LIOUVILLE THEOREMS AND EXTERIOR BOUNDARY VALUE PROBLEMS

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INTRODUCTION

The following facts are well known for the Laplace equation

$$\Delta u = 0 \tag{0.1}$$

in R^n:

1) when $n = 2$, any positive superharmonic function equals a constant, but this is not the case when $n \geq 3$;

2) for any n, any bounded solution of Eq. (0.1) equals a constant, as does any solution of (0.1) with a finite Dirichlet ingegral $D(u)= \int_{R^n}|\nabla u|^2 dx < \infty$;

3) when $n = 2$, the solution of the exterior Dirichlet boundary value problem for Eq. (0.1) is unique in the class of bounded functions, as well as in the class $D(u) < \infty$, but this is not the case for $n \geq 3$;

4) for any n, the solution of the above exterior problem with the additional condition that the flow equal 0 is unique in the class of bounded functions, as well as in the class $D(u) < \infty$..

The present paper considers Eq. (0.1) on an arbitrary Riemannian manifold; in this case, Δ is the Laplace-Beltrami operator. Kondrat'ev called our attention to the following problem: for what manifolds are properties 3) and 4) satisfied, i.e., when nothing need be required for single-valued solvability of the exterior Dirichlet problem in the class $\sup|u| < \infty$ of $D(u) < \infty$ and when is the additional requirement that the solution flow across the boundary equal zero needed? We will establish for arbitrary manifolds the connection between the validity of Liouville theorems of type 1), 2) and single-valued solvability of exterior boundary value problems. The exact formulations are given in §§1, 2, and 4. We will not touch upon the geometric conditions under which the Liouville theorems are satisfied, as quite a large number of publications have dealt with this topic (see, e.g., [1-5]).

The following decomposition principle follows from the fact that validity of a Liouville theorem is equivalent to unique solvability of some exterior boundary value problem: if the Riemannian metric or topology of a manifold is altered on some compact, the validity of the Liouville theorem does not break down (the manifolds under consideration are assumed to be connected).

Most of this paper will consider the following equation with small terms, which is more general than (0.1):

$$Lu = \Delta u + (b(x), \nabla u) + c(x)u = 0, \tag{0.2}$$

where $b(x)$ is a smooth vector field and $c(x)$ is a smooth function, $c(x) \leq 0$. The aforementioned decomposition principle is also valid for Eq. (0.2): local changes in the coefficients $b(x)$ and $c(x)$ have no influence on the Liouville theorems, except for the case where a nonzero coefficient of $c(x)$ is converted to one identical to zero (these two cases differ in the fact that the Liouville theorems are formulated differently for them).

As is well known (see [6]), existence of a nonzero bounded solution of the equation $\Delta u - \lambda u = 0$, $\lambda = \text{const} > 0$, is equivalent to stochastic incompleteness of the manifold under consideration (a manifold is stochastically complete if a Wiener process on it is unique, which is equivalent to stating that the solution of the Cauchy problem for the thermal conductivity equation $\partial u/\partial t - \Delta u = 0$ is unique in the class of bounded functions, e.g., R^n is stochastically complete and the open ball in R^n is not). The decomposition principle is thus also valid for stochastic completeness.

The following notation will be employed throughout the paper: M is a Riemannian manifold and ∂M is the edge of manifold M (possibly empty); if the edge is nonempty, then we will consider only those solutions of (0.2) that are smooth up to the edge and satisfy the Neumann condition $\partial u/\partial \nu = 0$ on ∂M, where ν is a normal to ∂M. Further, Δ, ∇ are the Laplacian and gradient on M and $\{\Omega_k\}$ is an increasing sequence of precompact open subsets of M having smooth boundaries (if ∂M is nonempty, then $\partial \Omega_k$ and ∂M are assumed to be transversal) and exhausting M.

§1. LIOUVILLE THEOREM FOR SUPERSOLUTIONS

A function $u \in C^2(M)$ is called superharmonic if $\Delta u \leq 0$ (as well as $\partial u/\partial \nu \geq 0$ on ∂M, where ν is an interior normal to ∂M). Manifolds on which any superharmonic function bounded from below equals a constant are called parabolic manifolds. For example, R^2 is parabolic and R^3 is not. For more on geometric conditions for parabolicity, see [1,2]. Manifold M is known (see [7]) to be parabolic when and only when the Laplace equation has no positive fundamental solution, which is in turn equivalent to the statement that the Wiener capacity of any compact in M equals zero (see [2]). Hence it readily follows that two manifolds isometric outside of compacts are simultaneously both parabolic or both not parabolic, i.e., the decomposition principle for parabolicity.

