Random walks on graphs with regular volume growth

Thierry Coulhon
Université de Cergy-Pontoise
95011 Cergy-Pontoise Cedex, France
email: coulhon@u-cergy.fr

Alexander Grigoryan*
Imperial College
London SW7 2BZ, United Kingdom
email: a.grigoryan@ic.ac.uk

June 14, 1997

Contents

*Supported by the EPSRC Fellowship, B/94/AF/1782

1 Introduction

Consider a nearest neighbourhood random walk on an infinite graph Γ. Assume that the transition probability of the random walk is given by a Markov kernel $p(x,y)$ $(x,y\in\Gamma)$ which is reversible with respect to a positive measure $m(x)$ on Γ. The main purpose of this paper is to reveal which assumptions on the graph structure and on $p(x,y)$ ensure the following upper bound of $p_k(x,y)$ - the convolution powers of $p(x,y)$:

$$p_k(x,y) \leq \frac{Cm(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \quad \forall x,y\in\Gamma,\ k\in\mathbb{N}^*. \tag{UE}$$

Here $d(x,y)$ is the combinatorial distance between the vertices x,y (that is the length of the shortest path between x,y, each edge being counted with the weight 1), $V(x,r)$ is the volume of the ball $B(x,r):=\{y\,|\,d(x,y)\leq r\}$, that is $V(x,r)=\sum_{y\in B(x,r)} m(y)$. The constants C,c are positive.

The upper estimate (UE) is inspired by similar results for Brownian motion on Riemannian manifolds. Let $h_t(x,y)$ be the heat kernel on a complete non-compact Riemannian manifold M which is the transition density of the Brownian motion on M. A theorem by Li and Yau [31] says that if M has non-negative Ricci curvature then

$$h_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-c\frac{d^2(x,y)}{t}\right) \quad \forall x,y\in M,\ t>0 \tag{1.1}$$

where now $d(x,y)$ denotes the geodesic distance between the points x,y, and $V(x,r)$ the Riemannian volume of the geodesic ball $B(x,r)$. Moreover, the same lower bound of $h_t(x,y)$ is valid too, but with different constants C,c. Let us mention also that Gushin [26] obtained a similar estimate in certain unbounded regions of the Euclidean space, with the Neumann boundary conditions.

The visible difference in (UE) and (1.1) - the presence of the factor $m(y)$ in (UE) - is not essential, it just reflects the fact that $p_k(x,y)$ is not an exact analogue of $h_t(x,y)$. The exact analogue would be the kernel $h_k(x,y)=\frac{p_k(x,y)}{m(y)}$ which is also symmetric in x,y as $h_t(x,y)$. However, we shall keep considering p_k because of its probabilistic significance.

Non-negativeness of Ricci curvature is a sufficient but by far not a necessary condition for the estimate (1.1). The criterion for (1.1) was proved by the second author [23] in terms of a certain isoperimetric inequality which we shall call here *a relative Faber-Krahn inequality.*

The main purpose of this paper is to prove that the upper bound (UE) together with the doubling property (see below for the definition) is equivalent to the relative Faber-Krahn inequality on the graph Γ.

Let us now introduce the necessary definitions and notation to state the results exactly. Let Γ be a locally finite graph. Write $x \sim y$ if $x, y \in \Gamma$ are neighbours. A path of length n between x and y in Γ is a sequence x_i, $0 = 1, ..., n$ such that $x_0 = x$, $x_n = y$ and $x_i \sim x_{i+1}$, $i = 0, ..., n-1$. We shall assume that Γ is connected, i.e. there exists a path between any two points of Γ. Let d be the natural metric on Γ: $d(x, y)$ is the minimal length of a path between x and y. Denote by $B(x, r)$ the closed ball of center $x \in \Gamma$ and radius $r > 0$.

Let p be a Markov kernel on Γ, reversible with respect to a measure m:

$$m(x) > 0, \ \forall x \in \Gamma,$$

$$p(x, y) \geq 0, \ p(x, y)m(x) = p(y, x)m(y), \ \forall x, y \in \Gamma,$$

$$\sum_{y \in \Gamma} p(x, y) = 1, \ \forall x \in \Gamma.$$

We shall denote

$$\mu_{xy} := p(x, y)m(x) = \mu_{yx}.$$

Both $m(x)$ and $p(x, y)$ can be recovered from μ_{xy} :

$$m(x) = \sum_{y \sim x} \mu_{xy}$$

and $p(x, y) = \frac{\mu_{xy}}{m(x)}$.

We shall assume for convenience that $p(x, y) = 0$ if $d(x, y) \geq 2$, but our methods can treat the case of finite range, i.e. there exists $n_0 \in \mathbb{N}^*$ such that

$$p(x, y) = 0 \text{ if } d(x, y) \geq n_0.$$

Define $p_1(x, y) = p(x, y)$ and

$$p_k(x, y) = \sum_{z \in \Gamma} p_{k-1}(x, z)p(z, y), \ k \geq 2.$$

The volume $|\Omega|$ of a subset Ω of Γ will be defined by

$$|\Omega| = m(\Omega) = \sum_{x \in \Omega} m(x).$$

Denote as above by $V(x, r)$ the volume $|B(x, r)|$ of the ball $B(x, r)$. The ℓ^p norms on Γ will be taken with respect to the measure m.

We shall say that (Γ, m) has *regular volume growth*, or satisfies the doubling property, if there exists b such that

$$V(x, 2r) \leq b\, V(x, r), \ \forall\, x \in \Gamma, \, r > 0. \tag{D}$$

Along with the estimate (UE), we will consider its on-diagonal version

$$p_k(x,x) \le \frac{C\,m(x)}{V(x,\sqrt{k})},\ \forall x \in \Gamma, k \in \mathbb{N}^*, \tag{DUE}$$

as well as the on-diagonal lower bound[1]

$$p_{2k}(x,x) \ge \frac{c\,m(x)}{V(x,\sqrt{k})},\ \forall x \in \Gamma, k \in \mathbb{N}^*. \tag{DLE}$$

For $f \in \mathbb{R}^\Gamma$, define the length of its gradient by

$$|\nabla f|(x) = \left(\frac{1}{2}\sum_{y\in\Gamma, x\sim y} |f(x)-f(y)|^2 p(x,y)\right)^{1/2}.$$

Note that

$$\||\nabla f|\|_2^2 = \frac{1}{2}\sum_{x,y\in\Gamma} |f(x)-f(y)|^2 \mu_{xy}.$$

Let us denote by Δ the discrete Laplace operator on Γ associated with the kernel $p(x,y)$ i.e.

$$\Delta u(x) = \sum_y p(x,y)u(y) - u(x)$$

and by P the corresponding Markov operator

$$Pu(x) = \sum_y p(x,y)u(y).$$

The characterisation of (UE) will be given in terms of (D) and of *a relative Faber-Krahn inequality* which states that, for any ball $B(x,r)$, $x \in \Gamma$, $r \ge 1/2$, and for any non-empty subset $\Omega \subset B(x,r)$,

$$\lambda_1(\Omega) \ge \frac{a}{r^2}\left(\frac{V(x,r)}{|\Omega|}\right)^\nu, \tag{FK}$$

where the positive constants a, ν are the same for all balls (here $\lambda_1(\Omega)$ - the first Dirichlet eigenvalue of the Laplace operator in Ω - has the usual variational definition, see §2).

Theorem 1.1 *For a reversible nearest neighbourhood random walk on the locally finite[2] graph Γ, the following properties are equivalent:*

1. *The relative Faber-Krahn inequality (FK).*

[1] We cannot claim in general any lower bound for $p_k(x,x)$ with *odd* k because $p_k(x,x)$ may simply vanish for such k as in the case of the simple random walk in $\mathbb{Z}^d$.

[2] The condition (D) implies that Γ is locally *uniformly* finite - see Lemma 4.2 below.

2. *The full upper estimate (UE) in conjunction with the doubling property (D).*

3. *The on-diagonal upper estimate (DUE) in conjunction with the doubling property (D).*

Moreover, each of them implies the on-diagonal lower estimate (DLE).

Let us assume that the volume is uniform and polynomial, i.e. there exists $C > 0$ such that

$$C^{-1}r^D \leq V(x,r) \leq Cr^D,$$

with some $D > 2$. It follows from the work of Varopoulos [39] (see also [3], [12] and [13]), that (DUE) is equivalent to the following Sobolev inequality

$$\|f\|_{\frac{2D}{D-2}} \leq C'\|\nabla f\|_2, \text{ for every finitely supported } f \in \mathbb{R}^\Gamma,$$

which is in turn equivalent to the uniform Faber-Krahn inequality

$$\lambda_1(\Omega) \geq c|\Omega|^{-D/2}, \ \forall\, \Omega \text{ finite subset of } \Gamma,$$

see [5],[8] (also [9] for generalisations to uniform but non-polynomial volume growths). Then from (DUE) one can get (UE) (see [27])

When $V(x,r)$ is uniform, i.e. does not essentially depend on x, what one has to estimate in order to get (DUE) is

$$\sup_{x\in\Gamma} \frac{p_k(x,x)}{m(x)} = \|P^k\|_{1\to\infty}$$

as a function of k, which can be done by functional analytic methods. This approach is no more at hand if the only available information on the volume growth is (D). One has to come back to methods closer to those in [23], but then the technical problems raised by the fact that time and space are discrete are non-negligible.

We propose here a strategy that enables us to overcome these difficulties: we manage to prove a discrete time parabolic Cacciopoli inequality, we deduce by iteration as in [23] a mean value inequality. The last step towards the upper bound in [23] involved an integrated maximum principle that is for the time being not available in the discrete setting (though one can think from the techniques of [21] and of the present paper that it is not out of reach); here, we use instead a technique introduced by Davies [17] that relies on Gaffney's lemma.

An additional technical assumption whose importance appeared in [12] and even more in [21] is the following: for some $\alpha > 0$

$$p(x,x) \geq \alpha \quad \forall x \in \Gamma. \tag{α}$$

At first sight, it looks disappointing because for the standard random walk in $\mathbb{Z}^d$, we have $p(x,x) = 0$. However, for the same random walk $p_2(x,x) = (2d)^{-1} > 0$ which satisfies (α). The trick is to prove first the necessary estimates for the Markov kernel

$p'(x,y) = p_2(x,y)$, and then to extend them to $p(x,y)$ by an additional argument. One can easily show that (D) implies the existence of $\alpha > 0$ such that $p_2(x,x) \geq \alpha$ for all $x \in \Gamma$.

The proof of Theorem 1.1 is presented in the following parts of the paper:

- $(FK) \Longrightarrow (D)$: Proposition 2.1;
- $(FK) \Longrightarrow (UE)$: Theorem 5.2 (depends heavily on Theorem 4.1 and Lemma 5.1);
- (DUE) and $(D) \Longrightarrow (FK)$: Theorem 5.4;
- (UE) and $(D) \Longrightarrow (DLE)$: Theorem 6.1.

Finally, among the graphs with regular volume growth, the class of those that satisfy (UE) and (DLE) is characterised by (FK). The subclass of those that satisfy in addition an optimal off-diagonal estimate can also be characterised in geometric terms.

One says that Γ satisfies the Poincaré inequality if there exists $C > 0$ and $C' \geq 1$ such that

$$\sum_{y \in B(x,r)} |f(y) - f_r(x)|^2 m(y) \leq Cr^2 \sum_{B(x,C'r)} |\nabla f|^2(y) m(y),\ \forall\, f \in \mathbb{R}^\Gamma,\ \forall r > 0, \qquad (P)$$

where

$$f_r(x) := \frac{1}{V(x,r)} \sum_{y \in B(x,r)} f(y)m(y).$$

See [14], §5 for combinatorial conditions that ensure (P).

Delmotte has proved in [21]:

Theorem 1.2 *Assume that the graph Γ satisfies the property (α). Assume in addition that $p(x,y) \geq \alpha$ if $x \sim y$. Then the conjunction of the doubling property (D) and of the Poincaré inequality (P) is equivalent to*

$$\frac{c\, m(y)}{V(x,\sqrt{k})} \exp\left(-C\frac{d^2(x,y)}{k}\right) \leq p_k(x,y) \leq \frac{C\, m(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right), \qquad (1.2)$$

$\forall\, x, y \in \Gamma, k \in \mathbb{N}^*$ *such that* $d(x,y) \leq k$ (note that if $d(x,y) > k$ then $p_k(x,y) = 0$).

Delmotte's strategy is the following: using Moser's iteration process, he proves a parabolic Harnack principle for the continuous time process p_t associated with p_k. The estimates follow for p_t, then a careful pointwise comparison between p_k and p_t gives the theorem. This is not straightforward because the continuous time kernel p_t has "abnormal" (non-Gaussian) asymptotics discovered by Pang [34] and E.B.Davies [18].

Our relative Faber-Krahn inequality follows from (D) and (P) (but the converse is false). Therefore we get the conclusion of Theorem 1.2, except for the off-diagonal lower bound, under strictly weaker assumptions, without going through comparison with a continuous time process (except, locally, in Section 5) nor parabolic Harnack inequality.

Finally, the present work together with [21] gives a (nearly) complete analogue of the theory of heat kernels on Riemannian manifolds with the doubling property as developed in [23], [37], [36], [10].

Acknowledgments. This work was done when the second author enjoyed the hospitality of the University of Cergy Pontoise. He gratefully acknowledges support of this university.

Thanks are due to Thierry Delmotte for carefully reading the manuscript.

2 Faber-Krahn inequalities

Denote by $c_0(\Omega)$ the space of functions on Γ vanishing outside a subset Ω of Γ and by $c_0(\Gamma)$ the space of finitely supported functions on Γ. Define

$$\lambda_1(\Omega) = \inf \left\{ \frac{\| |\nabla f| \|_2^2}{\|f\|_2^2}; f \in c_0(\Omega) \right\}.$$

One says that Γ satisfies a relative Faber-Krahn inequality (see [23]) if there exists $a > 0$, $\nu > 0$ such that, for every $x \in \Gamma$, $r \geq 1/2$, and for every non-empty finite subset Ω of Γ contained in $B(x,r)$,

$$\lambda_1(\Omega) \geq \frac{a}{r^2} \left(\frac{V(x,r)}{|\Omega|} \right)^{\nu}. \qquad (FK)$$

It is straightforward that (FK) is implied by the Nash type inequality

$$\|f\|_2^{\nu+1} \leq Cr \left(V(x,r)\right)^{-\nu/2} \|f\|_1^{\nu} \| |\nabla f| \|_2,\ \forall\, f \in c_0(B(x,r)),\ \forall r \geq 1/2. \qquad (N)$$

Indeed, (N) may be written

$$\frac{C^{-2}}{r^2} \left(V(x,r)\right)^{\nu} \left(\frac{\|f\|_2^2}{\|f\|_1^2} \right)^{\nu} \leq \frac{\| |\nabla f| \|_2^2}{\|f\|_2^2}.$$

Now, if f is supported in Ω,

$$\frac{\|f\|_2^2}{\|f\|_1^2} \geq \frac{1}{|\Omega|}$$

and one gets (FK) by taking the infimum in the right hand side.

