

Asymptotic behavior of the heat semigroup on certain Riemannian manifolds

Alexander Grigor'yan, Effie Papageorgiou and Hong-Wei Zhang

Abstract We show that, on a complete, connected and non-compact Riemannian manifold of non-negative Ricci curvature, the solution to the heat equation with L^1 initial data behaves asymptotically as the mass times the heat kernel. In contrast to the previously known results in negatively curved contexts, the radially assumption on the initial data is not required. Similar long-time convergence results remain valid on more general manifolds satisfying the Li-Yau two-sided estimate of the heat kernel. Moreover, we provide a counterexample such that this asymptotic phenomenon fails in sup norm on manifolds with two Euclidean ends.

1 Introduction

Let \mathcal{M} be a Riemannian manifold of dimension $n \geq 2$ and Δ be the Laplace-Beltrami operator on \mathcal{M} . It is well understood that the long-time behavior of solutions to the heat equation

$$\partial_t u(t, x) = \Delta_x u(t, x), \quad u(0, x) = u_0(x), \quad t > 0, \quad x \in \mathcal{M} \quad (1)$$

is very much related to the global geometry of \mathcal{M} . This applies also to the heat kernel $h_t(x, y)$ that is the minimal positive fundamental solution of the heat equation or, equivalently, the integral kernel of the heat semigroup $\exp(t\Delta)$ (see for instance [7, 17, 12]).

If the initial function u_0 belongs to the space $L^p(\mathcal{M}, \mu)$ with $p \in [1, \infty)$ (where μ is the Riemannian measure on \mathcal{M}) then the Cauchy problem (1) has a unique solution u such that $u(t, \cdot) \in L^p$ for any $t > 0$, and this solution is given by

$$u(t, x) = \int_{\mathcal{M}} h_t(x, y) u_0(y) d\mu(y). \quad (2)$$

The same is true for the case $p = \infty$ provided \mathcal{M} is stochastically complete. Hence, by a solution of (1) we always mean the function (2).

The aim of this paper is to investigate the connection between the long-time behavior of the solution $u(t, x)$ of (1) and that of the heat kernel $h_t(x, y)$. Let the initial function u_0 belong to $L^1(\mathcal{M})$, and denote by

Alexander Grigor'yan
Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
e-mail: grigor@math.uni-bielefeld.de

Effie Papageorgiou
Department of Mathematics and Applied Mathematics, University of Crete, Crete, Greece
e-mail: papageoeffie@gmail.com

Hong-Wei Zhang
Department of Mathematics: Analysis, Logic and Discrete Mathematics Ghent University, Belgium
e-mail: hongwei.zhang@ugent.be

$M = \int_{\mathcal{M}} u_0(x) d\mu(x)$ its mass. In the case when $\mathcal{M} = \mathbb{R}^n$ with the Euclidean metric, the heat kernel is given by

$$h_t(x, y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$$

and the solution to (1) satisfies as $t \rightarrow \infty$

$$\|u(t, \cdot) - Mh_t(\cdot, x_0)\|_{L^1(\mathbb{R}^n)} \rightarrow 0 \quad (3)$$

and

$$t^{\frac{n}{2}} \|u(t, \cdot) - Mh_t(\cdot, x_0)\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0. \quad (4)$$

By interpolation, a similar convergence holds with respect to any L^p norm when $1 < p < \infty$:

$$t^{\frac{n}{2p'}} \|u(t, \cdot) - Mh_t(\cdot, 0)\|_{L^p(\mathbb{R}^n)} \rightarrow 0$$

where p' is the Hölder conjugate of p .

Note that (3) and (4) hold for *any* choice of x_0 , which means that in the long run the solution $u(t, x)$ and the heat kernel $h_t(x, x_0)$ “forget” about the initial function u_0 resp. initial point x_0 . We refer to a recent survey [20] for more details about this property in the Euclidean setting.

The convergence properties (3) and (4) have an interesting probabilistic meaning. Let $\{X_t\}$ be Brownian motion on \mathcal{M} whose transition density is $h_t(x, y)$. Then (4) means, in particular, that X_t eventually “forgets” about its starting point x_0 , which corresponds to the fact that X_t escapes to ∞ rotating chaotically in angular direction.

The situation is drastically different in hyperbolic spaces. It was shown by Vázquez [21] that (3) fails for a general initial function $u_0 \in L^1(\mathbb{H}^n)$ but is still true if u_0 is spherically symmetric around x_0 . Similar results were obtained in [2] in a more general setting of symmetric spaces of non-compact type by using tools of harmonic analysis. Note that these spaces have nonpositive sectional curvature. Recall that in hyperbolic spaces Brownian motion X_t tends to escape to ∞ along geodesics, which means that it “remembers” at least the direction of the starting point x_0 .