In this section, we will prove the decomposition principle for existence of positive nontrivial supersolutions of the equation

$$Lu = \Delta u + (b(x), \nabla u) = 0. \tag{1.1}$$

The reason why we will not consider the complete operator (0.2) here is as follows: Eq. (0.2) always has a positive solution (see [8,9]) and, if $c(x) \not\equiv 0$, this solution obviously does not equal a constant. The exterior Dirichlet problem for Eq. (0.2) will be considered in §2.

Any function $u \in C^2(M)$ such that $Lu \leq 0$ (and $\partial u/\partial \nu > 0$ in the case of a nonempty edge ∂M) will be called a supersolution of Eq. (1.1).

We denote by $g_\Omega(x, y)$ the Green function of operator L in an open precompact domain $\Omega \subset M$, having the smooth boundary $\partial\Omega$ (transversal to ∂M). By definition we have for each fixed $y \in \Omega$

$$Lg_\Omega(x, y) = -\delta_y(x),\ g_\Omega(x, y)|_{x\in\partial\Omega} = 0,\ \partial g_\Omega/\partial\nu|_{x\in\partial M} = 0.$$

The function g_Ω, considered as a function of y, satisfies the conjugate equation $L^* g_\Omega = -\delta_x(y)$ and the conjugate boundary conditions $g_\Omega = 0$ when $y \in \partial\Omega$, $\partial g_\Omega/\partial\nu^* = 0$ when $y \in \partial M$, where $L^* u = \Delta u - \nabla^*(ub)$, $\partial/\partial\nu^* = \partial/\partial\nu - (b, \nu)$. It is also known that $g_\Omega > 0$. As follows from the maximum principle, when domain Ω is enlarged, the function g_Ω increases. If Ω_k is the exhaustion of manifold M (see Introduction), then the Green function in domain Ω_k is denoted by g_k. The increasing sequence of functions $\{g_k\}$ either tends to ∞ for all x or is bounded for some x. By the known properties of elliptic equations, sequence $\{g_k\}$ then has a limit for all $x \neq y$. This limit will be called the Green function of operator L on manifold M and denoted by $g(x, y)$. In precisely the same manner, its existence for all $y \neq x$ follows from the existence of $\lim g_k(x, y)$ at some one y. It is readily seen that

$$Lg = -\delta_y(x),\ L^* g = -\delta_x(y),\ \partial g/\partial\nu|_{x\in\partial M} = 0,\ \partial g/\partial\nu^*|_{y\in\partial M} = 0.$$

It follows from the maximum principle that $g(x, y)$ exists when and only when a positive fundamental solution of Eq. (1.1) exists and the Green function is the least positive fundamental solution.

Theorem 1.1. Let Ω be an arbitrary precompact domain in M with a smooth boundary (transversal to ∂M). The following conditions are equivalent:

a) there exists on M a positive supersolution of Eq. (1.1) that does not equal a constant;

b) there exists in $M\setminus\bar{\Omega}$ a solution to the exterior boundary value problem $Lv=0$, $v|_{\partial\Omega}=0$, $\partial v/\partial\nu|_{\partial M}=0$, satisfying the restrictions $0<v<1$, $v\neq 0$;

c) there exists a Green function g(x, y).

Proof. a) → b). Let $\{\Omega_k\}$, $k=1, 2, \ldots$, be the exhaustion of manifold M. It can be assumed that $\Omega_k\supset\bar{\Omega}$. We solve the following boundary value problem in $\Omega_k\setminus\bar{\Omega}$:

$$Lw_k=0,\ w_k|_{\partial\Omega}=1,\ w_k|_{\partial\Omega_k}=0,\ \partial w_k/\partial\nu|_{\partial M}=0.$$

By the maximum principle, we have $0<w_k<1$, $w_{k+1}>w_k$. As $k\to\infty$, sequence w_k has the limit w satisfying the conditions: Lw = 0 outside Ω, $w|_{\partial\Omega}=1$, $\partial w/\partial\nu|_{\partial M}=0$, $0<w<1$. If we prove that $w\not\equiv 1$, then we take 1 - w as the function v sought. In order to prove $w\not\equiv 1$, we utilize the condition for existence on M of a positive supersolution $u\not\equiv$ const. Displacing function u by a constant, it can be assumed that inf u = 0. By the strict maximum principle for supersolutions, we have u > 0. For some constant C, we therefore have $Cu|_{\partial\Omega}>1$. Then, by the maximum principle $Cu>w_k$, whence $Cu\geq w$. Consequently $\inf w=0$, $w\not\equiv 1$.

b) → c). We denote by v_k the solution of the following boundary value prblem: $Lv_k=0$ in Ω_k, $v_k|_{\partial\Omega_k}=v$, $\partial v_k/\partial\nu|_{\partial M}=0$. Utilizing integral representation of the solution and Green's formula, we have

$$v_k(x)=\int_{\partial\Omega_k} v(y)\frac{\partial g_k}{\partial\nu^*}dy=\int_{\partial\Omega\cup\partial\Omega_k\cup\partial M}\frac{\partial v}{\partial\nu}g_k\,dy-$$
$$-\int_{\partial\Omega\cup\partial M} v(y)\frac{\partial g_k}{\partial\nu^*}dy+\int_{\Omega_k\setminus\Omega}(Lv\cdot g_k-L^*g_k\cdot v)\,dy.$$