In fact, (N) and (FK) turn out to be equivalent, and also equivalent to the Sobolev type inequality

$$\|f\|_q \leq Cr \left(V(x,r)\right)^{-\nu/2} \| |\nabla f| \|_2,\ \forall\, f \in c_0(B(x,r)),\ \forall r \geq 1/2, \qquad (S)$$

where $q = \frac{2}{1-\nu}$ (one can always take $\nu < 1$). That (S) implies (N) follows from Hölder:

$$\|f\|_2^{\nu+1} \leq \|f\|_q \|f\|_1^{\nu}.$$

The implication from (FK) to (S) follows from truncation techniques: see the series of papers [5], [23], [8], [1], and also [20, p.31]. We shall not need this fact here.

Proposition 2.1 *(FK) implies*

$$\frac{V(x,r)}{V(x,s)} \leq C(a,\nu)\left(\frac{r}{s}\right)^{2/\nu}, \ \forall r \geq s > 0, \tag{VR}$$

and in particular (D) with the constant b depending only on a and ν.

Proof: The following argument is adapted from [5], p.222. Note first that if $r < 1/2$ then (VR) is obvious. Suppose from now on that $r \geq 1/2$. Take $\Omega = B(x,s)$, $s \leq r$, and apply (FK). One gets

$$\frac{a}{r^2}\left(\frac{V(x,r)}{V(x,s)}\right)^{\nu} \leq \lambda_1(B(x,s)). \tag{2.1}$$

For $r = 1$ and $s \in]0,1[$, (2.1) yields

$$a\left(\frac{V(x,1)}{m(x)}\right)^{\nu} \leq \lambda_1(\{x\}) = 1.$$

Thus

$$V(x,1) \leq C\, m(x), \ \forall x \in \Gamma. \tag{2.2}$$

Note that (2.2) gives

$$m(y) \leq C\, m(x), \ \forall x, y \in \Gamma, x \sim y, \tag{2.3}$$

and that (2.3) and (2.2) together imply that there exists $N \in \mathbb{N}^*$ such that any $x \in \Gamma$ has at most N neighbours (one says that Γ is locally uniformly finite).

Let us come back to (2.1). One has

$$\lambda_1(B(x,s)) \leq \frac{\||\nabla f|\|_2^2}{\|f\|_2^2},$$

where $f(y) = (s - d(x,y))_+$. Since $|\nabla f|$ is zero outside $B(x,s+1)$, $|\nabla f|(x) \leq 1$ if $x \in B(x,s+1)$, and $f \geq s/2$ in $B(x,s/2)$, one can estimate the right hand side by

$$\frac{4V(x,s+1)}{s^2 V(x,s/2)}.$$

Now thanks to (2.2), $V(x,s+1) \leq C\, V(x,s)$, where C only depends on a and ν.

This yields

$$\frac{a}{r^2}\left(\frac{V(x,r)}{V(x,s)}\right)^{\nu} \leq \frac{4CV(x,s)}{s^2V(x,s/2)}$$

that is

$$V(x,s) \geq \left(\frac{s^2 a_r}{4C}\right)^{\frac{1}{1+\nu}} (V(x,s/2))^{\frac{1}{1+\nu}}, \tag{2.4}$$

where $a_r = \frac{a}{r^2}V(x,r)^{\nu}$. Let us replace s by $s/2^i$ in (2.4) and iterate. One gets

$$V(x,s) \geq \left(\frac{s^2 a_r}{4C}\right)^{\sum_{i=1}^{j}\frac{1}{(1+\nu)^i}} \left(V(x,s/2^j)\right)^{\frac{1}{(1+\nu)^j}}, \tag{2.5}$$

Next observe that $V(x,s/2^j) \geq m(x)$ and that $\sum_{i=1}^{+\infty}\frac{1}{(1+\nu)^i} = 1/\nu$. Therefore letting j go to infinity in (2.5) gives

$$V(x,s) \geq \left(\frac{s^2 a_r}{4C}\right)^{1/\nu},$$

which is nothing but (VR) with $C(a,\nu) = (4C/a)^{1/\nu}$. ■

Remark: In view of Proposition 2.1, we can state Theorem 1.1 in the following way: assuming a priori (D), the relative Faber-Krahn inequality (FK) is equivalent to each of the estimates (UE), (DUE) and implies (DLE).

Let us gather now some consequences of (D) itself.

Lemma 2.2 *Assume that (D) holds on Γ, then:*

1. *For all $r \geq s > 0$, for all $x \in \Gamma$ and $y \in B(x,r)$*

$$\frac{V(x,r)}{V(y,s)} \leq C(b)\left(\frac{r}{s}\right)^{\theta} \tag{2.6}$$

where $\theta > 0$ depends on the constant b in (D). In particular, for any $y \in B(x,r)$

$$\frac{V(x,r)}{m(y)} \leq C(b)\, r^{\theta}. \tag{2.7}$$

2. *For a large enough constant $C' = C'(b)$, for all $x \in \Gamma$ and for all $r \geq 1$*

$$V(x, C'r) \geq 2V(x,r). \tag{2.8}$$

Remark: The inequality (2.8) is opposite to the doubling property, and at first sight it might look wrong. Indeed, if the graph Γ is *finite* then it is not true. The proof uses essentially that Γ is infinite. A similar property for the continuous setting is well known [26], [22].

Proof of Lemma 2.2. 1. If $y = x$ then (2.6) follows by iterating (D) $\lceil \log_2 \frac{r}{s} \rceil$ times (where $\lceil \cdot \rceil$ is the ceiling function, i.e. $\lceil x \rceil$ is the smallest integer greater than x). It yields the exponent $\theta = \log_2 b$. If $y \neq x$ then we have

$$\frac{V(x,r)}{V(y,s)} \leq \frac{V(y,r+s)}{V(y,s)} \leq \text{const} \left(\frac{r+s}{s}\right)^{\theta} \leq \text{const}\, 2^{\theta} \left(\frac{r}{s}\right)^{\theta}.$$

The inequality (2.7) follows from (2.6) by letting $s = \frac{1}{2}$.

2. It is sufficient to prove (2.8) for integer r. Since the graph Γ is infinite and connected, there is a point $y \in \Gamma$ such that $d(x,y) = 3r$ (see fig. 1).

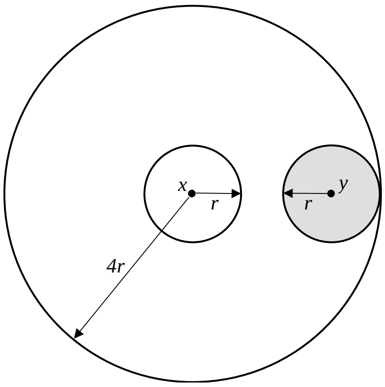

Figure 1: Choosing point y

Since $y \in B(x,3r)$, we have by (2.6)

$$V(y,r) \geq \varepsilon V(x,3r)$$

for a (small) positive constant ε. Therefore, we obtain

$$V(x,4r) \geq V(x,r) + V(y,r) \geq (1+\varepsilon)\, V(x,r).$$

By iterating this inequality sufficiently many times, we get (2.8). ■

We are now going to check that (FK) follows from (D) and (P). In the setting of Riemannian manifolds, this has been proved in [23]. A shorter proof has been given in [36]. This is the one we shall follow here. Note that the inequality stated in Theorem 2.1 of the latter reference is nothing but the analogue of (S), with an additional term in the right hand side that can be disposed of by applying (P) once again. In our setting, the scheme of [36] has been worked out in [20]. Again, an additional term is obtained first, that one gets rid of by Poincaré. We would like to point out that one does not really need (P) to get the inequality in the desired form. In fact, the inequality with an additional term self improves as soon as the graph is infinite (or the manifold is non-compact), by using Lemma 2.2. As a consequence, one can formulate a slightly stronger statement, namely that (FK) follows from (D) and an integrated form of (P) called a pseudo-Poincaré inequality:

$$\|f - f_r\|_2 \leq Cr \||\nabla f|\|_2,\ \forall f \in c_0(\Gamma),\ \forall r > 0. \tag{PP}$$

Notice that even if at first sight they look similar, (PP) is substantially different from (P): the summations are taken on the whole Γ and not on a ball of radius r, and the argument in f_r is not fixed, but moves together with the one in f.

It is well-known that (P) and (D) imply (PP) (see [36, Lemma 2.4], [20, Lemme 4.2]). Property (PP) is somewhat more difficult to handle but sometimes easier to prove than (P) (see [14]).

Proposition 2.3 *(D) and (PP) imply (FK).*

Proof: The first step is to prove

$$\|f\|_2^{\nu+1} \leq C \frac{r}{(V(x,r))^{\nu/2}} \|f\|_1^\nu \left(\||\nabla f|\|_2 + r^{-1}\|f\|_2 \right), \forall f \in c_0(B(x,r)), \forall r > 0 \tag{2.9}$$

for some $\nu > 0$, as in [36] (or [20], but there (P) is used).

Fix $r > 0$. Let f be supported in $B(x,r)$ and $s > 0$. Write

$$\|f\|_2 \leq \|f - f_s\|_2 + \|f_s\|_2.$$

If $s \leq r$, one checks easily using (2.6) that

$$\|f_s\|_2 \leq (V(x,r))^{-1/2} \left(\frac{r}{s}\right)^{\theta/2} C\|f\|_1,$$

([36, Lemma 2.3], [20, Lemme 4.1]), therefore, by (PP),

$$\|f\|_2 \leq C \left(s\||\nabla f|\|_2 + (V(x,r))^{-1/2} \left(\frac{r}{s}\right)^{\theta/2} \|f\|_1 \right),$$

if $0 < s \leq r$, and for all $s > 0$,

$$\|f\|_2 \leq C \left(s \left(\||\nabla f|\|_2 + r^{-1}\|f\|_2\right) + (V(x,r))^{-1/2} \left(\frac{r}{s}\right)^{\theta/2} \|f\|_1 \right),$$

Next choose s such that

$$s \left(\||\nabla f|\|_2 + r^{-1}\|f\|_2\right) = (V(x,r))^{-1/2} \left(\frac{r}{s}\right)^{\theta/2} \|f\|_1,$$

i.e.

$$s = r^{\frac{\theta}{2+\theta}} (V(x,r))^{-\frac{1}{2+\theta}} \|f\|_1^{\frac{2}{2+\theta}} \left(\||\nabla f|\|_2 + r^{-1}\|f\|_2\right)^{\frac{-2}{2+\theta}}.$$

One gets

$$\|f\|_2 \leq 2Cr^{\frac{\theta}{2+\theta}} (V(x,r))^{-\frac{1}{2+\theta}} \|f\|_1^{\frac{2}{2+\theta}} \left(\||\nabla f|\|_2 + r^{-1}\|f\|_2\right)^{\frac{\theta}{2+\theta}}$$

which is (2.9) for $\nu = 2/\theta$.

Let $A > 1$ be a number to be chosen later. Since $c_0(B(x,r)) \subset c_0(B(x,Ar))$, one can also write

$$\|f\|_2^{\nu+1} \leq \frac{CAr}{(V(x,Ar))^{\nu/2}} \|f\|_1^{\nu} \left(\||\nabla f|\|_2 + A^{-1}r^{-1}\|f\|_2\right), \ \forall r > 0, \forall f \in c_0(B(x,r)). \tag{2.10}$$

Since f is supported in $B(x,r)$,

$$\|f\|_1 \leq (V(x,r))^{1/2} \|f\|_2,$$

and (2.10) gives

$$\|f\|_2^{\nu+1} \leq \frac{CAr}{(V(x,Ar))^{\nu/2}} \|f\|_1^{\nu} \||\nabla f|\|_2 + C (V(x,Ar))^{-\nu/2} (V(x,r))^{\nu/2} \|f\|_2^{\nu+1}. \tag{2.11}$$

Now, by iterating (2.8), one can choose A so large that

$$C (V(x,Ar))^{-\nu/2} (V(x,r))^{\nu/2} \leq 1/2, \ \forall r \geq 1/2.$$

Therefore (2.11) implies

$$\|f\|_2^{\nu+1} \leq 2CAr (V(x,Ar))^{-\nu/2} \|f\|_1^{\nu} \||\nabla f|\|_2, \ \forall r \geq 1/2, \forall f \in c_0(B(x,r)). \tag{2.12}$$

One deduces easily (N) therefore (FK). ■

The above discussion shows that the class of weighted graphs satisfying (FK) contains the class of those satisfying (D) and (P). In fact, the latter is strictly larger. For example, two copies of $\mathbb{Z}^2$ joined by a single edge (take $\mu \equiv 1$) satisfy (D) and (FK) but not (P), and one can easily imagine how to generalise this construction, by gluing a finite number of graphs satisfying (D) and (FK), with in addition the same volume growth.

It is also clear that the weighted graphs obtained by discretisation from manifolds with locally bounded geometry and satisfying continuous analogues of (D) and (FK) also satisfy such conditions (see the techniques in [7]).

2.1 Relative isoperimetric inequalities

This section can be skipped in a first reading, because it will not be used in the rest of the article. It makes the connection between relative Faber-Krahn inequalities and relative isoperimetric inequalities, which are more obviously geometric.

If Ω is a subset of Γ, define its boundary $\partial\Omega$ by

$$\partial\Omega = \{x \in \Omega; \ \exists y \in \Gamma \setminus \Omega, \ y \sim x\}.$$

Define

$$|\partial\Omega|_s = \sum_{x \in \Omega, y \notin \Omega} \mu_{xy}.$$

We shall say that Γ satisfies a relative isoperimetric inequality if there exists $a' > 0$, $\nu > 0$ such that, for every $x \in \Gamma$, $r \geq 1/2$, and for every non-empty finite subset Ω of Γ contained in $B(x, r)$,

$$\frac{|\partial\Omega|_s}{|\Omega|} \geq \frac{a'}{r}\left(\frac{V(x,r)}{|\Omega|}\right)^{\nu/2}. \tag{I}$$

Note that if one consider the Cheeger's constant

$$h(\Omega) = \inf_{\omega\subset\Omega} \frac{|\partial\omega|_s}{|\omega|},$$

then (I) may be reformulated as

$$h(\Omega) \geq \frac{a'}{r}\left(\frac{V(x,r)}{|\Omega|}\right)^{\nu/2}.$$

Now Cheeger's inequality says that

$$\lambda_1(\Omega) \geq \frac{h^2(\Omega)}{4},$$

therefore (I) implies (FK), but is likely to be stronger (see [11] for a related discussion).

One says that Γ satisfies the L^1 Poincaré inequality if there exists $C > 0$ and $C' \geq 1$ such that

$$\sum_{y\in B(x,r)} |f(y) - f_r(x)|m(y) \leq Cr \sum_{y\in B(x,C'r)} |\nabla f|(y)m(y),\ \forall f \in \mathbb{R}^\Gamma,\ \forall r > 0. \tag{P_1}$$

(P_1) is strictly stronger than (P) (see [28], 6.19 for a continuous example that can be made discrete using the techniques in [16]). The combinatorial conditions given in [14], §5 imply (P_1). Together with (D), (P_1) implies the L^1 pseudo-Poincaré inequality:

$$\|f - f_r\|_1 \leq Cr\||\nabla f|\|_1,\ \forall f \in c_0(\Gamma),\ \forall r > 0 \tag{PP_1}$$

([14] §3, lemme). We shall prove:

Proposition 2.4 *(D) and (PP_1) imply (I).*

This result is somewhat implicit in [36]. Our proof uses the method of [14].

