POINTWISE ESTIMATES OF SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS AND INEQUALITIES

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Abstract. We obtain sharp pointwise estimates for positive solutions to the equation $-Lu+Vu^q=f$, where L is an elliptic operator in divergence form, $q\in\mathbb{R}\setminus\{0\}$, $f\geq 0$ and V is a function that may change sign, in a domain Ω in $\mathbb{R}^n$, or in a weighted Riemannian manifold.

Contents

1. Introduction

Consider the following elliptic differential equation

$$-Lu+V(x)\,u^q=f \tag{1.1}$$

in an open connected set $\Omega\subseteq\mathbb{R}^n$, where q is a non-zero real number,

$$L=\sum_{i,j=1}^{n}\partial_{x_i}\left(a_{ij}(x)\,\partial_{x_j}\right) \tag{1.2}$$

is a divergence form elliptic operator with smooth coefficients $a_{ij}=a_{ji}$, V and f are continuous functions in Ω, and $f\geq 0$, $f\not\equiv 0$. Note that $V(x)$ can be signed and we do not impose any explicit boundary condition on V.

Assuming that u is a nonnegative (or positive in the case $q<0$) solution, our purpose is to obtain pointwise estimates of u in terms of the function h that is the minimal positive solution in Ω of the equation $-Lh=f$. It is not obvious at all, that u should satisfy any bound via h, but nevertheless the following is true.

Date: November 8, 2015.

2010 *Mathematics Subject Classification.* Primary 35J61. Secondary 58J05.

Key words and phrases. Green's functions, Doob's transform, superharmonic functions, semilinear equations, Schrödinger operators, weighted manifolds.

The first author is supported in part by SFB 701 of the German Research Council. The second author is supported in part by NSF grant DMS-1161622.

Assume that the Dirichlet Green function of L in Ω exists and denote it by $G^{\Omega}(x,y)$. Set

$$h(x)=\int_{\Omega}G^{\Omega}(x,y)\,f(y)\,dy,$$

and assume that $h(x)<\infty$ for all $x\in\Omega$ (note also that $h(x)>0$ in Ω), and that the integral

$$\int_{\Omega}G^{\Omega}(x,y)\,h^{q}(y)\,V(y)\,dy \tag{1.3}$$

is well-defined. Our main Theorem 3.1 states that the following estimates hold for all $x\in\Omega$.

(i) If $q=1$ then

$$u(x)\geq h(x)\exp\left(-\frac{1}{h(x)}\int_{\Omega}G^{\Omega}(x,y)\,h(y)\,V(y)\,dy\right). \tag{1.4}$$

(ii) If $q>1$ then

$$u(x)\geq\frac{h(x)}{\left[1+(q-1)\frac{1}{h(x)}\int_{\Omega}G^{\Omega}(x,y)\,h^{q}(y)\,V(y)\,dy\right]^{\frac{1}{q-1}}}, \tag{1.5}$$

where the expression in square brackets is necessarily positive, that is,

$$-(q-1)G^{\Omega}(h^{q}V)(x)<h(x). \tag{1.6}$$

(iii) If $0<q<1$ then

$$u(x)\geq h(x)\left[1-(1-q)\frac{1}{h(x)}\int_{\Omega^{+}}G^{\Omega}(x,y)\,h^{q}(y)\,V(y)\,dy\right]_{+}^{\frac{1}{1-q}}, \tag{1.7}$$

where

$$\Omega^{+}=\{x\in\Omega:\ u(x)>0\}.$$

In this case we assume that the integral in (1.7) is well-defined instead of (1.3).

(iv) If $q<0$, $u>0$ in Ω, and in addition $u(y)\to0$ as $y\to\partial\Omega$ or $|y|\to\infty$, then (1.6) holds and

$$u(x)\leq h(x)\left[1-(1-q)\frac{1}{h(x)}\int_{\Omega}G^{\Omega}(x,y)\,h^{q}(y)\,V(y)\,dy\right]^{\frac{1}{1-q}}. \tag{1.8}$$

Let us emphasize that in the case (iv) we obtain an upper bound for u in contrast to the lower bound in the cases (i)-(iii).

