Lower estimates for a perturbed Green function

Alexander Grigor'yan
Universität Bielefeld
Postfach 100131
33501 Bielefeld, Germany
grigor@math.uni-bielefeld.de

Wolfhard Hansen
Universität Bielefeld
Postfach 100131
33501 Bielefeld, Germany
hansen@math.uni-bielefeld.de

February 2007

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1 Introduction

Let us consider a stationary Schrödinger equation in $\mathbb{R}^n$, $n > 2$,

$$\Delta u - u\mu = 0 \tag{1.1}$$

where μ is a measure on $\mathbb{R}^n$, which is considered as a perturbation to the Laplace operator. Let $g(x, y)$ be the Green function of the Laplacian, that is $g(x, y) =$

$c_n |x-y|^{2-n}$. If μ belongs to the local Kato class, that is, for any bounded open set $A \subset \mathbb{R}^n$ the function

$$x \mapsto \int_A g(x,y)\, d\mu(y)$$

is finite and continuous, then the equation (1.1) has also a positive Green function ${}^{\mu}g(x,y)$, which is continuous off the diagonal (see for example [7]). The question of obtaining estimates for the perturbed Green function ${}^{\mu}g(x,y)$ was addressed in a large number of publications.

Here we present a general method of obtaining lower estimates for ${}^{\mu}g$ (note that since $\mu \geq 0$, the comparison principle implies that ${}^{\mu}g \leq g$). The following statement is a particular case of our main Theorem 4.3. Denote by $B(x,R)$ the ball in $\mathbb{R}^n$ of radius R centered at point x, and set

$$\Gamma^{\mu}(x,R) := \int_{B(x,R)} g(x,z) d\mu(z). \tag{1.2}$$

Theorem 1.1 *If μ belongs to the local Kato class, then the perturbed Green function ${}^{\mu}g$ satisfies the following estimate, for all distinct $x,y \in \mathbb{R}^n$ and $R \geq 3|x-y|$,*

$$\frac{{}^{\mu}g(x,y)}{g(x,y)} \geq \exp\left(-C - C\Gamma^{\mu}(x,R) - C\Gamma^{\mu}(y,R)\right), \tag{1.3}$$

where $C = C(n) > 0$.

Although the lower bound for ${}^{\mu}g$ obtained by (1.3) may not be optimal, the power of Theorem 1.1 is in its generality: it gives some lower bound for *any* μ as above, and in many cases it allows to correctly identify the range of x,y where

$${}^{\mu}g(x,y) \simeq g(x,y) \tag{1.4}$$

(the sign $\simeq$ means that the ratio of the left- and right hand sides is bounded from above and below by positive constants). For example, if

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} g(x,z) d\mu(z) < \infty \tag{1.5}$$

then (1.3) implies that (1.4) holds for all $x \neq y$. This result is not new, though: under the same hypothesis (1.5) or a similar one it was proved in many places, see for example [7], [8], [10], [35], [37] (see also [16] and [41, Theorem A] for similar results for heat kernels).

Let $d\mu = V(x)\,dx$ and assume that the potential V satisfies the estimate

$$V(x) \leq \frac{V_0}{1+|x|^{\gamma}}, \tag{1.6}$$

where γ and V_0 are positive constants. Consider the following cases.

1. If $\gamma > 2$ then (1.5) holds and hence ${}^{\mu}g$ satisfies (1.4) for all $x \neq y$.

2. If $\gamma = 2$ then (1.3) yields

$$\frac{{}^{\mu}g(x,y)}{g(x,y)} \geq c \min\left(\frac{\langle x\rangle}{\langle y\rangle}, \frac{\langle y\rangle}{\langle x\rangle}\right)^{\tau}, \tag{1.7}$$

where $\langle x\rangle := 1 + |x|$ and τ and c are positive constants. In particular, ${}^{\mu}g$ satisfies (1.4) provided $\langle x\rangle \simeq \langle y\rangle$.

3. If $\gamma < 2$ then (1.3) yields

$$\frac{{}^{\mu}g(x,y)}{g(x,y)} \geq c \exp\left(-C\,|x-y|^{2-\gamma}\left(\frac{|x-y|}{\langle x\rangle + \langle y\rangle}\right)^{\theta}\right), \tag{1.8}$$

where $\theta = \min(\gamma, n-2)$ and $C, c > 0$. In particular, ${}^{\mu}g$ satisfies (1.4) provided

$$|x-y| \leq C\left(\langle x\rangle + \langle y\rangle\right)^{\eta},$$

where $\eta = \frac{\theta}{\theta+2-\gamma}$.

In the second case, if $V(x) = V_0\,|x|^{-2}$ for $|x| > 1$, then then following estimate of ${}^{\mu}g$ was proved in [16]:

$$\frac{{}^{\mu}g(x,y)}{g(x,y)} \simeq \min\left(\frac{\langle y\rangle}{\langle x\rangle}, \frac{\langle x\rangle}{\langle y\rangle}\right)^{\tau}, \tag{1.9}$$

where the exponent τ is determined by

$$\tau = -\left(\frac{n}{2} - 1\right) + \sqrt{\left(\frac{n}{2} - 1\right)^2 + V_0}.$$

An upper bound in (1.9) was also proved by Murata [35] with the same τ. Hence, (1.7) yields a qualitatively correct lower bound but without explicit value of τ (our method is too robust to give a sharp value of τ).

In the third case, for the potential $V(x) = V_0\,|x|^{-\gamma}$, a sharp upper bound for ${}^{\mu}g$ was proved by Murata [35], which in the range $|x| \geq 2\,|y| \geq 2$ amounts to

$${}^{\mu}g\,(x,y) \leq \frac{C}{|x|^a\,|y|^b} \exp\left(-c\,|x|^{1-\gamma/2}\right),$$

for some positive a, b depending on n and γ, and which was shown to be asymptotically correct as $|x| \to \infty$ (a similar but not so sharp upper bound for ${}^{\mu}g$ was obtained in [39]). For comparison, our estimate (1.8) gives in this range

$${}^{\mu}g\,(x,y) \geq \frac{c}{|x|^{n-2}} \exp\left(-C\,|x|^{2-\gamma}\right).$$

As we see, the exponent $2-\gamma$ is twice larger than the one sharp one. Nevertheless, we believe that the estimate (1.8) is still the best available lower bound for the potential (1.6).

The structure of the paper is as follows. In Section 2 we introduce the setting of $\mathcal{P}$-harmonic Bauer spaces, which is natural for our problem, and prove a sufficient condition for the existence of a perturbed Green function (Theorem 2.12). In Section 3, we prove the following general lower bound for the perturbed Green function (Theorem 3.3):

$$\frac{{}^{\mu}g(x,y)}{g(x,y)} \geq \exp\left(-\frac{\int_X g(x,z)g(z,y)d\mu(z)}{g(x,y)}\right), \tag{1.10}$$

where X is the underlying space. Under some additional hypotheses on $g(x,y)$ introduced in Section 4 (and which are satisfied in $\mathbb{R}^n$), the estimate (1.10) implies (1.3) (Theorem 4.3).

Section 5 contains specific examples of application of Theorem 4.3, including Theorems 5.8 and 5.10, which in particular cover the setting when the underlying metric space is Ahlfors regular, the Green function $g(x,y)$ decays polynomially in distance, and the perturbation potential $V(x)$ is similar to (1.6). Apart from the Green function of the Laplace operator in $\mathbb{R}^n$, this includes Green functions on a large class of fractal spaces.

NOTATION. Letters C, c denote unimportant positive constants whose value may change at each occurrence, unless otherwise stated.

2 Existence of the perturbed Green function

2.1 Harmonic spaces and Green functions

To work in reasonable generality we shall assume in the following that $(X, \mathcal{H})$ is a Bauer space with a Green function. We cite here the main properties of such spaces and refer the reader to [9] for a detailed account.

Let X be a locally connected locally compact topological space with countable base. For every open subset U of X, let $\mathcal{B}(U)$ (resp. $\mathcal{C}(U)$) be the set of all Borel measurable (resp. continuous real) functions on U. Given a set $\mathcal{F}(U)$ of functions on U, $\mathcal{F}^+(U), \mathcal{F}_b(U), \mathcal{F}_c(U)$ will be the sets of all functions in $\mathcal{F}(U)$ that are respectively non-negative, bounded, or with compact support. Let $\mathcal{U}_c$ denote the family of all open relatively compact subsets of X.

A *harmonic sheaf* on X is a map $\mathcal{H}$ which to every open subset U of X assigns a linear subspace $\mathcal{H}(U)$ of $\mathcal{C}(U)$ such that the following properties hold:

- For any two open subsets U, V of X, $U \subset V \Longrightarrow \mathcal{H}(U) \subset \mathcal{H}(V)$.
- For any family $(U_i)_{i \in I}$ of open subsets and any numerical function h on $U := \bigcup_{i \in I} U_i$,
$$h|_{U_i} \in \mathcal{H}(U_i) \quad (i \in I) \quad \Longrightarrow \quad h \in \mathcal{H}(U).$$

The elements of $\mathcal{H}(U)$ are called *harmonic functions* on U.

A set $V \in \mathcal{U}_c$ is called *regular* if every $f \in C(\partial V)$ possesses a unique continuous extension $H_V f$ on $\overline{V}$ such that $H_V f$ is harmonic on V and $H_V f \geq 0$ if $f \geq 0$.

The couple $(X, \mathcal{H})$ is called a *Bauer* space if the following properties are satisfied:

- For every $x \in X$, there exists a harmonic function h defined in a neighbourhood of x such that $h(x) \neq 0$.

- There exists a base $\mathcal{V}$ of regular sets such that $U \cap V \in \mathcal{V}$ for any $U, V \in \mathcal{V}$.

- (*Bauer's convergence axiom*) For any increasing sequence (h_n) of non-negative harmonic functions on an open set U,

$$h := \sup h_n \text{ is locally bounded in } U \quad \Longrightarrow \quad h \in \mathcal{H}(U).$$

Alongside Bauer's convergence axiom, sometimes we use the following stronger hypotheses:

- (*Doob's convergence axiom*) For any increasing sequence (h_n) of non-negative harmonic functions on an open set U,

$$h := \sup h_n \text{ is finite on a dense subset of } U \quad \Longrightarrow \quad h \in \mathcal{H}(U).$$

- (*Brelot's convergence axiom*) For any increasing sequence (h_n) of non-negative harmonic functions on a connected open set U,

$$h := \sup h_n \text{ is finite at a point in } U \quad \Longrightarrow \quad h \in \mathcal{H}(U).$$

A *Brelot* space is a Bauer space satisfying Brelot's convergence axiom.