Proof: It follows from (D) and Lemma 2.2, 1 that, if $f \in c_0(B(x,r))$ and $s \leq r$,

$$\|f_s\|_\infty \leq \frac{b}{V(x,s)}\|f\|_1 \leq C'\left(\frac{r}{s}\right)^\theta \frac{\|f\|_1}{V(x,r)}. \tag{2.13}$$

¿From Lemma 2.2, 2, one can choose $A \geq 1$ so large that

$$\frac{V(x,r)}{V(x,Ar)} \leq \frac{1}{4C'},\ \forall r \geq 1/2.$$

Let $f \in c_0(B(x,r)) \setminus \{0\}$ and $s > 0$. Write

$$\begin{aligned} m(\{|f| \geq \|f\|_\infty/2\}) &\leq m(\{|f - f_s| \geq \|f\|_\infty/4\}) + m(\{|f_s| > \|f\|_\infty/4\}) \\ &\leq \frac{4}{\|f\|_\infty}\|f - f_s\|_1 + m(\{|f_s| > \frac{\|f\|_\infty}{4}\}). \end{aligned}$$

Now apply (PP_1); one gets

$$m(\{|f| \geq \|f\|_\infty/2\}) \leq \frac{Cs}{\|f\|_\infty}\||\nabla f|\|_1 + m(\{|f_s| > \frac{\|f\|_\infty}{4}\}) \tag{2.14}$$

Set

$$C'\left(\frac{Ar}{s_0}\right)^\theta \frac{\|f\|_1}{V(x,Ar)} = \frac{\|f\|_\infty}{4},$$

i.e.

$$s_0 = Ar\left(\frac{4C'\|f\|_1}{V(x,Ar)\|f\|_\infty}\right)^{1/\theta}.$$

Now $\|f\|_1 \leq \|f\|_\infty V(x,r)$, therefore owing to the choice of A, $s_0 \leq Ar$, and one can apply (2.13) in $B(x,Ar)$ to get

$$\|f_{s_0}\|_\infty \leq C'\left(\frac{Ar}{s_0}\right)^\theta \frac{\|f\|_1}{V(x,Ar)} = \|f\|_\infty/4$$

therefore

$$m(\{|f_{s_0}| > \|f\|_\infty/4\}) = 0.$$

Now apply (2.14):

$$m(\{|f| \geq \|f\|_\infty/2\}) \leq \frac{Cs_0}{\|f\|_\infty}\||\nabla f|\|_1. \tag{2.15}$$

Replacing s_0 by its value, we get

$$m(\{|f| \geq \|f\|_\infty/2\}) \leq \frac{C''r}{\|f\|_\infty^{1+\frac{1}{\theta}}}\left(\frac{\|f\|_1}{V(x,Ar)}\right)^{1/\theta}\||\nabla f|\|_1.$$

Taking $f = 1_\Omega$, one has $\|f\|_\infty = 1$,

$$m(\{|f| \geq \|f\|_\infty/2\}) = \|f\|_1 = |\Omega|,$$

and using the fact that

$$\||\nabla 1_\Omega|\|_1 \leq C|\partial\Omega|_s,$$

one gets (I) with $\nu = 2/\theta$. ■

3 Calculus on graphs

We have collected here the definitions and some useful properties of the discrete differentiation and integration as well as some spectral properties of the Laplace operator. They all are so elementary that it is easier to write them down than to refer the reader elsewhere.

3.1 Definitions and rules

Let f be a function on $\mathbb{N}\times\Gamma$ or on Γ. Depending on the context, we may abbreviate $f(k,x)$ to $f_k(x)$, f_k, $f(x)$ or even f.

1. Gradient
$$\nabla_{xy} f := f(y) - f(x)$$
and the "time derivative"
$$\partial_k f(x) := f(k+1,x) - f(k,x)\,.$$

2. Differentiation of a product:
$$\nabla_{xy}(fg) = (\nabla_{xy} f)g(y) + (\nabla_{xy} g)f(x)$$

3. Differentiation of a square
$$\nabla_{xy} f^2 = 2(\nabla_{xy} f)f(x) + (\nabla_{xy} f)^2.$$

4. The same formulas for the "time derivatives":
$$\partial_k\,(fg) = (\partial_k f)\,g_{k+1} + (\partial_k g)\,f_k$$
and
$$\partial_k\left(f^2\right) = 2\,(\partial_k f)\,f_k + (\partial_k f)^2\,.$$

5. Laplace operator:
$$\Delta f(x) := \sum_{y\in\Gamma} f(y)p(x,y) - f(x) = \sum_{y\in\Gamma} p(x,y)\nabla_{xy} f = \frac{1}{m(x)}\sum_{y\in\Gamma}(\nabla_{xy} f)\mu_{xy}\,.$$

6. Integration by parts: if one of the functions f, g on Γ has a finite support then
$$\sum_{x\in\Gamma} \Delta f(x)g(x)m(x) = -\frac{1}{2}\sum_{x,y\in\Gamma}(\nabla_{xy} f)\,(\nabla_{xy} g)\,\mu_{xy}. \tag{3.16}$$
(one half is no misprint - it appears because the summation on the right hand side is done twice over each edge).

3.2 Subsolutions

A function u on $\mathbb{N} \times \Gamma$ is called *a subsolution* of the heat equation if it satisfies the inequality

$$\partial_k u \leq \Delta u.$$

This can be rewritten as

$$u_{k+1}(x) \leq \sum_y p(x,y) u_k(y) = (P u_k)(x). \tag{3.17}$$

The following simple property of subsolutions will be frequently used.

Lemma 3.1 *If u is a subsolution and φ is a convex function on $\mathbb{R}$ then $\varphi(u)$ is also a subsolution.*

Indeed, by (3.17) and convexity of φ, we have

$$\varphi(u_{k+1}(x)) \leq \sum_y p(x,y) \varphi(u_k(y)), \tag{3.18}$$

and $\varphi(u)$ is a subsolution, too.

3.3 Spectrum of the Laplace operator

The first Dirichlet eigenvalue $\lambda_1(\Omega)$ in a finite set $\Omega \subset \Gamma$, can be defined in two equivalent ways:

1. the variational definition used in the previous section

$$\lambda_1(\Omega) := \inf_{f \in c_0(\Omega)} \frac{-\sum_x f \Delta f m}{\sum_x f^2 m} = \inf_{f \in c_0(\Omega)} \frac{\frac{1}{2}\sum_{x,y} |\nabla_{xy} f|^2 \mu_{xy}}{\sum_x f^2(x) m(x)}; \tag{3.19}$$

2. $\lambda_1(\Omega)$ is the smallest eigenvalue of the operator $-\Delta_\Omega$ where Δ_Ω is the restriction of Δ to $c_0(\Omega)$.

Let us note that the operator Δ_Ω is finitely dimensional and self-adjoint with respect to the inner product $(f,g) = \sum_x f(x) g(x) m(x)$. In particular, the spectrum of Δ_Ω is real.

Let us denote by $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ the maximum and the minimum eigenvalues of an operator A. The following are elementary properties of eigenvalues on graphs (see also [6]).

Lemma 3.2 *For any finite non-empty set $\Omega \subset \Gamma$*

$$0 \leq \lambda_{\min}(-\Delta_\Omega) \leq 1 \tag{3.20}$$

and

$$\lambda_{\max}(-\Delta_\Omega) + \lambda_{\min}(-\Delta_\Omega) \leq 2. \tag{3.21}$$

Proof. Non-negativeness of $\lambda_{\min}$ is seen from (3.19). It follows from the explicit formula for Δ_Ω

$$-\Delta_\Omega f(x) = f(x) - \sum_y p(x,y) f(y) \tag{3.22}$$

and from $p(x,y) \geq 0$ that

$$\operatorname{tr}(-\Delta_\Omega) \leq \#\Omega \tag{3.23}$$

whence

$$\lambda_{\min}(-\Delta_\Omega) \leq \frac{\operatorname{tr}(-\Delta_\Omega)}{\#\Omega} \leq 1.$$

To prove (3.21), let us denote by f an eigenfunction of $\lambda_{\max}(-\Delta_\Omega)$ that is $f \in c_0(\Omega)$, $f \not\equiv 0$ and

$$\lambda_{\max}(-\Delta_\Omega) = \frac{\frac{1}{2}\sum_{x,y}(\nabla_{xy} f)^2 \mu_{xy}}{\sum_x f^2(x) m(x)}.$$

Then by (3.19) applied to the function $|f|$, we have

$$\lambda_{\min}(-\Delta_\Omega) \leq \frac{\frac{1}{2}\sum_{x,y}(\nabla_{xy}|f|)^2 \mu_{xy}}{\sum_x f^2(x) m(x)}.$$

Since

$$(\nabla_{xy} f)^2 + (\nabla_{xy}|f|)^2 = [f(x) - f(y)]^2 + [|f(x)| - |f(y)|]^2 \leq 2\left[f^2(x) + f(y)^2\right]$$

then

$$\lambda_{\max}(-\Delta_\Omega) + \lambda_{\min}(-\Delta_\Omega) \leq \frac{\sum_{x,y}[f^2(x) + f(y)^2]\mu_{xy}}{\sum_x f^2(x) m(x)} = 2.$$

■

There is a simple connection between the spectra of Δ_Ω and P_Ω where P_Ω is the restriction of the Markov operator P to $c_0(\Omega)$. Indeed, (3.22) can be rewritten as $-\Delta_\Omega = Id - P_\Omega$ which implies

$$\operatorname{spec}(-\Delta_\Omega) = 1 - \operatorname{spec}(P_\Omega). \tag{3.24}$$

Corollary 3.3 *We have*

$$0 \leq \lambda_{\max}(P_\Omega) \leq 1$$

and

$$-\lambda_{\max}(P_\Omega) \leq \lambda_{\min}(P_\Omega) \leq \lambda_{\max}(P_\Omega).$$

4 Mean value inequality

Theorem 4.1 *Let the graph* Γ *satisfy* (FK)*. Let* $u(k,x)$ *be a non-negative subsolution on* $\mathbb{N}\times\Gamma$*, then for any* $z\in\Gamma$ *and* $T,R\in\mathbb{N}^*$

$$u(T,z)\leq\frac{\mathrm{const}_{a,\nu}\vartheta(T/R^2)}{2TV(z,R)}\sum_{k=0}^{2T}\sum_{x\in B(z,R)}u(k,x)m(x) \qquad (ML^1)$$

where

$$\vartheta\left(s\right):=\max\left(s,s^{-1/\nu}\right). \qquad (4.1)$$

Remark: We assume for simplicity that the subsolution u is defined in $\mathbb{N}\times\Gamma$. However, the theorem remains true if u is defined in $[0,2T]\times B(z,R)$ and the inequality (3.17) is satisfied for all $k\in[0,2T-1]$ and $x\in B(z,R-1)$. In this case, we have to assume (FK) only for balls $B(x,r)\subset B(z,R)$.

Let us note that $2TV(z,R)$ can be interpreted as the volume of the cylinder $\Psi:=[0,2T]\times B(z,R)$. Therefore, the right hand side of (ML^1) contains the arithmetic mean of u over this cylinder. If $T=R^2$ then we can rewrite (ML^1) in the form

$$u(T,z)\leq\frac{\mathrm{const}}{m\left(\Psi\right)}\sum_{k,x\in\Psi}u(k,x)m(x) \qquad (4.2)$$

which will only be used[3].

The proof of Theorem 4.1 is long and consists of several stages. The strategy is similar to [22, Theorem 3.1]. The idea of using the level sets was borrowed from [19] and [29, Theorem 6.1, p.102]. However, the discreteness of the time variable brings serious complications which can be overcome by using the new technical tools provided.

Let us briefly describe the scheme before going into detail. First, we will show that the hypothesis (α) may be assumed. This is done by switching to the kernel $p'=p_2$ for which the hypothesis (α) is implied by the doubling property (D). We will prove that (FK) is inherited by p', and once we have obtained the mean value property for p', we will transfer it back to p. Therefore, we may assume (α) from the first.

Second, we will prove a Cacciopoli type inequality which is, very roughly, the result of multiplying the heat equation by a cut-off function with subsequent integration by parts and certain estimates based on the Cauchy-Schwarz inequality. In the continuous time setting, this is well known and almost trivial. With a discrete time, the usual continuous tricks do not work! The crucial point which eventually makes everything work, is the hypothesis (α). As far as we know, the discrete time Cacciopoli inequality was not known before.

[3] Let us note that (ML^1) follows easily from its particular case (4.2) - see the end of the proof of Theorem 4.1.

The third step in the proof is the comparison of L^2-norms of the subsolutions in two cylinders $\Psi' \subset \Psi$ (see Fig. 2 below). This is the only place where the Faber-Krahn inequality is used directly. The input of (FK) is a function with a support in a given ball. This function will be $f = (u-\theta)_+ \varphi$ where φ is a standard cut-off function. Roughly speaking, the Cacciopoli inequality provides the lower bound of $\|f\|_2$ in terms of $\|\nabla f\|_2$ whereas the Faber-Krahn inequality (FK) yields the inequality in the opposite direction. The combination of both implies the comparison of the L^2-norms of the subsolutions in different cylinders. The crucial point in this scheme is a simple estimate of the measure of support of f by Chebyshev's inequality

$$m\,(\operatorname{supp} f) \leq \theta^{-2} \|u\|_2^2 . \tag{4.3}$$

The set $\operatorname{supp} f$ is used as Ω in (FK), and the inequality (4.3) results in an additional power of the L^2-norm of the solution in the smaller cylinder.

The fourth step consists of iterating the inequality of the previous step in the following way. We consider a shrinking sequence of cylinders $\{\Psi_n\}$ (see Fig. 5 below) and the numerical sequence I_n of the L^2-norms of $(u-\theta_n)_+$ in Ψ_n, where $\theta_n > 0$ is an increasing sequence in $[\theta, 2\theta]$. The previous step yields the upper bound of I_n via $I_{n-1}^{1+\nu}$, where the exponent $1+\nu$ appears due to the Faber-Krahn inequality and (4.3). By iterating this inequality sufficiently many times and by choosing θ properly, we obtain the L^2-version of (4.2)- the inequality (ML^2) below.

The final step is to pass from the L^2-mean value inequality to the L^1-version. This is done also by means of a certain iteration process the idea of which we have borrowed from [30].

4.1 L^2-mean value inequality

The L^2-analogue of (ML^1) is the following inequality for a non-negative subsolution u :

$$u^2(T,z) \leq \frac{\mathrm{const}_{a,\nu}\vartheta(T/R^2)}{2TV(z,R)} \sum_{k=0}^{2T} \sum_{x\in B(z,R)} u^2(k,x)m(x) \tag{ML^2}$$

The major part of the proof of Theorem 4.1 consists of proving (ML^2).

