

# Existence of positive solutions to some nonlinear equations on locally finite graphs

Alexander Grigor'yan

*Department of Mathematics, University of Bielefeld, Bielefeld 33501, Germany*

Yong Lin, Yunyan Yang

*Department of Mathematics, Renmin University of China, Beijing 100872, P. R. China*

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## Abstract

Let  $G = (V, E)$  be a locally finite graph, whose measure  $\mu(x)$  have positive lower bound, and  $\Delta$  be the usual graph Laplacian. Applying the mountain-pass theorem due to Ambrosetti-Rabinowitz, we establish existence results for some nonlinear equations, namely  $\Delta u + hu = f(x, u)$ ,  $x \in V$ . In particular, we prove that if  $h$  and  $f$  satisfy certain assumptions, then the above mentioned equation has strictly positive solutions. Also, we consider existence of positive solutions of the perturbed equation  $\Delta u + hu = f(x, u) + \epsilon g$ . Similar problems have been extensively studied on the Euclidean space as well as on Riemannian manifolds.

*Key words:* Variational method, Mountain-pass theorem, Semi-linear equation on graphs  
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## 1. Introduction

Let  $G = (V, E)$  be a locally finite graph, where  $V$  denotes the vertex set and  $E$  denotes the edge set. We say that a graph is locally finite if for any  $x \in V$ , there are only finite  $y$ 's such that  $xy \in E$ . For any edge  $xy \in E$ , we assume that its weight  $w_{xy} > 0$  and that  $w_{xy} = w_{yx}$ . Let  $\mu : V \rightarrow \mathbb{R}^+$  be a finite measure. For any function  $u : V \rightarrow \mathbb{R}$ , the  $\mu$ -Laplacian (or Laplacian for short) of  $u$  is defined as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)). \quad (1)$$

Here and throughout this paper,  $y \sim x$  stands for any vertex  $y$  with  $xy \in E$ . The associated gradient form reads

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)). \quad (2)$$

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*Email addresses:* grigor@math.uni-bielefeld.de (Alexander Grigor'yan), linyong01@ruc.edu.cn (Yong Lin), yunyanyang@ruc.edu.cn (Yunyan Yang)

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Write  $\Gamma(u) = \Gamma(u, u)$ . We denote the length of its gradient by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left( \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy} (u(y) - u(x))^2 \right)^{1/2}. \quad (3)$$

For any function  $g : V \rightarrow \mathbb{R}$ , an integral of  $g$  over  $V$  is defined by

$$\int_V g d\mu = \sum_{x \in V} \mu(x) g(x).$$

Let  $C_c(V)$  be the set of all functions with compact support, and  $W^{1,2}(V)$  be the completion of  $C_c(V)$  under the norm

$$\|u\|_{W^{1,2}(V)} = \left( \int_V (|\nabla u|^2 + u^2) d\mu \right)^{1/2}.$$

Clearly  $W^{1,2}(V)$  is a Hilbert space with the inner product

$$\langle u, v \rangle = \int_V (\Gamma(u, v) + uv) d\mu, \quad \forall u, v \in W^{1,2}(V).$$

Let  $h(x) \geq h_0 > 0$  for all  $x \in V$ . We define a space of functions

$$\mathcal{H} = \left\{ u \in W^{1,2}(V) : \int_V hu^2 d\mu < +\infty \right\} \quad (4)$$

with a norm

$$\|u\|_{\mathcal{H}} = \left( \int_V (|\nabla u|^2 + hu^2) d\mu \right)^{1/2}. \quad (5)$$

Obviously  $\mathcal{H}$  is also a Hilbert space with the inner product

$$\langle u, v \rangle_{\mathcal{H}} = \int_V (\Gamma(u, v) + huv) d\mu, \quad \forall u, v \in \mathcal{H}.$$

Let  $h : V \rightarrow \mathbb{R}$  and  $f : V \times \mathbb{R} \rightarrow \mathbb{R}$  be two functions. We say that  $u : V \rightarrow \mathbb{R}$  is a *solution* of the equation

$$-\Delta u + hu = f(x, u) \quad (6)$$

if (6) holds for all  $x \in V$ . We shall prove the following:

**Theorem 1.** *Let  $G = (V, E)$  be a locally finite graph. Assume that its weight satisfies  $w_{xy} = w_{yx}$  for all  $y \sim x \in V$ , and that its measure  $\mu(x) \geq \mu_{\min} > 0$  for all  $x \in V$ . Let  $h : V \rightarrow \mathbb{R}$  be a function satisfying the hypotheses*

*(H<sub>1</sub>) there exists a constant  $h_0 > 0$  such that  $h(x) \geq h_0$  for all  $x \in V$ ;*

*(H<sub>2</sub>)  $1/h \in L^1(V)$ .*

*Suppose that  $f : V \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following hypotheses:*

*(F<sub>1</sub>)  $f(x, s)$  is continuous in  $s$ ,  $f(x, 0) = 0$ , and for any fixed  $M > 0$ , there exists a constant  $A_M$  such that  $\max_{s \in [0, M]} f(x, s) \leq A_M$  for all  $x \in V$ ;*

*(F<sub>2</sub>) there exists a constant  $\theta > 2$  such that for all  $x \in V$  and  $s > 0$ ,*

$$0 < \theta F(x, s) = \theta \int_0^s f(x, t) dt \leq s f(x, s);$$

*(F<sub>3</sub>)  $\limsup_{s \rightarrow 0^+} \frac{2F(x, s)}{s^2} < \lambda_1 = \inf_{\int_V u^2 d\mu = 1} \int_V (|\nabla u|^2 + hu^2) d\mu$ .*

*Then the equation (6) has a strictly positive solution.*

There are other hypotheses on  $h$  and  $f$  such that (6) has a positive solution. In particular, we shall prove the following:

**Theorem 2.** *Let  $G = (V, E)$  be a locally finite graph. Assume that its weight satisfies  $w_{xy} = w_{yx}$  for all  $y \sim x \in V$ , and that its measure  $\mu(x) \geq \mu_{\min} > 0$  for all  $x \in V$ . Let  $h : V \rightarrow \mathbb{R}$  be a function satisfying  $(H_1)$  and*

*$(H'_2)$   $h(x) \rightarrow +\infty$  as  $\text{dist}(x, x_0) \rightarrow +\infty$  for some fixed  $x_0 \in V$ .*