For every open subset U of X, a lower semicontinuous function $s : U \to (-\infty, +\infty]$ is called *hyperharmonic on* U provided that $H_V s \leq s$ for every regular $V \in \mathcal{U}_c$. It is *superharmonic on* U if in addition the functions $H_V s$ are locally bounded on V. It follows from Doob's convergence axiom that a hyperharmonic function that is finite on a dense subset is superharmonic.

A superharmonic function $s \geq 0$ on U is called a *potential on* U if 0 is the largest harmonic minorant of s on U. We write $\mathcal{S}(U)$ (resp. $\mathcal{P}(U)$) for the set of all superharmonic functions (resp. potentials) on U. Every function $s \in \mathcal{S}^+(U)$ admits a unique decomposition $s = h + p$ such that $h \in \mathcal{H}^+(U)$ and $p \in \mathcal{P}(U)$ (Riesz decomposition).

A Bauer space $(X, \mathcal{H})$ is called $\mathcal{P}$*-harmonic* if there exists a potential $p > 0$ on X.

Definition 2.1 Given a Bauer space $(X, \mathcal{H})$, a function $g : X \times X \to [0, \infty]$ is called a *Green function* if the following conditions are satisfied:

(i) for every $y \in X$, $g^y := g(\cdot, y)$ is a potential on X, harmonic on $X \setminus \{y\}$;

(ii) for every $x \in X$, $g(x, \cdot)$ is continuous on $X \setminus \{x\}$;

(iii) for every continuous real potential p on X, there exists a Borel measure $\nu \geq 0$ on X such that

$$p = \int_X g^y d\nu(y).$$

We shall assume in the sequel that we have a Green function g on a Bauer space $(X, \mathcal{H})$. Then the space $(X, \mathcal{H})$ is $\mathcal{P}$-harmonic if and only if the family of all sets $\{g^y > 0\}_{y \in X}$ covers X.

It is known that if $(X, \mathcal{H})$ is a connected Brelot space then $g^y > 0$ on X, g^y is locally bounded on $X \setminus \{y\}$, and g is lower semicontinuous on $X \times X$. In particular, in such a space the existence of a Green function obviously implies $\mathcal{P}$-harmonicity.

In examples below X is an open subset of $\mathbb{R}^n$ unless specified otherwise.

Example 2.2 Let a_{ij}, b_i, c be measurable real functions on X. Consider the operator

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c, \tag{2.1}$$

assuming that the matrix $(a_{ij}(x))_{i,j=1}^{n}$ is a symmetric and uniformly elliptic, b_i and c are bounded, and $c \leq 0$. It follows from [34] that continuous weak solutions to the equation $Lu = 0$ form a Brelot space.

Example 2.3 Let a_{ij}, b_i, c be continuous real functions on X. Consider the operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \tag{2.2}$$

assuming that the matrix $(a_{ij}(x))_{i,j=1}^{n}$ is symmetric and positive definite, and $c \leq 0$. It follows from [30] that continuous solutions to the equation $Lu = 0$ form a Brelot space (see [29]).

Example 2.4 Consider in $X' := \mathbb{R} \times X$ a parabolic operator $\frac{\partial}{\partial t} - L$ where L is one of operators (2.1) or (2.2). Then continuous solutions to the equation $\left(\frac{\partial}{\partial t} - L\right) u = 0$ form a Bauer space. Moreover, a Doob convergence axiom is also satisfied in this case.

Example 2.5 Let

$$L = \frac{1}{2} \sum_{k=1}^{r} A_k^2 + A_0 \tag{2.3}$$

where $A_0, A_1, \ldots, A_r$ are $\mathcal{C}^\infty$-vector fields on X. If the family $\{A_k\}_{k=1}^{r}$ satisfies the *Hörmander condition* then $\mathcal{C}^\infty$-solutions to the equation $Lu = 0$ form a Bauer space with Doob convergence axiom. Moreover, if $A_0 = 0$ then one obtains a Brelot space (see [3], [4], [5]).

Example 2.6 Let L be the Laplace-Beltrami operator on a Riemannian manifold X. The solutions to the equation $Lu = 0$ form a Brelot space, so that $\mathcal{P}$-harmonicity of this space is equivalent to the existence of a positive Green function. Various necessary and sufficient conditions for that can be found in [14].

For any Borel measure μ on X consider operator G^μ that acts on functions as follows:

$$G^\mu f := \int_X g^y f(y) d\mu(y). \tag{2.4}$$

For any $f \in \mathcal{B}^+(X)$, $G^\mu f$ is a hyperharmonic function. If $G^\mu f$ is locally bounded then $G^\mu f$ is a potential (under Doob's convergence axiom it suffices to know that the set $\{G^\mu f < \infty\}$ is dense in X, and under Brelot's convergence axiom is suffices to know that $G^\mu f \not\equiv \infty$).

We will also use the notation

$$g^\mu := G^\mu 1 = \int_X g^y \, d\mu(y). \tag{2.5}$$

Let us endow $\mathcal{B}_b(X)$ with the sup-norm $\|\cdot\|_\infty$. Clearly, if g^μ is bounded then G^μ is a positive bounded operator in $\mathcal{B}_b(X)$ whose norm is equal to $\|g^\mu\|_\infty$.

If $(X,\mathcal{H})$ is a $\mathcal{P}$-harmonic Bauer space then there are harmonic kernels H_U for *all* open subsets $U \subset X$ yielding generalized solutions to the Dirichlet problem. Given a Green function g on $(X,\mathcal{H})$, for every open subset $U \subset X$ we obtain a corresponding Green function g_U for U by

$$g_U^y = g^y - H_U g^y \quad \text{for all } y \in U. \tag{2.6}$$

If V is an open subset of U, then $H_V H_U = H_U$ and hence

$$g_V^y = g_U^y - H_V g_U^y \quad \text{for all } y \in V. \tag{2.7}$$

In particular, this implies that $g_V \le g_U$. Note that g_U is symmetric provided g is symmetric (see [25], [27]).

The operator G_U^μ and the function g_U^μ are defined similarly to (2.4) and (2.5), respectively.

Definition 2.7 A Radon measure $\mu \ge 0$ on X is called a (local g-) *Kato measure* if, for every compact set $A \subset X$, the function $G^\mu 1_A$ is finite and continuous (in particular, $G^\mu 1_A$ is a potential).

Let $\mathcal{M}_s(X)$ denote the set of all countable sums of Kato measures with compact supports. All Kato measures belong to $\mathcal{M}_s(X)$. So, $\mathcal{M}_s(X)$ is the set of countable sums of Kato measures. Measures $\mu \in \mathcal{M}_s(X)$ do not charge semipolar sets (see, for example, [3, Lemma VI.5.15]).

It is worth mentioning that, for any $f \in \mathcal{B}^+(X)$ and $\mu \in \mathcal{M}_s(X)$, $G^\mu f$ is a countable sum of continuous real potentials each of them being harmonic outside a compact set. A consequence of that is the following *domination principle*: if $\mu \in \mathcal{M}_s(X)$ is supported by a Borel set A and $s \in \mathcal{S}^+(X)$ then

$$s \ge g^\mu \quad \text{on } A \quad \Longrightarrow \quad s \ge g^\mu \quad \text{on } X. \tag{2.8}$$

2.2 A perturbed Green function

Let g be a Green function for a Bauer space $(X,\mathcal{H})$.

Definition 2.8 Given a measure $\mu \in \mathcal{M}_s(X)$, a function ${}^\mu g : X \times X \to [0,\infty]$ will be called *a perturbed Green function* if, for every $y \in X$, ${}^\mu g^y$ is a non-negative Borel function on X satisfying

$${}^\mu g^y + G^\mu\left({}^\mu g^y\right) = g^y. \tag{2.9}$$

Note that a perturbed Green function is not necessarily a Green function in the sense of Definition 2.1.

The motivation behind the identity (2.9) is as follows. Let X be a domain in $\mathbb{R}^d$ and g be the classical Green function of the Laplace operator Δ in X. Then applying Δ to (2.9) (in the distributional sense) we obtain

$$\Delta({}^\mu g^y) - {}^\mu g^y \, \mu = -\delta^y,$$

whence it follows that ${}^{\mu}g$ is a Green function of the operator $\Delta - \mu$. Similarly, for other linear second order elliptic or parabolic operators.

It is well known that if $(X, \mathcal{H})$ is a Brelot space and if μ is a Kato measure then a perturbed Green function ${}^{\mu}g$ exists. Moreover, in this case ${}^{\mu}g$ is a Green function (in the sense of Definition 2.1) of a *perturbed* harmonic space $(X, {}^{\mu}\mathcal{H})$ (see [6], [26]). For a general $\mu \in \mathcal{M}_s(X)$ this is not necessarily the case. For any $\mu \in \mathcal{M}_s(X)$, (2.9) implies that ${}^{\mu}g^y \leq g^y$. Being the difference of g^y and $G^{\mu}({}^{\mu}g^y)$, the function ${}^{\mu}g^y$ is finely continuous on $X \setminus \{y\}$.

We need some preparation for the proof of the existence of a perturbed Green function. The following lemma was proved in [18, Lemma 4.1] (actually, it was stated for Brelot spaces and for a smaller class of measures but the proof given in [18, Lemma 4.1] works as well in our setting).

Lemma 2.9 *Let P be a polar subset of X and $\mu \in \mathcal{M}_s(X)$. Let $s \in \mathcal{S}^+(X)$ and let f be a real Borel function on X be such that $G^{\mu}|f| < \infty$ on $X \setminus P$. Then:*

$$f + G^{\mu}f \leq s \quad \text{on } X \setminus P \quad \Longrightarrow \quad G^{\mu}f \leq s \quad \text{on } X \setminus P.$$

Corollary 2.10 *Under the hypotheses of* Lemma 2.9*,*

$$f + G^{\mu}f = s \quad \text{on } X \setminus P \quad \Longrightarrow \quad f \geq 0 \quad \text{on } X \setminus P.$$

In particular,

$$f + G^{\mu}f = 0 \quad \text{on } X \setminus P \quad \Longrightarrow \quad f = 0 \quad \text{on } X \setminus P.$$

Proof. By Lemma 2.9 we have $G^{\mu}f \leq s$ which together with $f + G^{\mu}f = s$ yields $f \geq 0$. The second claim obviously follows from the first one. ■

It follows from Lemma 2.9 that the operator $I + G^{\mu}$ is injective on $\mathcal{B}_b(X)$ provided g^{μ} is bounded. In fact, then $I + G^{\mu}$ is even invertible in $\mathcal{B}_b(X)$ (see, for example, [3, II.7.4]).