We start with the trivial situation when $T \leq K$ or $R \leq K$ where K may be chosen to be any fixed number. For technical reason we will take $K = 16$. If $R \leq K$ then we note that by the doubling property, the measure $V(z,R)$ is of order $m(z)$ whereas the term $\vartheta(T/R^2)$ is of order T. Therefore, (ML^2) takes the form

$$u^2(T,z) \leq \frac{\mathrm{const}_{a,\nu}}{m\,(z)} \sum_{k=0}^{2T} \sum_{x\in B(z,R)} u^2(k,x)m(x) \tag{4.4}$$

which is trivially true if we restrict summation on the right hand side to the single point T,z.

Let now $T \le K$, then we have by (4.1) $\vartheta(T/R^2) \sim R^{2/\nu}$. By the comparison of volumes of the balls,

$$V(z,R) \le \mathrm{const} R^{2/\nu} m(z).$$

Therefore, (ML^2) amounts to (4.4) again.

Thus, we may assume in the sequel that $T > K$ and $R > K$.

4.2 Hypothesis (α)

Let us emphasize that we do not assume the hypothesis (α) in Theorem 4.1. However, we cannot proceed without (α). We first prove

Lemma 4.2 *The doubling property (D) on Γ implies that the kernel $p'(x,y) := p_2(x,y)$ satisfies (α).*

This was basically proved in [21], but the argument is very simple, and we reproduce it here for the sake of completeness. Let us first note that for any neighbours $x, y \in \Gamma$

$$m(y) \le bm(x) \tag{4.5}$$

where b is the constant from (D). Indeed, we have by (D)

$$m(y) \le V(x,1) \le bV(x, \frac{1}{2}) = bm(x).$$

Of course, (4.5) also means that $m(x) \le bm(y)$.

Let N_x denote the number of neighbours of x, then (4.5) implies

$$N_x \le \frac{V(x,1)}{\min_{y\sim x} m(y)} \le b^2$$

so that the graph Γ is locally uniformly finite.

Finally, we have

$$\begin{aligned} p'(x,x) &= p_2(x,x) = \sum_{y\sim x} p(x,y)p(y,x) \\ &= \sum_{y\sim x} \frac{\mu_{xy}^2}{m(x)m(y)} \ge b^{-1} \sum_{y\sim x} \frac{\mu_{xy}^2}{m^2(x)} \\ &\ge b^{-1} N_x^{-1} \left(\sum_{y\sim x} \frac{\mu_{xy}}{m(x)} \right)^2 = b^{-1}N_x^{-1} \ge b^{-3}. \end{aligned}$$

Therefore, p' satisfies (α) with $\alpha = b^{-3}$. ∎

Next we will prove (ML^2) for the kernel p' and then pass to p. To do that carefully, let us introduce the graph Γ' which coincides with Γ as a set of vertices, and $x \sim y$ in Γ' if and only if $d(x,y) \le 2$ in Γ. We endow Γ' with the measure $m' = m$ and with the kernel $p' = p_2$. Obviously, p' is a nearest neighbourhood Markov kernel on Γ', reversible with respect to the measure m'. We will mark by an apostrophe all notation which relates to the graph Γ' as opposed to those on Γ.

Lemma 4.3 *Let Ω be a finite subset of Γ and Ω_0 be the set of the interior points of* Ω (that is, those points from Ω which have no neighbours in Γ outside Ω). *Assume that Ω_0 is non-empty, then*

$$\lambda_1'(\Omega_0) \geq \lambda_1(\Omega) \,. \tag{4.6}$$

Remark: Let us emphasize that no assumptions on graph Γ or on the kernel P are required for (4.6) except for the fact that P is Markov, reversible and is of a nearest neighbourhood.

Proof. In view of (3.24), the inequality (4.6) is equivalent to

$$\lambda_{\max}\left(P'_{\Omega_0}\right) \leq \lambda_{\max}(P_\Omega) \,.$$

Observe that $P'_{\Omega_0} = P^2|_{\Omega_0} = (P_\Omega)^2|_{\Omega_0}$ (we cannot write $P^2|_{\Omega_0} = (P_{\Omega_0})^2$ because of the influence of the boundary) and that restricting the operator P^2_Ω to Ω_0 can only diminish its $\lambda_{\max}$. Hence, we have

$$\lambda_{\max}\left(P'_{\Omega_0}\right) \leq \lambda_{\max}\left(P^2_\Omega\right) \,,$$

and (4.6) will follow from

$$\lambda_{\max}\left(P^2_\Omega\right) \leq \lambda_{\max}(P_\Omega) \,. \tag{4.7}$$

To verify (4.7), we note that

$$\lambda_{\max}\left(P^2_\Omega\right) = \max\left\{\lambda^2_{\max}(P_\Omega)\,, \lambda^2_{\min}(P_\Omega)\right\}$$

and thus, (4.7) reduces to two inequalities

$$\begin{aligned} \lambda^2_{\max}(P_\Omega) &\leq \lambda_{\max}(P_\Omega) \\ \lambda^2_{\min}(P_\Omega) &\leq \lambda_{\max}(P_\Omega) \end{aligned}$$

which follow from $\lambda_{\max}(P_\Omega) \in [0,1]$ and $\lambda_{\min}(P_\Omega) \in [-\lambda_{\max}(P_\Omega)\,, \lambda_{\max}(P_\Omega)]$ (see Corollary 3.3). ■

Now we are ready to prove

Proposition 4.4 *We have:*

1. *The doubling property (D) on Γ implies that on Γ'.*

2. *The relative Faber-Krahn inequality (FK) on Γ implies that on Γ'.*

3. *The mean value inequality (ML^2) on Γ' implies that on Γ.*

The first statement is a simple consequence of the fact that the distances and the measures on Γ and Γ' are finitely proportional.

The second statement is basically a consequence of Lemma 4.3 and of the fact that (FK) implies the doubling property. Indeed, given a finite non-empty set $\Omega \subset B'(x,n)$ with $n \in \mathbb{N}^*$ we denote by $\widetilde{\Omega}$ its 1-neighbourhood in Γ and observe that $\widetilde{\Omega} \subset B(x, 2n+1) \subset B(x, 3n)$. We obtain by Lemma 4.3 and (FK)

$$\lambda_1'(\Omega) \geq \lambda_1\left(\widetilde{\Omega}\right) \geq \frac{a}{(3n)^2}\left(\frac{V(x,3n)}{m\left(\widetilde{\Omega}\right)}\right)^{\nu} \geq \frac{ab^{-\nu}}{9n^2}\left(\frac{V'(x,n)}{m'(\Omega)}\right)^{\nu},$$

where b is the constant from the doubling property (D) which we have applied in the last inequality:

$$m'\left(\widetilde{\Omega}\right) = m\left(\widetilde{\Omega}\right) \leq \sum_{x\in\Omega} V(x,1) \leq b\sum_{x\in\Omega} m(x) = bm'(\Omega).$$

Let us prove the third statement of Proposition 4.4. Given a Γ-subsolution u, let us define $w(k,x) = u(2k,x)$. Obviously, w is a Γ'-subsolution and hence, the mean-value inequality (ML^2) holds for w. In terms of u, it means that (ML^2) is satisfied for any *even* T and any *even* R (moreover, the summation on the right hand side of (ML^2) is assumed for even k's which we extend at once to all k's).

Extension to odd R is straightforward.. Let T be odd. Then $T-1$ is even, and we apply the mean value inequality (ML^2) in the cylinder $[0, T-1] \times B(y, R-1)$. Since $T \sim T-1$ and $R \sim R-1$ (as was explained above, we may assume T and R to be large enough), we can write

$$u(T-1,y) \leq \frac{\mathrm{const}\vartheta\left(T/R^2\right)}{2TV(y,R-1)} \sum_{k=0}^{2T} \sum_{x\in B(y,R-1)} u^2(k,x)m(x).$$

Now, we have

$$\begin{aligned}
u(T,z) &\leq \sum_{y\sim z} p(z,y)u(T-1,y) \\
&\leq \max_{y\sim z} u(T-1,y) \\
&\leq \frac{\mathrm{const}\vartheta\left(T/R^2\right)}{2TV(z,R)} \sum_{k=0}^{2T} \sum_{x\in B(z,R)} u^2(k,x)m(x)
\end{aligned}$$

which was to be proved. ∎

Proposition (4.4) justifies the following strategy of proof of (ML^2). We will proceed further assuming (FK) and (α) (the latter will be used in the next subsection in a derivation of a Cacciopoli inequality). After we have proved (ML^2) for this setting, we argue as follows: Let now Γ satisfy only (FK). Since (FK) implies (D), then by Lemma 4.2 and Proposition 4.4, the graph Γ' possesses both (FK) and (α). We conclude that Γ' possesses (ML^2), and by Proposition 4.4, so does the graph Γ.

4.3 Discrete Cacciopoli inequality

The following Proposition requires no assumption on Γ except for (α) which we assume henceforth. Its continuous analogues are well known in various forms; see, for example, [33], [26], [22] (Lemma 3.1).

Proposition 4.5 *There exist $c, A > 0$ depending on α such that, for every non-negative subsolution f and any function φ with finite support*

$$\sum \partial_k(f^2)\varphi^2 m + c\sum_{x,y} |\nabla_{xy}(f\varphi)|^2\mu_{xy} \leq A\sum_{x,y} |\nabla_{xy}\varphi|^2 f^2\mu_{xy}. \tag{4.8}$$

Proof: Let us start with a lemma.

Lemma 4.6 *For every non-negative subsolution f,*

$$\partial_k(f^2) \leq 2(\Delta f)f + (\Delta f)^2. \tag{4.9}$$

Proof of the lemma: The claimed inequality can be rewritten as

$$f_{k+1}^2 \leq f_k^2 + 2(\Delta f)f_k + (\Delta f)^2.$$

Now

$$f_k^2 + 2(\Delta f)f_k + (\Delta f)^2 = (f_k + \Delta f)^2 = (Pf)^2,$$

therefore, since f is non-negative, what we have to prove is simply

$$f_{k+1} \leq Pf,$$

which is true because f is a subsolution. ∎

Lemma 4.6 gives

$$\sum \partial_k(f^2)\varphi^2 m \leq 2\sum(\Delta f)f\varphi^2 m + \sum(\Delta f)^2\varphi^2 m. \tag{4.10}$$

We apply the integration by parts formula (3.16) to the first term on the right-hand side:

$$2\sum(\Delta f)f\varphi^2 m = -\sum_{x,y} \nabla_{xy}f\nabla_{xy}(f\varphi^2)\mu_{xy}.$$

The second term can be estimated in the following way:

$$\begin{aligned}
\sum(\Delta f)^2\varphi^2 m &= \sum_x \left(\sum_{y\neq x}(\nabla_{xy}f)p(x,y)\right)^2 \varphi^2(x)m(x) \\
&\leq \sum_x \left(\sum_{y\neq x} p(x,y)\right)\left(\sum_y (\nabla_{xy}f)^2 p(x,y)\right)\varphi^2(x)m(x) \\
&\leq (1-\alpha)\sum_{x,y} |\nabla_{xy}f|^2\varphi^2\mu_{xy}
\end{aligned}$$

where we used the hypothesis (α) that is $p(x,x) \geq \alpha$ and, hence,

$$\sum_{y \neq x} p(x,y) \leq 1 - \alpha.$$

This crucial trick comes from [21], §1.5.

Thus, we obtain from (4.10)

$$\sum \partial_k (f^2)\varphi^2 m \leq -\sum_{x,y} \nabla_{xy} f \nabla_{xy}(f\varphi^2)\mu_{xy} + (1-\alpha)\sum_{x,y} |\nabla_{xy} f|^2 \varphi^2 \mu_{xy},$$

and Proposition 4.5 will follow if we prove

$$\begin{aligned} & c\sum_{x,y} |\nabla_{xy}(f\varphi)|^2 \mu_{xy} + (1-\alpha)\sum_{x,y} |\nabla_{xy} f|^2 \varphi^2 \mu_{xy} \\ \leq \; & \sum_{x,y} (\nabla_{xy} f)\, \nabla_{xy}(f\varphi^2)\mu_{xy} + A\sum_{x,y} |\nabla_{xy}\varphi|^2 f^2 \mu_{xy}. \end{aligned} \tag{4.11}$$

We estimate the first term on the left hand side as follows:

$$|\nabla_{xy}(f\varphi)|^2 = ((\nabla_{xy} f)\varphi(y) + (\nabla_{xy}\varphi)f(x))^2 \leq 2(|\nabla_{xy} f|^2\varphi^2(y) + |\nabla_{xy}\varphi|^2 f^2(x)).$$

By interchanging x and y in the summation, we see that

$$\sum_{x,y} |\nabla_{xy} f|^2 \varphi^2(x)\mu_{xy} = \sum_{x,y} |\nabla_{xy} f|^2 \varphi^2(y)\mu_{xy}.$$

Thus, (4.11) amounts to

$$c'\sum_{x,y} |\nabla_{xy} f|^2 \varphi^2(y)\mu_{xy} \leq \sum_{x,y} \nabla_{xy} f \nabla_{xy}(f\varphi^2)\mu_{xy} + A'\sum_{x,y} |\nabla_{xy}\varphi|^2 f^2(x)\mu_{xy}, \tag{4.12}$$

where $c' = 2c + 1 - \alpha$ and $A' = A - 2c$; note that one can ensure $c' < 1$ and $A' > 0$ by choosing $c < \alpha/2$ and $A > \alpha$.

Next we have

$$\begin{aligned} \nabla_{xy}(f\varphi^2) &= (\nabla_{xy} f)\varphi^2(y) + (\nabla_{xy}\varphi^2)f(x) \\ &= (\nabla_{xy} f)\varphi^2(y) + 2\,(\nabla_{xy}\varphi)\,f(x)\varphi(x) + |\nabla_{xy}\varphi|^2 f(x) \end{aligned}$$

and (4.12) transforms to

$$\begin{aligned} 0 \;\leq\; & (1-c')\sum_{x,y} |\nabla_{xy} f|^2\varphi^2(x)\mu_{xy} + 2\sum_{x,y} (\nabla_{xy} f)(\nabla_{xy}\varphi)f(x)\varphi(x)\mu_{xy} \\ & + \sum_{x,y} \nabla_{xy} f |\nabla_{xy}\varphi|^2 f(x)\mu_{xy} + A'\sum_{x,y} |\nabla_{xy}\varphi|^2 f^2(x)\mu_{xy}. \end{aligned}$$

For a large enough A, we have the quadratic inequality

$$(1-c')|\nabla_{xy} f|^2\varphi^2(x) + 2(\nabla_{xy} f)(\nabla_{xy}\varphi)f(x)\varphi(x) + (A'-2)|\nabla_{xy}\varphi|^2 f^2(x) \geq 0,$$

so that we are left to show that

$$0 \leq \sum_{x,y} (\nabla_{xy} f) \, |\nabla_{xy}\varphi|^2 f(x)\mu_{xy} + 2\sum_{x,y} |\nabla_{xy}\varphi|^2 f^2(x)\mu_{xy}. \tag{4.13}$$

By interchanging x and y in the sum, we have

$$\begin{aligned} -\sum_{x,y} \nabla_{xy} f |\nabla_{xy}\varphi|^2 f(x)\mu_{xy} &= -\frac{1}{2}\sum_{x,y} \left((\nabla_{xy} f) f(x) + (\nabla_{yx} f) f(y)\right) |\nabla_{xy}\varphi|^2 \mu_{xy} \\ &= \frac{1}{2}\sum_{x,y} (f(x) - f(y))^2 \, |\nabla_{xy}\varphi|^2 \mu_{xy} \\ &\leq \sum_{x,y} \left(f^2(x) + f^2(y)\right) |\nabla_{xy}\varphi|^2 \mu_{xy} \\ &= 2\sum_{x,y} |\nabla_{xy}\varphi|^2 f^2(x)\mu_{xy} \end{aligned}$$

which coincides with (4.13). The proposition is proved. ■

Corollary 4.7 *Let $\eta(k,x)$ be a function on $\mathbb{N} \times \Gamma$ such that*

1. *for all $k \in \mathbb{N}$, η_k is supported by a finite set Ω (that is, it vanishes outside Ω);*
2. *$\eta(0,x) = 0$;*
3. *for some constant M*

$$|\nabla_{xy}\eta|^2 \leq M \quad \text{and} \quad \partial_k(\eta^2) \leq M. \tag{4.14}$$

Then for any non-negative subsolution $f(k,x)$ and $\tau \in \mathbb{N}$ we have

$$\sum_{x\in\Omega} f^2(\tau,x)\,\eta^2(\tau,x)\,m(x) + c\sum_{k=0}^{\tau-1}\sum_{x,y\in\Gamma} |\nabla_{xy}(f\eta)|^2 \mu_{xy} \leq 2AM\sum_{k=0}^{\tau}\sum_{x\in\widetilde{\Omega}} f^2(x)m(x) \tag{4.15}$$

where $\widetilde{\Omega}$ is the 1-neighbourhood of Ω, and c, A are the constants from Proposition 4.5.