Here ν everywhere denotes a normal directed inward into domain $\Omega_k\setminus\Omega$. Remarking that only $\int_{\partial\Omega}\frac{\partial v}{\partial\nu}g_k\,dy$, of the boundary integrals does not equal zero and also that $Lv=0$, $-L^*g_k=\delta_x(y)$, we obtain

$$v_k(x)=v(x)+\int_{\partial\Omega}\frac{\partial v}{\partial\nu}g_k\,dy$$

(it would be more accurate to delete a neighborhood of point x and then pass to the limit, but this procedure is standard and the details are omitted). It specifically follows from $v(x)>0$, $v_k(x)<1$ that

$$\int_{\partial\Omega}\frac{\partial v}{\partial\nu}g_k\,dy<1. \tag{1.2}$$

since $\partial v/\partial\nu>0$ on ∂Ω, it follows from (1.2) that $\lim_{k\to\infty} g_k(x,y)$ exists.

c) → a). For an arbitrary function $\varphi\in C_0^\infty(M)$, $\varphi>0$, $\varphi\not\equiv 0$, set $u(x)=\int_M g(x,y)\varphi(y)\,dy$. Since $Lu=-\varphi<0$, then u is a positive superharmonic function, $u\not\equiv$ const.

We will say, by analogy with the definition of a parabolic mainfold, that the pair (M, b) (where b(x) is a smooth vector field on M) is parabolic if any positive supersolution of Eq. (1.1) equals a constant.

Corollary 1.1. The following conditions are equivalent: a) the pair (M, b) is parabolic; b) the exterior boundary problem in $M\setminus\bar{\Omega}$ $Lu=0$, $u|_{\partial\Omega}=0$, $\partial u/\partial\nu=0$ on ∂M, has the unique bounded solution $u \equiv 0$.

Actually, according to clause b) of theorem 1.1, it is sufficient to verify that, if there exists a nonzero bounded solution u of this exterior boundary problem, then there also exists a bounded solution $v \geq 0$, $v \neq 0$ of the problem. Put $u^{\pm}=(1/2)(u\pm|u|)$. It can be assumed that $u^+ \not\equiv 0$ (otherwise $v = -u^-$). Construct the sequence $\{v_k\}$ of solutions of the following boundary problems: $Lv_k = 0$ in $\Omega_k\setminus\bar{\Omega}$, $v_k|_{\partial\Omega}=0$, $v_k|_{\partial\Omega_k}=u^+$, $\partial v_k/\partial\nu|_{\partial M}=0$. We successively deduce from the maximum principle that $v_k>0$, $v_k>u$, $v_k>u^+$, $v_{k+1}>v_k$, $v_k<\sup u$, so that the sequence $\{v_k\}$ has the limit $v=\lim\limits_{k\to\infty} v_k$, which is a solution of the exterior boundary problem and satisfies the conditions $u^+<v<\sup u$.

Corollary 1.2. Let (M_1, b_1) and (M_2, b_2) be two pairs such that there exists the isometry $i: M_1\setminus K_1\to M_2\setminus K_2$ of the exteriors of compacts K_1, K_2 that sends field b_1 into b_2. These pairs are then simultaneously both parabolic or both not parabolic.

§2. LIOUVILLE THEOREMS FOR BOUNDED SOLUTIONS

Theorem 2.1. Let L be operator (0.2) with $c(x)\leq 0$, $c(x)\not\equiv 0$; K is a compact in M. The following conditions are equivalent:

a) there exists on M a nonzero bounded solution u(x) of Eq. (0.2) (with the Neumann condition on ∂M)

b) there exists in $M\setminus K$ a bounded solution v(x) of Eq. (0.2) (with the Neumann condition on ∂M) that satisfies the following conditions for some precompact domains G_1, G_2, $K\subset G_1 \subset\subset G_2$,

$$M_1(v)<M_2(v)^+,\ m_1(v)>m_2(v)^-, \tag{2.1}$$

where $M_i(v)=\sup\limits_{\partial G_i} v$, $m_i(v)=\inf\limits_{\partial G_i} v$, $a^{\pm}=(1/2)(a\pm|a|)$.