*Suppose that  $f : V \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy  $(F_2)$ ,  $(F_3)$ , and*

*$(F'_1)$  for any  $s, t \in \mathbb{R}$ , there exists some constant  $L > 0$  such that*

$$|f(x, s) - f(x, t)| \leq L|s - t| \quad \text{for all } x \in V;$$

*Then the equation (6) has a strictly positive solution.*

We also consider the perturbation of (6), namely

$$-\Delta u + hu = f(x, u) + \epsilon g, \tag{7}$$

where  $\epsilon > 0$ ,  $g \in \mathcal{H}'$ , the dual space of  $\mathcal{H}$  defined by (4). Concerning this problem, we shall prove the following:

**Theorem 3.** *Let  $G = (V, E)$  be a locally finite graph. Assume that its weight satisfies  $w_{xy} = w_{yx}$  for all  $y \sim x \in V$ , and that its measure  $\mu(x) \geq \mu_{\min} > 0$  for all  $x \in V$ . Let  $h : V \rightarrow \mathbb{R}$  be a function satisfying  $(H_1)$  and  $(H_2)$ , and  $f : V \times \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying  $(F_1)$ ,  $(F_2)$ , and  $(F_3)$ . Suppose that  $g \in \mathcal{H}'$  satisfies  $g(x) \geq 0$  for all  $x \in V$  and  $g \not\equiv 0$ . Then there exists a constant  $\epsilon_0 > 0$  such that for any  $0 < \epsilon < \epsilon_0$ , the equation (7) has two distinct strictly positive solutions.*

**Theorem 4.** *Let  $G = (V, E)$  be a locally finite graph. Assume that its weight satisfies  $w_{xy} = w_{yx}$  for all  $y \sim x \in V$ , and that its measure  $\mu(x) \geq \mu_{\min} > 0$  for all  $x \in V$ . Let  $h : V \rightarrow \mathbb{R}$  be a function satisfying  $(H_1)$  and  $(H'_2)$ , and  $f : V \times \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying  $(F'_1)$ ,  $(F_2)$ , and  $(F_3)$ . Suppose that  $g \in \mathcal{H}'$  satisfies  $g(x) \geq 0$  for all  $x \in V$  and  $g \not\equiv 0$ . Then there exists a constant  $\epsilon_1 > 0$  such that for any  $0 < \epsilon < \epsilon_1$ , the equation (7) has two distinct strictly positive solutions.*

This kind of problems have been extensively studied in the Euclidean space, see for examples Alama-Li [4], Adimurthi [1], Adimurthi-Yadava[2], Adimurthi-Yang [3], Alves-Figueiredo [5], Cao [7], Ruf et al [8, 9], Ding-Ni [10], do Ó et al [11, 12, 13, 14], Jeanjean [16], Kryszewski-Szulkin [17], Panda [18], Yang [19, 20], and the references therein. For the Riemannian manifold case, we refer the reader to [15, 21, 22, 23].

The method of proving Theorems 1-4 is to use the critical point theory, in particular, the mountain-pass theorem. Though this idea has been used in the Euclidean space case and Riemannian manifold case, the Sobolev embedding in our setting is quite different from those cases. This let us assume different growth conditions on the nonlinear term  $f(x, u)$ . Our results closely resemble that of [14, 3, 19, 20, 21].

The remaining part of this paper is organized as follows: In Section 2, we prove two Sobolev embedding lemmas. In Sections 3 and 4, we prove Theorems 1 and 2 respectively. Finally, we prove Theorems 3 and 4 in Section 5.

## 2. Sobolev embedding

Let  $\mathcal{H}$  be defined by (4) and (5). To understand the function space  $\mathcal{H}$ , we have the following compact Sobolev embedding:

**Lemma 5.** *If  $\mu(x) \geq \mu_{\min} > 0$  and  $h$  satisfies  $(H_1)$  and  $(H_2)$ , then  $\mathcal{H}$  is weakly pre-compact and  $\mathcal{H}$  is compactly embedded into  $L^q(V)$  for all  $1 \leq q \leq +\infty$ . Namely, if  $u_k$  is bounded in  $\mathcal{H}$ , then up to a subsequence, there exists some  $u \in \mathcal{H}$  such that up to a subsequence,  $u_k \rightharpoonup u$  weakly in  $\mathcal{H}$  and  $u_k \rightarrow u$  strongly in  $L^q(V)$  for any fixed  $q$  with  $1 \leq q \leq +\infty$ .*

*Proof.* Suppose  $\mu(x) \geq \mu_{\min} > 0$ . It is easy to see that  $W^{1,2}(V) \hookrightarrow L^\infty(V)$  continuously. Hence interpolation implies that  $W^{1,2}(V) \hookrightarrow L^q(V)$  continuously for all  $2 \leq q \leq \infty$ . Suppose  $u_k$  is bounded in  $\mathcal{H}$ . Since  $h$  satisfies  $(H_1)$ , there holds  $\mathcal{H} \hookrightarrow W^{1,2}(V)$  continuously. Noting that  $W^{1,2}(V)$  is reflexive (every Hilbert space is reflexive), we have up to a subsequence,  $u_k \rightharpoonup u$  weakly in  $\mathcal{H}$ . In particular,

$$\lim_{k \rightarrow \infty} \int_V hu_k \varphi d\mu = \int_V hu \varphi d\mu, \quad \forall \varphi \in C_c(V).$$

This leads to  $\lim_{k \rightarrow +\infty} u_k(x) = u(x)$  for any fixed  $x \in V$ . We now prove  $u_k \rightarrow u$  in  $L^q(V)$  for all  $2 \leq q \leq \infty$ . Since  $u_k$  is bounded in  $\mathcal{H}$  and  $u \in \mathcal{H}$ , there exists some constant  $C_1$  such that

$$\int_V h(u_k - u)^2 d\mu \leq C_1. \quad (8)$$

Let  $x_0 \in V$  be fixed. For any  $\epsilon > 0$ , in view of  $(H_2)$ , there exists some  $R > 0$  such that

$$\int_{\text{dist}(x, x_0) > R} \frac{1}{h} d\mu < \epsilon^2.$$

Hence by the Hölder inequality,

$$\begin{aligned} \int_{\text{dist}(x, x_0) > R} |u_k - u| d\mu &= \int_{\text{dist}(x, x_0) > R} \frac{1}{\sqrt{h}} \sqrt{h} |u_k - u| d\mu \\ &\leq \left( \int_{\text{dist}(x, x_0) > R} \frac{1}{h} d\mu \right)^{1/2} \left( \int_{\text{dist}(x, x_0) > R} h |u_k - u|^2 d\mu \right)^{1/2} \\ &\leq \sqrt{C_1} \epsilon. \end{aligned} \quad (9)$$