Our main result is the following theorem that deals with manifolds of non-negative Ricci curvature. Denote by $B(x, r)$ the geodesic ball on \mathcal{M} of radius r centered at $x \in \mathcal{M}$ and set $V(x, r) = \mu(B(x, r))$.

Theorem 1 *Let \mathcal{M} be a complete, connected and non-compact Riemannian manifold of non-negative Ricci curvature. Fix a base point $x_0 \in \mathcal{M}$ and suppose that $u_0 \in L^1(\mathcal{M})$. Then the solution to the heat equation (1) satisfies as $t \rightarrow \infty$*

$$\|u(t, \cdot) - Mh_t(\cdot, x_0)\|_{L^1(\mathcal{M})} \rightarrow 0 \quad (5)$$

and

$$\| |u(t, \cdot) - Mh_t(\cdot, x_0)| V(\cdot, \sqrt{t}) \|_{L^\infty(\mathcal{M})} \rightarrow 0. \quad (6)$$

Remark 1 By interpolation between (5) and (6), we obtain for any $p \in (1, \infty)$

$$\| |u(t, \cdot) - Mh_t(\cdot, x_0)| V(\cdot, \sqrt{t})^{1/p'} \|_{L^p(\mathcal{M})} \rightarrow 0. \quad (7)$$

In Section 2 we will give a short review of different estimates of the heat kernel and reformulate Theorem 1 in more general ways. In Section 3 we prove Theorem 1. An essential idea of the proof is to describe the critical region where the heat kernel concentrates. In the last section, we show that an estimate as (6) fails on manifolds with two Euclidean ends.

2 Auxiliary results

2.1 Heat kernel estimates

From now on, \mathcal{M} denotes a complete, connected, non-compact Riemannian manifold of dimension $n \geq 2$. Let μ be the Riemannian measure on \mathcal{M} . Let $d(x, y)$ be the geodesic distance between two points $x, y \in \mathcal{M}$, and $V(x, r) = \mu(B(x, r))$ be the Riemannian volume of the geodesic ball $B(x, r)$ of radius r centered at $x \in \mathcal{M}$.

Throughout the paper we follow the convention that C, C_1, \dots denote large positive constants whereas c, c_1, \dots are small positive constants. These constants may depend on \mathcal{M} but do not depend on the variables x, y, t . Moreover, the notation $A \lesssim B$ between two positive expressions means that $A \leq CB$, and $A \asymp B$ means $cB \leq A \leq CB$.

We say that \mathcal{M} satisfies the volume doubling property if, for all $x \in \mathcal{M}$ and $r > 0$, we have

$$V(x, 2r) \leq C V(x, r). \quad (8)$$

It follows from (8) that there exist some positive constants $\nu, \nu' > 0$ such that

$$c \left(\frac{R}{r} \right)^{\nu'} \leq \frac{V(x, R)}{V(x, r)} \leq C \left(\frac{R}{r} \right)^{\nu} \quad (9)$$

for all $x \in \mathcal{M}$ and $0 < r \leq R$ (see for instance [12, Section 15.6]). Moreover, (9) implies that, for all $x, y \in \mathcal{M}$ and $r > 0$,

$$\frac{V(x, r)}{V(y, r)} \leq C \left(1 + \frac{d(x, y)}{r} \right)^{\nu}. \quad (10)$$

The integral kernel $h_t(x, y)$ of the heat semigroup $\exp(t\Delta)$ is the smallest positive fundamental solution to the heat equation (1). It is known that $h_t(x, y)$ is smooth in (t, x, y) , symmetric in x, y , and satisfies the semigroup identity (see for instance [19], [12]). Besides, for all $y \in \mathcal{M}$ and $t > 0$

$$\int_{\mathcal{M}} h_t(x, y) d\mu(x) \leq 1.$$

The manifold \mathcal{M} is called stochastically complete if for all $y \in \mathcal{M}$ and $t > 0$

$$\int_{\mathcal{M}} h_t(x, y) d\mu(x) = 1.$$

It is known that if \mathcal{M} is geodesically complete and, for some $x_0 \in \mathcal{M}$ and all large enough r ,

$$V(x_0, r) \leq e^{Cr^2},$$

then \mathcal{M} is stochastically complete. In particular, the volume doubling property (8) implies that \mathcal{M} is stochastically complete.