In fact, Theorem 3.1 holds in much higher generality, when Ω is any open subset of any weighted Riemannian manifold, L is the associated weighted Laplace operator, and equation (1.1) can be replaced by an inequality.

Equation (1.1) and its generalizations have attracted attention of many authors, investigating various aspects from the existence of positive solutions to pointwise estimates (see, for example, [1], [2], [7], [8], [26], [22], [24], [29], [28], [30], [31], etc). There is no possibility to give a detailed overview of the literature on this subject, which would have required a full size survey. We restrict our attention here to those earlier results that are most closely related to ours.

In the case $q=1$ estimate (1.4) was known before and is included here for the sake of completeness. For $V\geq0$ (1.4) was proved by Hansen and Ma [23, Prop. 1.9] using the tools of potential theory (see also [20]). For $V\leq0$ in domains Ω with boundary Harnack principle estimate (1.4) as well a matching upper estimate for u were obtained in [14], [15] using a completely different method (but without sharp constants).

For a general signed V in a relatively compact Ω estimate (1.4) can be obtained using the Feynman-Kac formula for Brownian motion and Jensen's inequality. This type of argument was implicit in [2], [8], [25, Prop. 2.5]. In the form (1.4) it was stated in [21]. However, neither the Feynman-Kac formula nor any of the cited above previous methods allows to treat the nonlinear case $q \neq 1$.

In the case $q > 1$ and $V \leq 0$ Kalton and the second author obtained in [27] the necessary condition (1.6), although without a sharp constant, and gave also a sufficient condition

$$-G^{\Omega}(h^q V)(x) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{q-1} h(x) \tag{1.9}$$

for the existence of a positive solution. Moreover, under (1.9) they obtained a two-sided estimate $u \simeq h$ for the minimal positive solution u of (1.1) in any domain Ω with the boundary Harnack principle (the sign $\simeq$ means that the ratio of both sides is bounded from above and below by positive constants).

In the case $q > 1$, $V \leq 0$, and $L = \Delta$, Brezis and Cabré [5] obtained the sharp necessary condition (1.6) for the existence of a positive solution in an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$, as well as the estimate $u \simeq h$ under (1.9). The proof of the necessary condition (1.6) in [5, Lemma 5.3] is based on a direct computation using the explicit form $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ of the Laplace operator. A much more expanded version of this computation will appear in our proof in Section 4 below.

The case $q > 1$, $V \equiv 1$, $f \equiv 0$ has been extensively studied, and we do not touch it here; we refer the reader to [12] and [28] as well as to the references therein.

In the case $0 < q < 1$, $V \leq 0$, and $L = \Delta$, Brezis and Kamin [6] obtained necessary and sufficient conditions for the existence of a bounded, positive solution of (1.1) in $\mathbb{R}^n$ and obtained certain pointwise bounds. Their lower bound is covered by our Theorem 3.3 below (see also [9], [10]).

In the case $q < 0$ [13], [17] obtained a sharp sufficient condition for the existence of a positive solution of (1.1) in the specific case where $V(x)$ depends only on the distance from x to $\partial\Omega$ and has a constant sign.

In the present paper we give a unified approach for treating all the values of $q \in \mathbb{R}\backslash\{0\}$, a general signed potential V, and a general divergence form operator L, not only in arbitrary domains of $\mathbb{R}^n$, but also on an arbitrary Riemannian manifold. Our estimates (i)-(iv) are new in this generality. In many cases these estimates happen to be sharp as one can see in examples in Section 9.

Let us briefly describe the idea of our proof. Assume for simplicity $L = \Delta$. Let $\{\Omega_k\}_{k=1}^{\infty}$ be an exhaustion of Ω by relatively compact open sets $\Omega_k \subset \Omega$ with smooth boundaries. We obtain first appropriate estimates for u in each Ω_k and then pass to the limit as $k \to \infty$. Define in Ω_k a new function h as the solution of the following boundary value problem

$$\begin{cases} -\Delta h = f, & \text{in } \Omega_k, \\ h = u, & \text{on } \partial\Omega_k. \end{cases}$$