Moreover, we have that up to a subsequence,

$$\lim_{k \rightarrow +\infty} \int_{\text{dist}(x, x_0) \leq R} |u_k - u| d\mu = 0. \quad (10)$$

Combining (9) and (10), we conclude

$$\liminf_{k \rightarrow +\infty} \int_V |u_k - u| d\mu = 0.$$

In particular, there holds up to a subsequence,  $u_k \rightarrow u$  in  $L^1(V)$ . Since

$$\|u_k - u\|_{L^\infty(V)} \leq \frac{1}{\mu_{\min}} \int_V |u_k - u| d\mu,$$

there holds for any  $1 < q < +\infty$ ,

$$\int_V |u_k - u|^q d\mu \leq \frac{1}{\mu_{\min}^{q-1}} \left( \int_V |u_k - u| d\mu \right)^q.$$

Therefore, up to a subsequence,  $u_k \rightarrow u$  in  $L^q(V)$  for all  $1 \leq q \leq +\infty$ .  $\square$

**Lemma 6.** *If  $\mu(x) \geq \mu_{\min} > 0$  and  $h$  satisfies  $(H_1)$  and  $(H'_2)$ , then  $\mathcal{H}$  is weakly pre-compact and  $\mathcal{H}$  is compactly embedded into  $L^q(V)$  for all  $2 \leq q \leq +\infty$ . Namely, if  $u_k$  is bounded in  $\mathcal{H}$ , then up to a subsequence, there exists some  $u \in \mathcal{H}$  such that  $u_k \rightarrow u$  weakly in  $\mathcal{H}$  and  $u_k \rightarrow u$  strongly in  $L^q(V)$  for all  $2 \leq q \leq +\infty$ .*

*Proof.* We only stress the difference from Lemma 8. By  $(H'_2)$ ,  $h(x) \rightarrow +\infty$  as  $\text{dist}(x, x_0) \rightarrow +\infty$ , there exists some  $R > 0$  such that

$$h(x) \geq \frac{2C_1}{\epsilon} \quad \text{when} \quad \text{dist}(x, x_0) > R.$$

This together with (8) gives

$$\int_{\text{dist}(x, x_0) > R} h(u_k - u)^2 d\mu \leq \frac{\epsilon}{2C_1} \int_{\text{dist}(x, x_0) > R} h(u_k - u)^2 d\mu \leq \epsilon. \quad (11)$$

Moreover, there holds up to a subsequence

$$\int_{\text{dist}(x, x_0) \leq R} h(u_k - u)^2 d\mu \rightarrow 0$$

Hence

$$\liminf_{k \rightarrow +\infty} \int_V h|u_k - u|^2 d\mu = 0.$$

Since the remaining part of the proof is completely analogous to that of Lemma 8, we omit the details here.  $\square$

### 3. Proof of Theorem 1

#### 3.1. Weak solution

We first define a *weak solution*  $u \in \mathcal{H}$  of the equation (6). If there holds

$$\int_V (\Gamma(u, \varphi) + hu\varphi) d\mu = \int_V f(x, u)\varphi d\mu, \quad \forall \varphi \in \mathcal{H},$$

then  $u$  is called a weak solution of (6). Note that  $C_c(V)$  is the set of all functions on  $V$  with compact support and it is dense in  $\mathcal{H}$ . If  $u$  is a weak solution, then integration by parts gives

$$\int_V (-\Delta u + hu)\varphi d\mu = \int_V f(x, u)\varphi d\mu \quad \forall \varphi \in C_c(V). \quad (12)$$

For any fixed  $y \in V$ , taking a test function  $\varphi : V \rightarrow \mathbb{R}$  in (12) with

$$\varphi(x) = \begin{cases} -\Delta u(y) + h(y)u(y) - f(y, u(y)), & x = y \\ 0, & x \neq y, \end{cases}$$

we have

$$-\Delta u(y) + h(y)u(y) - f(y, u(y)) = 0.$$

Since  $y$  is arbitrary, we conclude the following:

**Proposition 7.** *If  $u \in \mathcal{H}$  is a weak solution of (6), then  $u$  is also a point-wise solution of (6).*

This proposition implies that we can use the variational method to solve (6).

### 3.2. A reduction

For the proof of Theorem 1, we shall make the following *reduction*: We can assume  $f(x, s) \equiv 0$  for all  $s \leq 0$ . Moreover, we only need to find a nontrivial weak solution of (6).

For this purpose, we follow do Ó et al [11, 14] (see also [3, 19, 20]). Let

$$\tilde{f}(x, s) = \begin{cases} 0, & f(x, s) < 0 \\ f(x, s), & f(x, s) \geq 0. \end{cases}$$

If  $u \in \mathcal{H}$  is a nontrivial weak solution of

$$-\Delta u + hu = \tilde{f}(x, u) \quad \text{on } V, \quad (13)$$

where  $h$  satisfies  $(H_1)$  and  $(H_2)$ , and  $f$  satisfies  $(F_1) - (F_3)$ . Here and in the sequel, we say that  $u$  is a nontrivial solution if  $u \not\equiv 0$ . Testing the above equation by the negative part of  $u$ , namely  $u_- = \min\{u, 0\}$ , we have

$$\int_V (|\nabla u_-|^2 + hu_-^2) d\mu = \int_V u_- \tilde{f}(x, u) d\mu \leq 0.$$

In view of  $(H_1)$ , we have by the above inequality that  $u_- \equiv 0$ . Applying the maximum principle to (13), we have that  $u(x) > 0$  for all  $x \in V$ . This together with the hypothesis  $(H_2)$  leads to  $f(x, u) \geq 0$ . Hence  $\tilde{f}(x, u) = f(x, u)$  and  $u$  is a strictly positive solution of (6). Therefore, without loss of generality, we can assume  $f(x, s) \equiv 0$  for all  $s \leq 0$  in the proof of Theorem 1, and we only need to prove that (6) has a nontrivial weak solution.