Corollary 2.11 *If a perturbed Green function ${}^{\mu}g(x, y)$ exists then it is unique outside the diagonal $\{x = y\}$.*

Proof. Indeed, if $f \in \mathcal{B}^+(X)$ is another function satisfying (2.9), that is

$$f + G^{\mu}f = g^y \quad \text{on } X,$$

then noticing that the set $P := \{g^y = \infty\}$ is polar we conclude from (2.9) and Corollary 2.10 that $f = {}^{\mu}g^y$ on $X \setminus P$. Since $P \subset \{y\}$ we obtain that $f = {}^{\mu}g^y$ on $X \setminus \{y\}$. ■

Theorem 2.12 *Let $(X, \mathcal{H})$ be a $\mathcal{P}$-harmonic Bauer space with a Green function $g(x, y)$, and let $\mu \in \mathcal{M}_s(X)$ be such that $G^{\mu}(1_A g^y)$ is a potential for any compact set $A \subset X$ and any $y \in X$. Then there exists a perturbed Green function ${}^{\mu}g(x, y)$.*

Proof. Let us first prove the existence of ${}^{\mu}g$ under the stronger assumptions that there exists a positive bounded superharmonic function on X, and g^{μ} as well as all $G^{\mu}g^{y}$, $y \in X$, are potentials.

Let us show that g^{μ} is a countable sum of bounded continuous potentials. By definition of $\mathcal{M}_s(X)$, $\mu = \sum_{n=1}^{\infty} \mu_n$ where each μ_n is a Kato measure with compact support. Then $g^{\mu} = \sum_{n=1}^{\infty} g^{\mu_n}$ where each g^{μ_n} is continuous by definition and is bounded by the domination principle (2.8).

Hence, by [23, Proposition 1.2], there exists a (unique) kernel K such that

$$(I + G^{\mu})\, K = G^{\mu}. \tag{2.10}$$

Let P denote the polar set $\{G^{\mu}g^{y} = \infty\}$ so that $\mu(P) = 0$. By (2.10), $G^{\mu}Kg^{y} \le G^{\mu}g^{y} < \infty$ on $X \setminus P$, so $f := 1_{X\setminus P}(g^{y} - Kg^{y})$ satisfies $G^{\mu}|f| < \infty$ on $X \setminus P$ and

$$(I + G^{\mu})\, f = (I + G^{\mu})\, g^{y} - G^{\mu}g^{y} = g^{y} \quad \text{on } X \setminus P.$$

By Corollary 2.10, we obtain that $f \ge 0$ on $X \setminus P$, i.e., $g^{y} \ge Kg^{y}$ on $X \setminus P$.

Let $\varphi \in \mathcal{B}^{+}(X)$ be such that

$$\varphi + Kg^{y} = g^{y} \quad \text{on } X \setminus P.$$

Applying $I + G^{\mu}$ and using (2.10) we conclude that

$$(I + G^{\mu})\, \varphi = g^{y} \quad \text{on } X \setminus P.$$

By fine continuity, $G^{\mu}\varphi \le g^{y}$ on X. Finally, choose ${}^{\mu}g^{y} \in \mathcal{B}^{+}(X)$ such that

$${}^{\mu}g^{y} + G^{\mu}\varphi = g^{y} \quad \text{on } X,$$

and

$${}^{\mu}g^{y}(y) = \infty \quad \text{if } g^{y}(y) = \infty. \tag{2.11}$$

Since $G^{\mu}\varphi \le G^{\mu}g^{y} < \infty$ on $X \setminus P$, we see that $\varphi = {}^{\mu}g^{y}$ on $X \setminus P$ and hence $G^{\mu}\varphi = G^{\mu}({}^{\mu}g^{y})$ on X. Thus

$${}^{\mu}g^{y} + G^{\mu}({}^{\mu}g^{y}) = g^{y} \quad \text{on } X. \tag{2.12}$$

Now let us consider the general case. Fix a precompact open set $U \subset X$ and check that the harmonic space $(U, \mathcal{H}_U)$ satisfies the hypotheses of the first part of the proof. Indeed, by $\mathcal{P}$-harmonicity of $(X, \mathcal{H})$ there exists a positive continuous superharmonic function s_0 on X, so that $s_0|_U$ is a positive bounded superharmonic function on U. Since $G^{\mu}_U g^{y} \le G^{\mu}(1_U g^{y})$, $G^{\mu}_U g^{y}$ is a countable sum of potentials, and $G^{\mu}(1_U g^{y})$ is superharmonic by assumption, we conclude that every function $G^{\mu}_U g^{y}$, $y \in X$, is a potential (see [3, III.6.3]).

Let us show that g^{μ}_U is also a potential. By $\mathcal{P}$-harmonicity of $(X, \mathcal{H})$ the sets $\{g^{y} > 0\}_{y \in X}$ cover X. Since $\overline{U}$ is compact, there is a finite set $y_1, .., y_n$ of points in X such that the sets $\{g^{y_i} > 0\}$, $i = 1, 2, ..., n$, cover $\overline{U}$. Therefore, the potential $s := \sum_{i=1}^{n} g^{y_i}$ is strictly positive on $\overline{U}$, say $s \ge \varepsilon$ on $\overline{U}$ where $\varepsilon > 0$. Then

$$g^{\mu}_U \le G^{\mu}1_U \le \varepsilon^{-1} G^{\mu}(1_{\overline{U}} s) = \varepsilon^{-1} \sum_{i=1}^{n} G^{\mu}(1_{\overline{U}} g^{y_i}). \tag{2.13}$$

Since g_U^μ is a countable sum of potentials and the function $G^\mu\left(1_{\overline{U}} g^{y_i}\right)$, $i = 1, ..., n$, are superharmonic on U, we obtain as above that g_U^μ is a potential.

Hence, by the first part of the proof, there exists a perturbed Green function ${}^\mu g_U$. Let us show that $U \mapsto {}^\mu g_U$ is increasing. Fix precompact open sets $V \subset U$, a point $y \in V$, and set $v = {}^\mu g_V^y$ and $u = {}^\mu g_U^y$. Then

$$v + G_V^\mu v = g_V^y \tag{2.14}$$

and

$$u + G_V^\mu u + H_V G_U^\mu u = u + G_U^\mu u = g_U^y = g_V^y + H_V g_U^y. \tag{2.15}$$

The set $P := \{g^y = \infty\} \subset \{y\}$ is polar. Set $f = 1_{V \setminus P}(u - v)$ and observe that $G^\mu |f| < \infty$ on $V \setminus P$. Moreover, by (2.14) and (2.15),

$$f + G_V^\mu f = H_V g_U^y - H_V G_U^\mu u \quad \text{on } V \setminus P.$$

Since the right hand side here is a non-negative harmonic function on V, we obtain by Corollary 2.10 that $f \geq 0$, that is $v \leq u$ on $V \setminus P$. Since $v = \infty = u$ on P by (2.11), we have $v \leq u$ on V.

Finally, we set for all $y \in X$

$${}^\mu g^y = \sup_U {}^\mu g_U^y \,.$$

Passing to the limit in

$${}^\mu g_U^y + G_U^\mu\left({}^\mu g_U^y\right) = g_U^y$$

as $U \uparrow X$ we obtain (2.12), that is ${}^\mu g$ is the perturbed Green function. ∎

Remark 2.13 If one knows that ${}^\mu g^y \geq c g^y$ on $X \setminus \{y\}$ then there exists a unique finely continuous modification of ${}^\mu g^y$.

The next statement provides a sufficient condition for finiteness of $G^\mu g^y$ in $X \setminus \{y\}$. Set

$$\widetilde{g}(x, y) = g(x, y) + g(y, x)$$

and define $\widetilde{g}^\mu$ similarly to (2.5).

Lemma 2.14 (cf. [6]) *Let $(X, \mathcal{H})$ be a $\mathcal{P}$-harmonic Bauer space with a Green function g. If $\mu \in \mathcal{M}_s(X)$ has a compact support and $\widetilde{g}^\mu < \infty$ then $G^\mu g^y < \infty$ on $X \setminus \{y\}$ for every $y \in X$.*

Proof. Let A denote the support of μ and s be a positive continuous superharmonic function on X. Fix distinct points $x, y \in X$ and choose a relatively compact open neighborhood W of y which does not contain x. Then

$$\alpha := \sup\{g(x, z) : z \in W\} < \infty,$$

and by the boundary minimum principle,

$$g^y \leq \frac{\sup_{\partial W} g^y}{\inf_{\partial W} s} s \quad \text{on } X \setminus W,$$

whence

$$\beta := \sup\{g(z,y) : z \in A \setminus W\} \le \frac{\sup_{\partial W} g^y}{\inf_{\partial W} s} \sup_A s < \infty.$$

Thus,

$$\begin{aligned} G^\mu g^y(x) &= \int_A g(x,z)g(z,y)\,d\mu(z) \\ &\le \alpha \int_W g(z,y)\,d\mu(z) + \beta \int_{A\setminus W} g(x,z)d\mu(z) \\ &\le \alpha\widetilde{g}^\mu(y) + \beta\widetilde{g}^\mu(x) < \infty, \end{aligned}$$

which was to be proved. ■

Corollary 2.15 *Let $(X,\mathcal{H})$ be a $\mathcal{P}$-harmonic Bauer space with a Green function g such that Doob's convergence axiom is satisfied. If $\mu \in \mathcal{M}_s(X)$ and*

$$\widetilde{g}^{1_A\mu} < \infty \tag{2.16}$$

for any compact set $A \subset X$ then a perturbed Green function ${}^\mu g$ exists.

Proof. By Lemma 2.14 applied to measure $1_A\mu$, we have $G^{1_A\mu}g^y < \infty$ on $X\setminus\{y\}$, which implies by Doob's convergence axiom that $G^{1_A\mu}g^y$ is a potential. Hence, ${}^\mu g$ exists by Theorem 2.12. ■

Remark 2.16 Under a condition similar to (2.16) the existence of the perturbed Green function was proved in [6].

3 A general lower bound for the perturbed Green function

The setting of this section is the same as in the previous Section 2. The next result provides an extension of [23, Proposition 1.9] to positive superharmonic functions which may be unbounded.