Proof. Indeed, let us apply (4.8) for $\varphi(x) = \eta(k,x)$ and sum up over all $k = 0, \ldots \tau - 1$. We obtain

$$\sum_{k=0}^{\tau-1}\sum_{x\in\Gamma} \partial_k(f^2)\eta^2 m + c\sum_{k=0}^{\tau-1}\sum_{x,y} |\nabla_{xy}(f\eta)|^2 \mu_{xy} \leq A\sum_{k=0}^{\tau-1}\sum_{x,y} |\nabla_{xy}\eta|^2 f^2 \mu_{xy}. \tag{4.16}$$

Let us observe next that

$$\partial_k \left(f^2\right)\eta^2 = \partial_k\left(f^2\eta^2\right) - \partial_k\left(\eta^2\right) f_{k+1}^2$$

whence

$$\begin{aligned}\sum_{k=0}^{\tau-1}\sum_x \partial_k(f^2)\eta^2 m &= \sum_{k=0}^{\tau-1}\sum_x \left\{\partial_k\left(f^2\eta^2\right)-\partial_k\left(\eta^2\right)f_{k+1}^2\right\}m \\ &= \sum_x f^2(\tau,x)\eta^2(\tau,x)m(x)-\sum_{k=0}^{\tau-1}\sum_x \partial_k\left(\eta^2\right)f_{k+1}^2 m.\end{aligned}$$

Hence, we rewrite (4.16) as

$$\begin{aligned}&\sum_{x\in\Omega} f^2(\tau,x)\eta^2(\tau,x)m(x)+c\sum_{k=0}^{\tau-1}\sum_{x,y}|\nabla_{xy}(f\eta)|^2\mu_{xy} \\ \leq\ &\sum_{k=0}^{\tau-1}\sum_{x\in\Gamma}\partial_k\left(\eta^2\right)f_{k+1}^2 m + A\sum_{k=0}^{\tau-1}\sum_{x,y}|\nabla_{xy}\eta|^2 f^2\mu_{xy}. \end{aligned} \tag{4.17}$$

In the first term in (4.17), we just replace $\partial_k\left(\eta^2\right)$ by M and the domain of summation by $[0,\tau]\times\Omega$. In the second term, note that for any $x\notin\widetilde{\Omega}$, we have $|\nabla_{xy}\eta|\,\mu_{xy}=0$ (indeed, either $y\in\Omega$ and $\mu_{xy}=0$ or $y\notin\Omega$ and $\nabla_{xy}\eta=0$). Therefore, we can restrict the summation to $x\in\widetilde{\Omega}$. Next, we use $|\nabla_{xy}\eta|^2\leq M$ and $\sum_y\mu_{xy}=m(x)$, and obtain finally (4.15). ■

4.4 Comparison of L^2 norms of subsolutions in different cylinders

Given the integers t,t',r,r', and the point $z\in\Gamma$, let us introduce two cylinders in $\mathbb{N}\times\Gamma$

$$\Psi=[0,t]\times B(z,r),\quad \Psi'=[t',t-1]\times B(x,r')$$

We assume that $1\leq t'<t$ and $1\leq r'<r-1$. Let v be a non-negative subsolution. The purpose of this part of the proof is to compare the following sums

$$I:=\sum_{k,x\in\Psi}v^2(k,x)m(x)$$

and

$$I':=\sum_{k,x\in\Psi'}\left(v(k,x)-\theta\right)_+^2 m(x)$$

where $\theta>0$ is a constant.

Obviously, we have $I'\leq I$. However, we will need to know that I' is essentially smaller than I. This may be achieved by varying two parameters: θ and

$$D:=\min\left(\left(r-r'\right)^2,t'\right). \tag{4.18}$$

In the lemma below, it is essential to expose dependence of the estimate on the parameters θ,D. Also, this lemma is the only point where we apply the Faber-Krahn

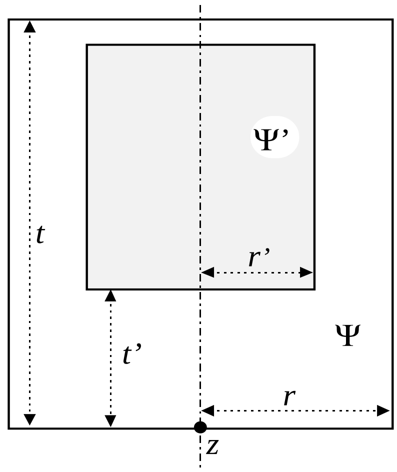

Figure 2: Cylinders Ψ and Ψ'

inequality (FK). Actually, the statement does not depend on the particular form of (FK), and to emphasize that (and to simplify notation) let us assume instead of (FK) that for any set $\Omega \subset B(z,r)$

$$\lambda_1(\Omega) \geq \Lambda(m(\Omega)) \tag{4.19}$$

with a decreasing positive function Λ. The inequality (FK) is a particular case of (4.19) with the function

$$\Lambda(\xi) = \frac{a}{r^2} V(z,r)^{\nu} \xi^{-\nu}.$$

Lemma 4.8 *Under the above assumptions, we have the inequality*

$$I' \leq \frac{C'I}{D\Lambda\left(\frac{C'I}{D\theta^2}\right)} \tag{4.20}$$

where $C' = C'(\alpha)$ is a large constant.

Proof. We will first apply the inequality (4.15) to $f = v$ and to the function η of the following form:

$$\eta(k,x) = \eta_1(x)\eta_2(k). \tag{4.21}$$

Let $r'' := \lceil (r + r')/2 \rceil$ and define the functions η_1, η_2 as follows

$$\eta_1(x) := \begin{cases} 1, & \text{if} \quad d(x,z) \leq r'' \\ 0, & \text{if} \quad d(x,z) \geq r \\ \frac{r - d(x,z)}{r - r''}, & \text{otherwise} \end{cases}.$$

and

$$\eta_2(k) := \begin{cases} 1, & \text{if} \quad k \geq t' \\ \frac{k}{t'}, & \text{if} \quad k < t' \end{cases}. \tag{4.22}$$

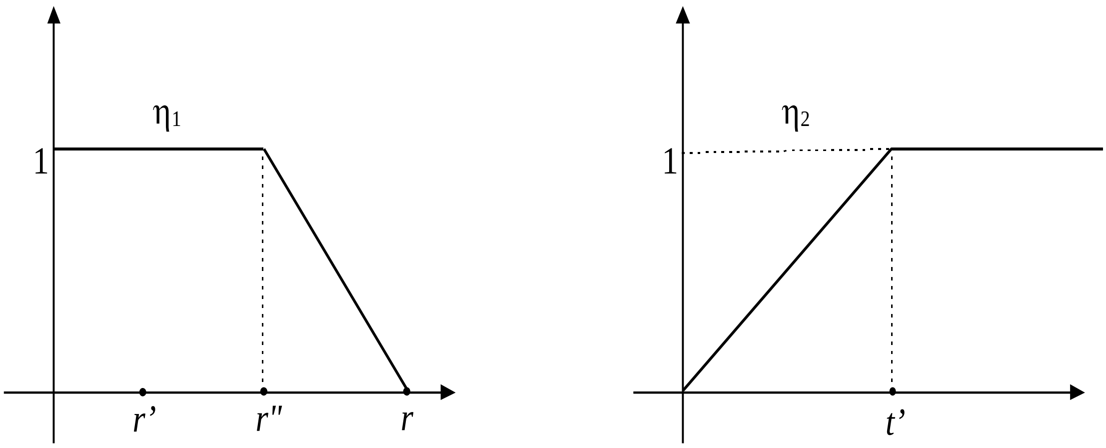

Figure 3: Functions η_1 and η_2

Let us note that

$$\left|\nabla_{xy}\eta_1\right| \le \frac{1}{r-r''} \le \frac{3}{r-r'}$$

and

$$\partial_k\left(\eta_2^2\right) = 2\left(\partial_k\eta_2\right)\eta_2 + \left(\partial_k\eta_2\right)^2 \le \frac{2}{t'} + \frac{1}{t'^2} \le \frac{3}{t'}.$$

Therefore, we obtain

$$\left|\nabla_{xy}\eta\right|^2 \le 9D^{-1} \quad \text{and} \quad \partial_k\left(\eta^2\right) \le 3D^{-1}$$

where D is defined by (4.18), and we can apply (4.15) with $M = 9D^{-1}$ and $\Omega = B(z, r-1)$.

This time we need to estimate by (4.15) only the first term on the left hand side of (4.15). So, we have for any $\tau \in [0, t]$

$$\sum_{x\in B(z,r-1)} v^2(\tau,x)\,\eta^2(\tau,x)\,m(x) \le 18AD^{-1}\sum_{k=0}^{\tau}\sum_{B(z,r)} v^2 m$$

or, since $\eta = 1$ in $B(z, r'')$,

$$\sum_{x\in B(z,r'')} v^2(\tau,x)\,m(x) \le 18AD^{-1}I. \tag{4.23}$$

Now we apply (4.15) once again but for a different set of functions. Namely, let $f = w := (v-\theta)_+$ and define η again by (4.21) with the same η_2 from (4.22) but with the slightly different η_1 :

$$\eta_1(x) := \begin{cases} 1, & \text{if } \ d(x,z) \le r' \\ 0, & \text{if } \ d(x,z) \ge r'' \\ \frac{r'-d(x,z)}{r'-r''}, & \text{otherwise} \end{cases}.$$

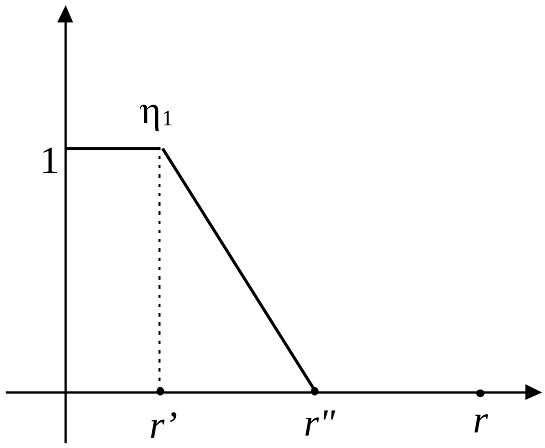

Figure 4: Function η_1

This time we will estimate the second term on the right-hand side of (4.15). We also take $\tau = t$, $\Omega = B(z, r'' - 1)$ and use the same M as above:

$$\sum_{k=0}^{t-1} \sum_{x,y} |\nabla_{xy}(w\eta)|^2 \mu_{xy} \leq 18c^{-1}AD^{-1} \sum_{k=0}^{t} \sum_{x \in B(z,r'')} w^2(x)m(x).$$

Since $w \leq v$ then we have

$$\sum_{k=0}^{t-1} \sum_{x,y} |\nabla_{xy}(w\eta)|^2 \mu_{xy} \leq 18c^{-1}AD^{-1}I. \tag{4.24}$$

Now we will estimate from below the sum in (4.24) by using the Faber-Krahn inequality (4.19). To that end, let us denote for any $k \in \mathbb{N}$

$$\Omega_k := \{x \in B(z, r'') \mid w(k,x) > 0\}.$$

Obviously, the function ηw_k is supported by Ω_k. Therefore, by (4.19)

$$\sum_{x,y} |\nabla_{xy}(w\eta)|^2 \mu_{xy} \geq \Lambda\left(m\left(\Omega_k\right)\right) \sum_{x} |w\eta|^2 m. \tag{4.25}$$

On the other hand, we can estimate $m\left(\Omega_k\right)$ by (4.23) and by Chebyshev's inequality. Indeed, since $v > \theta$ on Ω_k then

$$m\left(\Omega_k\right) \leq \theta^{-2} \sum_{x \in B(z,r'')} v^2\left(\tau, x\right) m\left(x\right) \leq 18\theta^{-2}AD^{-1}I. \tag{4.26}$$

By substituting this estimate into (4.25), we see that

$$\sum_{x,y} |\nabla_{xy}(w\eta)|^2 \mu_{xy} \geq \Lambda\left(\frac{18AI}{\theta^2 D}\right) \sum_{x} |w\eta|^2 m.$$

Let us sum up this inequality for k from t' to $t-1$ and note that for k in this range and for $x \in B(z, r')$, we have $\eta(k, x) = 1$:

$$\sum_{k=t'}^{t-1} \sum_{x,y} |\nabla_{xy}(w\eta)|^2 \mu_{xy} \geq \Lambda\left(\frac{18AI}{\theta^2 D}\right) \sum_{k=t'}^{t-1} \sum_{x \in B(z,r')} w^2 m = \Lambda\left(\frac{18AI}{\theta^2 D}\right) I'.$$

By comparison with (4.24), we conclude

$$\Lambda\left(\frac{18AI}{\theta^2 D}\right) I' \leq 18c^{-1}AD^{-1}I.$$

whence (4.20) follows. ■

4.5 Proof of L^2-mean value inequality

We will prove here (ML^2) for the main case when R and T are large, say

$$R > 16 \quad \text{and} \quad T > 16. \tag{4.27}$$

Let us introduce the sequence of the cylinders $\Psi_n = [T - T_n, 2T - n] \times B(z, R_n)$ where $\{R_n\}, \{T_n\}$ are strictly decreasing sequences of positive integers satisfying so far the assumptions: $R_0 = R$, $T_0 = T$ and

$$R_n < R_{n-1} - 1, \quad 2T - n > T - T_n \quad (\Leftrightarrow T_n > n - T). \tag{4.28}$$

Clearly, the number of such cylinders is finite, we will specify it later. Let us define

$$D_n = \min\left(T_{n-1} - T_n, \, (R_{n-1} - R_n)^2\right).$$

Let us fix some $\theta > 0$ and introduce another sequence

$$\theta_n = \theta\left(2 - 2^{-n}\right)$$

so that $\theta_0 = \theta$ and $\theta_n \uparrow 2\theta$.