Proof. a) $\to$ b). This is obvious: set $v = u$ and condition (2.19) is then satisfied by the maximum principle for (0.2).

b) $\to$ a). Continue function v by some smooth image on the entire M (it is permissible to change v in the neighborhood of K). Let $Lv = f$, where $\operatorname{supp} f\subset G_1$. Let $\{\Omega_k\}$ be the exhaustion of manifold M. It can be assumed that $\Omega_k\supset\bar{G}_2$. Solve in Ω_k the boundary value problems: $Lu_k=0$, $u_k|_{\partial\Omega_k}=v$, $\partial u_k/\partial\nu=0$ on ∂M. Since $\sup|u_k|$

$< \sup|v|$, sequence $\{u_k\}$ is compact. Let u(x) be the limiting function. It is sufficient to show that $u \not\equiv 0$. Assume that $u \equiv 0$. Set $w_k = v - u_k$. Then

$$Lw_k = f,\ w_k|_{\partial\Omega_k} = 0,\ \partial w_k/\partial\nu|_{\partial M} = 0 \qquad (2.2)$$

and $w_k \to v$ when $k \to \infty$. Therefore, with sufficiently large k,

$$M_1(w_k) < M_2(w_k)^+,\ m_1(w_k) > m_2(w_k)^-. \qquad (2.3)$$

Applying the maximum principle to function w_k in the domain $\Omega_k \setminus \bar{G}_1$ (in which $Lw_k = 0$), we obtain $M_1(w_k)^+ > M_2(w_k)$, which together with (2.3) gives $M_2(w_k) < 0$, < 0. In precisely the same manner, we obtain $m_1(w_k) > 0$, which is not possible simultaneously with $M_1(w_k) < 0$.

Theorem 2.2. Let L be the operator (1.1). The following conditions are equivalent (see the preceding theorem for notation:

a) there exists on M a bounded (positive) solution u(x) of Eq. (1.1) that does not equal a constant (with the Neumann condition on ∂M);

b) there exists in $M \setminus K$ a bounded (positive) solution v(x) of Eq. (1.1) (with the Neumann condition on ∂M) that satisfies the conditions

$$M_1(v) < M_2(v),\ m_1(v) > m_2(v). \qquad (2.4)$$

Proof. a) → b) is obvious. b) → a). Retaining the notation used in the proof of theorem 2.1, we will show that $u \not\equiv$ const (still considering the case of bounded solutions). If $u \equiv$ const, then for the function $w = v - u$, condition (2.4) and also condition (2.3) are satisfied, which leads to a contradiction.

Now let the function v be positive; we have to construct a positive solution u on M. The entire preceding proof holds with one exception: it is not clear why sequence $\{u_k\}$ should have a limiting function (no boundedness). We utilize the Green function from §1 to prove convergence of $\{u_k\}$. It follows from (2.2) that $w_k(x) = \int_{\Omega_k} g_k(x, y) f(y)\, dy$. Therefore, if $\lim_{k\to\infty} g_k(x, y)$ exists, then $\lim_{k\to\infty} w_k$ exists and thereby $\lim_{k\to\infty} u_k$ also exists. By virtue of theorem 1.1 it is sufficient to prove that for some domain Ω there exists a solution to the exterior boundary value problem $LW = 0$ in $M \setminus \bar{\Omega}$, $W|_{\partial\Omega} = 1$, $\partial W/\partial\nu = 0$ on ∂M with $0 < W < 1$, $W \not\equiv 1$. We first find solutions W_k to the following problems: $LW_k = 0$ in $\Omega_k \setminus \bar{\Omega}$, $W_k|_{\partial\Omega} = 1$, $W_k|_{\partial\Omega_k} = 0$, $\partial W_k/\partial\nu = 0$ on M and we set $W = \lim W_k$. We will prove that $W \not\equiv 1$. Choose domain Ω such that $G_1 \subset\subset \Omega \subset\subset G_2$. The function v from condition b) can be assumed to satisfy the condition $\inf_{\partial\Omega} v = 1$. By the maximum principle, we have $v > W_k$, $v > W$. The function v(x) in the layer $\bar{G}_2 \setminus G_1$ attains its minimum on ∂G_2 by (2.4). It follows from $\inf_{\partial\Omega} v = 1$ and the maximum principle that $\inf_{\partial G_1} v < 1$. Therefore $\inf_{\partial G_1} W < 1$, $W \not\equiv 1$. The theorem has been proved.

Corollary 2.1. If manifolds M_1 and M_2 are isometric outside some compact, then they both simultaneously are or are not stochastically complete.

Corollary 2.2. Let Ω be a precompact open submanifold of M with smooth boundary $\partial\Omega$ not intersecting ∂M and $M\setminus\bar{\Omega}$ be connected. Let L be operator (0.2), $c(x) \leq 0$; either $c(x) \not\equiv 0$ in $M\setminus\bar{\Omega}$, or $c(x) \equiv 0$ in M. The following conditions are equivalent:

a) there exists on M a bounded (positive*) solution of Eq. (0.2) that does not equal a constant (with the Neumann condition on ∂M);

b) there exists in $M\setminus\bar{\Omega}$ a bounded (positive*) solution of the exterior Neumann problem

$$Lv=0,\ \partial v/\partial\nu|_{\partial\Omega}=0,\ \partial v/\partial\nu|_{\partial M}=0.$$

Actually, we consider the set $M\setminus\Omega$ to be a manifold with edge $\partial\Omega\cup\partial M$. Since the neighborhoods of ∞ for manifolds M and $M\setminus\Omega$ coincide, as a consequence of theorems 2.1 and 2.2, the Liouville theorems are simultaneously satisfied or not satisfied on M and $M\setminus\Omega$.

