)+\alpha(u_i,\varphi)=(f,\varphi), \tag{4.19}$$

for any $\varphi \in \mathcal{F}(\Omega_i)$. Choosing here $\varphi = u_i$, we obtain

$$\mathcal{E}(u_i, u_i) + \alpha(u_i, u_i) = (f, u_i).$$

Fix $k > i$ and observe that the function $\varphi = u_k - 2u_i$ belongs to $\mathcal{F}(\Omega_k)$. Therefore, by the analogous equation for u_k, we obtain

$$\mathcal{E}(u_k, u_k - 2u_i) + \alpha(u_k, u_k - 2u_i) = (f, u_k - 2u_i).$$

Adding up the above two lines yields

$$\mathcal{E}(u_k) + \mathcal{E}(u_i) - 2\mathcal{E}(u_k, u_i) + \alpha\left(\|u_k\|^2 + \|u_i\|^2 - 2(u_k, u_i)\right) = (f, u_k - u_i),$$

whence

$$\mathcal{E}(u_k - u_i) + \alpha\|u_k - u_i\|^2 = (f, u_k - u_i) \leq \|f\|\|u_k - u_i\|.$$

Since $\|u_k - u_i\| \to 0$ as $k, i \to \infty$, we conclude that also $\mathcal{E}(u_k - u_i) \to 0$ and, hence, $\mathcal{E}_1(u_k - u_i) \to 0$. Therefore, the sequence $\{u_i\}$ is Cauchy in $\mathcal{F}(\Omega)$ and, hence, converges in $\mathcal{F}(\Omega)$. Since its limit in $L^2(\Omega)$ is u, we conclude that the limit of $\{u_i\}$ in $\mathcal{F}(\Omega)$ is also u. In particular, $u \in \mathcal{F}(\Omega)$.

Now we can show that $u = R_\alpha^\Omega f$. Fix a function $\varphi \in \mathcal{F}_0(\Omega)$ and observe that the support of φ is contained in Ω_i when i is large enough. Therefore, (4.19) holds for this φ for all large enough i. Passing to the limit as $i \to \infty$, we obtain that the same equation holds for u instead of u_i, that is,

$$\mathcal{E}(u, \varphi) + \alpha(u, \varphi) = (f, \varphi). \tag{4.20}$$

Since $\mathcal{F}_0(\Omega)$ is dense in $\mathcal{F}(\Omega)$, this identity holds for all $\varphi \in \mathcal{F}(\Omega)$. Since the function $R_\alpha^\Omega f$ belongs to $\mathcal{F}(\Omega)$ and also satisfies (4.20), we obtain that the function $v = u - R_\alpha f$ belongs to $\mathcal{F}(\Omega)$ and satisfies the identity

$$\mathcal{E}(v, \varphi) + \alpha(v, \varphi) = 0,$$

for all $\varphi \in \mathcal{F}(\Omega)$. Setting $\varphi = v$, we obtain $v = 0$, which finishes the proof. □

The following statement is a modification of Lemma 4.12 in the case of a local form, where the hypotheses $u \geq 0$ in M can be relaxed to $u \geq 0$ in Ω.

Lemma 4.14. *Let the Dirichlet form $(\mathcal{E}, \mathcal{F})$ be local. Let $\Omega \subset M$ be an open set, $\lambda > 0$, and let a function $u \in \mathcal{F} \cap L^\infty(M)$ be non-negative in Ω and satisfy weakly in Ω the inequality*

$$-\Delta u + \lambda u \geq f, \tag{4.21}$$

where $0 \leq f \in L^2(\Omega)$. Then

$$u \geq R_\lambda^\Omega f.$$

Proof. It suffices to prove that

$$u \geq R_\lambda^U f,$$

for any open set $U \Subset \Omega$ and then take an exhaustion of Ω by such sets U and pass to the limit by Lemma 4.13. Let φ be a cut-off function of the pair (U, Ω). Then $\varphi u \in \mathcal{F}$ and, since φu is supported in Ω, it follows that $\varphi u \in \mathcal{F}(\Omega)$. Observe that $\varphi u \geq 0$. Let us apply Lemma 4.12 to the function φu instead of u and in the space Ω instead of M. For that, we need to verify that the following inequality holds weakly in U:

$$-\Delta(\varphi u) + \lambda(\varphi u) \geq f.$$

Indeed, for any $0 \leq \psi \in \mathcal{F}(U)$, we have

$$\mathcal{E}(\varphi u, \psi) + \lambda(\varphi u, \psi) = \mathcal{E}((\varphi - 1)u, \psi) + \mathcal{E}(u, \psi) + \lambda(u, \psi) \geq (f, \psi),$$

where we have used

$$\mathcal{E}((\varphi - 1)u, \psi) = 0,$$

which is true by the locality of the form $(\mathcal{E}, \mathcal{F})$, and

$$\mathcal{E}(u, \psi) + \lambda(u, \psi) \geq (f, \psi),$$

which is true by (4.21). By Lemma 4.12, we conclude that in U

$$u = \varphi u \geq R_\lambda^U f,$$

which was to be proved. □

Corollary 4.15. *Assume that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is local. Let $\Omega \subset M$ be a precompact open set and $\lambda > 0$. If a function $w \in \mathcal{F} \cap L^\infty(M)$ is such that $0 \leq w \leq 1$ in Ω and w satisfies weakly in Ω the inequality*

$$-\Delta w + \lambda w \leq 0, \tag{4.22}$$

then

$$w \leq 1 - \lambda R_\lambda^\Omega \mathbf{1}_\Omega \text{ in } \Omega. \tag{4.23}$$

Proof. Let φ be a cut-off function of the pair (Ω, M) and consider the function $u = \varphi - w$ (see Fig. 8). Clearly, $u \in \mathcal{F} \cap L^\infty(M)$ and $u \geq 0$ in Ω. Let us show that u satisfies weakly in Ω the

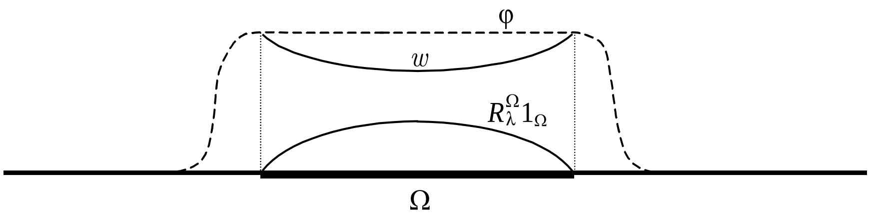

FIGURE 8. Functions w, φ, $R_\lambda^\Omega \mathbf{1}_\Omega$

inequality

$$-\Delta u + \lambda u \geq \lambda.$$

Indeed, for any $0 \leq \psi \in \mathcal{F}(\Omega)$, we have

$$\mathcal{E}(u, \psi) + \lambda(u, \psi) = \mathcal{E}(\varphi, \psi) + \lambda(\varphi, \psi) - (\mathcal{E}(w, \psi) + \lambda(w, \psi)) \geq \lambda(1, \psi),$$

where we have used that $\mathcal{E}(\varphi, \psi) \geq 0$ by Lemma 4.2, $(\varphi, \psi) = (1, \psi)$, and

$$\mathcal{E}(w, \psi) + \lambda(w, \psi) \leq 0$$

by (4.22). By Lemma 4.14, we conclude that

$$u \geq \lambda R_\lambda^\Omega \mathbf{1}_\Omega,$$

whence it follows that in Ω

$$w = \varphi - u = 1 - u \leq 1 - \lambda R_\lambda^\Omega \mathbf{1}_\Omega,$$

proving (4.23). □

4.3. **Comparisons lemmas for the heat semigroup.** In this subsection, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(M)$.

Lemma 4.16. *Let U be an open subset of M, and $0 \leq f \in L^2(U)$. If $u : \mathbb{R}_+ \to \mathcal{F}$ is a weak non-negative supersolution to the heat equation in $\mathbb{R}_+ \times U$ and*

$$u(t, \cdot) \xrightarrow{L^2(U)} f \text{ as } t \to 0 \tag{4.24}$$

then, for all $t > 0$,

$$u(t, \cdot) \geq P_t^U f. \tag{4.25}$$

Proof. Function $P_t^U f$ is a weak solution to the heat equation in $\mathbb{R}_+ \times U$ (cf. Example 4.10), and satisfies the initial condition (4.24). Hence, for the difference $w = P_t^U f - u$, we have

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w \leq 0 \text{ weakly in } \mathbb{R}_+ \times U, \\ w_+ (t, \cdot) \in \mathcal{F}(U) \ \text{ for any } t > 0, \\ w(t, \cdot) \stackrel{L^2(U)}{\longrightarrow} 0 \ \text{ as } t \to 0, \end{cases}$$

where the middle condition follows from $w(t, \cdot) \leq P_t^U f \in \mathcal{F}(U)$ and Lemma 4.4. By Proposition 4.11, we conclude that $w \leq 0$, whence (4.25) follows. □

In particular, if Ω is an open set containing U then applying Lemma 4.16 to $u = P_t^\Omega f$ we obtain $P_t^\Omega f \geq P_t^U f$.