Proposition 3.1 *Let $(X,\mathcal{H})$ be a $\mathcal{P}$-harmonic Bauer space with Green function g. Let $\mu \in \mathcal{M}_s(X)$ and $f \in \mathcal{B}^+(X)$ be such that the function $s := f + G^\mu f$ is superharmonic on X. Then*

$$f \ge s \exp\left(-\frac{G^\mu s}{s}\right) \quad \text{on } \{0 < s < \infty\}. \tag{3.1}$$

Remark 3.2 Inequality (3.1) has appeared before in literature in various settings under more restrictive hypotheses. In [11, Proposition 11], it was proved in the case when the function s is harmonic and bounded in a domain of $\mathbb{R}^m$. In [23, Proposition 1.9], (3.1) was proved in an abstract setting in the case when the function s is superharmonic and bounded (assuming also that 1 is a superharmonic function and measure μ is Kato). The proof is based on the following observation. Consider the resolvent operator

$$R_\alpha = (I + \alpha G^\mu)^{-1} G^\mu,$$

where $\alpha > 0$, which obviously satisfies the identity

$$(I + \alpha G^\mu)^{-1} = I - \alpha R_\alpha.$$

On the other hand, if s is bounded and superharmonic then, for a fixed x, the function

$$\varphi(\alpha) = (s - \alpha R_\alpha s)(x)$$

is non-negative and completely monotone, which follows from the identity

$$\varphi^{(n)}(\alpha) = (-1)^n n! R_\alpha^n (s - \alpha R_\alpha s)(x)$$

(see, for example, [21, p. 656]). Hence, $\varphi(\alpha)$ is log-convex, which yields the inequality

$$\varphi(1) \geq \exp\left(\frac{\varphi'(0)}{\varphi(0)}\right) \varphi(0)$$

that is equivalent to (3.1).

We will use the inequality (3.1) for bounded s to handle unbounded superharmonic functions.

Proof. Consider first the case when 1 is a superharmonic function on X. Let ν be any Kato measure on X with compact support such that $0 \leq \nu \leq \mu$. Then g^ν is bounded and continuous, hence $I + G^\nu$ is an invertible operator on $\mathcal{B}_b(X)$ and

$$K := (I + G^\nu)^{-1} G^\nu$$

defines a kernel on X (see, for example, [7, Proposition 2.5]). Define a sequence $(s_n)_{n\in\mathbb{N}}$ by

$$s_n = \min(s, n),$$

Since s_n is bounded, we get by the above mentioned result of [23, Proposition 1.9] the inequality (3.1) for s_n instead of s, that is,

$$(I + G^\nu)^{-1} s_n \geq s_n \exp\left(-\frac{G^\nu s_n}{s_n}\right). \tag{3.2}$$

Since (s_n) increases to s, we obtain that

$$K s_n \underset{n\to\infty}{\longrightarrow} K s = K(I + G^\mu) f \geq K(I + G^\nu) f = G^\nu f. \tag{3.3}$$

The identity

$$(I + G^\nu)^{-1} = (I + G^\nu)^{-1} \left((I + G^\nu) - G^\nu\right) = I - K$$

and (3.3) imply that

$$\lim_{n\to\infty} (I + G^\nu)^{-1} s_n = \lim_{n\to\infty} (s_n - K s_n) \leq s - G^\nu f \quad \text{on } \{0 < s < \infty\}. \tag{3.4}$$

Clearly, we have also

$$\lim_{n\to\infty} \exp(-\frac{G^\nu s_n}{s_n}) = \exp(-\frac{G^\nu s}{s}) \quad \text{on } \{0 < s < \infty\},$$

whence, combining with (3.2) and (3.4), we conclude that

$$s - G^\nu f \geq s \exp\left(-\frac{G^\nu s}{s}\right) \quad \text{on } \{0 < s < \infty\}.$$

The proof in this case is finished by letting $\nu \uparrow \mu$.

In the general case when 1 is not necessarily a superharmonic function, fix any positive continuous real superharmonic function w on X, which exists by $\mathcal{P}$-harmonicity of $(X, \mathcal{H})$. Consider a new sheaf $\mathcal{H}_w$ where all functions are obtained from those from $\mathcal{H}$ by dividing by w. Then the function g_w given by

$$g_w(x,y) = \frac{g(x,y)}{w(x)\,w(y)}$$

is a Green function in the $\mathcal{P}$-harmonic Bauer space $(X, \mathcal{H}_w)$. It is easy to verify that for $\nu = w^2\mu$

$$\frac{1}{w} G^\mu f = G_w^\nu\left(\frac{f}{w}\right).$$

The assumption $f + G^\mu f = s$ implies that

$$\frac{f}{w} + G_w^\nu\left(\frac{f}{w}\right) = \frac{s}{w},$$

and we conclude by the first part of the proof that

$$\frac{f}{w} \geq \frac{s}{w} \exp\left(-\frac{G_w^\nu(s/w)}{s/w}\right) = \frac{s}{w} \exp\left(-\frac{G^\mu s}{s}\right),$$

whence the claim follows. ■

Now we can prove the main result of this section.

Theorem 3.3 *Let $(X, \mathcal{H})$ be a $\mathcal{P}$-harmonic Bauer space with a Green function g. If $\mu \in \mathcal{M}_s(X)$ and a perturbed Green function ${}^\mu g$ exists then*

$$1 \geq \frac{{}^\mu g(x,y)}{g(x,y)} \geq \exp\left(-\frac{\int_X g(x,z)g(z,y)d\mu(z)}{g(x,y)}\right), \tag{3.5}$$

for all $x, y \in X$ such that $0 < g(x,y) < \infty$.

Proof. The upper bound in (3.5) follows from (2.9). To prove the lower bound, fix $y \in X$ and denote $s := g^y$ and $f := {}^\mu g^y$. Then, by (2.9), $f + G^\mu f = s$. Since s is superharmonic, by Proposition 3.1 the functions f and s satisfy (3.1) whence

$${}^\mu g^y \geq g^y \exp\left(-\frac{G^\mu g^y}{g^y}\right) \quad \text{on } \{0 < g^y < \infty\}, \tag{3.6}$$

which is exactly (3.5). ■

Corollary 3.4 *Let $(X, \mathcal{H})$ be a connected Brelot space with a Green function g. If $\mu \in \mathcal{M}_s(X)$ and $G^\mu\left(1_A g^y\right) \not\equiv \infty$ for any precompact set $A \subset X$ and any $y \in X$ then the perturbed Green function ${}^\mu g$ exists and satisfies the estimate*

$$1 \geq \frac{{}^\mu g^y}{g^y} \geq \exp\left(-\frac{G^\mu g^y}{g^y}\right) \quad \text{on } X \setminus \{y\}\,. \tag{3.7}$$

Proof. Recall that a connected Brelot space with a Green function is $\mathcal{P}$-harmonic. The assumption $G^\mu\left(1_A g^y\right) \not\equiv \infty$ and Brelot's convergence axiom imply that $G^\mu\left(1_A g^y\right)$ is a potential. By Theorem 2.12 the perturbed Green function exists. Finally, (3.7) follows from (3.6) and the fact that $0 < g^y < \infty$ on $X \setminus \{y\}$. ■

Remark 3.5 If for some $C > 0$ and all $y \in X$ we have

$$G^\mu g^y \leq C g^y$$

then (3.5) or (3.7) yields that ${}^\mu g \simeq g$. In various particular cases this was proved before in [7], [10], [35], [36], [37] for elliptic equations and in [32], [33], [40] for parabolic equations. Other conditions implying ${}^\mu g \simeq g$ can be found in [24], [38], [42], [43].

4 Lower bound for the perturbed Green function in Brelot spaces

In this section, we assume that (X, d) is a connected, locally compact, separable, metric space. Let $(X, \mathcal{H})$ be a Brelot space possessing a Green function $g\,(x, y)$, which is jointly continuous in $X \times X \setminus \text{diag}$. In particular, $g > 0$ on $X \times X$, and the space $(X, \mathcal{H})$ is $\mathcal{P}$-harmonic.

For any $x \in X$ and $r > 0$, denote by $B(x, r)$ the d-ball of radius r centered at x, that is

$$B(x, r) := \{z \in X \,:\, d(x, z) < r\}\,.$$

Consider the following conditions which in general may be true or not.

(A) For some constant $C > 0$ and for all $x, y, z \in X$,

$$\min(g(x, z), g(z, y)) \leq C g(x, y). \tag{4.1}$$

(B) For some constants $c, \delta \in (0, 1)$ and for all $r > 0$ and $x, y \in X$,

$$x \in B\left(y, \delta r\right) \quad \Longrightarrow \quad g_{B(y,r)}(x, y) \geq c g(x, y). \tag{4.2}$$

For example, in $\mathbb{R}^n$ (4.2) holds with any $\delta < 1$. The next statement provides a simple sufficient condition for (A) and (B) (a more general result will be proved in Proposition 5.1).

Lemma 4.1 (cf. [18, Proposition 9.2]) *Let 1 be a superharmonic function, and assume that there exists a constant $\sigma > 0$ such that for all distinct $x, y \in X$*

$$g(x,y) \simeq d(x,y)^{-\sigma}. \tag{4.3}$$

Then both (A) and (B) are satisfied.

Proof. By the triangle inequality, we have

$$\begin{aligned} \frac{1}{g(x,y)} &\simeq d(x,y)^{\sigma} \le (d(x,z) + d(z,y))^{\sigma} \\ &\le C_{\sigma}\left(d(x,z)^{\sigma} + d(z,y)^{\sigma}\right) \simeq \frac{1}{g(x,z)} + \frac{1}{g(z,y)}, \end{aligned}$$

whence (A) follows.

To prove (B) denote $U = B(y,r)$ and observe that by (2.6)

$$g^{y} = g_{U}^{y} + H_{U} g^{y}. \tag{4.4}$$

By the maximum principle (which holds due to the assumption that 1 is a superharmonic function),

$$\sup_{U} H_{U} g^{y} = \sup_{\partial U} g^{y},$$

whence we obtain the following inequality in U

$$g_{U}^{y} \ge g^{y} - \sup_{\partial U} g^{y}. \tag{4.5}$$

By (4.3) we have $\sup_{\partial U} g^{y} \le C r^{-\sigma}$, whence we obtain for any $x \in B(y,r)$

$$g_{U}(x,y) \ge g(x,y) - C r^{-\sigma}.$$

If $x \in B(y, \delta r)$ where δ is small enough then $g(x,y) \ge 2C r^{-\sigma}$ whence we obtain

$$g_{U}(x,y) \ge \frac{1}{2} g(x,y).$$

■

Remark 4.2 For further discussions about property (A) see [19], [20], [21], [22].

Let us introduce the following notation

$$\Gamma^{\mu}(x,R) := \int_{B(x,R)} (g(x,z) + g(z,x))\, d\mu(z) = \widetilde{g}^{1_{B(x,R)}\mu}(x). \tag{4.6}$$

In particular, $\Gamma^{\mu}(x,\infty) = \widetilde{g}^{\mu}(x)$. The following theorem is our main result.