When the Ricci curvature of \mathcal{M} is non-negative, the following two-sided estimates of the heat kernel were proved by Li and Yau [15]:

$$\frac{c_1}{V(x, \sqrt{t})} \exp\left(-C_1 \frac{d^2(x, y)}{t}\right) \leq h_t(x, y) \leq \frac{C_2}{V(x, \sqrt{t})} \exp\left(-c_2 \frac{d^2(x, y)}{t}\right). \quad (11)$$

Besides, Li and Yau proved in [15] also the following gradient estimate for any positive solution $u(t, x)$ of the heat equation $\partial_t u = \Delta u$ on $\mathbb{R}_+ \times \mathcal{M}$:

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{C}{t} \quad (12)$$

with $C = \frac{n}{2}$. By a result of [11], the upper bound of the heat kernel in (11), that is,

$$h_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right), \quad (13)$$

implies the following estimate of the time derivative:

$$\left|\frac{\partial h_t}{\partial t}(x, y)\right| \leq \frac{C}{t V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right). \quad (14)$$

It follows from (12) that

$$|\nabla u|^2 \leq u \partial_t u + \frac{C}{t} u^2$$

and, hence,

$$|\nabla u| \leq \sqrt{u |\partial_t u|} + \frac{C}{\sqrt{t}} u.$$

Applying this for the function $u(t, y) = h_t(x, y)$ and combining with (13) and (14), we obtain that

$$|\nabla_y h_t(x, y)| \leq \frac{C}{\sqrt{t} V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right). \quad (15)$$

It is known that the upper bound (13) of h_t (and, hence, its consequence (14)) holds on a larger class of Riemannian manifolds satisfying a so called *relative Faber-Krahn inequality* (see, for example, [12]). But the estimate (15) of the gradient ∇h_t is much more subtle and requires more serious hypotheses, for example, non-negative Ricci curvature as we consider here.

Let us observe that the gradient estimate (15) is a particular (and limiting) case of the following Hölder estimate:

$$|h_t(x, y) - h_t(x, z)| \leq \left(\frac{d(y, z)}{\sqrt{t}}\right)^\theta \frac{C}{V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right), \quad (16)$$

for some $0 < \theta \leq 1$, and all $t > 0$, $x, y, z \in \mathcal{M}$ such that $d(y, z) \leq \sqrt{t}$. It is important to mention that (16) is a consequence of the two-sided estimate (11) alone (for some $0 < \theta < 1$, see [17, Theorem 5.4.12]), and, hence, (16) holds on more general classes of manifolds than (15).

Finally, let us observe that on spaces of essentially negative curvature the above estimates of the heat kernel typically fail: for example, these are hyperbolic spaces [6], non-compact symmetric spaces [1], asymptotically hyperbolic manifolds [5], and fractal-like manifolds [3].

2.2 Alternative statements of the main theorem

Now let us reformulate Theorem 1 in a bit more general way.

Theorem 2 *Let \mathcal{M} be a geodesically complete non-compact manifold that satisfies the following conditions:*

- *the volume doubling condition (8);*
- *the upper bound (13) of the heat kernel;*
- *the Hölder regularity (16) of the heat kernel.*

Then the conclusions of Theorem 1 hold.

Apart from manifolds with non-negative Ricci curvature, the above-described manifolds cover many other examples. Let us recall that, on a complete Riemannian manifold, the following three properties are equivalent:

- The two-sided estimate (11) of the heat kernel;
- The uniform parabolic Harnack inequality:

$$\sup_{(\frac{T}{4}, \frac{T}{2}) \times B(x, \frac{r}{2})} u(t, x) \leq C \sup_{(\frac{3T}{4}, T) \times B(x, \frac{r}{2})} u(t, x), \quad (17)$$

where $u(t, x)$ is a non-negative solution of the heat equation $\partial_t u = \Delta u$ in a cylinder $(0, T) \times B(x, r)$ with $x \in \mathcal{M}$, $r > 0$ and $T = r^2$.

- The conjunction of the volume doubling property (8) and the Poincaré inequality:

$$\int_{B(x, r)} |f - f_B|^2 d\mu \leq C r^2 \int_{B(x, r)} |\nabla f|^2 d\mu, \quad (18)$$

for all $x \in \mathcal{M}$, $t > 0$, and bounded Lipschitz functions f in $B(x, r)$. Here, f_B is the mean of f over $B(x, r)$.

See, for instance, [9, 10, 16, 17] for more details. Manifolds satisfying these equivalent conditions include complete manifolds with non-negative Ricci curvature, connected Lie groups with polynomial volume growth, co-compact covering manifolds whose deck transformation has polynomial growth, and many others. We refer to [18, pp. 417–418] for a list of examples. These equivalent conditions yield particularly the Hölder regularity (16) for some $0 < \theta < 1$, see [17, Theorem 5.4.12]. Hence, we obtain following corollary.

Corollary 1 *Let \mathcal{M} be a geodesically complete non-compact manifold that satisfies one of the following equivalent conditions:*

- *the two-sided estimate (11) of the heat kernel;*
- *the uniform parabolic Harnack inequality (17);*
- *the conjunction of the volume doubling property (8) and the Poincaré inequality (18).*

Then the conclusions of Theorem 1 hold.