Lemma 4.17. *If $\{\Omega_i\}_{i=1}^\infty$ is an increasing sequence of open subsets of M and $\Omega = \bigcup_{i=1}^\infty \Omega_i$ then, for any $t > 0$ and any $0 \leq f \in L^2(\Omega)$,*

$$P_t^{\Omega_i} f \stackrel{\text{a.e.}}{\longrightarrow} P_t^\Omega f \text{ as } i \to \infty. \tag{4.26}$$

Proof. Assume first that $f \in L^2(\Omega_1)$. The sequence of functions $\{P_t^{\Omega_i} f\}_{i=1}^\infty$ is increasing and is bounded by $P_t^\Omega f$. Hence, for any $t > 0$, the sequence $\{P_t^{\Omega_i} f\}$ converges almost everywhere to a measurable function u_t on Ω such that

$$0 \leq u_t \leq P_t^\Omega f.$$

We need to show that $u_t = P_t^\Omega f$. Since $u_t \in L^2(\Omega)$, the dominated convergence theorem implies that

$$P_t^{\Omega_i} f \stackrel{L^2(\Omega)}{\longrightarrow} u_t.$$

Since the semigroup $\left\{P_t^\Omega\right\}_{t\geq 0}$ is strongly continuous, the function $t \mapsto P_t^\Omega f$ is continuous as a path in $L^2(\Omega)$, for all $t \geq 0$. Let us prove that the path $t \mapsto u_t$ is continuous in $L^2(\Omega)$. For all $s > 0$ and $t \geq 0$, we have

$$\left\| P_{t+s}^{\Omega_i} f - P_t^{\Omega_i} f \right\| = \left\| P_t^{\Omega_i} \left(P_s^{\Omega_i} f - f \right) \right\| \leq \left\| P_s^{\Omega_i} f - f \right\|.$$

Since $P_s^{\Omega_1} f \leq P_s^{\Omega_i} f \leq P_s^\Omega f$, it follows that

$$\left\| P_{t+s}^{\Omega_i} f - P_t^{\Omega_i} f \right\| \leq \left\| P_s^{\Omega_1} f - f \right\| + \left\| P_s^\Omega f - f \right\|.$$

Letting $i \to \infty$, we obtain

$$\| u_{t+s} - u_t \| \leq \left\| P_s^{\Omega_1} f - f \right\| + \left\| P_s^\Omega f - f \right\| \to 0 \text{ as } s \to 0,$$

which means that u_t is right continuous. If $t > s > 0$ then we have

$$\left\| P_{t-s}^{\Omega_i} f - P_t^{\Omega_i} f \right\| = \left\| P_{t-s}^{\Omega_i} \left(f - P_s^{\Omega_i} f \right) \right\| \leq \left\| P_s^{\Omega_i} f - f \right\|.$$

Arguing as above, we obtain that u_t is also left continuous.

Fix a non-negative function $\varphi \in \mathcal{F} \cap C_0(\Omega)$ and observe that $\varphi \in \mathcal{F}(\Omega_i)$ for large enough i. It follows from (4.16) and the monotone convergence theorem that, for any $\alpha > 0$,

$$(R_\alpha^{\Omega_i} f, \varphi) = \int_0^\infty e^{-\alpha t} (P_t^{\Omega_i} f, \varphi) dt \longrightarrow \int_0^\infty e^{-\alpha t} (u_t, \varphi) \, dt,$$

as $i \to \infty$. On the other hand, by Lemma 4.13,

$$(R_\alpha^{\Omega_i} f, \varphi) \longrightarrow \left(R_\alpha^\Omega f, \varphi \right) = \int_0^\infty e^{-\alpha t} \left(P_t^\Omega f, \varphi \right) dt,$$

whence it follows that

$$\int_0^\infty e^{-\alpha t} (u_t, \varphi) \, dt = \int_0^\infty e^{-\alpha t} \left(P_t^\Omega f, \varphi \right) dt.$$

Since $(u_t,\varphi)\le\left(P_t^{\Omega}f,\varphi\right)$ and the functions (u_t,φ), $\left(P_t^{\Omega}f,\varphi\right)$ are continuous in t, this identity is only possible when

$$(u_t,\varphi)=\left(P_t^{\Omega}f,\varphi\right)\text{ for all }t>0. \tag{4.27}$$

It follows that $u_t=P_t^{\Omega}f$, which was claimed.