Theorem 4.3 *Let μ be a Radon measure on X such that $\Gamma^{\mu}(x,R)<\infty$ for all $x\in X$ and $R\in(0,+\infty)$, and $\Gamma^{\mu}(x,R)\to 0$ as $R\to 0$ locally uniformly in x. Then the perturbed Green function ${}^{\mu}g$ exists.*

If in addition the hypothesis (A) holds then, for all $x\neq y$,

$$1\geq\frac{{}^{\mu}g(x,y)}{g(x,y)}\geq\exp\left(-C\left(\widetilde{g}^{\mu}(x)+\widetilde{g}^{\mu}(y)\right)\right), \tag{4.7}$$

where the constant C is the same as in (A). In particular, if $\sup\widetilde{g}^{\mu}<\infty$ then ${}^{\mu}g(x,y)\simeq g(x,y)$ for all $x\neq y$.

If the hypotheses (A) and (B) hold then, for all $x,y\in X$ and $R>0$ such that

$$0<d(x,y)<\frac{\delta}{2}R,$$

the following estimate takes place

$$1\geq\frac{{}^{\mu}g(x,y)}{g(x,y)}\geq c\exp\left(-\frac{C}{c}\left(\Gamma^{\mu}(x,R)+\Gamma^{\mu}(y,R)\right)\right), \tag{4.8}$$

where the constants C,c,δ are the same as in (A) and (B).

Remark 4.4 Let us note that if μ is Lebesgue measure on $\mathbb{R}^n$ then ${}^{\mu}g$ exists but the estimate (4.7) is void because the right hand side of (4.7) is 0. On the other hand, (4.8) gives a non-trivial lower bound for ${}^{\mu}g$ (see Section 5 for details).

Proof. If we know already that $\mu\in\mathcal{M}_s$ then, by Corollary 2.15, to prove the existence of a perturbed Green function, it suffices to verify that $\widetilde{g}^{\mathbf{1}_A\mu}<\infty$ for any compact set $A\subset X$. Indeed, for any $x\in X$ there exists $R>0$ such that $A\subset B(x,R)$, whence

$$\widetilde{g}^{\mathbf{1}_A\mu}(x)=\int_A\widetilde{g}(x,z)\,d\mu(z)\leq\Gamma^{\mu}(x,R)<\infty. \tag{4.9}$$

To prove that $\mu\in\mathcal{M}_s$ it suffices to show that the function $g^{\mathbf{1}_A\mu}$ is finite and continuous for any compact set $A\subset X$. The finiteness follows from (4.9). The continuity is easily proved by the following argument taken from [7, p.129]. Fix $x\in X$, $\varepsilon>0$, and let $A'=A\cap B(x,\varepsilon)$, $A''=A\setminus B(x,\varepsilon)$. Then, for any $y\in B(x,\varepsilon)$,

$$g^{\mathbf{1}_{A'}\mu}(y)\leq\int_{B(x,\varepsilon)}g(y,z)\,d\mu(z)\leq\int_{B(y,2\varepsilon)}g(y,z)\,d\mu(z)\leq\sup_{u\in B(x,\varepsilon)}\Gamma^{\mu}(u,2\varepsilon),$$

which tends to 0 as $\varepsilon\to 0$. Moreover, $g^{\mathbf{1}_{A''}\mu}(y)$ is continuous on $B(x,\varepsilon)$, since g is continuous outside the diagonal. Thus, the function $g^{\mathbf{1}_A\mu}=g^{\mathbf{1}_{A'}\mu}+g^{\mathbf{1}_{A''}\mu}$ is continuous at x.

If (A) holds then rewrite (4.1) as follows:

$$\frac{g(x,z)g(z,y)}{g(x,y)}\leq C\left(g(x,z)+g(z,y)\right), \tag{4.10}$$

for all $x, y, z \in X$ such that $x \neq y$ (recall that $g > 0$ on $X \times X$). By Theorem 3.3 and (4.10), we obtain that

$$\begin{aligned}
\frac{{}^{\mu}g(x,y)}{g(x,y)} &\geq \exp\left(-\frac{\int_X g(x,z)g(z,y)\,d\mu(z)}{g(x,y)}\right) \\
&\geq \exp\left(-C\int_X \left(g(x,z)+g(z,y)\right)\,d\mu(z)\right) \\
&\geq \exp\left(-C\left(\widetilde{g}^{\mu}(x)+\widetilde{g}^{\mu}(y)\right)\right),
\end{aligned}$$

which proves (4.7)

Assume now that (B) holds as well. Fix $y \in X$, $R > 0$, and set $U = B(y, R/2)$. Applying estimate (3.5) of Theorem 3.3 to the harmonic space $(U, \mathcal{H}_U)$ with the Green function g_U and replacing g_U by g where the inequality allows, we obtain

$$\frac{{}^{\mu}g(x,y)}{g_U(x,y)} \geq \exp\left\{-\frac{\int_U g(x,z)g(z,y)\,d\mu(z)}{g_U(x,y)}\right\},$$

for any $x \in U \setminus \{y\}$. Let $0 < d(x,y) < \delta R/2$. Then the hypothesis (B) implies that $g_U(x,y) \geq c g(x,y)$ whence

$$\begin{aligned}
\frac{{}^{\mu}g(x,y)}{g(x,y)} &\geq c\exp\left(-\frac{1}{c}\frac{\int_U g(x,z)g(z,y)\,d\mu(z)}{g(x,y)}\right) \\
&\geq c\exp\left(-\frac{C}{c}\int_U \left(g(x,z)+g(z,y)\right)\,d\mu(z)\right) \\
&\geq c\exp\left(-\frac{C}{c}\left(\Gamma^{\mu}(x,R)+\Gamma^{\mu}(y,R)\right)\right),
\end{aligned}$$

where in the last line we have replaced U by the larger sets $B(x,R)$ and $B(y,R)$. ■

5 Specific lower bounds for the perturbed Green function

The setting in this section is the same as in the previous Section 4, and we assume in addition that 1 is a superharmonic function. The purpose of this section is to obtain, on the one hand, simple sufficient conditions for (A) and (B), and, on the other hand, explicit lower bounds for ${}^{\mu}g$.

5.1 Sufficient conditions for (A) and (B)

The following statements generalizes [18, Propositions 9.2, 9.3].

Proposition 5.1 *Assume that $J(x,r)$ is a positive function on $X \times (0,+\infty)$ such that, for some constants $\varepsilon > 0$, $C > 1$, and $N \geq 2$,*

(J1) *$J(x,r) \leq CJ(x,t)$ for all $x \in X$ and $0 < t \leq Nr$;*

(J2) $J(x,r) \geq (1+\varepsilon) J(x,Nr)$ *for all* $x \in X$, $r>0$,

(J3) $J(x,r) \leq CJ(y,r)$ *whenever* $d(x,y) \leq r$.

If, for all $x, y \in X$,

$$g(x,y) \simeq J(x,r) \quad \textit{where } r = d(x,y)\,, \tag{5.1}$$

then the hypotheses (A) *and* (B) *are satisfied.*

Example 5.2 All hypotheses of Proposition 5.1 are satisfied if for some $\sigma > 0$

$$g(x,y) \simeq d(x,y)^{-\sigma} \tag{5.2}$$

since we can take $J(x,r) = r^{-\sigma}$. Hence, (5.2) implies (A) and (B), which we already have seen in Lemma 4.1. Different examples will be given below after Lemma 5.3.

Proof. Let us set

$$r_1 := d(y,z),\ r_2 := d(x,z),\ r_3 := d(x,y).$$

where $x, y, z \in X$ are given points (see 1).

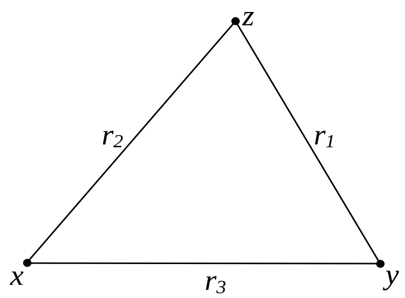

Figure 1: Triangle xyz

Let us prove that

$$\min(J(x,r_2), J(z,r_1)) \leq C^3 J(x,r_3), \tag{5.3}$$

which implies (A) by (5.1). Observe that $r_1 + r_2 \geq r_3$ and hence either $r_1 \geq \frac{1}{2} r_3$ or $r_2 \geq \frac{1}{2} r_3$.

Suppose first that $r_2 \geq \frac{1}{2} r_3$. Then, by (J1)

$$J(x,r_2) \leq CJ(x,r_3),$$

whence (5.3) follows. Suppose now $r_1 \geq \frac{1}{2} r_3$. Using (J3) and (J1), we obtain

$$J(z,r_1) \leq CJ(y,r_1) \leq C^2 J(y,r_3) \leq C^3 J(x,r_3),$$

which was to be proved.

Let us now prove (B). It suffices to show that there exist $\lambda, \delta \in (0,1)$ such that, for all $y \in X$ and $r > 0$,

$$g^y - g^y_{B(y,r)} \leq \lambda g^y \quad \text{in } B(y, \delta r). \tag{5.4}$$

Set $U := B(y,r)$ and recall that by (4.5) the following inequality holds in U:

$$g^y - g^y_U \leq \sup_{\partial U} g^y, \tag{5.5}$$

so that (5.4) amounts to

$$\sup_{\partial U} g^y \leq \lambda \inf_{B(y,\delta r)} g^y. \tag{5.6}$$

For any $z \in \partial U$ and $x \in U$ we have $d(x,z) \leq 2r$ and hence by (J1) and (J3)

$$g(z,y) \simeq J(z,r) \leq CJ(z,2r) \leq C^2 J(x,2r) \leq C^3 J(x,r),$$

whence

$$\sup_{z \in \partial U} g^y \leq C' \inf_{x \in U} J(x,r). \tag{5.7}$$

If $x \in B(y, \delta r)$ then by (5.1) and (J1),

$$g(x,y) \geq cJ(x, \delta r),$$

for $c > 0$. Let us take $\delta := N^{-k}$ for some positive integer k. Then by (J2)

$$g(x,y) \geq cJ(x, N^{-k} r) \geq c(1+\varepsilon)^k J(x,r)$$

whence

$$\inf_{B(y,\delta r)} g^y \geq c(1+\varepsilon)^k \inf_{x \in U} J(x,r).$$

Comparing with (5.7), we obtain (5.6) with $\lambda = \text{const}\,(1+\varepsilon)^{-k}$, which can be made < 1 by taking k large enough. ■

Lemma 5.3 *Let $\nu(x,r)$ be a positive function on $X \times (0,+\infty)$ with the following properties:*

- *for all $x \in X$ and $0 < r \leq R$,*

$$A_1 \left(\frac{R}{r}\right)^{\alpha} \leq \frac{\nu(x,R)}{\nu(x,r)} \leq A_2 \left(\frac{R}{r}\right)^{\alpha'} \tag{5.8}$$

  for some constants $A_1, A_2 > 0$ and

$$\alpha' \geq \alpha > \beta; \tag{5.9}$$

- *if $B(y,r) \subset B(x,R)$ and $r \leq R$ then*

$$\nu(y,r) \leq \nu(x,R); \tag{5.10}$$

- *for all distinct $x, y \in X$ and a constant $\beta > 0$*

$$g(x,y) \simeq \int\limits_{d(x,y)}^{\infty} \frac{t^{\beta-1}dt}{\nu(x,t)}\,; \tag{5.11}$$

Then the hypotheses (A) and (B) are satisfied.