In fact, we do not need a differential operator like the Laplacian (or sub-Laplacian). All we need is a function $h_t(x, y)$ satisfying certain estimates, and the result can be formulated as a property of integral operators. For example, our method can be applied on a metric space (X, d, μ) of homogeneous type, assuming that one has a semigroup of self-adjoint linear operators acting on $L^2(X, \mu)$ and admitting a good integral kernel, see for instance [8].

2.3 Two preliminary lemmas

We prove Theorem 2 in the next section. In the next two lemmas we describe some consequences of the hypotheses of Theorem 2, in particular, the critical annulus where the heat kernel concentrates.

Lemma 1 *Under the volume doubling condition (8), we have, for any $c > 0$, $x_0 \in \mathcal{M}$, and $t > 0$,*

$$\int_{\mathcal{M}} \exp\left(-c \frac{d(x, x_0)^2}{t}\right) \frac{d\mu(x)}{V(x_0, \sqrt{t})} \lesssim 1 \quad (19)$$

and, for any $N \in \mathbb{N}$ and all $r \geq \sqrt{t}$,

$$\int_{B(x_0, r)^c} \exp\left(-c \frac{d(x, x_0)^2}{t}\right) \frac{d\mu(x)}{V(x_0, \sqrt{t})} \lesssim \left(\frac{r}{\sqrt{t}}\right)^{-N}. \quad (20)$$

Proof. Let us prove first (20). Using (9), we have

$$\begin{aligned}
& \frac{1}{V(x_0, \sqrt{t})} \int_{d(x, x_0) \geq r} \exp\left(-c \frac{d(x, x_0)^2}{t}\right) d\mu(x) \\
&= \frac{1}{V(x_0, \sqrt{t})} \sum_{j=0}^{\infty} \int_{2^j r \leq d(x, x_0) < 2^{j+1} r} \exp\left(-c \frac{d(x, x_0)^2}{t}\right) d\mu(x) \\
&\lesssim \sum_{j=0}^{\infty} \frac{V(x_0, 2^{j+1} r)}{V(x_0, \sqrt{t})} \exp\left(-c \frac{(2^j r)^2}{t}\right) \\
&\lesssim \sum_{j=0}^{\infty} \left(\frac{2^{j+1} r}{\sqrt{t}}\right)^{\nu} \exp\left(-c \frac{2^{2j} r^2}{t}\right).
\end{aligned}$$

Since $\exp(-s) \leq (N!) s^{-N}$ for any $s > 0$ and $N \in \mathbb{N}$, we obtain that

$$\begin{aligned}
\int_{B(x_0, r)^c} \exp\left(-c \frac{d(x, x_0)^2}{t}\right) \frac{d\mu(x)}{V(x_0, \sqrt{t})} &\lesssim \sum_{j=0}^{\infty} \left(\frac{r}{\sqrt{t}}\right)^{\nu} 2^{\nu j - 2Nj} \left(\frac{t}{r^2}\right)^N \\
&\lesssim \left(\frac{r}{\sqrt{t}}\right)^{\nu - 2N},
\end{aligned}$$

which proves (20).

In order to prove (19) we apply (20) with $r = \sqrt{t}$ and $N = 1$ and obtain

$$\begin{aligned}
& \int_{\mathcal{M}} \exp\left(-c \frac{d(x, x_0)^2}{t}\right) \frac{d\mu(x)}{V(x_0, \sqrt{t})} \\
&= \left(\int_{B(x_0, \sqrt{t})^c} + \int_{B(x_0, \sqrt{t})} \right) \exp\left(-c \frac{d(x, x_0)^2}{t}\right) \frac{d\mu(x)}{V(x_0, \sqrt{t})} \\
&\lesssim 1 + \int_{B(x_0, \sqrt{t})} \frac{d\mu(x)}{V(x_0, \sqrt{t})} = 2.
\end{aligned}$$

■

The next lemma describes the annulus where the heat kernel concentrates. Let us fix a point $x_0 \in \mathcal{M}$ and a positive function $\varphi(t)$ such that $\varphi(t) \rightarrow 0$ and $\varphi(t)\sqrt{t} \rightarrow \infty$ as $t \rightarrow \infty$. For any $t > 0$, define the following annulus in \mathcal{M} :

$$\Omega_t = \left\{ x \in \mathcal{M} \mid \varphi(t)\sqrt{t} \leq d(x, x_0) \leq \frac{\sqrt{t}}{\varphi(t)} \right\}. \quad (21)$$