### 3.3. Functional framework

We define a functional on  $\mathcal{H}$  by

$$J(u) = \frac{1}{2} \int_V (|\nabla u|^2 + hu^2) d\mu - \int_V F(x, u) d\mu, \quad (14)$$

where  $h$  satisfies  $(H_1)$  and  $(H_2)$ ,  $F(x, s) = \int_0^s f(x, t) dt$  is the primitive function of  $f$ , and  $f$  satisfies  $(F_1)$ ,  $(F_2)$  and  $(F_3)$ . We need to describe the geometry profile of  $J$ . Firstly we have

**Lemma 8.** *There exists some nonnegative function  $u \in \mathcal{H}$  such that  $J(tu) \rightarrow -\infty$  as  $t \rightarrow +\infty$ .*

*Proof.* By  $(F_2)$ , there exist positive constants  $c_1$  and  $c_2$  such that  $F(x, s) \geq c_1 s^\theta - c_2$  for all  $(x, s) \in V \times [0, +\infty)$ . Let  $x_0 \in V$  be fixed. Take a function

$$u(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \neq x_0. \end{cases}$$

Then we have

$$\begin{aligned}
J(tu) &= \frac{t^2}{2} \sum_{x \sim x_0} \mu(x) |\nabla u|^2(x) + \frac{t^2}{2} \mu(x_0) h(x_0) - \mu(x_0) F(x_0, t) \\
&\leq \frac{t^2}{2} \sum_{x \sim x_0} \mu(x) |\nabla u|^2(x) + \frac{t^2}{2} \mu(x_0) h(x_0) - c_1 t^\theta \mu(x_0) + c_2 \mu(x_0) \\
&\rightarrow -\infty
\end{aligned}$$

as  $t \rightarrow +\infty$ , since  $\theta > 2$  and  $V$  is locally finite.  $\square$

Secondly we have the following:

**Lemma 9.** *There exist positive constants  $\delta$  and  $r$  such that  $J(u) \geq \delta$  for all functions  $u$  with  $\|u\|_{\mathcal{H}} = r$ , where  $\|\cdot\|_{\mathcal{H}}$  is defined as in (5).*

*Proof.* By  $(F_3)$ , there exist positive constants  $\tau$  and  $\varrho$  such that if  $|s| \leq \varrho$ , then

$$F(x, s) \leq \frac{\lambda_1 - \tau}{2} s^2.$$

By  $(F_2)$ , we have  $F(x, s) > 0$  for all  $s > 0$ . Note also that  $F(x, s) \equiv 0$  for all  $s \leq 0$ . It follows that if  $|s| \geq \varrho$ , then

$$F(x, s) \leq \frac{1}{\varrho^3} s^3 F(x, s).$$

For all  $(x, s) \in V \times \mathbb{R}$ , there holds

$$F(x, s) \leq \frac{\lambda_1 - \tau}{2} s^2 + \frac{1}{\varrho^3} s^3 F(x, s).$$

In view of Lemma 5, for any function  $u$  with  $\|u\|_{\mathcal{H}} \leq 1$ , we have that  $\|u\|_{L^\infty(V)} \leq C_2 \|u\|_{\mathcal{H}}$  and  $\|u\|_{L^3(V)} \leq C_3 \|u\|_{\mathcal{H}}$  for constants  $C_2$  and  $C_3$ , and that

$$\int_V u^3 F(x, u) d\mu \leq \left( \max_{(x,s) \in V \times [0, C_2]} F(x, s) \right) \int_V |u|^3 d\mu \leq C_4 \|u\|_{\mathcal{H}}^3,$$

where  $(F_1)$  is employed, and  $C_4$  is some constant depending only on  $C_1, C_2, C_3$ , and  $A_{C_2}$ . Hence we have for any  $u$  with  $\|u\|_{\mathcal{H}} \leq 1$ ,

$$\begin{aligned}
J(u) &\geq \frac{1}{2} \|u\|_{\mathcal{H}}^2 - \frac{\lambda_1 - \tau}{2} \int_V u^2 d\mu - \frac{C_4}{\varrho^3} \|u\|_{\mathcal{H}}^3 \\
&\geq \left( \frac{1}{2} - \frac{\lambda_1 - \tau}{2\lambda_1} \right) \|u\|_{\mathcal{H}}^2 - \frac{C_4}{\varrho^3} \|u\|_{\mathcal{H}}^3 \\
&= \left( \frac{\tau}{2\lambda_1} - \frac{C_4}{\varrho^3} \|u\|_{\mathcal{H}} \right) \|u\|_{\mathcal{H}}^2.
\end{aligned}$$

Setting  $r = \min\{1, \tau\varrho^3/(4\lambda_1 C_4)\}$ , we have  $J(u) \geq \tau r^2/(4\lambda_1)$  for all  $u$  with  $\|u\|_{\mathcal{H}} = r$ . This completes the proof of the lemma.  $\square$

**Lemma 10.** *If  $h$  satisfies  $(H_1)$  and  $(H_2)$ ,  $f$  satisfies  $(F_1)$  and  $(F_2)$ , then  $J$  satisfies the  $(PS)_c$  condition for any  $c \in \mathbb{R}$ . Namely, if  $(u_k) \subset \mathcal{H}$  is such that  $J(u_k) \rightarrow c$  and  $J'(u_k) \rightarrow 0$ , then there exists some  $u \in \mathcal{H}$  such that up to a subsequence,  $u_k \rightarrow u$  in  $\mathcal{H}$ .*

*Proof.* Note that  $J(u_k) \rightarrow c$  and  $J'(u_k) \rightarrow 0$  as  $k \rightarrow +\infty$  are equivalent to

$$\frac{1}{2}\|u_k\|_{\mathcal{H}}^2 - \int_V F(x, u_k) d\mu = c + o_k(1) \quad (15)$$

$$\left| \langle u_k, \varphi \rangle_{\mathcal{H}} - \int_V f(x, u_k) \varphi d\mu \right| = o_k(1) \|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H}. \quad (16)$$

Here and in the sequel,  $o_k(1) \rightarrow 0$  as  $k \rightarrow +\infty$ . Taking  $\varphi = u_k$  in (16), we have

$$\|u_k\|_{\mathcal{H}}^2 = \int_V f(x, u_k) u_k d\mu + o_k(1) \|u_k\|_{\mathcal{H}}. \quad (17)$$