Remark 5.4 The condition $\alpha > \beta$ implies that the integral (5.11) converges.

Proof. Let us show that in fact

$$g(x,y) \simeq \frac{r^{\beta}}{\nu(x,r)}, \tag{5.12}$$

where $r = d(x,y)$. Indeed, we have

$$\int_r^{\infty} \frac{t^{\beta-1}dt}{\nu(x,t)} \geq \int_r^{2r} \frac{t^{\beta-1}dt}{\nu(x,t)} \geq \frac{r^{\beta}}{\nu(x,2r)} \geq A_2^{-1}2^{-\alpha'}\frac{r^{\beta}}{\nu(x,r)},$$

so we are left to prove the upper bound

$$\int_r^{\infty} \frac{t^{\beta-1}dt}{\nu(x,t)} \leq C\frac{r^{\beta}}{\nu(x,r)}.$$

The latter is equivalent to

$$\int_r^{\infty} \frac{\nu(x,r)}{\nu(x,t)} t^{\beta-1}dt \leq Cr^{\beta},$$

which in view of (5.8) amounts to

$$A_1^{-1}\int_t^{\infty} \left(\frac{r}{t}\right)^{\alpha} t^{\beta-1}dt \leq Cr^{\beta}$$

and which holds because $\alpha > \beta$.

To finish the proof, it suffices to verify that the function

$$J(x,r) := \frac{r^{\beta}}{\nu(x,r)}$$

satisfies the hypotheses (J1) – (J3) of Proposition 5.1, which trivially follows from (5.8)–(5.10). ■

Example 5.5 If

$$\nu(x,r) \simeq r^{\alpha} \tag{5.13}$$

where $\alpha > \beta$ then (5.11) amounts to

$$g(x,y) \simeq d(x,y)^{-(\alpha-\beta)}. \tag{5.14}$$

We already have seen that (5.14) implies (A) and (B) (see Lemma 4.1 and Proposition 5.1).

Example 5.6 Let X be a non-compact geodesically complete Riemannian manifold and $\nu(x,R)$ be the Riemannian volume of the geodesic ball $B(x,R)$ so that (5.10) is automatically satisfied. Under certain hypotheses on X (in particular, if the Ricci curvature of X is non-negative), the heat kernel $p_t(x,y)$ of X, that is, the minimal positive fundamental solution to the heat equation on X, satisfies the Gaussian estimate

$$p_t(x,y) \simeq \frac{1}{\nu(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \tag{5.15}$$

for all $x,y \in X$, $t>0$, where the constant c may be different for upper and lower bounds (see [12], [13], [16], [31]). Since the Green function $g(x,y)$ of the Laplace-Beltrami operator is related to the heat kernel by

$$g(x,y) = \int_0^\infty p_t(x,y)\,dt, \tag{5.16}$$

integrating (5.15) in t, one obtains (5.11) with $\beta = 2$. Note that inequalities (5.8) follow from (5.15) with *some* $0<\alpha\le\alpha'$ (see [12]). Hence, if (5.15) holds and in addition $\alpha>2$ then the hypotheses (A) and (B) are satisfied.

Example 5.7 Let (X,d,ν) be a metric measure space and let $\nu(x,r) = \nu(B(x,r))$ where $B(x,r)$ is a d-ball. In this case, (5.10) is automatically satisfied. Let in addition $(\mathcal{E},\mathcal{F})$ be a Dirichlet form in $L^2(X,\nu)$ whose generator Δ has a continuous stochastically complete heat kernel $p_t(x,y)$. Assume in addition that $p_t(x,y)$ satisfies the following *sub-Gaussian* estimate

$$p_t(x,y) \simeq \frac{1}{\nu(x,t^{1/\beta})} \exp\left(-\left(\frac{d^\beta(x,y)}{ct}\right)^{\frac{1}{\beta-1}}\right), \tag{5.17}$$

for all $x,y\in X$, $t>0$, where $\beta>1$ is a parameter called the *walk dimension*. The estimate (5.17) holds on a large variety of fractal spaces (see for example [1], [2], [15], [17], [28]). Since the Green function $g(x,y)$ of Δ is related to the heat kernel by (5.16), integrating (5.17) in t, one obtains (5.11). It is possible to prove that (5.17) implies (5.8) with some $0<\alpha\le\alpha'$. If in addition $\alpha>\beta$ then all the assumptions of Lemma 5.3 are satisfied and hence the hypotheses (A) and (B) hold.

5.2 Radial perturbations

We say that a positive function $h(r)$ on $\mathbb{R}_+$ has the *doubling property* if $h(r_1) \le Ch(r_2)$ for all $r_1, r_2 \in \mathbb{R}_+$ such that $\frac{1}{2} \le \frac{r_1}{r_2} \le 2$.

Theorem 5.8 *Let ν be a Borel measure on X such that the function $\nu(x,r) := \nu(B(x,r))$ is finite, positive and, for some $0<\alpha\le\alpha'$,*

$$C^{-1}\left(\frac{R}{r}\right)^\alpha \le \frac{\nu(x,R)}{\nu(x,r)} \le C\left(\frac{R}{r}\right)^{\alpha'} \text{ whenever } x\in X \text{ and } 0<r<R.$$

Assume that, for some $0<\beta<\alpha$, the Green function $g(x,y)$ satisfies the following estimate

$$g(x,y) \simeq \frac{r^\beta}{\nu(x,r)} \quad \text{where } r = d(x,y), \tag{5.18}$$

for all distinct $x, y \in X$. Let μ be a Radon measure on X such that $d\mu = f(x)\,d\nu$ where function $f(x)$ is bounded on any ball. Fix a reference point $o \in X$, set $|x| = d(x, o)$, and

$$F(R) = \sup_{\frac{R}{2} \le |x| \le 2R} |f(x)|.$$

Then the perturbed Green function ${}^{\mu}g$ exists and satisfies the estimate

$${}^{\mu}g(x, y) \simeq g(x, y),$$

for all distinct $x, y \in X$ such that

$$d(x, y) \le \varepsilon \min\left(\langle x\rangle, \langle y\rangle, F(|x|)^{-1/\beta}, F(|y|)^{-1/\beta}\right), \tag{5.19}$$

where $\langle x\rangle = 1 + |x|$ and $\varepsilon > 0$ depends on the constants from the hypotheses.

Remark 5.9 If $f(x) \simeq |x|^{-\gamma}$ for large r where $\gamma \le \beta$ then (5.19) is equivalent to

$$d(x, y) \le \varepsilon \min(\langle x\rangle, \langle y\rangle)^{\eta},$$

where $\eta = \gamma/\beta$. Some improvement of this result will be obtained in Corollary 5.11 below, although under a bit more restrictive hypotheses.

Proof. By Lemma 5.3, the hypotheses (A) and (B) are satisfied. In order to apply Theorem 4.3, we need to estimate the function

$$\Gamma^{\mu}(x, R) = \int_{B(x,R)} \widetilde{g}(x, z)\, f(z)\, d\nu(z). \tag{5.20}$$

Fix $x \in X$ and $R > 0$. Observe that the function

$$g(r) := \frac{r^{\beta}}{\nu(x, r)}$$

is doubling and $\widetilde{g}(x, z) \simeq g(\rho)$ where $\rho = d(x, z)$. For any positive function h on $\mathbb{R}_+$ with doubling property we have

$$\begin{aligned} \int_{B(x,R)} h(\rho)\, d\nu(z) &= \sum_{k=0}^{\infty} \int_{B(x,2^{-k}R)\setminus B\left(x,2^{-(k+1)}R\right)} h(\rho)\, d\nu(z) \\ &\le C \sum_{k=0}^{\infty} h\left(2^{-k}R\right) \nu\left(x, 2^{-k}R\right) \\ &\le C \int_0^R h(r)\, \nu(x, r) \frac{dr}{r}, \end{aligned} \tag{5.21}$$

whence

$$\int_{B(x,R)} \widetilde{g}(x, z)\, d\nu(z) \le C \int_{B(x,R)} g(\rho)\, d\nu(z) \le C \int_0^R g(r)\, \nu(x, r) \frac{dr}{r} = CR^{\beta}. \tag{5.22}$$

Since $f(z)$ is bounded on any ball, this implies that $\Gamma^\mu(x,R)$ is finite and $\Gamma^\mu(x,R) \to 0$ as $R \to 0$ locally uniformly in x. Hence, Theorem 4.3 applies and yields that ${}^\mu g$ exists and admits the estimate (4.8).

Let us show that

$$R \le \min\left(\frac{1}{4}\langle x\rangle, F(|x|)^{-1/\beta}\right) \quad \Longrightarrow \quad \Gamma^\mu(x,R) \le C. \tag{5.23}$$

Assume first $|x| \ge 1$. Then $\langle x\rangle \le 2|x|$ and hence $R \le \frac{1}{2}|x|$. Therefore, for any $z \in B(x,R)$, we have $\frac{1}{2}|x| \le |z| \le \frac{3}{2}|x|$ and hence $f(z) \le F(|x|)$. By (5.20) and (5.22), we obtain

$$\Gamma^\mu(x,R) \le C\int_{B(x,R)} \widetilde{g}(x,z)\, F(|x|)\, d\nu(z) \le CR^\beta F(|x|),$$

whence the claim follows because $RF(|x|)^{1/\beta} \le 1$. If $|x| < 1$ then $R < 1$ and, hence, $\Gamma^\mu(x,R)$ is bounded by (5.22).