Lemma 2 Under the hypotheses (8) and (13), we have for all large enough t

$$\int_{\mathcal{M} \setminus \Omega_t} h_t(x, x_0) d\mu(x) \lesssim \varphi(t)^{\nu'} \quad (22)$$

where ν' is the exponent from (9). Consequently,

$$\int_{\Omega_t} h_t(x, x_0) d\mu(x) \longrightarrow 1 \quad \text{as } t \longrightarrow \infty. \quad (23)$$

Proof. Since \mathcal{M} is stochastically complete, (23) follows from (22) and $\varphi(t) \rightarrow 0$. Since

$$\mathcal{M} \setminus \Omega_t = B(x_0, \varphi(t)\sqrt{t}) \cup B^c(x_0, \sqrt{t}/\varphi(t)),$$

we estimate the integrals over $B(x_0, \varphi(t)\sqrt{t})$ and $B^c(x_0, \sqrt{t}/\varphi(t))$ separately. Assume that t is large enough so that $\varphi(t) < 1$. Using (13) and (9) we obtain

$$\int_{d(x, x_0) < \varphi(t)\sqrt{t}} h_t(x, x_0) d\mu(x) \lesssim \frac{V(x_0, \varphi(t)\sqrt{t})}{V(x_0, \sqrt{t})} \lesssim \varphi(t)^{\nu'}.$$

Using (13) and (20) with $r = \frac{\sqrt{t}}{\varphi(t)}$ and any $N \in \mathbb{N}$, we obtain

$$\begin{aligned} \int_{d(x, x_0) > \frac{\sqrt{t}}{\varphi(t)}} h_t(x, x_0) d\mu(x) &\lesssim \int_{B(x_0, r)^c} \frac{d\mu(x)}{V(x_0, \sqrt{t})} \exp\left(-c \frac{d(x, x_0)^2}{t}\right) \\ &\lesssim \left(\frac{r}{\sqrt{t}}\right)^{-N} = \varphi(t)^N, \end{aligned}$$

whence the claim follows. ■

3 Proof of the main theorem

We start the proof of Theorem 2 with continuous compactly supported initial data and prove the asymptotic properties of the solution in the L^1 norm, working separately outside and inside the critical region Ω_t . Then, we show that these properties remain valid for all L^1 initial data by using a density argument. The L^∞ convergence is proved in the same spirit.

Proposition 1 *Let \mathcal{M} be a manifold defined as in Theorem 2. Let $x_0 \in \mathcal{M}$ and $u_0 \in C_c(B(x_0, a))$ for some $a > 0$. Assume that $\varphi(t)$ is a positive function such that $\varphi(t) \rightarrow 0$ and $\varphi(t)\sqrt{t} \rightarrow \infty$ as $t \rightarrow \infty$. Then the solution (2) satisfies*

$$\|u(t, \cdot) - Mh_t(\cdot, x_0)\|_{L^1(\mathcal{M} \setminus \Omega_t)} \lesssim \varphi(t)^{\nu'} \quad (24)$$

and

$$\|u(t, \cdot) - Mh_t(\cdot, x_0)\|_{L^1(\Omega_t)} \lesssim t^{-\frac{\theta}{2}}, \quad (25)$$

where Ω_t is defined by (21), ν' and θ are exponents from (9) and (16), and

$$M = \int_{\mathcal{M}} u_0(x) d\mu(x).$$

Consequently,

$$\|u(t, \cdot) - Mh_t(\cdot, x_0)\|_{L^1(\mathcal{M})} \lesssim t^{-\eta} \quad (26)$$

for any $\eta < \min(\nu', \theta)/2$.

Proof. By the upper bound (13) of the heat kernel and Lemma 2 we have

$$\|h_t(\cdot, x_0)\|_{L^1(\mathcal{M} \setminus \Omega_t)} \lesssim \varphi(t)^{\nu'}$$

so that (24) will follow if we prove that

$$\|u(t, \cdot)\|_{L^1(\mathcal{M} \setminus \Omega_t)} \lesssim \varphi(t)^{\nu'}. \quad (27)$$

We write

$$\begin{aligned} \|u(t, \cdot)\|_{L^1(\mathcal{M} \setminus \Omega_t)} &= \int_{\mathcal{M} \setminus \Omega_t} \left| \int_{B(x_0, a)} h_t(x, y) u_0(y) d\mu(y) \right| d\mu(x) \\ &\leq \int_{B(x_0, a)} |u_0(y)| \left\{ \int_{\mathcal{M} \setminus \Omega_t} h_t(x, y) d\mu(x) \right\} d\mu(y). \end{aligned}$$