In view of  $(H_2)$ , we have by combining (15) and (16) that

$$\begin{aligned} \|u_k\|_{\mathcal{H}}^2 &= 2 \int_V F(x, u_k) d\mu + 2c + o_k(1) \\ &\leq \frac{2}{\theta} \int_V f(x, u_k) u_k d\mu + 2c + o_k(1) \\ &= \frac{2}{\theta} \|u_k\|_{\mathcal{H}}^2 + o_k(1) \|u_k\|_{\mathcal{H}} + 2c + o_k(1). \end{aligned}$$

Since  $\theta > 2$ ,  $u_k$  is bounded in  $\mathcal{H}$ . By  $(H_1)$  and  $(H_2)$ , the Sobolev embedding (Lemma 5) implies that up to a subsequence,  $u_k \rightharpoonup u$  weakly in  $\mathcal{H}$ ,  $u_k \rightarrow u$  in  $L^q(V)$  for any  $1 \leq q \leq +\infty$ . It follows that

$$\left| \int_V f(x, u_k) (u_k - u) d\mu \right| \leq C \int_V |u_k - u| d\mu = o_k(1).$$

Replacing  $\varphi$  by  $u_k - u$  in (16), we have

$$\langle u_k, u_k - u \rangle_{\mathcal{H}} = \int_V f(x, u_k) (u_k - u) d\mu + o_k(1) \|u_k - u\|_{\mathcal{H}} = o_k(1). \quad (18)$$

Moreover, since  $u_k \rightharpoonup u$  weakly in  $\mathcal{H}$ , there holds

$$\langle u, u_k - u \rangle_{\mathcal{H}} = o_k(1).$$

This together with (18) leads to  $\|u_k - u\|_{\mathcal{H}} = o_k(1)$ , or equivalently  $u_k \rightarrow u$  in  $\mathcal{H}$ .  $\square$

### 3.4. Completion of the proof of Theorem 1

*proof of Theorem 1.* By Lemmas 8, 9 and 10,  $J$  satisfies all the hypothesis of the mountain-pass theorem:  $J \in C^1(\mathcal{H}, \mathbb{R})$ ;  $J(0) = 0$ ;  $J(u) \geq \delta > 0$  when  $\|u\|_{\mathcal{H}} = r$ ;  $J(u^*) < 0$  for some  $u^* \in \mathcal{H}$  with  $\|u^*\|_{\mathcal{H}} > r$ ;  $J$  satisfies the Palais-Smale condition. Using the mountain-pass theorem due to Ambrosetti-Rabinowitz [6], we conclude that

$$c = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J(u)$$



is the critical point of  $J$ , where

$$\Gamma = \{\gamma \in C([0, 1], \mathcal{H}) : \gamma(0) = 0, \gamma(1) = u^*\}.$$

In particular, there exists some  $u \in \mathcal{H}$  such that  $J(u) = c$ . Clearly the Euler-Lagrange equation of  $u$  is (6), or equivalently,  $u$  is a weak solution of (6). Since

$$J(u) = c \geq \delta > 0,$$

we have that  $u \neq 0$ . Recalling the previous reduction (Section 3.2), we finish the proof of the theorem.  $\square$

#### 4. Proof of Theorem 2

The proof of Theorem 2 is analogous to that of Theorem 1. The difference is that hypotheses  $(H_2)$  and  $(F'_1)$  are replaced by  $(H'_2)$  and  $(F'_1)$  respectively. Let  $J : \mathcal{H} \rightarrow \mathbb{R}$  be defined by (14). The geometry of the functional  $J$  is described as below.

**Lemma 11.** *If  $h$  satisfies  $(H_1)$  and  $(H'_2)$ ,  $f$  satisfies  $(F'_1)$  and  $(F_2)$ , then  $J$  satisfies the  $(PS)_c$  condition for any  $c \in \mathbb{R}$ . Namely, if  $(u_k) \subset \mathcal{H}$  is such that  $J(u_k) \rightarrow c$  and  $J'(u_k) \rightarrow 0$ , then there exists some  $u \in \mathcal{H}$  such that up to a subsequence,  $u_k \rightarrow u$  in  $\mathcal{H}$ .*

*Proof.* Similar to the proof of Lemma 10, it follows from  $J(u_k) \rightarrow c$  and  $J'(u_k) \rightarrow 0$  that (15) and (16) holds, and  $u_k$  is bounded in  $\mathcal{H}$ . By  $(H_1)$  and  $(H'_2)$ , the Sobolev embedding (Lemma 6) implies that  $u_k \rightarrow u$  weakly in  $\mathcal{H}$ ,  $u_k \rightarrow u$  in  $L^q(V)$  for any  $2 \leq q \leq +\infty$ . By  $(F'_1)$ , we have

$$|f(x, u_k)| = |f(x, u_k) - f(x, 0)| \leq L|u_k|.$$

Hence

$$\begin{aligned} \left| \int_V f(x, u_k)(u_k - u) d\mu \right| &\leq L \int_V |u_k(u_k - u)| d\mu \\ &\leq L \left( \int_V u_k^2 d\mu \right)^{1/2} \left( \int_V |u_k - u|^2 d\mu \right)^{1/2} \\ &= o_k(1). \end{aligned}$$

Taking  $\varphi$  by  $u_k - u$  in (16), we have

$$\langle u_k, u_k - u \rangle_{\mathcal{H}} = \int_V f(x, u_k)(u_k - u) d\mu + o_k(1) \|u_k - u\|_{\mathcal{H}} = o_k(1). \quad (19)$$

On the other hand, we have by  $u_k \rightarrow u$  weakly in  $\mathcal{H}$  that  $\langle u, u_k - u \rangle_{\mathcal{H}} = o_k(1)$ . This together with (19) leads to  $u_k \rightarrow u$  in  $\mathcal{H}$ .  $\square$

*Proof of Theorem 2.* By Lemmas 8, 9 and 11,  $J$  satisfies all the hypothesis of the mountain-pass theorem:  $J \in C^1(\mathcal{H}, \mathbb{R})$ ;  $J(0) = 0$ ;  $J(u) \geq \delta > 0$  when  $\|u\|_{\mathcal{H}} = r$ ;  $J(u_1) < 0$  for some  $u_1 \in \mathcal{H}$  with  $\|u_1\|_{\mathcal{H}} > r$ ;  $J$  satisfies the Palais-Smale condition. Using the mountain-pass theorem due to Ambrosetti-Rabinowitz [6], we conclude that

$$c = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J(u)$$

is the critical point of  $J$ , where

$$\Gamma = \{\gamma \in C([0, 1], \mathcal{H}) : \gamma(0) = 0, \gamma(1) = u_1\}.$$