Corollary 2.3. Let Ω be a precompact open set with smooth boundary $\partial\Omega$ transversal to ∂M and let $L=\Delta+(b,\nabla)$. The following conditions are equivalent:

a) any bounded solution of the equation $Lu = 0$ (with the Neumann condition on ∂M) equals a constant;

b) the exterior problem in $M\setminus\bar{\Omega}$ $Lv=0,\ v|_{\partial\Omega}=0,\ \partial v/\partial\nu=0$ on ∂M, has a unique solution $v \equiv 0$. In the class of bounded functions with zero flow $p(v)=\int_{\partial\Omega}\partial v/\partial\nu=0$.

Proof. a) $\Rightarrow$ b). Let v be a nonzero bounded solution of the problem in clause b), with $p(v) = 0$. Then function v satisfies condition (2.4) for arbitrary precompact domains G_1, G_2 such that $G_2\supset\supset G_1\supset\supset\Omega$. Actually, if, e.g., $M_2(v)<M_1(v)$, then, since function v attains its maximum at the boundary in the domain $\bar{G}_2\setminus\Omega$, we have $M_2(v) \leq 0$. We then have $v \leq 0$ everywhere in $G_2\setminus\bar{\Omega}$ and, from the lemma on the normal derivative, $\partial v/\partial\nu<0$ on $\partial\Omega$ (where ν is an exterior normal to $\partial\Omega$), which contradicts $p(v) = 0$.

b) $\Rightarrow$ a). Let u be a nontrivial bounded solution of the equation $Lu = 0$ on M (with the Neumann condition on ∂M). We will show how to construct a bounded solution different from u for the exterior boundary problem: $Lw = 0$ in $M\setminus\bar{\Omega}$, $w|_{\partial\Omega}=u$, $\partial w/\partial\nu=0$ on ∂M, $p(w)=p(u)$. Then the difference $v = u - w$ is the function sought.

Consider the sequence of boundary value problems in $\Omega_k\setminus\bar{\Omega}$:

*Positive solutions are considered only in the case $c(x) \equiv 0$.

$$Lw_k = 0,\ w_k|_{\partial\Omega} = u,\ w_k|_{\partial\Omega_k} = c_k,\ \partial w_k/\partial\nu|_{\partial M} = 0, \tag{2.5}$$

where c_k is a constant that can be determined from the condition $p(w_k) = p(u)$. It can be assumed that $c_k \in [\inf_{\partial\Omega} u, \sup_{\partial\Omega} u]$. Actually, if $c_k > \sup_{\partial\Omega} u$, and $w_k > u$ and, since $w_k = u$ on $\partial\Omega$, $p(w_k) > p(u)$. Analogously, if $c_k < \inf_{\partial\Omega} u$, then $p(w_k) < p(u)$. For some $c_k \in [\inf_{\partial\Omega} u, \sup_{\partial\Omega} u]$, we therefore have $p(w_k) = p(u)$. By virtue of the above restrictions on c_k, the sequence $\{w_k\}$ is bounded and therefore compact. Let w be its limiting function. We will prove that $w \not\equiv u$. If G is a precompact domain, $G \supset\supset \Omega$, then by the maximum principle we have $\sup_{\partial G} u > \sup_{\partial\Omega} u$ for the function u in G. For the function w_k, we have $\sup_{\partial G} w_k < \sup_{G\setminus\bar\Omega} w_k < \sup_{\partial\Omega} u$. As $k \to \infty$, we obtain $\sup_{\partial G} w < \sup_{\partial\Omega} u$. Thus, $w \not\equiv u$.

Corollary 2.4. Let $L = \Delta + (b(x), \nabla) + c(x)$, where $c(x) \le 0$, $c(x) \not\equiv 0$. The following conditions are equivalent:

a) any bounded solution of the equation $Lu = 0$ on M (with the Neumann condition on ∂M) equals zero;

b) the exterior problem $Lv = 0$ in $M \setminus \bar\Omega$, $v|_{\partial\Omega} = 0$, $\partial v/\partial\nu = 0$ on ∂M has the unique bounded solution $v \equiv 0$ (here Ω is the same as in corollary 2.3).