Notice that $x \in \mathcal{M} \setminus \Omega_t$ and $y \in B(x_0, a)$ imply $x \in \mathcal{M} \setminus \tilde{\Omega}_{t,y}$, where

$$\tilde{\Omega}_{t,y} = \left\{ x \in \mathcal{M} \mid 2\varphi(t)\sqrt{t} \leq d(x, y) \leq \frac{1}{2} \frac{\sqrt{t}}{\varphi(t)} \right\}$$

provided t is large enough. Indeed, if $x \in \tilde{\Omega}_{t,y}$ then

$$d(x, x_0) \leq d(x, y) + d(y, x_0) \leq \frac{1}{2} \frac{\sqrt{t}}{\varphi(t)} + a \leq \frac{\sqrt{t}}{\varphi(t)}$$

and

$$d(x, x_0) \geq d(x, y) - d(y, x_0) \geq 2\varphi(t)\sqrt{t} - a \geq \varphi(t)\sqrt{t}$$

for t large enough, since $\varphi(t) \rightarrow 0$ and $\varphi(t)\sqrt{t} \rightarrow \infty$ as $t \rightarrow \infty$. It follows that $x \in \Omega_t$.

Applying (22) with $\tilde{\Omega}_{t,y}$ instead of Ω_t we obtain

$$\int_{\mathcal{M} \setminus \Omega_t} h_t(x, y) \, d\mu(x) \leq \int_{\mathcal{M} \setminus \tilde{\Omega}_{t,y}} h_t(x, y) \, d\mu(x) \lesssim \varphi(t)^{\nu'},$$

whence (27) follows.

Now, let us turn to (25). Observe that

$$\begin{aligned} u(t, x) - Mh_t(x, x_0) &= \int_{\mathcal{M}} u_0(y) (h_t(x, y) - h_t(x, x_0)) \, d\mu(y) \\ &= \int_{B(x_0, a)} u_0(y) (h_t(x, y) - h_t(x, x_0)) \, d\mu(y). \end{aligned} \quad (28)$$

One the one hand, we deduce from the Hölder regularity (16) that

$$\begin{aligned} &|u(t, x) - Mh_t(x, x_0)| \\ &\leq \frac{C}{V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, x_0)}{t}\right) \int_{B(x_0, a)} \left(\frac{d(x_0, y)}{\sqrt{t}}\right)^\theta |u_0(y)| \, d\mu(y) \\ &\lesssim t^{-\frac{\theta}{2}} \frac{1}{V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, x_0)}{t}\right), \end{aligned} \quad (29)$$

for some $0 < \theta \leq 1$ and for t large enough such that $d(x_0, y) \leq a \leq \sqrt{t}$. On the other hand, (10) implies that

$$\begin{aligned} \frac{1}{V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, x_0)}{t}\right) &\lesssim \left(\frac{d(x, x_0)}{\sqrt{t}} + 1\right)^\nu \frac{1}{V(x_0, \sqrt{t})} \exp\left(-c \frac{d^2(x, x_0)}{t}\right) \\ &\lesssim \frac{1}{V(x_0, \sqrt{t})} \exp\left(-c \frac{d^2(x, x_0)}{2t}\right). \end{aligned} \quad (30)$$

Substituting (30) into the right-hand side of (29) and integrating in x over Ω_t , we obtain by (19)

$$\int_{\Omega_t} |u(t, x) - Mh_t(x, x_0)| \, d\mu(x) \lesssim t^{-\frac{\theta}{2}} \int_{\Omega_t} \frac{d\mu(x)}{V(x_0, \sqrt{t})} \exp\left(-c \frac{d^2(x, x_0)}{2t}\right) \lesssim t^{-\frac{\theta}{2}}.$$

Finally, (26) follows by adding up (24) and (25) with $\varphi(t) = t^{\varepsilon - \frac{\theta}{2}}$ with small enough ε . ■

Next, we prove our main theorem.

Proof of Theorem 2. Given $u_0 \in L^1(\mathcal{M})$, fix $\varepsilon > 0$ and choose $\tilde{u}_0 \in C_c(\mathcal{M})$ such that

$$\|u_0 - \tilde{u}_0\|_{L^1(\mathcal{M})} < \frac{\varepsilon}{3}.$$

Let us prove first (5). Setting $\tilde{M} = \int_{\mathcal{M}} \tilde{u}_0(y) d\mu(y)$ we have

$$|M - \tilde{M}| \leq \int_{\mathcal{M}} d\mu(y) |u_0(y) - \tilde{u}_0(y)| = \|u_0 - \tilde{u}_0\|_{L^1(\mathcal{M})} < \frac{\varepsilon}{3}. \quad (31)$$

It follows that

$$\|Mh_t(\cdot, x_0) - \tilde{M}h_t(\cdot, x_0)\|_{L^1(\mathcal{M})} \leq |M - \tilde{M}| \|h_t(\cdot, x_0)\|_{L^1(\mathcal{M})} < \frac{\varepsilon}{3}. \quad (32)$$