In particular, (6) has a weak solution  $u \in \mathcal{H}$ . Noting that  $J(u) = c \geq \delta > 0$ , we know that  $u$  is nontrivial. In view of the previous reduction (Section 3.2), this completes the proof of the theorem.  $\square$

## 5. Positive solutions of the perturbed equation

In this section, we prove Theorems 3 and 4. In view of (7), when  $\epsilon > 0$ ,  $g \geq 0$  and  $g \neq 0$ , similarly as in Section 3.2, we can assume  $f(x, s) \equiv 0$  for all  $s \in (-\infty, 0]$ . Moreover, we only need to find two distinct weak solutions in each case. Indeed if  $u$  is a weak solution of (7) with  $\epsilon > 0$ ,  $g \geq 0$  and  $g \neq 0$ , then obviously  $u \neq 0$ , and thus the maximum principle implies that  $u$  is a strictly positive point-wise solution of (7).

### 5.1. Proof of Theorem 3

To prove Theorem 3, we define a functional on  $\mathcal{H}$  by

$$J_\epsilon(u) = \frac{1}{2}\|u\|_{\mathcal{H}}^2 - \int_V F(x, u)d\mu - \epsilon \int_V gud\mu,$$

where  $\epsilon > 0$  and  $g \in \mathcal{H}'$ . The geometric profile of  $J_\epsilon$  is described by the following two lemmas.

**Lemma 12.** *For any  $\epsilon > 0$ , there exists some  $u \in \mathcal{H}$  such that  $J_\epsilon(tu) \rightarrow -\infty$  as  $t \rightarrow +\infty$ .*

*Proof.* An obvious analog of the proof of Lemma 8.  $\square$

**Lemma 13.** *There exists some  $\epsilon_1 > 0$  such that if  $0 < \epsilon < \epsilon_1$ , there exist constants  $r_\epsilon > 0$  and  $\delta_\epsilon > 0$  such that  $J_\epsilon(u) \geq \delta_\epsilon$  for all  $u \in \mathcal{H}$  with  $\frac{1}{2}r_\epsilon \leq \|u\|_{\mathcal{H}} \leq 2r_\epsilon$ . Furthermore,  $r_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ .*

*Proof.* By  $(F_3)$ , we can find positive constants  $\tau$  and  $\varrho$  such that for all  $(x, s) \in V \times \mathbb{R}$ , there holds

$$F(x, s) \leq \frac{\lambda_1 - \tau}{2}s^2 + \frac{s^3}{\varrho^3}F(x, s).$$

For any  $u \in \mathcal{H}$  with  $\|u\|_{\mathcal{H}} \leq 1$ , we have by Lemma 5 that  $\|u\|_{L^\infty(V)} \leq C$  for some constant  $C$ , and that there exists another constant (still denoted by  $C$ ) such that

$$\begin{aligned} J_\epsilon(u) &\geq \frac{1}{2}\|u\|_{\mathcal{H}}^2 - \frac{\lambda_1 - \tau}{2\lambda_1}\|u\|_{\mathcal{H}}^2 - C\|u\|_{\mathcal{H}}^3 - \epsilon\|g\|_{\mathcal{H}'}\|u\|_{\mathcal{H}} \\ &= \|u\|_{\mathcal{H}} \left( \frac{\tau}{2\lambda_1}\|u\|_{\mathcal{H}} - C\|u\|_{\mathcal{H}}^2 - \epsilon\|g\|_{\mathcal{H}'} \right). \end{aligned} \quad (20)$$

Take

$$r_\epsilon = \sqrt{\epsilon}, \quad \delta_\epsilon = \frac{\tau\epsilon}{16\lambda_1}, \quad \epsilon_1 = \min \left\{ \frac{1}{4}, \frac{\tau^2}{64\lambda_1^2(4C + \|g\|_{\mathcal{H}'})^2} \right\}.$$

Then if  $0 < \epsilon < \epsilon_1$ , we have  $J_\epsilon(u) \geq \delta_\epsilon$  for all  $u \in \mathcal{H}$  with  $\frac{1}{2}r_\epsilon \leq \|u\|_{\mathcal{H}} \leq 2r_\epsilon$ . Obviously  $r_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ .  $\square$

We now prove that  $J_\epsilon$  satisfies the Palais-Smale condition.

**Lemma 14.** Let  $\epsilon \in \mathbb{R}$  be fixed. If  $h$  satisfies  $(H_1)$  and  $(H_2)$ ,  $f$  satisfies  $(F_1)$  and  $(F_2)$ , then  $J_\epsilon$  satisfies the  $(PS)_c$  condition for any  $c \in \mathbb{R}$ . Namely, if  $(v_k) \subset \mathcal{H}$  is such that  $J_\epsilon(v_k) \rightarrow c$  and  $J'_\epsilon(v_k) \rightarrow 0$ , then there exists some  $v \in \mathcal{H}$  such that  $v_k \rightarrow v$  in  $\mathcal{H}$ .

*Proof.* Clearly, the hypotheses  $J_\epsilon(u_k) \rightarrow c$  and  $J'_\epsilon(u_k) \rightarrow 0$  are equivalent to the following:

$$\frac{1}{2} \int_V (|\nabla v_k|^2 + h v_k^2) d\mu - \int_V F(x, v_k) d\mu - \epsilon \int_V g v_k d\mu \rightarrow c \quad \text{as } k \rightarrow +\infty, \quad (21)$$

$$\left| \int_V (\Gamma(v_k, \varphi) + h v_k \varphi) d\mu - \int_V f(x, v_k) \varphi d\mu - \epsilon \int_V g \varphi d\mu \right| \leq \epsilon_k \|\varphi\|_{\mathcal{H}}, \quad \forall \varphi \in \mathcal{H}, \quad (22)$$

where  $\epsilon_k \rightarrow 0$  as  $k \rightarrow +\infty$ . Taking  $\varphi = v_k$  in (22), we have

$$\|v_k\|_{\mathcal{H}}^2 = \int_V f(x, v_k) v_k d\mu + \epsilon \int_V g v_k d\mu + o_k(1) \|v_k\|_{\mathcal{H}}.$$