Proof. a) $\to$ b). Theorem 2.1 is applied directly.

b) $\to$ a). Let u be a nonzero bounded solution of the equation $Lu = 0$ (with the Neumann condition on ∂M). If $u|_{\partial\Omega} = 0$, then we set $v = u$. It can otherwise be assumed that $\sup_{\partial\Omega} u > 1$. We solve the sequence of boundary value problems (2.5), putting $c_k = 1$. The inequality $\sup w < \sup_{\partial\Omega} u$ is then satisfied for limiting function w. Since $\sup_{\partial\Omega} u < \sup_{M\setminus\Omega} u$, then $w \not\equiv u$ and the function $v = u - w$ is what is sought.

As follows from theorems 2.1 and 2.2, the bilateral Liouville theorems do not break down when there is a local change in the metric or topology of manifold M or in the coefficients $b(x)$, $c(x)$, albeit with the following restriction on the change in the coefficient $c(x)$: the relations $c \equiv 0$, $c \not\equiv 0$ should not be violated (these cases are handled by different theorems). Actually, let $c \not\equiv 0$, $c \in C_0^\infty(M)$. Validity of the bilateral Lioxville theorem for the operator $L = \Delta + c(x)$ is then equivalent to parabolicity of M (see corollaries 2.4 and 1.1). If there is a local change in the coefficient c that makes it identical to zero, operator L is converted to Δ and the bilateral Liouville theorem for Δ is not equivalent to parabolicity of M (e.g., if $M = R^3$).

The unilateral Liouville theorem for the operator $\Delta + (b(x), \nabla)$ does not break down under any local changes in the manifold or the coefficient $b(x)$.

§3. LIOUVILLE THEOREMS UNDER CHANGE IN ABSORPTION COEFFICIENT

Let $L=\Delta+(b(x),\nabla)$. We will here consider the question of the existence of nonzero bounded solutions of the equation $Lu-c(x)u=0$ with different coefficients $c(x)$, $c(x)\geq 0$, $c(x)\not\equiv 0$.

Theorem 3.1. Let $0<c_2(x)<Ac_1(x)$, where $A=\text{const}>0$, $c_2\not\equiv 0$. Then, if the bilateral Liouville theorem is satisfied for the equation $Lv-c_2(x)v=0$, it is also satisfied for the equation $Lu-c_1(x)u=0$.

Comment. The boundedness condition c_2/c_1 is essential. For example, if in R^3 $c_2(x)$ has a compact carrier and c_1 = const > 0, then the Liouville theorem is not satisfied for the operator $\Delta-c_2(x)$ (see the end of §2) but is satisfied for $\Delta - c_1$ (sinece R^3 is stochastically complete; see the Introduction).

Proof. Let there exist a bounded solution $u\not\equiv 0$ of the equation $Lu-c_1(x)u=0$ (with the Neumann condition on ∂M). Initially let $c_2(x)<c_1(x)$. We will prove that there exists a positive bounded solution of the equation $Lv-c_2(x)v=0$ (with the Neumann condition on ∂M). It is constructed as the limit of the solutions of the following boundary value problems in Ω_k: $Lv_k-c_2v_k=0$, $v_k|_{\partial\Omega_k}=u^+$, $\partial v_k/\partial\nu=0$ on ∂M. We can assume without loss of generality that $u^+\not\equiv 0$. Obviously, $0<v_k<\sup u$. Since $Lv_k-c_1v_k<Lv_k-c_2v_k=0$, $v_k|_{\partial\Omega_k}>u$, then by the maximum principle $v_k>u$, $v_k>\max(0,u)=$ $=u^+$. The limiting function of the sequence $\{v_k\}$ is therefore bounded and positive.

Now let $c_2(x)<Ac_1(x)$, $A>1$. By virtue of the preceding argument, the function u can be considered positive. We will assume that there exists a positive bounded solution of the equation $Lv-Ac_1(x)v=0$. Then, by what has been proved above, the equation $Lv-c_2(x)=0$ will also have such a solution.

Let v_k be the solution of the boundary value problem in Ω_k:

$$Lv_k-Ac_1(x)v_k=0,\quad v_k|_{\partial\Omega_k}=u,\quad \partial v_k/\partial\nu|_{\partial M}=0.$$

Since $0<v_k<\sup u$, there exists a limiting function v, $0<v<\sup u$. We will prove that $v>0$. Consider the following functions w_k, $\bar{u}_k$, $\bar{v}_k$, which are solutions to the boundary value problems in the domain Ω_k: $Lw_k=0$, $w_k|_{\partial\Omega_k}=u$, $L\bar{u}_k=-c_1(x)u$, $\bar{u}_k|_{\partial\Omega_k}=$ $=0$, $L\bar{v}_k=-Ac_1v_k$, $\bar{v}_k|_{\partial\Omega_k}=0$ (the Neumann condition on ∂M is also assumed). Obviously, $w_k-\bar{u}_k=u$, $w_k-\bar{v}_k=v_k$. The following inequalities arise from the maximum principle: $\bar{u}_k>0$, $\bar{v}_k>0$, $w_k=u+\bar{u}_k>u$. We will prove that $\bar{v}_k<A\bar{u}_k$. Let us first remark that $v_k\leq u$, since $A>1$, $Lv_k-c_1(x)v_k>0$. Hence it follows that $L(A\bar{u}_k)=-Ac_1u<-Ac_1v_k=$ $=L\bar{v}_k$, and, since the boundary conditions for the functions $A\bar{u}_k$ and $\bar{v}_k$ are identical, then $A\bar{u}_k>\bar{v}_k$.