The condition (5.19) implies $d(x,y) \le 4\varepsilon R$ where

$$R = \min\left(\frac{1}{4}\langle x\rangle, \frac{1}{4}\langle y\rangle, F(|x|)^{-1/\beta}, F(|x|)^{-1/\beta}\right).$$

Assuming that $\varepsilon < \frac{\delta}{8}$ where δ is the constant from hypothesis (B), we obtain $d(x,y) < \frac{\delta}{2}R$ and hence, by (4.8) and (5.23), ${}^\mu g(x,y) \ge cg(x,y)$, which was to be proved. ∎

Theorem 5.10 *Let ν be a Borel measure on X such that for all $x \in X$ and $t > 0$,*

$$\nu(B(x,r)) \simeq v(r),$$

where $v(r)$ is a positive continuous function in $\mathbb{R}_+$ satisfying the estimates

$$C^{-1}\left(\frac{R}{r}\right)^\alpha \le \frac{v(R)}{v(r)} \le C\left(\frac{R}{r}\right)^{\alpha'} \tag{5.24}$$

for all $0 < r < R$, where $0 < \alpha \le \alpha'$. Assume that the Green function $g(x,y)$ satisfies the estimate

$$g(x,y) \simeq \frac{r^\beta}{v(r)}, \quad \textit{where } r = d(x,y), \tag{5.25}$$

for all distinct $x,y \in X$, where β is a constant such that

$$0 < \beta < \alpha.$$

Fix a reference point $o \in X$ and assume that μ is Radon measure on X such that

$$\frac{d\mu}{d\nu}(x) \le f(|x|) \quad \textit{for all } x \in X, \tag{5.26}$$

where $|x| := d(x,o)$ and $f(s)$ is a positive decreasing function on $\mathbb{R}_+$ satisfying the doubling condition.

Then a perturbed Green function ${}^\mu g$ exists and satisfies the following estimates, for some positive constants C, c and for all distinct points $x, y \in X$.

(*i*) *If*

$$\int_1^\infty f(r)\, r^{\beta-1}dr < \infty$$

then

$$^{\mu}g(x,y) \simeq g(x,y). \tag{5.27}$$

(*ii*) *If* $f(r) \simeq r^{-\beta}$ *for large* r *then*

$$1 \geq \frac{^{\mu}g(x,y)}{g(x,y)} \geq c \min\left(\frac{\langle y\rangle}{\langle x\rangle}, \frac{\langle x\rangle}{\langle y\rangle}\right)^{\tau}, \tag{5.28}$$

with some constant $\tau > 0$, *where* $\langle x\rangle := 1 + |x|$.

(*iii*) *If* $f(r) \simeq r^{-\gamma}$ *for large* r *where* $0 < \gamma < \beta$ *then*

$$1 \geq \frac{^{\mu}g(x,y)}{g(x,y)} \geq c \exp\left[-C\, d(x,y)^{\beta-\gamma}\left(\frac{d(x,y)}{\langle x\rangle + \langle y\rangle}\right)^{\theta}\right], \tag{5.29}$$

where $\theta = \min(\gamma, \alpha - \beta)$.

Corollary 5.11 *In the setting of* Theorem 5.10, *let* $f(r) \simeq r^{-\gamma}$ *for large* r, *where* $\gamma > 0$.

(*i*) *If* $\gamma > \beta$ *then* (5.27) *holds for all distinct* x, y.

(*ii*) *If* $\gamma = \beta$ *then* (5.27) *holds provided* $\langle x\rangle \simeq \langle y\rangle$.

(*iii*) *If* $\gamma < \beta$ *then* (5.27) *holds provided*

$$d(x,y) \leq C\left(\langle x\rangle + \langle y\rangle\right)^{\eta}, \tag{5.30}$$

where $\eta = \frac{\theta}{\theta+\beta-\gamma} = \min\left(\frac{\gamma}{\beta}, \frac{\alpha-\beta}{\alpha-\gamma}\right)$ (see Fig. 2).

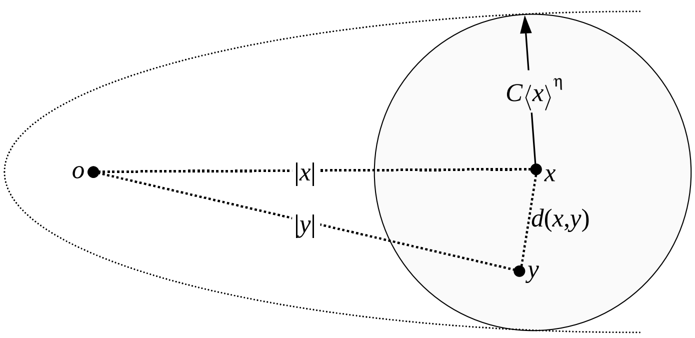

Figure 2: For all $y \in B(x, C\langle x\rangle^{\eta})$, we have $^{\mu}g(x,y) \simeq g(x,y)$

Proof of Theorem 5.10. Obviously, the hypotheses of Theorem 5.8 are satisfied, and hence the perturbed Green function $^{\mu}g$ exists and satisfies the estimate

(4.8) of Theorem 4.3. All we need is to obtain good enough estimates for $\Gamma^{\mu}(x,R)$ to use in (4.8).

Set $g(r)=r^{\beta}/v(r)$ so that $\widetilde{g}(x,y)\simeq g(r)$ where $r=d(x,y)$. Given $x\in X$ and $R>0$, split the integration in (4.6) into two domains: $|z|<d(x,z)=:\rho$ and $|z|\geq\rho$ (see Fig. 3) and set

$$\Gamma_1 := \int\limits_{B(x,R)\cap\{|z|<\rho\}} \widetilde{g}(x,z)d\mu(z) \tag{5.31}$$

$$\Gamma_2 := \int\limits_{B(x,R)\cap\{|z|\geq\rho\}} \widetilde{g}(x,z)d\mu(z) \tag{5.32}$$

so that $\Gamma^{\mu}(x,R)=\Gamma_1+\Gamma_2$.

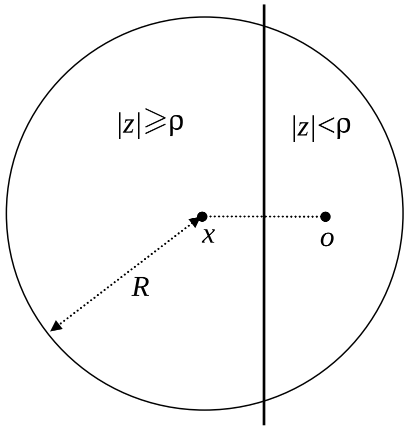

Figure 3: Estimating Γ_1 and Γ_2

Estimate of Γ_1. If $z\in B(x,R)$ and $|z|<\rho$ then $|z|<R$ whence

$$\Gamma_1\leq \int\limits_{B(o,R)\cap\{|z|<\rho\}} \widetilde{g}(x,z)d\mu(z)\simeq \int\limits_{B(o,R)\cap\{|z|<\rho\}} g(\rho)d\mu(z)$$

It follows from the triangle inequality that $\rho\simeq|x|+|z|$ and hence

$$g(\rho)\simeq g(|x|+|z|).$$

Using this, (5.26), and (5.21), we obtain

$$\Gamma_1\leq C\int_{B(o,R)} g(|x|+|z|)\,f(|z|)\,d\nu(z)\leq C\int_0^R g(|x|+r)\,f(r)\,v(r)\,\frac{dr}{r}.$$

Estimate of Γ_2. If $z\in B(x,R)$ and $|z|\geq\rho$ then by the triangle inequality $|z|\simeq|x|+\rho$ and hence

$$\frac{d\mu}{d\nu}(z)\leq f(|z|)\simeq f(|x|+\rho).$$

Therefore, by (5.32) and (5.21),

$$\begin{aligned}\Gamma_2 &\leq C\int_{B(x,R)} f(|x|+\rho)\, g(\rho)\, d\nu(z)\\ &\leq C\int_0^R f(|x|+r)\, g(r)\, v(r)\,\frac{dr}{r}\\ &\leq C\int_0^R f(|x|+r)\, r^{\beta-1}dr.\end{aligned}$$

Combining the estimates of Γ_1 and Γ_2, we obtain

$$\Gamma^\mu(x,R) \leq C\int_0^R f(r)\, g(|x|+r)\, v(r)\,\frac{dr}{r} + C\int_0^R f(|x|+r)\, r^{\beta-1}dr. \tag{5.33}$$

Consider some consequences of (5.33). It follows from the definition of $g(r)$ and $\alpha>\beta$ that

$$g(|x|+r)\, v(r) \leq Cg(r)\, v(r) = Cr^\beta.$$

Since f is decreasing, we have

$$f(|x|+r) \leq f(r).$$

Hence, we obtain from (5.33) that

$$\Gamma^\mu(x,R) \leq C\int_0^R f(r)\, r^{\beta-1}dr.$$

Consequently, if

$$\int_0^\infty f(r)\, r^{\beta-1}dr < \infty, \tag{5.34}$$

then $\Gamma^\mu(x,R) \leq \mathrm{const}$ and hence, for all $x\neq y$,

$${}^\mu g(x,y) \simeq g(x,y),$$

which settles part (i).