In view of  $(F_2)$ , this together with (21) leads to

$$\begin{aligned} \|v_k\|_{\mathcal{H}}^2 &= 2c + 2 \int_V F(x, v_k) d\mu + 2\epsilon \int_V g v_k d\mu + o_k(1) \\ &\leq 2c + \frac{2}{\theta} \int_V f(x, v_k) v_k d\mu + 2\epsilon \int_V g v_k d\mu + o_k(1) \\ &= 2c + \frac{2}{\theta} \|v_k\|_{\mathcal{H}}^2 + 2\epsilon \left(1 - \frac{1}{\theta}\right) \int_V g v_k d\mu + o_k(1) \|v_k\|_{\mathcal{H}} + o_k(1) \\ &\leq 2c + \frac{2}{\theta} \|v_k\|_{\mathcal{H}}^2 + \left(2\epsilon \left(1 - \frac{1}{\theta}\right) \|g\|_{\mathcal{H}'} + o_k(1)\right) \|v_k\|_{\mathcal{H}} + o_k(1). \end{aligned}$$

Since  $\theta > 2$ , we can see from the above inequality that  $v_k$  is bounded in  $\mathcal{H}$ . By Lemma 5, there exists some  $v \in \mathcal{H}$  such that up to a subsequence,  $v_k \rightharpoonup v$  weakly in  $\mathcal{H}$ , and  $v_k \rightarrow v$  strongly in  $L^q(V)$  for any  $1 \leq q \leq +\infty$ . Taking  $\varphi = v_k - v$  in (22), we have

$$\langle v_k, v_k - v \rangle_{\mathcal{H}} = \int_V f(x, v_k)(v_k - v) d\mu + \epsilon \int_V g(v_k - v) d\mu + o_k(1) \|v_k - v\|_{\mathcal{H}}. \quad (23)$$

Since  $v_k \rightharpoonup v$  weakly in  $\mathcal{H}$  and  $g \in \mathcal{H}'$ , there holds

$$\lim_{k \rightarrow +\infty} \int_V g(v_k - v) d\mu = 0. \quad (24)$$

In view of  $(H_1)$ , we can see that  $|f(x, v_k)| \leq C$  for some constant  $C$  since  $v_k$  is uniformly bounded. Hence we estimate

$$\left| \int_V f(x, v_k)(v_k - v) d\mu \right| \leq C \int_V |v_k - v| d\mu = o_k(1). \quad (25)$$

Inserting (24) and (25) into (23), we obtain

$$\langle v_k, v_k - v \rangle_{\mathcal{H}} = o_k(1). \quad (26)$$

Moreover, it follows from  $v_k \rightharpoonup v$  weakly in  $\mathcal{H}$  that  $\langle v, v_k - v \rangle_{\mathcal{H}} = o_k(1)$ . This together with (26) leads to  $v_k \rightarrow v$  in  $\mathcal{H}$ , and ends the proof of the lemma.  $\square$

For the first weak solution of (7), we have the following:

**Proposition 15.** Let  $\epsilon_1$  be given as in Lemma 13. When  $0 < \epsilon < \epsilon_1$ , (7) has a mountain-pass type solution  $u_M$  verifying that  $J_\epsilon(u_M) = c_M$ , where  $c_M > 0$  is a min-max value of  $J_\epsilon$ .

*Proof.* By Lemmas 12, 13 and 14,  $J_\epsilon$  satisfies all the hypothesis of the mountain-pass theorem:  $J_\epsilon \in C^1(\mathcal{H}, \mathbb{R})$ ;  $J_\epsilon(0) = 0$ ;  $J_\epsilon(u) \geq \delta_\epsilon > 0$  when  $\|u\|_{\mathcal{H}} = r_\epsilon$ ;  $J_\epsilon(\tilde{u}) < 0$  for some  $\tilde{u} \in \mathcal{H}$  with  $\|\tilde{u}\|_{\mathcal{H}} > r_\epsilon$ . Using the mountain-pass theorem due to Ambrosetti-Rabinowitz [6], we conclude that

$$c_M = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J_\epsilon(u)$$

is the critical point of  $J_\epsilon$ , where

$$\Gamma = \{\gamma \in C([0, 1], \mathcal{H}) : \gamma(0) = 0, \gamma(1) = \tilde{u}\}.$$

In particular, (7) has a weak solution  $u_M \in \mathcal{H}$  verifying  $J(u) = c_M \geq \delta_\epsilon > 0$ .  $\square$

**Lemma 16.** Assume  $h$  satisfies  $(H_1)$  and  $(H_2)$ ,  $g \not\equiv 0$  and  $(F_1)$  holds. There exist  $\tau_0 > 0$  and  $v \in \mathcal{H}$  with  $\|v\|_{\mathcal{H}} = 1$  such that  $J_\epsilon(tv) < 0$  for all  $0 < t < \tau_0$ . Particularly

$$\inf_{\|u\|_{\mathcal{H}} \leq \tau_0} J_\epsilon(u) < 0.$$

*Proof.* We first claim that the equation

$$-\Delta v + hv = g \quad \text{in } V \tag{27}$$

has a solution  $v \in \mathcal{H}$ . To see this, we minimize the functional

$$J_g(v) = \frac{1}{2} \int_V (|\nabla v|^2 + hv^2) dv_g - \int_V gvd\mu.$$

For any  $v \in \mathcal{H}$ , we have

$$\left| \int_V gvd\mu \right| \leq \|g\|_{\mathcal{H}'} \|v\|_{\mathcal{H}} \leq \frac{1}{4} \|v\|_{\mathcal{H}}^2 + \|g\|_{\mathcal{H}'}^2. \tag{28}$$

Hence  $J_g$  has a lower bound on  $\mathcal{H}$ . Denote

$$\lambda_g = \inf_{v \in \mathcal{H}} J_g(v).$$

Take  $v_k \in \mathcal{H}$  such that  $J_g(v_k) \rightarrow \lambda_g$ . In view of (28),  $v_k$  is bounded in  $\mathcal{H}$ . Then by the Sobolev embedding (Lemma 5), we can find some  $v \in \mathcal{H}$  such that  $v_k \rightarrow v$  weakly in  $\mathcal{H}$ . Hence

$$\|v\|_{\mathcal{H}} \leq \liminf_{k \rightarrow +\infty} \|v_k\|_{\mathcal{H}} = \lambda_g,$$

and  $v$  is a minimizer of  $J_g$ . The Euler-Lagrange equation of  $v$  is exactly (27). Since  $g \not\equiv 0$ , it follows that

$$\int_V gvd\mu = \|v\|_{\mathcal{H}}^2 > 0. \tag{29}$$

Secondly, we consider the derivative of  $J_\epsilon(tv)$  as follows.

$$\frac{d}{dt} J_\epsilon(tv) = t \|v\|_{\mathcal{H}}^2 - \int_V f(x, tv) v d\mu - \epsilon \int_V gvd\mu. \tag{30}$$

Since  $f(x, 0) = 0$ , we have by inserting (29) into (30),

$$\left. \frac{d}{dt} J_\epsilon(tv) \right|_{t=0} < 0.$$

This gives the desired result.  $\square$

The second weak solution of (7) can be found in the following way.