Finally, consider an arbitrary non-negative function $f\in L^2(\Omega)$. Fix $k\in\mathbb{N}$ and set $f_k=f|_{\Omega_k}$. By the previous part of the proof, we have

$$P_t^{\Omega_i}f_k\overset{L^2(\Omega)}{\longrightarrow}P_t^{\Omega}f_k\text{ as }i\to\infty.$$

For any $i>k$, we have

$$\begin{aligned}\left\|P_t^{\Omega}f-P_t^{\Omega_i}f\right\| &\le \left\|P_t^{\Omega}f-P_t^{\Omega}f_k\right\|+\left\|P_t^{\Omega}f_k-P_t^{\Omega_i}f_k\right\|+\left\|P_t^{\Omega_i}f_k-P_t^{\Omega_i}f\right\|\\ &\le 2\left\|f-f_k\right\|+\left\|P_t^{\Omega}f_k-P_t^{\Omega_i}f_k\right\|\end{aligned}$$

whence it follows that

$$\limsup_{i\to\infty}\left\|P_t^{\Omega}f-P_t^{\Omega_i}f\right\|\le 2\left\|f-f_k\right\|.$$

Letting $k\to\infty$, we obtain $\lim_{i\to\infty}\left\|P_t^{\Omega}f-P_t^{\Omega_i}f\right\|=0$, whence (4.26) follows. □

Lemma 4.18. *For any two open subsets $U\subset\Omega$ of M, for any compact set $K\subset U$, for any $0\le f\in L^2(M)$ and all $t>0$,*

$$\operatorname*{essup}_{\Omega}\left(P_t^{\Omega}f-P_t^{U}f\right)\le\sup_{s\in[0,t]}\operatorname*{essup}_{\Omega\setminus K}P_s^{\Omega}f. \tag{4.28}$$

In particular, applying (4.28) for $\Omega=M$, we obtain, for any $f\in L^2(M)$, $t>0$ and almost all $x\in M$,

$$P_tf(x)-P_t^{U}f(x)\le\sup_{s\in[0,t]}\operatorname*{essup}_{K^c}P_sf. \tag{4.29}$$

Proof. Let $\{\Omega_i\}_{i=1}^{\infty}$ be an increasing sequence of precompact open sets that exhausts Ω and $\{U_i\}_{i=1}^{\infty}$ be a similar sequence to exhaust U and such that $K\subset U_i\subset\Omega_i$ for all i. By (4.26) and $P_t^{U_i}f\le P_t^{U}f$, it suffices to prove that

$$\operatorname*{essup}_{\Omega_i}\left(P_t^{\Omega_i}f-P_t^{U_i}f\right)\le\sup_{s\in[0,t]}\operatorname*{essup}_{\Omega_i\setminus K}P_s^{\Omega_i}f.$$

Hence, renaming Ω_i to Ω and U_i to U, we can assume in the sequel that U and Ω are precompact.

Fix some $T>0$ and set

$$m=\sup_{s\in[0,T]}\operatorname*{essup}_{\Omega\setminus K}P_s^{\Omega}f.$$

If $m\equiv\infty$ then (4.28) is trivially satisfied for $t=T$. Assuming in the sequel that $m<\infty$, choose a cut-off function φ of the couple (Ω,M), and consider the function

$$u_t=P_t^{\Omega}f-P_t^{U}f-m\varphi. \tag{4.30}$$

It suffices to prove that $u_T\le 0$ in Ω. In fact, we shall prove that $u_t\le 0$ in Ω for all $t\in[0,T]$.