Let x_0 be a point at which $u(x_0)>\sup u-\varepsilon$. Then $w_k(x_0)>\sup u-\varepsilon$, $\bar{u}_k(x_0)=w_k(x_0)-$

$- u(x_0) < \sup u - u(x_0) < \varepsilon$, $\bar{v}_k(x_0) < A\varepsilon$, $v_k(x_0) = w_k(x_0) - \bar{v}_k(x_0) > \sup u - (A+1)\varepsilon$. As $k \to \infty$, we therefore obtain $v(x_0) > \sup u - (A+1)\varepsilon$, which is greater than zero at sufficiently small ε. The theorem has been proved.

Theorem 3.1.can be employed to investigate Liouville theorems on the Riemannian product $M \times N$, where N is a compact manifold (in this case either ∂M or ∂N is empty). For example, if the bilateral (unilateral) Liouville theorem for the equation $Lu = 0$ and the bilateral Liouville theorem for the equation $Lu - u = 0$ are satisfied on M, then by theorem 3.1, the Liouville theorem is satisfied for the equation $Lu - \lambda u = 0$, where λ is any number ≥ 0. Utilizing this fact for $\lambda = \lambda_k$, i.e, the eigenvalues of the Laplace operator Δ_N on manifold N, the validity of the bilateral (unilateral) Liouville theorem for the operator $L + \Delta_N$ on manifold $M \times N$ is easily proved.

§4. D-LIOUVILLE THEOREM

We will consider in this section solutions of the Laplacian $\Delta u = 0$ with $D(u) = \int_M |\nabla u|^2 dx < \infty$. The assertion that any harmonic function on M (with the Neumann condition on ∂M) with a finite Dirichlet integral equals a constant will be called the D-Liouville theorem.

Theorem 4.1. Let Ω be a precompact domain with smooth boundary $\partial\Omega$ transversal to ∂M. The following conditions are equivalent:

a) the D-Liouville theorem is valid on M;

b) the exterior boundary value problem $v = 0$ in $M \setminus \bar{\Omega}$, $v|_{\partial\Omega} = 0$, $\partial u/\partial\nu = 0$ on M has the unique solution $v \equiv 0$ in the class of functions satisfying the conditions $D(v) < \infty$, $p(v) = 0$.

Proof. a) $\to$ b). Let v(x) be a nonzero solution of the problem in clause b) with the restrictions indicated. We will prove that there then exists a nonzero solution $\bar{v}$ of this problem that is still bounded. Our argument is similar to the proof of a theorem of Ahlfors (see [7]). As follows from $v|_{\partial\Omega} = 0$, $p(v) = 0$, the function v takes both positive and negative values on ∂G, where G is an arbitrary precompact domain, $G \supset\supset \Omega$. Let N be a number so large that $\int_{|v|>N} |\nabla v|^2 dx < \varepsilon$, where ε is chosen smaller. We denote the N-slice of function v as V and solve the following boundary value problems in $\Omega_k \setminus \bar{\Omega}$: $\Delta u_k = 0$, $u_k|_{\partial\Omega} = 0$, $u_k|_{\partial\Omega_k} = V^+$, $\Delta w_k = 0$, $w_k|_{\partial\Omega} = 0$, $w_k|_{\partial\Omega_k} = V^-$ (u_k and w_k also satisfy the Neumann condition on ∂M).

The sequence $\{u_k\}$ is positive and bounded, so that there exists for it a limiting function u satisfying in $M \setminus \bar{\Omega}$ the conditions $\Delta u = 0$, $u|_{\partial\Omega} = 0$, $\partial u/\partial\nu|_{\partial M} = 0$, $0 < u < N$. We will prove that $u \not\equiv 0$. For this purpose, consider the differences $U_k = v - u_k$, $U = v - u$. The function U_k is the solution of the boundary value problem in $\Omega_k \setminus \bar{\Omega}$: $\Delta U_k = 0$, $U_k|_{\partial\Omega} = 0$, $U_k|_{\partial\Omega_k} = v - V^+$, $\partial U_k/\partial\nu|_{\partial M} = 0$. Since $v - V^+$ also satisfies the boundary conditions indicated, by the variational property of the Dirichlet problem, we have

$$D(U_k) < \int_{\Omega_k \setminus \Omega} |\nabla(v - V^+)|^2 dx < \int_M |\nabla v^-|^2 dx + \int_{\{v > N\}} |\nabla v|^2 dx < D(v^-) + \varepsilon.$$

Therefore $D(U) < D(v^-) + \varepsilon$. If $\varepsilon < D(v^+)$, thence $D(U) < D(v)$, $U \not\equiv v$, $u \not\equiv 0$.