Let $\tilde{u}(t, x)$ be the semigroup solution to the heat equation with the initial data \tilde{u}_0 . Then we have

$$\begin{aligned} & \|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{L^1(\mathcal{M})} \\ & \leq \int_{\mathcal{M}} |u_0(y) - \tilde{u}_0(y)| \left\{ \int_{\mathcal{M}} d\mu(x) h_t(x, y) \right\} d\mu(y) < \frac{\varepsilon}{3}. \end{aligned} \quad (33)$$

By the estimate (26) of Proposition 1, we have, for sufficiently large t ,

$$\|\tilde{u}(t, \cdot) - \tilde{M}h_t(\cdot, x_0)\|_{L^1(\mathcal{M})} < \frac{\varepsilon}{3}. \quad (34)$$

Combining (32), (33) and (34) together, we obtain

$$\|u(t, \cdot) - Mh_t(\cdot, x_0)\|_{L^1(\mathcal{M})} < \varepsilon$$

which finishes the proof of (5).

Let us turn to the sup norm convergence (6). Since \tilde{u}_0 is compactly supported, we can apply (29) which gives for all x

$$|\tilde{u}(t, x) - \tilde{M}h_t(x, x_0)| \leq \frac{C}{t^{\theta/2}V(x, \sqrt{t})} \leq \frac{\varepsilon}{V(x, \sqrt{t})} \quad (35)$$

for sufficiently large t . Next, we have

$$\begin{aligned} |u(t, x) - \tilde{u}(t, x)| & \leq \int_{\mathcal{M}} h_t(x, y) |u_0(y) - \tilde{u}_0(y)| d\mu(y) \\ & \leq \frac{C \|u_0 - \tilde{u}_0\|_{L^1}}{V(x, \sqrt{t})} \lesssim \frac{\varepsilon}{V(x, \sqrt{t})} \end{aligned} \quad (36)$$

and

$$|Mh_t(x, x_0) - \tilde{M}h_t(x, x_0)| = |M - \tilde{M}| h_t(x, x_0) \lesssim \frac{\varepsilon}{V(x, \sqrt{t})}. \quad (37)$$

Putting together (35), (36) and (37), we conclude that

$$|u(t, x) - Mh_t(x, x_0)| \lesssim \frac{\varepsilon}{V(x, \sqrt{t})}$$

whence (6) follows. ■

4 A counterexample

Let $n \geq 3$. Consider a manifold $\mathcal{M} = \mathbb{R}^n \# \mathbb{R}^n$ that is a connected sum of two copies of \mathbb{R}^n . This means that \mathcal{M} is a union of a compact part K and two Euclidean ends $E^\pm = \mathbb{R}^n \setminus B(0, R)$ (see Fig. 1). This is a classical setting where the two-sided heat kernel estimate (11) fails, see for instance [13].

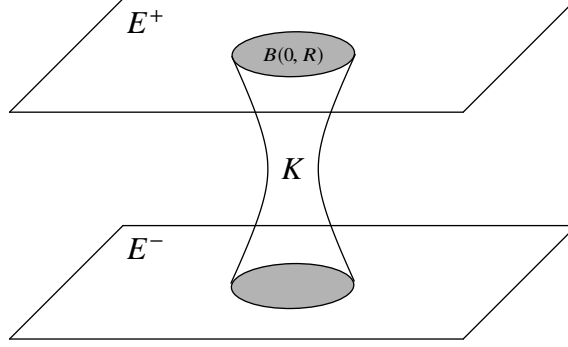


Fig. 1 Manifold $\mathcal{M} = \mathbb{R}^n \# \mathbb{R}^n$.

Note that $V(x, R) \asymp R^n$ for all $x \in \mathcal{M}$, and the heat kernel on \mathcal{M} satisfies the upper bound (13) (see, for instance, [13, Corollary 4.6, p.1946]). However, the essential hypotheses, Hölder regularity (16) or pointwise gradient estimate (15), fail in the present setting (and so does the Poincaré inequality, [14]). Indeed, it was shown in [4, Proposition 6.1] that, for large t ,

$$\sup_{x, y \in \mathcal{M}} |\nabla_y h_t(x, y)| \geq c t^{-n/2},$$

which is incompatible with (15).