Given a non-negative subsolution u, let us define the sums

$$I_n := \sum_{k,x \in \Psi_n} (u(k,x) - \theta_n)_+^2 .$$

We would like to apply (4.20) to compare I_{n-1} and I_n. By Lemma 3.1, the function $v := (u - \theta_{n-1})_+$ is a subsolution, and we can apply (4.20) to compare the L^2-norm of v in Ψ_{n-1} to that of $(u - \theta_n)_+ = (v - (\theta_n - \theta_{n-1}))_+$ in Ψ_n. Finally, the function Λ can be taken as

$$\Lambda(\xi) = \beta \xi^{-\nu}$$

where

$$\beta = \frac{a}{R^2} V(z, R)^\nu \tag{4.29}$$

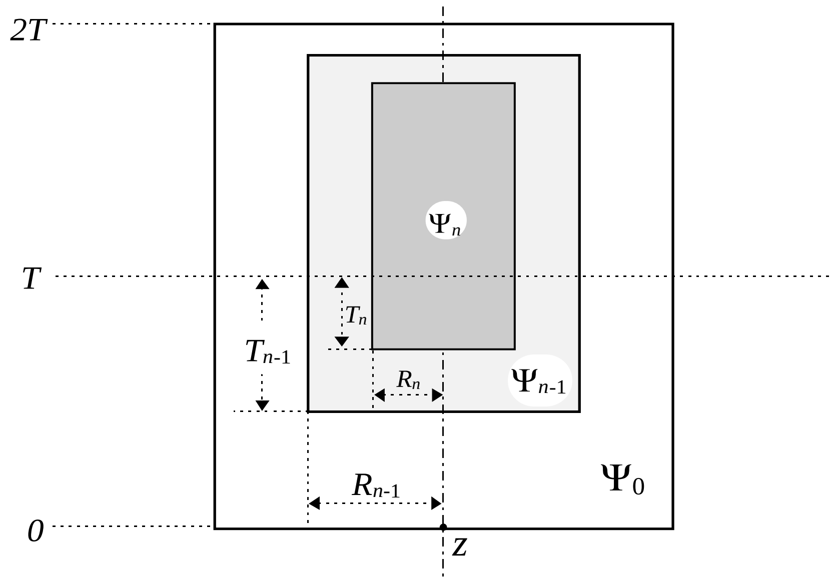

Figure 5: Sequence of cylinders Ψ_n

will be treated so far as a constant.

So, we have by (4.20)

$$I_n \leq \frac{C' I_{n-1}}{\beta D_n \left(\frac{C' I_{n-1}}{D_n(\theta_n - \theta_{n-1})^2}\right)^{-\nu}}$$

or

$$I_n \leq \frac{\mathrm{const}\, I_{n-1}^{1+\nu}}{\beta D_n^{1+\nu} 4^{-\nu n} \theta^{2\nu}} \tag{4.30}$$

where we denote by const any constant depending only on ν.

At this point we have to choose $\{R_n\}, \{T_n\}$ and, thus, $\{D_n\}$. Let us define them inductively: $R_0 = R$, $T_0 = T$ and

$$R_n = \lceil R_{n-1}/2 \rceil \quad \text{and} \quad T_n = \lceil T_{n-1}/2 \rceil . \tag{4.31}$$

Let us define the number N as the maximal integer satisfying the requirements

$$R_N > 2, \qquad T_N > 2 \quad \text{and} \quad N < T \tag{4.32}$$

Such a number does exist, for example, $N = 1$ satisfies (4.32) (because $R > 16$ and $T > 16$). The conditions (4.32) guarantee in particular that the sequences $\{R_n\}$ and $\{T_n\}$ are strictly decreasing up to $n = N$, and (4.28) holds for $n \leq N$. Hence, we have (4.30) for all $n \leq N$.

Let us estimate N and the gaps D_n. Let us denote

$$D := \min(T, R^2).$$

By (4.31), we have

$$R_n \geq 2^{-n}R \quad \text{and} \quad T_n \geq 2^{-n}T.$$

Since $N+1$ does not satisfy (4.32) then

$$R \leq 2^{N+2} \quad \text{or} \quad T \leq 2^{N+2} \quad \text{or } T \leq N$$

which implies

$$N \geq \min\left(T,\, \log_2 T,\, \log_2 R\right) - 2 \geq \log_2 D - 2. \tag{4.33}$$

Next, for any $n = 1, 2, ...N$, we have

$$R_{n-1} - R_n = R_{n-1} - \lceil R_{n-1}/2 \rceil \geq \frac{1}{4} R_{n-1}$$

(the latter follows from $R_{n-1} \geq R_{N-1} \geq 4$) and similarly $T_{n-1} - T_n \geq \frac{1}{4} T_{n-1}$. Therefore

$$D_n = \min\left(T_{n-1} - T_n,\, (R_{n-1} - R_n)^2\right) \geq \frac{1}{16} \min\left(T_{n-1}, R_{n-1}^2\right) \geq 4^{-n-1} \min\left(T, R^2\right)$$

or

$$D_n \geq 4^{-n-1} D. \tag{4.34}$$

By applying (4.34), we can now rewrite (4.30) as follows

$$I_n \leq \frac{\exp\left(bn\right) I_{n-1}^{\gamma}}{\beta D^{\gamma} \theta^{2\nu}} \tag{4.35}$$

where $\gamma = 1 + \nu$ and $b = \log\left(4^{\nu} 4^{1+\nu}\right)$. For simplicity, we have absorbed the constant const into β. By iterating (4.35), we obtain

$$I_N \leq \frac{\exp\left(b \sum_{i=1}^{N} i \gamma^{N-i}\right) I^{\gamma^N}}{\left(\beta D^{\gamma} \theta^{2\nu}\right)^{1+\gamma+\gamma^2+...+\gamma^{N-1}}}$$

or

$$\begin{aligned} I_N^{\gamma^{-N}} &\leq \frac{\exp\left(b \sum_{i=1}^{\infty} i \gamma^{-i}\right) I}{\left(\beta D^{1+\nu} \theta^{2\nu}\right)^{\nu^{-1}(1-\gamma^{-N})}} \\ &\leq \frac{\mathrm{const} I}{\beta^{\frac{1}{\nu}} D^{\frac{1}{\nu}+1} \theta^{2(1-\gamma^{-N})}} \beta^{\nu^{-1}\gamma^{-N}} D^{\left(1+\nu^{-1}\right)\gamma^{-N}}. \end{aligned} \tag{4.36}$$

Let us first estimate the last two factors from (4.36). Let us recall that (FK) implies

$$\frac{V(z, R)}{m(z)} \leq \mathrm{const}\, R^{2/\nu}$$

whence we obtain by (4.29)

$$\beta \le \frac{\text{const}}{R^2}\left(R^{2/\nu} m(z)\right)^{\nu} = \text{const}\, m(z)^{\nu}.$$

Therefore, we conclude

$$\beta^{\nu^{-1}\gamma^{-N}} \le \text{const}\, m(z)^{\gamma^{-N}}. \tag{4.37}$$

To estimate the second factor, we note that by (4.33)

$$\gamma^N = \exp\left(N\log\gamma\right) \ge \text{const}\exp\left(\frac{1}{2}\log\gamma\log_2 D\right) = \text{const}\, D^{\varepsilon}$$

with $\varepsilon = \frac{1}{2}\log_2\gamma > 0$. So, we have

$$D^{\left(1+\nu^{-1}\right)\gamma^{-N}} \le \exp\left(\frac{\left(1+\nu^{-1}\right)\log D}{\text{const}\, D^{\varepsilon}}\right) \le \text{const}.$$

We rewrite then (4.36) as follows

$$I_N^{\gamma^{-N}} \le \frac{\text{const} I}{\beta^{\frac{1}{\nu}} D^{\frac{1}{\nu}+1}\theta^{2(1-\gamma^{-N})}} m(z)^{\gamma^{-N}}. \tag{4.38}$$

Since $T \in [T - T_N, 2T - N]$ by (4.32) and $\theta_N \le 2\theta$, we have obviously

$$I_N \ge \left(u(T,z) - 2\theta\right)_+^2 m(z).$$

By substituting this into (4.38), we see that both $m(z)$ cancel, and we obtain

$$\left(u(T,z) - 2\theta\right)_+^{2\gamma^{-N}} \theta^{2\left(1-\gamma^{-N}\right)} \le \frac{\text{const} I}{\beta^{\frac{1}{\nu}} D^{\frac{1}{\nu}+1}}. \tag{4.39}$$

Finally, we take $\theta = \frac{1}{3}u(T,z)$ and observe another miracle that both exponents γ^{-N} cancel in (4.39). After substituting the value of β from (4.29) and $D = \min(T, R^2)$ we obtain

$$\begin{aligned} u(T,z)^2 &\le \frac{\text{const}_{a,\nu} R^{2/\nu}}{V(z,R)\min\left(T,R^2\right)^{1+1/\nu}} \sum_{k,x\in\Psi} u^2 \\ &= \frac{\text{const}_{a,\nu}}{m\left(\Psi\right) T^{-1} \min\left(T^{1+1/\nu}R^{-2/\nu}, R^2\right)} \sum_{k,x\in\Psi} u^2 \end{aligned}$$

which together with definition (4.1) of the function ϑ, implies (ML^2).

4.6 From L^2 to L^1 mean value inequality

The last part of the proof of Theorem 4.1 consists of iterating in a certain manner the L^2-mean value inequality (ML^2) in order to obtain the L^1-version. The idea of this method is borrowed from [30] where it was applied in the context of Riemannian manifolds for the elliptic mean value property. However, its implementation for the discrete situation is technically more involved. The fact that the L^2(and L^p with $p > 1$)-mean value inequality implies the L^1 analogue, is a very general property which is apparently valid in the setting of metric measure spaces. To emphasise that and to use convenient notation, we introduce the space $X = \mathbb{Z} \times \Gamma$ and will interpret the time cylinders on Γ as balls in X.

We will denote pairs $(k, x) \in X$ by single letters, say, ξ, η etc. For any $r \geq 0$ and any point $\xi = (k; x) \in X$, we denote by $\Psi(\xi, r)$ the cylinder $[k - r^2, k + r^2] \times B(x, r)$ which will be referred to as a "ball" in X. It is easy to check that if $\eta \in \Psi(\xi, r)$ then $\Psi(\eta, r') \subset \Psi(\xi, r + r')$(these "balls" are actually metric balls of a parabolic distance on X).

We also extend the measure m to X by setting $m(\xi) = m(x)$. Let us observe that the volume regularity property is inherited by X in the form

$$\frac{m(\Psi(\xi, R))}{m(\Psi(\xi, r))} \leq \text{const} \left(\frac{R}{r}\right)^{\sigma}, \quad \forall R \geq r > 0 \tag{4.40}$$

where $\sigma = 2 + 2/\nu$.

Let us denote by X^+ the product $\mathbb{N} \times \Gamma$, and let u be a non-negative subsolution on X^+. By the L^2-mean value property (ML^2) , we have for any "ball" $\Psi(\xi, r) \subset X^+$

$$u^2(\xi) \leq \frac{\text{const}}{m(\Psi)} \sum_{\eta \in \Psi} u^2(\eta)\, m(\eta). \tag{4.41}$$

We shall prove that for any "ball" $\Psi(\zeta, R) \subset X^+$ with an integer radius R

$$u(\zeta) \leq \frac{\text{const}}{m(\Psi)} \sum_{\eta \in \Psi} u(\eta)\, m(\eta). \tag{4.42}$$

After that, we will prove (ML^1) for an arbitrary cylinder and thus will complete the proof of Theorem 4.1.

Let us introduce the sequence of increasing concentric "balls"

$$\Psi_n = \Psi(\zeta, R - r_n)$$

where $r_0 = \lfloor R/2 \rfloor$ and $r_n = \lfloor r_{n-1}/2 \rfloor$ for $n \geq 1$. We continue this sequence while $r_n \geq 1$. Let N be the maximal n with this property. Since $r_n \geq r_{n-1}/4$ and thus $r_n \geq R/4^{n+1}$, we have clearly

$$N \geq \log_4 R - 2. \tag{4.43}$$

For any point $\xi \in \Psi_{n-1}$, $n \leq N$, let us consider the ball $\Psi(\xi, r_n) \subset \Psi_n$ (the inclusion follows from $r_n \leq \frac{1}{2} r_{n-1}$).

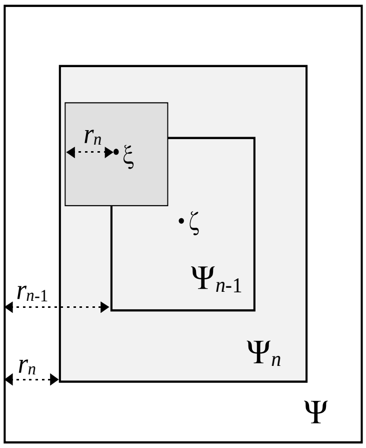

Figure 6: "Balls" Ψ_n

We have by (4.41)

$$
\begin{aligned}
u^2(\xi) &\leq \frac{\text{const}}{m(\Psi(\xi, r_n))} \sum_{\eta \in \Psi(\xi, r_n)} u^2(\eta)\, m(\eta) \\
&\leq \max u(\Psi_n) \frac{\text{const}}{m(\Psi(\xi, r_n))} \sum_{\eta \in \Psi_n} u(\eta)\, m(\eta) \\
&\leq \max u(\Psi_n) \frac{\text{const} 3^{\sigma n}}{m(\Psi)} \sum_{\eta \in \Psi} u(\eta)\, m(\eta) \qquad (4.44)
\end{aligned}
$$

where we have used in addition the volume regularity property:

$$
\frac{m(\Psi)}{m(\Psi(\xi, r_n))} \leq \frac{m(\Psi(\xi, 2R))}{m(\Psi(\xi, r_n))} \leq \text{const} \left(\frac{2R}{r_n}\right)^{\sigma} \leq \text{const}\, 4^{\sigma n}.
$$

Let us set $M_n := \max u(\Psi_n)$ and

$$
A := \frac{\text{const}}{m(\Psi)} \sum_{\eta \in \Psi} u(\eta)\, m(\eta).
$$

We conclude from (4.44) that for $b = \log 4^{\sigma}$ and $n = 1, 2, ...N$

$$
M_{n-1}^2 \leq \exp(nb)\, A M_n.
$$

By iterating this inequality, we obtain

$$
M_0^{2^N} \leq \exp\left(b \sum_{i=1}^{N} i 2^{N-i}\right) A^{1+2+2^2+...2^{N-1}} M_N
$$

and

$$
u(\zeta) \leq M_0 \leq \text{const}\, A^{1-2^{-N}} M_N^{2^{-N}} = \text{const}\, A \left(\frac{M_N}{A}\right)^{2^{-N}}. \qquad (4.45)
$$

Let us finally show that

$$\left(\frac{M_N}{A}\right)^{2^{-N}} \leq \text{const}\,.$$

Let ξ_0 be the point in Ψ_N where u attains its maximum so that $M_N = u(\xi_0)$. Then obviously

$$u(\xi_0)\, m(\xi_0) \leq \sum_{\eta\in\Psi} u(\eta)\, m(\eta) = \text{const } m(\Psi)\, A.$$

Therefore, by using the volume regularity (4.40)

$$M_N \leq \frac{\text{const}\, m(\Psi)\, A}{m(\xi_0)} \leq \text{const}\, R^{\sigma} A.$$

By combining this with the estimate (4.43) for N in the form $2^N \geq \text{const}\, R^{1/2}$, we see that

$$\left(\frac{M_N}{A}\right)^{2^{-N}} \leq \text{const } \left(R^{\sigma}\right)^{2^{-N}} = \text{const } \exp\left(\frac{\text{const }\log R}{R^{1/2}}\right) \leq \text{const}$$

which concludes the proof of (4.42).