In exactly the same manner, if $\varepsilon < D(v^-)$, then the sequence $\{w_k\}$ has a limiting function $w \not\equiv 0$ such that $\Delta w = 0$ in $M \setminus \bar{\Omega}$, $w|_{\partial\Omega} = 0$, $\partial w/\partial\nu|_{\partial M} = 0$, $0 > w > -N$. Obviously $p(u) > 0$, $p(w) < 0$. We will find a constant $c > 0$ such that $p(u + cw) = 0$, and prove that the function $\bar{v} = u + cw$ is the one sought. In order to do so, we must verify that $\bar{v} \not\equiv 0$, $D(\bar{v}) < \infty$. It can be assumed without loss of generality that $c \leq 1$. The function $\bar{v}$ is limiting for the sequence $\{v_k\}$ determined from the conditions $\Delta v_k = 0$ in $\Omega_k \setminus \bar{\Omega}$, $v_k|_{\partial\Omega} = 0$, $v_k|_{\partial\Omega_k} = V^+ + cV^-$, $\partial v_k/\partial\nu|_{\partial M} = 0$. For the difference $v - v_k$, we have

$$D(v - v_k) < \int_{\Omega_k \setminus \Omega} |\nabla(v - V^+ - cV^-)|^2 dx < (1-c)^2 \int_M |\nabla V^-|^2 dx + \varepsilon < D(v^-) + \varepsilon.$$

Therefore $D(v - \bar{v}) < D(v^-) + \varepsilon < D(v)$, $\bar{v} \not\equiv 0$. It also follows from the variational properties of the Dirichlet problem that $D(v_k) < D(V^+ + cV^-) < D(v)$, whence $D(\bar{v}) < \infty$.

According to corollary 2.3, there exists on M a nontrivial bounded harmonic function u. We will verify that $D(u) < \infty$. Actually, this function is constructed as the limit of the solutions of the following problems: $\Delta u_k = 0$ in Ω_k, $u_k|_{\partial\Omega_k} = \bar{v}$, $\partial u_k/\partial\nu|_{\partial M} = 0$. Since $D(u_k) < D(\bar{v})$, thence $D(u) < \infty$.

b) → a). If u is a nontrivial harmonic function with $D(u) < \infty$, then it can be assumed to be bounded (as in the preceding proof). The balance of the argument duplicates the proof of corollary 2.3, except that, instead of (2.5), it is necessary to solve the following boundary value problem: $\Delta w_k = 0$ in $\Omega_k \setminus \bar{\Omega}$, $w_k|_{\partial\Omega} = u$, $\partial w_k/\partial\nu|_{\partial\Omega_k} = 0$, $\partial w_k/\partial\nu|_{\partial M} = 0$. We have from variational considerations $D(w_k) < D(u)$, so that we also obtain $D(w) < \infty$ for the limiting function w.

Corollary 4.1. If manifolds M_1, M_2 are isometric outside compacts, then the D-Liouville theorems are simultaneously satisfied or not satisfied on them.

Corollary 4.2. Let the boundary of domain Ω not intersect edge ∂M. The D-Liouville theorem is then satisfied on M when and only when the exterior boundary value problem $\Delta u = 0$ in $M \setminus \bar{\Omega}$, $\partial u/\partial\nu|_{\partial\Omega} = 0$, $\partial u/\partial\nu|_{\partial M} = 0$ has the unique solution $u \equiv 0$ ($M \setminus \bar{\Omega}$ is connected) in the class $D(u) < \infty$.

Corollary 4.3. Manifold M is parabolic when and only when the exterior boundary value problem in $M \setminus \bar{\Omega}$ $\Delta u = 0$, $u|_{\partial\Omega} = 0$, $\partial u/\partial\nu|_{\partial M} = 0$ has the unique solution $u \equiv 0$ in the class $D(u) < \infty$.

Actually, if there exists a nonzero solution of this problem, then, as can be seen from the proof of theorem 4.1, it can be assumed to be bounded. By corollary 1.1, manifold M is not parabolic. Conversely, if M is not parabolic, then $D(v_k) = p(v_k)$, is obviously satisfied for functions v_k determined from the conditions $\Delta v_k = 0$ in $\Omega_k \setminus \bar{\Omega}$, $v_k|_{\partial\Omega} = 0$, $v_k|_{\partial\Omega_k} = 1$, $\partial v_k/\partial\nu|_{\partial M} = 0$, so that we have $D(v) < p(v) < \infty$ for the limiting function v.

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