In general, we can estimate the integrals in (5.33) as follows. By (5.24), we know that $g(|x|+r) \leq Cg(|x|)$ if $r\leq|x|$ and $g(|x|+r)\leq Cg(r)$ if $r\geq|x|$. Therefore,

$$\int_0^R f(r)g(|x|+r)v(r)\,\frac{dr}{r} \leq Cg(|x|)\int_0^{|x|\wedge R} f(r)v(r)\,\frac{dr}{r} + C\int_{|x|\wedge R}^R f(r)r^{\beta-1}\,dr. \tag{5.35}$$

Moreover, since f is decreasing,

$$\int_0^R f(|x|+r)r^{\beta-1}\,dr \leq \frac{1}{\beta}f(|x|)(|x|\wedge R)^\beta + \int_{|x|\wedge R}^R f(r)r^{\beta-1}\,dr. \tag{5.36}$$

Adding (5.35) and (5.36), we obtain by (5.33) that

$$\Gamma^\mu(x,R) \leq Cg(|x|)\int_0^R f(r)\, v(r)\,\frac{dr}{r} + Cf(|x|)\, R^\beta, \quad \text{if } |x|\geq R. \tag{5.37}$$

Furthermore, by (5.24),

$$\frac{g(|x|)}{f(|x|)|x|^\beta}\int_0^{|x|} f(r)v(r)\,dr \ge \int_0^{|x|}\frac{v(r)}{v(|x|)}\frac{dr}{r} \ge c\int_0^{|x|}\left(\frac{r}{|x|}\right)^{\alpha'}\frac{dr}{r} \ge c$$

so that

$$g\left(|x|\right)\int_0^{|x|} f(r)v(r)\,dr \ge cf(|x|)|x|^\beta.$$

Thus, the estimates (5.35) and (5.36) yield that

$$\Gamma^\mu\left(x,R\right) \le Cg\left(|x|\right)\int_0^{|x|} f\left(r\right)v\left(r\right)\frac{dr}{r} + C\int_{|x|}^R f\left(r\right)r^{\beta-1}dr \quad \text{if } |x|<R. \tag{5.38}$$

Now assume that $f\left(r\right) \le Cr^{-\beta}$ for large r. Then the first integral in (5.37) and (5.38) can be estimated by

$$\frac{|x|^\beta}{v\left(|x|\right)}\int_0^{|x|} f\left(r\right)v\left(r\right)\frac{dr}{r} \le C+C\int_1^{|x|\vee 1}\frac{v\left(r\right)}{v\left(|x|\right)}\left(\frac{|x|}{r}\right)^\beta\frac{dr}{r} \le C+C\int_1^{|x|\vee 1}\left(\frac{r}{|x|}\right)^{\alpha-\beta}\frac{dr}{r} \le C,$$

whence in the both cases

$$\Gamma^\mu\left(x,R\right) \le C + C\log_+\frac{R}{\langle x\rangle},$$

where $\langle x\rangle = 1+|x|$. Taking in (4.8) $R = 3\delta^{-1}d\left(x,y\right)$ and using this estimate of $\Gamma^\mu\left(x,R\right)$ we obtain

$$\frac{{}^\mu g(x,y)}{g(x,y)} \ge c\left(1+\frac{R}{\langle x\rangle}\right)^{-\tau}\left(1+\frac{R}{\langle y\rangle}\right)^{-\tau},$$

for some $\tau>0$. If $|x|\ge|y|$ then $R\le C\,|x|$ and the first factor here is estimated by a constant, whereas for the second factor we have

$$1+\frac{R}{\langle y\rangle} = \frac{\langle y\rangle + R}{\langle y\rangle} \simeq \frac{\langle x\rangle}{\langle y\rangle}$$

whence

$$\frac{{}^\mu g(x,y)}{g(x,y)} \ge c\left(\frac{\langle y\rangle}{\langle x\rangle}\right)^\tau.$$

Similarly, one treats the case $|x|\le|y|$, whence (5.28) follows.

Suppose next that $f\left(r\right) \le Cr^{-\gamma}$ for large r, where $0<\gamma<\beta$. Then, for any $s\ge 1$,

$$\int_0^s f\left(r\right)v\left(r\right)\frac{dr}{r} \le C+\int_1^s\frac{v\left(r\right)}{r^\gamma}\frac{dr}{r} = C+\int_1^s\frac{v\left(r\right)}{r^\beta}r^{\beta-\gamma-1}dr \le C+C\frac{v\left(s\right)}{s^\beta}s^{\beta-\gamma} \le C+C\frac{v\left(s\right)}{s^\gamma},$$

whereas for $0<s<1$

$$\int_0^s f\left(r\right)v\left(r\right)\frac{dr}{r} \le C.$$

Therefore, in the case $|x| < R$, we obtain by (5.38) that

$$\Gamma^{\mu}(x,R) \leq C + C\frac{|x|^{\beta}}{v(|x|)}\int_{1}^{|x|}\frac{v(r)}{r^{\gamma}}\frac{dr}{r} + CR^{\beta-\gamma} \leq C + C|x|^{\beta-\gamma} + CR^{\beta-\gamma} \leq C + CR^{\beta-\gamma},$$

and in the case $|x| \geq R$, (5.37) yields

$$\begin{aligned}
\Gamma^{\mu}(x,R) &\leq C\frac{|x|^{\beta}}{v(|x|)}\int_{0}^{R}\frac{v(r)}{r^{\gamma}}\frac{dr}{r} + C|x|^{-\gamma}R^{\beta} \\
&\leq C + \frac{|x|^{\beta}}{v(|x|)}\frac{v(R)}{R^{\gamma}} + C|x|^{-\gamma}R^{\beta} \\
&\leq C + CR^{\beta-\gamma}\left(\frac{|x|^{\beta}}{R^{\beta}}\frac{v(R)}{v(|x|)} + \frac{R^{\gamma}}{|x|^{\gamma}}\right) \\
&\leq C + CR^{\beta-\gamma}\left(\left(\frac{R}{|x|}\right)^{\alpha-\beta} + \left(\frac{R}{|x|}\right)^{\gamma}\right) \\
&\leq C + CR^{\beta-\gamma}\left(\frac{R}{|x|}\right)^{\theta},
\end{aligned}$$

where $\theta = \min(\alpha-\beta,\gamma)$. So, in both cases, we have

$$\Gamma^{\mu}(x,R) \leq C + R^{\beta-\gamma}\min\left(1,\frac{R}{|x|}\right)^{\theta} \leq C + CR^{\beta-\gamma}\left(\frac{R}{|x|+R}\right)^{\theta}. \tag{5.39}$$

Taking in (4.8) $R = \frac{3}{\delta}d(x,y)$ and observing that $R+|x| \simeq |x|+|y|$, we obtain from (5.39)

$$\frac{{}^{\mu}g(x,y)}{g(x,y)} \geq c\exp\left(-CR^{\beta-\gamma}\left(\frac{R}{|x|+|y|}\right)^{\theta}\right). \tag{5.40}$$

Finally, let us show why $|x|+|y|$ can be replaced by $\langle x\rangle + \langle y\rangle$. If $|x|+|y| \geq 1$ then $|x|+|y| \simeq \langle x\rangle + \langle y\rangle$ and all is clear. If $|x|+|y| \leq 1$ then $d(x,y) \leq |x|+|y| \leq 1$ whence it follows that both R and $\frac{R}{|x|+|y|}$ are bounded and hence $\frac{{}^{\mu}g(x,y)}{g(x,y)} \geq c$. Therefore, any negative expression inside the exponential in (5.40) will do. ■

5.3 An example with singular measure

Let S be a hyperplane in $\mathbb{R}^n$, $n > 2$, passing through the origin, and consider the measure[1] $\mu = f(x)\delta_S$, where f is a continuous function on S. Given $x \in \mathbb{R}^n$ and $R > 0$, let x' be the orthogonal projection of x onto S and set

$$R_x = \begin{cases} \sqrt{R^2 - |x-x'|^2}, & \text{if } |x-x'| < R, \\ 0, & \text{if } |x-x'| \geq R \end{cases}$$

(see Fig. 4).

[1]Here δ_S is the singular measure on $\mathbb{R}^n$ which is supported on S and its restriction on S is the $(n-1)$-dimensional Lebesgue measure.

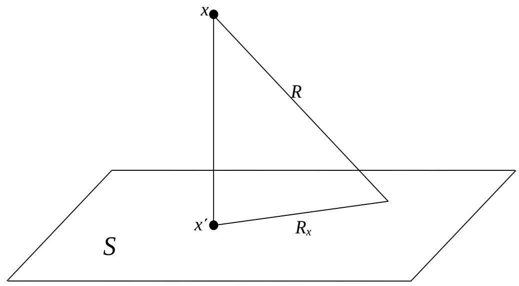

Figure 4:

Clearly, we have

$$\Gamma^{\mu}(x,R)=\Gamma^{\mu}(x',R_x).$$

In order to estimate $\Gamma^{\mu}(x',r)$, we can apply the estimates obtained in the proof of Theorems 5.8 and 5.10 to the case $X=S$, $v(r)=cr^{n-1}$, and $g(r)=cr^{2-n}$ so that $\alpha=n-1$ and $\beta=1<\alpha$. Therefore, as in the proof of Theorem 5.8, $\Gamma^{\mu}(x',r)$ is finite and goes to 0 as $r\to 0$ locally uniformly in x', and hence, by Theorem 4.3, ${}^{\mu}g$ exists and satisfies the estimate (4.8), which can be rewritten as follows:

$$\frac{{}^{\mu}g(x,y)}{g(x,y)}\geq \exp\left(-C-C\Gamma^{\mu}(x',R_x)-C\Gamma^{\mu}(y',R_y)\right). \tag{5.41}$$

Assuming further that $f(x)\simeq |x|^{-\gamma}$ for large $|x|$, we obtain as in Theorem 5.10 the following:

(i) If $\gamma>1$ then $\Gamma^{\mu}(x',r)\leq C$ and hence $\Gamma^{\mu}(x,R)\leq C$ for all $x\in\mathbb{R}^n$ and $R>0$, which implies ${}^{\mu}g(x,y)\simeq g(x,y)$ for all distinct $x,y\in\mathbb{R}^n$.

(ii) If $\gamma=1$ then

$$\Gamma^{\mu}(x',r)\leq C+C\log_{+}\frac{r}{\langle x'\rangle}.$$

Taking $R=3|x-y|$ we obtain from (5.41)

$$\begin{aligned}\frac{{}^{\mu}g(x,y)}{g(x,y)} &\geq \exp\left(-C-C\log_{+}\frac{R_x}{\langle x'\rangle}-C\log_{+}\frac{R_y}{\langle y'\rangle}\right)\\ &\geq c\left(1+\frac{R_x}{\langle x'\rangle}\right)^{-\tau}\left(1+\frac{R_y}{\langle y'\rangle}\right)^{-\tau},\end{aligned} \tag{5.42}$$

for some $c,\tau>0$. The dependence on x,y here is not transparent. Of course, if $x,y\in S$ then (5.42) amounts to (5.28). Consider another interesting case when x and y are on the opposite sides from S. In this case,

$$|x-y|\geq |x-x'|+|y-y'|$$

whence $R_x\simeq R\simeq R_y$ and

$$\frac{{}^{\mu}g(x,y)}{g(x,y)}\geq c\left(1+\frac{|x-y|}{\langle x'\rangle}\right)^{-\tau}\left(1+\frac{|x-y|}{\langle y'\rangle}\right)^{-\tau}.$$

(iii) If $0 < \gamma < 1$ then

$$\Gamma^{\mu}(x', r) \leq C + Cr^{1-\gamma} \left(\frac{r}{|x'| + r} \right)^{\theta},$$

where $\theta = \min(n-2, \gamma)$. In the case when x and y are on the opposite sides from S and $|x'| \geq |y'|$, we obtain from (5.41)

$$\frac{{}^{\mu}g(x,y)}{g(x,y)} \geq \exp\left(-C - C|x-y|^{1-\gamma} \left(\frac{|x-y|}{|x'| + |x-y|} \right)^{\theta} \right).$$

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