We are left to extend the L^1-mean value property from the "balls" $\Psi(\zeta, R)$ to arbitrary cylinders as in (ML^1). Given the cylinder $\Psi := [0, 2T] \times B(z, R)$, let us denote $\zeta = (T, z)$ and consider the "ball" $\Psi(\zeta, r)$ with maximal integer radius r which lies in Ψ. We have by (4.42)

$$u(T, z) \leq \frac{\text{const}}{r^2 V(z, r)} \sum_{k=0}^{2T} \sum_{x\in B(z,R)} u(k, x) m(x)\,,$$

and (ML^1) will follow from

$$\frac{TV(z, R)}{r^2 V(z, r)} \leq \text{const } \max\left(\frac{T}{R^2}, \left(\frac{R^2}{T}\right)^{1/\nu}\right).$$

The latter inequality follows from the regularity of volume by considering two cases: $r = R$ and $r = \sqrt{T}$. ■

5 Upper bound

We shall state now a discrete version of a result of Davies ([17], Theorem 2), that follows from what he calls Gaffney's Lemma ([17], Lemma 1). Note that such a result holds for any reversible Markov chain with finite range (with constants depending on the range), without any further assumption.

Lemma 5.1 *There exist universal constants $C, c > 0$ such that, for all finite subsets E, F of Γ,*

$$\sum_{x\in E}\sum_{y\in F} p_k(x,y)m(x) \leq Ce^{-c\frac{d^2(E,F)}{k}}|E|^{1/2}|F|^{1/2}, \quad \forall k\in\mathbb{N}^*,$$

where $d(E,F) = \min_{x\in E, y\in F} d(x,y)$.

Proof: Let $s\in\mathbb{R}$. Consider the operator P_s on $\mathbb{R}^\Gamma$ with kernel

$$p^s(x,y) = e^{sd(x,F)}p(x,y)e^{-sd(y,F)}.$$

The iterated operator P_s^k has obviously the kernel

$$p_k^s(x,y) = e^{sd(x,F)}p_k(x,y)e^{-sd(y,F)}.$$

It is proved in [27], Lemma 2.4 (there F is reduced to a point, but the proof works in our situation), that

$$\|P_s^k\|_{2\to 2} \leq Ce^{Cs^2k}, \ \forall s>0, \ k\in\mathbb{N}^*,$$

where C does not depend on F. This is the discrete version of Gaffney's Lemma. Note that the proof in [27] goes through the comparison with a continuous time semigroup, which we avoided so far in this paper. It would be nice to have a direct proof of this fact.

Now

$$\begin{aligned}
\sum_{x\in E}\sum_{y\in F} p_k(x,y)m(x) &= \sum_{x,y\in\Gamma} 1_E(x)p_k(x,y)1_F(y)m(x)\\
&= \sum_{x,y\in\Gamma} 1_E(x)p_k(x,y)e^{-sd(y,F)}1_F(y)m(x)\\
&= \sum_{x,y\in\Gamma} 1_E(x)e^{-sd(x,F)}p_k^s(x,y)1_F(y)m(x)\\
&\leq e^{-sd(E,F)}\sum_{x,y\in\Gamma} 1_E(x)p_k^s(x,y)1_F(y)m(x)\\
&= e^{-sd(E,F)}(1_E, P_s^k 1_F)\\
&\leq Ce^{Cs^2k-sd(E,F)}\|1_E\|_2\|1_F\|_2.
\end{aligned}$$

The result follows upon choosing $s = d(E,F)/2Ck$. ■

Note that if one takes both E and F reduced to a point, the above lemma gives the following universal off-diagonal estimate for reversible Markov chains due to Varopoulos ([40], see also [4])

$$p_k(x,y) \leq C\sqrt{\frac{m(y)}{m(x)}}e^{-c\frac{d^2(x,y)}{k}}, \ \forall\, x,y\in\Gamma, \ k\in\mathbb{N}^*. \tag{5.1}$$

We are now in a position to get (UE) from Theorem 4.1.

Theorem 5.2 *(FK) implies*

$$p_k(x,y) \le \frac{Cm(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right), \quad \forall x,y \in \Gamma,\ k \in \mathbb{N}^* \tag{UE}$$

where $C, c > 0$ depend on a, ν.

Proof: It will be more convenient to work with the kernel

$$h_k(x,y) = \frac{p_k(x,y)}{m(y)}$$

because it is symmetric in x, y and satisfies the heat equation with respect to each couple (k,x), (k,y). We will prove that

$$h_k(x,y) \le \frac{\text{const}}{\sqrt{V(x,\sqrt{k})V(y,\sqrt{k})}} \exp\left(-c\frac{d^2}{k}\right) \tag{5.2}$$

where $d = d(x,y)$.

It is standard how to get rid of $V(y,\sqrt{k})$ once we know (5.2). Indeed, we have by the volume comparison condition for any $\varepsilon > 0$

$$\frac{V(x,\sqrt{k})}{V(y,\sqrt{k})} \le \frac{V(y,d+\sqrt{k})}{V(y,\sqrt{k})} \le \text{const}\left(\frac{d+\sqrt{k}}{\sqrt{k}}\right)^{2/\nu} \le \text{const}_\varepsilon \exp\left(\varepsilon\frac{d^2}{k}\right)$$

whence we can replace $V(y,\sqrt{k})$ in (5.2) by $V(x,\sqrt{k})$ at the expense of slightly decreasing c, and obtain (UE).

To prove (5.2), let us first consider the case $k < 3$ (or any other integer instead of 3). By (5.1) and (D), we obtain

$$\begin{aligned} h_k(x,y) &\le \frac{C}{\sqrt{m(x)m(y)}} \exp\left(-c\frac{d^2(x,y)}{k}\right) \\ &\le \frac{Cb^2}{\sqrt{V(x,\sqrt{k})V(y,\sqrt{k})}} \exp\left(-c\frac{d^2(x,y)}{k}\right). \end{aligned}$$

Let now $k \ge 3$. We have by (ML^1) for all $x, y \in \Gamma$

$$h_k(x,y) \le \frac{\text{const}\,\vartheta\left(l/r^2\right)}{lV(x,r)} \sum_{i=k-l}^{k+l} \sum_{x' \in B(x,r)} h_i(x',y)m(x') \tag{5.3}$$

where r and l are so far positive integers such that $3l \le k$ (here we use that $k \ge 3$), and ϑ is defined by (4.1). In the same way,

$$h_i(x',y) \le \frac{\text{const}\,\vartheta\left(l/r^2\right)}{lV(y,r)} \sum_{j=i-l}^{i+l} \sum_{y' \in B(y,r)} h_j(x',y')m(y') \tag{5.4}$$

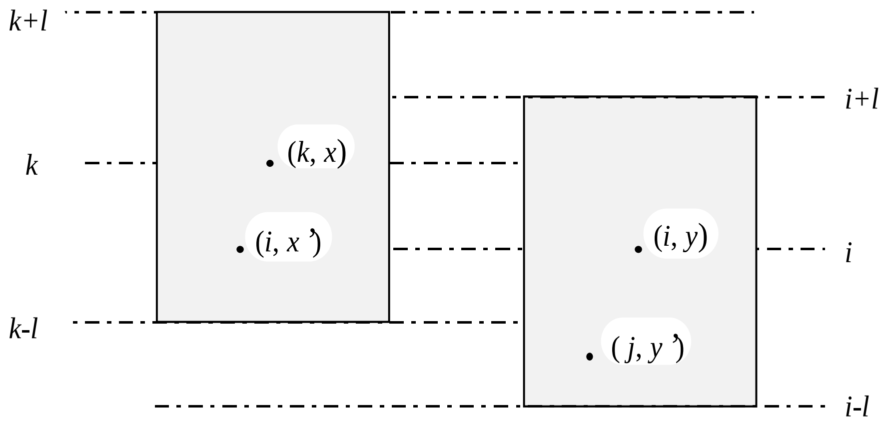

Figure 7: Cylinders $[k-l, k+l] \times B(x,r)$ and $[i-l, i+l] \times B(y,r)$

By combining (5.3) and (5.4) together and by extending the exterior summation in (5.4) to $j \in [k-2l,\, k+2l]$, we obtain

$$h_k(x,y) \leq \frac{\mathrm{const}\,\vartheta^2\,(l/r^2)}{l^2 V(x,r)V(y,r)} \sum_{i=k-l}^{k+l} \sum_{j=k-2l}^{k+2l} \sum_{x' \in B(x,r)} \sum_{y' \in B(y,r)} h_j(x',y')m(x')m(y'). \quad (5.5)$$

By Lemma 5.1, applied to $E = B(x,r)$ and $F = B(y,r)$

$$\sum_{x' \in B(x,r)} \sum_{y' \in B(y,r)} h_j(x',y')m(x')m(y') \leq \mathrm{const}\exp\left(-c\frac{(d-2r)_+^2}{j}\right) V(x,r)^{\frac{1}{2}} V(y,r)^{\frac{1}{2}} \quad (5.6)$$

where $d = d(x,y)$. Since both i and j in (5.5) vary in the range $\sim k$ and l can be taken of the order k then by changing the constant c in (5.6) we obtain

$$\begin{aligned} h_k(x,y) &\leq \frac{\mathrm{const}\,\vartheta^2\,(l/r^2)}{l^2 V(x,r)V(y,r)} l^2 \exp\left(-c\frac{(d-2r)_+^2}{k}\right) V(x,r)^{\frac{1}{2}} V(y,r)^{\frac{1}{2}} && (5.7) \\ &\leq \frac{\mathrm{const}\,\vartheta^2\,(k/r^2)}{\sqrt{V(x,r)V(y,r)}} \exp\left(-c\frac{(d-2r)_+^2}{k}\right). && (5.8) \end{aligned}$$

Finally, we have to choose r. Take $r = \left\lceil \sqrt{k} \right\rceil$, then both $V(x,r)$, $V(y,r)$ can be replaced by $V(x,\sqrt{k})$, $V(y,\sqrt{k})$, and $\vartheta^2\,(k/r^2)$ is bounded by a constant. If $d \geq 3r$ then $d - 2r \geq \frac{1}{3}d$ and we obtain (5.2) from (5.8) by changing the constant $c > 0$. If $d \leq 3r$ then we get from (5.8)

$$h_k(x,y) \leq \frac{\mathrm{const}}{\sqrt{V(x,\sqrt{k})V(y,\sqrt{k})}} \leq \frac{\mathrm{const}}{\sqrt{V(x,\sqrt{k})V(y,\sqrt{k})}} \exp\left(-c\frac{d^2}{k}\right)$$

because $d^2 \leq \text{const}\, k$. Thus, we have obtained (5.2) in all cases. ■

The above estimate yields a weak form of the law of the iterated logarithm. Let X_k be the random variable with values in Γ that is the position after k steps of the random walk governed by p and started at X_0. We can state

Corollary 5.3 *Assume that the graph Γ satisfies (FK), then for the random walk X_k one has almost surely*

$$\limsup_{k\to\infty} \frac{d(X_k, X_0)}{\sqrt{k \log\log k}} \leq C \tag{5.9}$$

for a finite constant C depending on a and ν.

Indeed, by Theorem 1.1, (FK) implies (UE) and (D) which as we have seen above, imply the estimate

$$\sum_{y\notin B(x,r)} p_k(x,y) \leq \text{const}\, e^{\frac{-r^2}{Ck}}. \tag{5.10}$$

It is well-known that (5.10) is enough to run the standard probabilistic argument and to deduce (5.9) (see [27], Theorem 9.1).

Theorem 5.4 *The doubling property (D) and the on-diagonal upper bound (DUE)*

$$p_k(x,x) \leq \frac{Cm(x)}{V(x,\sqrt{k})}, \ \ \forall x \in \Gamma, k \in \mathbb{N}^* \tag{DUE}$$

imply (FK).

Proof: Fix a ball $B(z,r)$ of an integer radius $r > 0$ and a non-empty set $\Omega \subset B(z,r)$. Our goal is to prove the following estimate:

$$\lambda_1(\Omega) \geq \frac{a}{r^2}\left(\frac{V(z,r)}{|\Omega|}\right)^{\nu} \tag{FK}$$

where $\nu = 2/\theta$ (θ is the exponent from (2.6)) and $a > 0$ depends on the doubling constant b.

Let us first observe that if $\lambda_1(\Omega) \geq \frac{1}{2}$ then we have nothing to prove. Indeed, take a point $y \in \Omega$, then by (2.7) and by $\theta\nu = 2$

$$\frac{1}{r^2}\left(\frac{V(z,r)}{|\Omega|}\right)^{\nu} \leq \frac{1}{r^2}\left(\frac{V(z,r)}{m(y)}\right)^{\nu} \leq \text{const}\, \frac{r^{\theta\nu}}{r^2} = \text{const}\,,$$

and (FK) follows for a small enough a. Next we assume that $\lambda_1(\Omega) < \frac{1}{2}$. We will use the operator P_Ω and start with the observation that for any $k \in \mathbb{N}^*$

$$\operatorname{tr} P_\Omega^k \leq \sum_{x\in\Omega} p_k(x,x)\,.$$

Together with (DUE), this yields

$$\operatorname{tr} P_\Omega^k \leq \sum_{x\in\Omega} \frac{Cm(x)}{V(x,\sqrt{k})} \leq \frac{C\,|\Omega|}{\inf_{x\in\Omega} V(x,\sqrt{k})}$$

For any $k \in \mathbb{N}^*$,

$$[\lambda_{\max}(P_\Omega)]^k \leq \lambda_{\max}\left(P_\Omega^k\right) \leq \operatorname{tr} P_\Omega^k$$

whence

$$\lambda_{\max}(P_\Omega) \leq \left\{\frac{C\,|\Omega|}{\inf_{x\in\Omega} V(x,\sqrt{k})}\right\}^{1/k}. \tag{5.11}$$

Next we apply the inequality $1-\xi \geq \frac{1}{2}\log\frac{1}{\xi}$ which is valid for all $\xi \in [\frac{1}{2},1]$. By letting $\xi = \lambda_{\max}(P_\Omega)$ we see that

$$\lambda_1(\Omega) = 1 - \lambda_{\max}(P_\Omega) \geq \frac{1}{2}\log\frac{1}{\lambda_{\max}(P_\Omega)}.$$

Combining with (5.11), we obtain

$$\lambda_1(\Omega) \geq \frac{1}{2k}\log\frac{\inf_{x\in\Omega} V(x,\sqrt{k})}{C\,|\Omega|}. \tag{5.12}$$

We will deduce (FK) from (5.12) by choosing an appropriate k. We will do it in two steps. In the first step, let us find k so that

$$\frac{\inf_{x\in\Omega} V(x,\sqrt{k})}{C\,|\Omega|} \geq \varepsilon \tag{5.13}$$

where $\varepsilon > 0$ but may be small. In the second step, we will increase ε. Set first

$$k = \lceil r^2 \left(\frac{|\Omega|}{V(z,r)}\right)^{2/\theta} \rceil. \tag{5.14}$$

Then k is a positive integer, and

$$r\left(\frac{|\Omega|}{V(z,r)}\right)^{1/\theta} \leq \sqrt{k} \leq r. \tag{5.15}$$

For any $x \in B(z,r)$, we have by (2.6) and (5.15)

$$\frac{|\Omega|}{V(x,\sqrt{k})} = \frac{V(z,r)}{V(x,\sqrt{k})}\frac{|\Omega|}{V(z,r)} \leq \mathrm{const}\left(\frac{r}{\sqrt{k}}\right)^\theta \left(\frac{\sqrt{k}}{r}\right)^\theta = \mathrm{const}$$

whence (5.13) follows.