For any $t\in[0,T]$, we have $u_t\in\mathcal{F}$ and $u_t\le 0$ in $M\setminus K$, the latter being true by the definition of m. It follows that $(u_t)_+=0$ in $M\setminus K$ and, hence, $(u_t)_+\in\mathcal{F}(U)$. By Proposition 4.11, in order to prove that $u_t\le 0$ in U, it suffices to verify that u_t is a weak subsolution to the heat equation in $(0,T)\times U$ and that $(u_t)_+ \to 0$ in $L^2(U)$ as $t\to 0$. Indeed, for any $0\le\psi\in\mathcal{F}(U)$, we have

$$\begin{aligned}\left(\frac{\partial u_t}{\partial t},\psi\right) &= \left(\frac{\partial}{\partial t}P_t^{\Omega}f-\frac{\partial}{\partial t}P_t^{U}f,\ \psi\right)\\ &= -\mathcal{E}\left(P_t^{\Omega}f-P_t^{U}f,\ \psi\right)\\ &= -\mathcal{E}\left(u_t,\psi\right)-m\mathcal{E}(\varphi,\psi)\le -\mathcal{E}\left(u_t,\psi\right),\end{aligned}$$

where we have used the identity (4.14) and the fact that $\mathcal{E}(\varphi,\psi)\ge 0$ by Lemma 4.2. Hence, u_t is a weak subsolution to the heat equation in $(0,T)\times U$.

Since $P_t^{\Omega}f$ and $P_t^{U}f$ tend to f in the norm of $L^2(U)$ as $t\to 0$ and $\varphi\equiv 1$ in U, it follows from (4.30) that $u_t\to -m$ in $L^2(U)$ as $t\to 0$. Hence, $(u_t)_+\to 0$ in $L^2(U)$ as $t\to 0$, which finishes the proof. □

References

1. G. Allain, *Sur la représentation des formes de Dirichlet*, Ann. Inst. Fourier, Grenoble **25** (1975), 1–10.
2. M. Barlow, *Diffusions on fractals*, Lect. Notes Math., vol. **1690**, pp. 1–121, Springer, 1998.
3. M. Barlow, T. Coulhon, and T. Kumagai, *Characterization of sub-gaussian heat kernel estimates on strongly recurrent graphs*, Comm. Pure Appl. Math. **58** (2005), 1642–1677.
4. M. Barlow, A. Grigor'yan, and T. Kumagai, *Heat kernel upper bounds for jump processes and the first exit time*, J. Reine Angew. Math. (to appear).
5. M. Biroli and U. Mosco, *A Saint-Venant type principle for Dirichlet forms on discontinuous media*, Annali di Matematica pura ed applicata (IV) **CLXIX** (1995), 125–181.
6. E.A. Carlen, S. Kusuoka, and D.W. Stroock, *Upper bounds for symmetric Markov transition functions*, Ann. Inst. H. Poincaré-Probab. Statist. **23** (1987), 245–287.
7. Z.Q. Chen and T. Kumagai, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Relat. Fields (to appear).
8. T. Coulhon, *Ultracontractivity and Nash inequalities*, J. Funct. Anal. **141** (1996), 510–539.
9. E.B. Davies, *Heat kernels and spectral theory*, Cambridge Univ. Press, 1989.
10. M. Fukushima, Y. Oshima Y., and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Walter de Gruyter, Berlin, 1994.
11. A. Grigor'yan, *Heat kernel upper bounds on a complete non-compact manifold*, Rev. Mat. Iberoamericana **10** (1994), 395–452.
12. ———, *Heat kernels and function theory on metric measure spaces*, Contemporary Math. **338** (2003), 143–172.
13. ———, *Heat kernel upper bounds on fractal spaces*, preprint (2006).
14. A. Grigor'yan and A. Telcs, *Sub-gaussian estimates of heat kernels on infinite graphs*, Duke Math. J. **109** (2001), 451–510.
15. J. Hu, *An analytical approach to heat kernel estimates on strongly recurrent metric spaces*, Proc. Edin. Math. Soc. **51** (2008), 171–199.
16. J. Kigami, *Local Nash inequality and inhomogeneity of heat kernels*, Proc. London Math. Soc. **89** (2004), 525–544.
17. K. Kuwae, *Maximum principles for subharmonic functions via local semi-Dirichlet forms*, Canad. J. Math. (to appear).
18. M. Tomisaki, *Comparison theorems on Dirichlet norms and their applications*, Forum Math. **2** (1990), 277–295.
19. N.Th. Varopoulos, *Hardy-Littlewood theory for semigroups*, J. Funct. Anal. **63** (1985), 240–260.

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.
E-mail address: `grigor@math.uni-bielefeld.de`

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
E-mail address: `hujiaxin@mail.tsinghua.edu.cn`