**Proposition 17.** *Let  $\epsilon_1 > 0$  be given as in Lemma 13. Let  $\epsilon$ ,  $0 < \epsilon < \epsilon_1$ , be fixed. Then there exists a function  $u_0 \in \mathcal{H}$  with  $\|u_0\|_{\mathcal{H}} \leq 2r_\epsilon$  such that*

$$J_\epsilon(u_0) = c_\epsilon = \inf_{\|u\|_{\mathcal{H}} \leq 2r_\epsilon} J_\epsilon(u),$$

where  $r_\epsilon$  is given as in Lemma 13, and  $c_\epsilon < 0$ . Moreover,  $u_0$  is a strictly positive solution of (7).

*Proof.* Let  $\epsilon$ ,  $0 < \epsilon < \epsilon_1$ , be fixed. In view of (20),  $J_\epsilon$  has a lower bound on the set

$$\mathcal{B}_{2r_\epsilon} = \{u \in \mathcal{H} : \|u\|_{\mathcal{H}} \leq 2r_\epsilon\}.$$

This together with Lemma 16 implies that

$$c_\epsilon = \inf_{\|u\|_{\mathcal{H}} \leq 2r_\epsilon} J_\epsilon(u) < 0.$$

Take a sequence of functions  $(u_k) \subset \mathcal{H}$  such that  $\|u_k\|_{\mathcal{H}} \leq 2r_\epsilon$  and  $J_\epsilon(u_k) \rightarrow c_\epsilon$  as  $k \rightarrow +\infty$ . It follows from Lemma 5 that up to a subsequence,  $u_k \rightharpoonup u_0$  weakly in  $\mathcal{H}$  and  $u_k \rightarrow u_0$  strongly in  $L^q(V)$  for all  $1 \leq q \leq +\infty$ . In view of  $(F_1)$ , there exists some constant  $C$  such that

$$|F(x, u_k) - F(x, u_0)| \leq C|u_k - u_0|,$$

which leads to

$$\lim_{k \rightarrow +\infty} \int_V F(x, u_k) d\mu = \int_V F(x, u_0) d\mu. \quad (31)$$

Since  $u_k \rightharpoonup u_0$  weakly in  $\mathcal{H}$ , we obtain

$$\|u_0\|_{\mathcal{H}} \leq \limsup_{k \rightarrow +\infty} \|u_k\|_{\mathcal{H}} \quad (32)$$

and

$$\lim_{k \rightarrow +\infty} \int_V g u_k d\mu = \int_V g u_0 d\mu. \quad (33)$$

Combining (31), (32), and (33), we obtain  $\|u_0\|_{\mathcal{H}} \leq 2r_\epsilon$  and

$$J_\epsilon(u_0) \leq \limsup_{k \rightarrow +\infty} J_\epsilon(u_k) = c_\epsilon.$$

Therefore  $u_0$  is the minimizer of  $J_\epsilon$  on the set  $\mathcal{B}_{2r_\epsilon}$ . By Lemma 13, we conclude that

$$\|u_0\|_{\mathcal{H}} < r_\epsilon/2.$$

For any fixed  $\varphi \in C_c(V)$ , we define a smooth function  $\zeta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\zeta(t) = J_\epsilon(u_0 + t\varphi).$$

Clearly, there exists a sufficiently small  $\tau_1 > 0$  such that  $u_0 + t\varphi \in \mathcal{B}_{2r_\epsilon}$  for all  $t \in (-\tau_1, \tau_1)$ . Hence  $\zeta(0) = \min_{t \in (-\tau_1, \tau_1)} \zeta(t)$ , and thus  $\zeta'(0) = 0$ , namely

$$\int_V (\Gamma(u_0, \varphi) + hu_0\varphi) d\mu - \int_V f(x, u_0)\varphi d\mu - \epsilon \int_V g\varphi d\mu = 0.$$

This implies that  $u_0$  is a weak solution of (7). This completes the proof of the proposition.  $\square$

*Completion of the proof of Theorem 3.* Let  $u_M$  and  $u_0$  be two solutions of (7) given as in Propositions 15 and 17 respectively. Noting that  $J_\epsilon(u_M) = c_M > 0$  and  $J_\epsilon(u_0) = c_\epsilon < 0$ , we finish the proof of Theorem 3.  $\square$

## 5.2. Proof of Theorem 4

*Proof of Theorem 4.* The proof is completely analogous to that of Theorem 3. We only stress their essential differences. During the process of finding the mountain-pass type solution, we use Lemma 6 instead of Lemma 5, and use  $(H_1)$ ,  $(H'_2)$ ,  $(F'_1)$  and  $(F_2)$  to prove that  $J_\epsilon$  satisfies the Palais-Smale condition. We only need to concern (25): By  $(F'_1)$ , we have

$$|f(x, u_k)| = |f(x, u_k) - f(x, 0)| \leq L|u_k|,$$

which together with the Hölder inequality implies that

$$\left| \int_V f(x, u_k)(u_k - u) d\mu \right| \leq L \left( \int_V u_k^2 d\mu \right)^{1/2} \left( \int_V |u_k - u|^2 d\mu \right)^{1/2} = o_k(1).$$

While during the process of finding the solution of negative energy, we need to prove (31) by  $(F'_1)$  instead of  $(F_1)$ , namely

$$\begin{aligned} \left| \int_V (F(x, u_k) - F(x, u_0)) d\mu \right| &\leq L \int_V |u_k - u_0| \max\{|u_k|, |u_0|\} d\mu \\ &\leq L \left( \int_V (u_k^2 + u_0^2) d\mu \right)^{1/2} \left( \int_V |u_k - u_0|^2 d\mu \right)^{1/2} \\ &= o_k(1). \end{aligned}$$

We omit the details, but leave it to the interested readers.  $\square$

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