Let us verify that also the conclusion (6) of Theorem 1 fails in this setting. For that consider on E^+ for any $t > 1$ a point x_t with $|x_t| = \sqrt{t}$ and with a fixed direction $\omega = x_t/\sqrt{t}$. It was proved in [4, Proposition 6.1] that, for any $x \in K$,

$$\lim_{t \rightarrow \infty} t^{\frac{n}{2}} h_t(x_t, x) = \Phi(x) \quad (38)$$

where $\Phi(x)$ is a positive harmonic function on \mathcal{M} that tends to a positive constant as $x \rightarrow \infty$ at the end E^+ and tends to 0 as $x \rightarrow \infty$ at the end E^- . Let us fix two points x_1 and x_2 in K such that $\Phi(x_1) \neq \Phi(x_2)$. Then (38) implies that

$$t^{\frac{n}{2}} (h_t(x_t, x_1) - h_t(x_t, x_2)) \longrightarrow (\Phi(x_1) - \Phi(x_2)) \quad \text{as } t \longrightarrow \infty \quad (39)$$

while (6) would imply that

$$t^{\frac{n}{2}} \|h_t(\cdot, x_1) - h_t(\cdot, x_2)\|_{L^\infty(\mathcal{M})} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty,$$

hence,

$$t^{\frac{n}{2}} (h_t(x_t, x_1) - h_t(x_t, x_2)) \longrightarrow 0,$$

which contradicts to (39).

Acknowledgements The first author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 317210226 - SFB 1283. The second author is supported by the Hellenic Foundation for Research and Innovation, Project HFRI-FM17-1733. The last author acknowledges financial support from the Methusalem Programme *Analysis and Partial Differential Equations* (Grant number 01M01021) during his postdoc stay at Ghent University.

References

1. Anker, J.-Ph., Ostellari, P.: The heat kernel on noncompact symmetric spaces. In: Lie Groups And Symmetric Spaces, pp. 27–46. Amer. Math. Soc., Providence, RI (2003)
2. Anker, J.-Ph., Papageorgiou, E., Zhang, H.-W.: Asymptotic behavior of solutions to the heat equation on noncompact symmetric spaces. *J. Funct. Anal.* **284**, Paper No. 109828 (2023)
3. Barlow, M.: Diffusions on fractals. In: Lectures On Probability Theory And Statistics (Saint-Flour, 1995), pp. 1–121. Springer, Berlin (1998)
4. Carron, G., Coulhon, T., Hassell, A.: Riesz transform and L^p -cohomology for manifolds with Euclidean ends. *Duke Math. J.* **133**, 59–93 (2006)
5. Chen, X., Hassell, A.: The heat kernel on asymptotically hyperbolic manifolds. *Comm. Partial Differential Equations* **45**, 1031–1071 (2020)
6. Davies, E., Mandouvalos, N.: Heat kernel bounds on hyperbolic space and Kleinian groups. *Proc. London Math. Soc.* (3) **57**, 182–208 (1988)
7. Davies, E.: Heat kernels and spectral theory. Cambridge University Press (1990)
8. Dziubański, J., Preisner, M.: Hardy spaces for semigroups with Gaussian bounds. *Ann. Mat. Pura Appl.* (4). **197**, 965–987 (2018)
9. Fabes, E., Stroock, D.: A new proof of Moser’s parabolic Harnack inequality using the old ideas of Nash. *Arch. Ration. Mech. Anal.* **96**, 327–338 (1986)
10. Grigor’yan, A.: The heat equation on noncompact Riemannian manifolds. *Mat. Sb.* **182**, 55–87 (1991)
11. Grigor’yan, A.: Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold. *J. Funct. Anal.* **127**, 363–389 (1995)
12. Grigor’yan, A.: Heat kernel and analysis on manifolds. AMS International Press (2009)
13. Grigor’yan, A., Saloff-Coste, L.: Heat kernel on manifolds with ends. *Ann. Inst. Fourier (Grenoble)* **59**, 1917–1997 (2009)
14. Grigor’yan, A., Ishiwata, S., Saloff-Coste, L.: Geometric analysis on manifolds with ends. In: Analysis And Partial Differential Equations On Manifolds, Fractals And Graphs, pp. 325–343, *Adv. Anal. Geom.* **3**, De Gruyter, Berlin (2021)
15. Li, P., Yau, S.: On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156**, 153–201 (1986)
16. Saloff-Coste, L.: A note on Poincaré, Sobolev, and Harnack inequalities. *Internat. Math. Res. Notices*, 27–38 (1992)
17. Saloff-Coste, L.: Aspects of Sobolev-type inequalities. Cambridge University Press (2002)
18. Saloff-Coste, L.: The heat kernel and its estimates. In: Probabilistic Approach To Geometry, pp. 405–436. *Adv. Stud. Pure Math.* **57**, Math. Soc. Japan, Tokyo (2010)
19. Strichartz, R.: Analysis of the Laplacian on the complete Riemannian manifold. *J. Functional Analysis* **52**, 48–79 (1983)
20. Vázquez, J.: Asymptotic behaviour methods for the heat equation. Convergence to the Gaussian. preprint, arXiv:1706.10034
21. Vázquez, J.: Asymptotic behaviour for the heat equation in hyperbolic space. To appear in *Comm. Analysis and Geometry*, arXiv:1811.09034