By the second part of Lemma 2.2, we can increase k by a (big) constant factor so that the volume $V(x,\sqrt{k})$ increases by a large enough constant factor, to make

ε in (5.13) greater than e. For this (revised) k, the logarithm in (5.12) is at least 1, and we obtain

$$\lambda_1(\Omega) \geq \frac{1}{2k}. \tag{5.16}$$

The final step is to show

$$k \leq \text{const}\, r^2 \left(\frac{|\Omega|}{V(z,r)}\right)^{2/\theta} \tag{5.17}$$

whence (FK) will follow. The inequality (5.17) will be a trivial consequence of (5.14) if we show that the argument of the ceiling function in (5.14) is separated from 0 so that taking the ceiling function increases it at most by a constant factor. This follows from (2.7):

$$r^2 \left(\frac{|\Omega|}{V(z,r)}\right)^{2/\theta} \geq r^2 \left(\frac{m(x)}{V(z,r)}\right)^{2/\theta} \geq \text{const} > 0$$

where x is any point in Ω. ■

Together with Proposition 2.1 and Theorem 5.2, Theorem 5.4 gives the two equivalencies in Theorem 1.1.

Theorems 5.2, 5.4 give as a by-product that, together with (D), (DUE) implies (UE). In the case of manifolds, this was proved in [25]. What is missing here for a direct proof is a discrete version of the integrated maximum principle (see [24]).

Also, (UE) and (D), therefore (DUE) and (D), or (FK), imply the L^p boundedness of Riesz transforms, $1 \leq p \leq 2$ (see [35]).

6 On-diagonal lower bound

It is well-known that an on-diagonal lower bound easily follows from a full off-diagonal upper bound, see for example [41, pp.369-370] [38], [15, Lemma 2.3], [10], [2].

Theorem 6.1 *(UE) and (D) imply*

$$p_{2k}(x,x) \geq \frac{c\, m(x)}{V(x,\sqrt{k})},\ \forall\, x \in \Gamma, k \in \mathbb{N}^*. \tag{DLE}$$

Proof: First one checks easily using (UE) and (D) that

$$\sum_{y \notin B(x,r)} p_k(x,y) \leq C\, e^{\frac{-r^2}{Ck}},\ \forall\, x \in \Gamma, k \in \mathbb{N}^*, r > 0.$$

Indeed

$$\begin{aligned}
\sum_{y \notin B(x,r)} p_k(x,y) &\leq C \sum_{y \notin B(x,r)} \frac{m(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \\
&= C \sum_{i=1}^{+\infty} \sum_{y \in B(x,2^{i+1}r) \setminus B(x,2^i r)} \frac{m(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \\
&\leq C \sum_{i=1}^{+\infty} \frac{V(x,2^{i+1}r)}{V(x,\sqrt{k})} \exp\left(-c\frac{2^{2i}r^2}{k}\right) \\
&\leq C \sum_{i=1}^{+\infty} \left(\frac{2^{i+1}r}{\sqrt{k}}\right)^{\theta} \exp\left(-c\frac{2^{2i}r^2}{k}\right) \\
&\leq C' e^{\frac{-r^2}{Ck}}.
\end{aligned}$$

It follows that, for A large enough,

$$\sum_{y \notin B(x,A\sqrt{k})} p_k(x,y) \leq C\, e^{\frac{-A^2}{C}} \leq 1/2,\ \forall\, x \in \Gamma, k \in \mathbb{N}^*,$$

thus

$$1/2 \leq \sum_{y \in B(x,A\sqrt{k})} p_k(x,y),\ \forall\, x \in \Gamma, k \in \mathbb{N}^*.$$

Write

$$\begin{aligned}
p_{2k}(x,x) &= \sum_z p_k(x,z) p_k(z,x) \\
&= m(x) \sum_z \frac{p_k^2(x,z)}{m(z)} \\
&\geq m(x) \sum_{z \in B(x,A\sqrt{k})} \frac{p_k^2(x,z)}{m(z)} \\
&\geq \frac{m(x)}{\sum_{z \in B(x,A\sqrt{k})} m(z)} \sum_{z \in B(x,A\sqrt{k})} p_k(x,z) \\
&\geq \frac{m(x)}{2V(x,A\sqrt{k})}.
\end{aligned}$$

Remark: The estimate in Theorem 6.1 can also be obtained for odd k if one assumes in addition hypothesis (α). Indeed then

$$p_{2k+1}(x,x) = \sum_{y \in \Gamma} p_{2k}(x,y) p(y,x) \geq p_{2k}(x,x) p(x,x) \geq \alpha p_{2k}(x,x).$$

Theorem 6.1 finishes the proof of Theorem 1.1.

Note that if one only assumes (D), one still has a weaker lower bound

$$p_k(x,x) \geq \frac{c\,m(x)}{V(x, \sqrt{k \log k})}, \ \forall\, x \in \Gamma, k \geq 2,$$

see [32, thm.3, (i)]. One may wonder if (D) alone does not give (DLE). Indeed, if one thinks in terms of anti-isoperimetric inequalities (see [10]), the anti-Faber-Krahn inequality corresponding to (FK) is automatic: if $\Omega = B(x,r)$, then

$$\lambda_1(\Omega) \leq \frac{C}{r^2} \left(\frac{V(x,r)}{|\Omega|} \right)^{\nu}$$

since, because of (D),

$$\lambda_1(B(x,r)) \leq \frac{C}{r^2}$$

(see [10, Lemma 2.6]).

Another open question is whether (D) alone does not give a weak upper bound, better than

$$p_k(x,x) \leq \frac{Cm(x)}{\sqrt{k}},$$

which holds true as soon as Γ is infinite. Such an improvement was obtained in [14], Théorème 7, in the case where the volume growth is uniformly polynomial. The basic observation is that the following weak form of Poincaré is always true:

$$\sum_{y \in B(x,r)} |f(y) - f_r(x)|^2 m(y) \leq 2rV(x,r) \sum_{B(x,C'r)} |\nabla f|^2(y) m(y), \, \forall\, f \in \mathbb{R}^{\Gamma}, \, \forall r > 0.$$

Can one deduce from there a lower estimate for $\lambda_1(\Omega)$ and an upper bound of $p_k(x,x)$?

One may also ask for a full analogue of [10, Theorem 7.2], namely, fix a point x in Γ and deduce the lower bound on $p_{2k}(x,x)$ from the upper bound and the doubling property for balls centered at x. Again, this would follow from a discrete integrated maximum principle.

References

[1] **Bakry D., Coulhon T., Ledoux M., Saloff-Coste L.**, Sobolev inequalities in disguise, *Indiana Univ. Math. J.*, **124** (1995) 4, 1033-1074.

[2] **Benjamini I., Chavel I., Feldman E.A.**, Heat kernel lower bounds on Riemannian manifolds using the old ideas of Nash, *Proceedings of London Math. Soc.*, **72** (1996) 215-240.

[3] **Carlen E.A., Kusuoka S., Stroock D.W.**, Upper bounds for symmetric Markov transition functions, *Ann. Inst. H. Poincaré Probab. Statist.*, **23** (1987) 2, suppl. 245-287.

[4] **Carne K.**, A transmutation formula for Markov chains, *Bull. Sc. Math., 2ème série*, **109** (1985) 399-403.

[5] **Carron G.**, Inégalités isopérimétriques de Faber-Krahn et conséquences, *in:* "Actes de la table ronde de géométrie différentielle (Luminy, 1992)", Collection SMF Séminaires et Congrès 1, 1996. 205–232.

[6] **Chung F.R.K.**, "Spectral Graph Theory", CBMS Regional Conference Series in Mathematics 92, AMS publications, 1996.

[7] CouSCinf

[8] **Coulhon T.**, Espaces de Lipschitz et inégalités de Poincaré, *J. Funct. Anal.*, **136** (1996) 1, 81-113.

[9] **Coulhon T.**, Ultracontractivity and Nash type inequalities, *J. Funct. Anal.*, **141** (1996) 2, 510-539.

[10] **Coulhon T., Grigor'yan A.**, On-diagonal lower bounds for heat kernels on non-compact manifolds and Markov chains, *Duke Math. J.*, **89** (1997) 1, 133-199.

[11] **Coulhon T., Ledoux M.**, Isopérimétrie, décroissance du noyau de la chaleur et transformations de Riesz: un contre-exemple, *Ark. Mat.*, **32** (1994) 63-77.

[12] **Coulhon T., Saloff-Coste L.**, Puissances d'un opérateur régularisant, *Ann. Inst. Henri Poincaré*, **26** (1990) 3, 419-436.

[13] **Coulhon T., Saloff-Coste L.**, Marches aléatoires non symétriques sur les groupes unimodulaires, *C.R. Acad. Sci. Paris, Sér. I*, **310** (1990) 627-630.

[14] **Coulhon T., Saloff-Coste L.**, Isopérimétrie pour les groupes et les variétés, *Revista Mathemática Iberoamericana*, **9** (1993) 2, 293-314.

[15] **Coulhon T., Saloff-Coste L.**, Minorations pour les chaînes de Markov unidimensionnelles, *Prob. Theory Relat. Fields*, **97** (1993) 423-431.

[16] **Coulhon T., Saloff-Coste L.**, Variétés riemanniennes isométriques à l'infini, *Revista Mathemática Iberoamericana*, **11** (1995) 3, 687-726.

[17] **Davies E.B.**, Heat kernel bounds, conservation of probability and the Feller property, *J. d'Analyse Math.*, **58** (1992) 99-119.

[18] **Davies E.B.**, Large deviations for heat kernels on graphs, *J. London Math. Soc. (2)*, **47** (1993) 65-72.

[19] **De Giorgi E.**, Sulla differenziabilita e l'analiticita delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat., ser.3*, **3** (1957) 25-43.

[20] **Delmotte T.**, Inégalité de Harnack elliptique sur les graphes, *Colloq. Math*, **72** (1997) 1, 19-37.

[21] **Delmotte T.**, Parabolic Harnack inequality and estimates of Markov chains on graphs, *Revista Mathemática Iberoamericana*, **15** (1999) 1, 181-232.

[22] **Grigor'yan A.**, The heat equation on non-compact Riemannian manifolds, (in Russian) *Matem. Sbornik*, **182** (1991) 1, 55-87. Engl. transl. *Math. USSR Sb.*, **72** (1992) 1, 47-77.

[23] **Grigor'yan A.**, Heat kernel upper bounds on a complete non-compact manifold, *Revista Mathemática Iberoamericana*, **10** (1994) 2, 395-452.

[24] **Grigor'yan A.**, Integral maximum principle and its applications, *Proc. Roy. Soc. Edinburgh*, **124A** (1994) 353-362.

[25] **Grigor'yan A.**, Gaussian upper bounds for the heat kernel on arbitrary manifolds, *J. Diff. Geom.*, **45** (1997) 33-52.

[26] **Gushchin A.K.**, Uniform stabilization of solutions of the second mixed problem for a parabolic equation, (in Russian) *Matem. Sbornik*, **119(161)** (1982) 4, 451-508. Engl. transl. *Math. USSR Sb.*, **47** (1984) 439-498.

[27] **Hebisch W., Saloff-Coste, L.**, Gaussian estimates for Markov chains and random walks on groups, *Ann. Prob.*, **21** (1993) 673–709.

[28] **Heinonen J., Koskela P.**, Quasiconformal maps in metric spaces with controlled geometry, preprint

[29] **Ladyzenskaja O.A., V.A. Solonnikov, Ural'ceva N.N.**, "Linear and quasilinear equations of parabolic type", Providence, Rhode Island, 1968.

[30] **Li P., Schoen R.**, L^p and mean value properties of subharmonic functions on Riemannian manifolds, *Acta Math.*, **153** (1984) 279-301.

[31] **Li P., Yau S.-T.**, On the parabolic kernel of the Schrödinger operator, *Acta Math.*, **156** (1986) 3-4, 153-201.

[32] **Lust-Piquard F.**, Lower bounds on $\|K^n\|_{1\to\infty}$ for some contractions K of $L^2(\mu)$, with some applications to Markov operators, *Math.Ann.*, **303** (1995) 699-712.

[33] **Moser J.**, A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.*, **17** (1964) 101-134. Correction: *Comm. Pure Appl. Math.*, **20** (1967) 231-236.

[34] **Pang M.M.H.**, Heat kernels on graphs, *J. London Math. Soc.*, (1993)

[35] **Russ E.**, Temporal regularity for random walks and Riesz transforms on graphs for $1 \leq p \leq 2$, preprint

[36] **Saloff-Coste L.**, A note on Poincaré, Sobolev, and Harnack inequalities, *Internat. Math. Res. Notices*, **2** (1992) 27-38.

[37] **Saloff-Coste L.**, Uniformly elliptic operators on Riemannian manifolds, *J. Diff. Geom.*, **36** (1992) 417-450.

[38] **Stroock D.W.**, Estimates on the heat kernel for the second order divergence form operators, *in:* "Probability theory. Proceedings of the 1989 Singapore Probability Conference held at the National University of Singapore, June 8-16 1989", Ed. L.H.Y. Chen, K.P. Choi, K. Hu and J.H. Lou, Walter De Gruyter, 1992. 29-44.

[39] **Varopoulos N.Th.**, Isoperimetric inequalities and Markov chains, *J. Funct. Anal.*, **63** (1985) 2, 215-239.

[40] **Varopoulos N.Th.**, Long range estimates for Markov chains, *Bull. Sc. Math., 2ème série*, **109** (1985) 113-119.

[41] **Varopoulos N.Th.**, Analysis on Lie groups, *J. Funct. Anal.*, **76** (1988) 346-410.