The following argument is used in the proof of Theorem 3.2 that treats (1.1) in relatively compact domains with the Dirichlet boundary condition. Assume first that $h \equiv 1$ (and then $f = 0$ in Ω_k). Fix a C^2 function ϕ on $\mathbb{R}$ (or on an interval in $\mathbb{R}$) with $\phi' > 0$ and consider the substitution

$$v = \phi^{-1}(u).$$

By the chain rule we have

$$\Delta u = \Delta \phi(v) = \phi'(v)\Delta v + \phi''(v)|\nabla v|^2,$$

which implies

$$\begin{aligned} -\Delta v+V\frac{\phi(v)^q}{\phi'(v)} &= -\frac{\Delta u-\phi''\left|\nabla v\right|^2}{\phi'}+V\frac{\phi(v)^q}{\phi'(v)} \\ &= -\frac{V\phi\left(v\right)^q-\phi''\left|\nabla v\right|^2}{\phi'}+V\frac{\phi(v)^q}{\phi'(v)} \\ &= \frac{\phi''}{\phi'}\left|\nabla v\right|^2. \end{aligned} \tag{1.10}$$

Now we choose ϕ to solve the initial value problem

$$\phi'\left(s\right)=\phi^q\left(s\right),\quad \phi\left(0\right)=1,$$

and obtain:

$$\phi\left(s\right)=\begin{cases} e^s, & q=1, \\ \left[(1-q)s+1\right]^{\frac{1}{1-q}}, & q\neq 1, \end{cases}$$

in the appropriate domains. In the case $q>0$ the function ϕ is convex, and we obtain from (1.10)

$$-\Delta v+V\geq 0. \tag{1.11}$$

Since on $\partial\Omega_k$ we have $v=\phi^{-1}\left(u\right)=\phi^{-1}\left(1\right)=0$, we obtain from (1.11) by the maximum principle that

$$v\left(x\right)\geq -\int_{\Omega_k}G^{\Omega_k}\left(x,y\right)V\left(y\right)dy.$$

Applying ϕ to both sides of this inequality gives an appropriate inequality for $u=\phi\left(v\right)$ in Ω_k.

In the case $q<0$ the function ϕ is concave, which leads to the opposite inequality for v and, hence, for u.

In the case of a general function h, consider a so-called h-transform (or Doob's transform [11]) of Δ:

$$\Delta^h=\frac{1}{h}\circ\Delta\circ h=\frac{1}{h^2}\operatorname{div}\left(h^2\nabla\right)+\frac{\Delta h}{h},$$

and the function $\widetilde{u}=\frac{u}{h}$. Then $\widetilde{u}$ solves the equation

$$-\Delta^h\widetilde{u}+h^{q-1}V\widetilde{u}^q=-\frac{\Delta h}{h}$$

with the boundary value $\widetilde{u}=1$ on $\partial\Omega_k$. Effectively the h-transform provides a reduction to the previous case, but for the operator Δ^h in place of Δ. The part $\frac{1}{h^2}\operatorname{div}\left(h^2\nabla\right)$ of this operator is a weighted Laplace operator, for which the same computation (1.10) using the chain rule works as for Δ. The part $\frac{\Delta h}{h}$ gives in the end an additional term

$$\frac{\Delta h}{h}\left(\frac{\phi(v)-1}{\phi'(v)}-v\right)$$

on the right hand side of (1.11) (cf. Lemma 4.2). In the case $q>1$ we obtain by the convexity of ϕ that the expression in parentheses is non-positive. Since $\Delta h=-f\leq 0$, the above term is non-negative which allows us to use the same argument as above. In the case $q<1$ this term is non-positive, which gives again a correct sign in the corresponding inequality.

The actual proof goes a bit differently as we have to overcome one more difficulty – a possibility of h vanishing on the boundary, which we have ignored in the above sketch (see Sections 5, 6).

The above argument allows a version that treats the case $f=0$ in (1.1) – see Theorem 3.3.

In Theorem 3.4 we provide complementary results: sufficient conditions for the existence of a positive solution u and two-sided estimates of u. Finally, Theorem 3.5 is an abstract version of Theorem 3.4 for solutions of integral equations.

The structure of the paper is as follows. In Section 2 we briefly describe the notion of the weighted manifold and the associated Laplace operator. In Section 3 we state our main results: Theorems 3.1, 3.2, 3.3, 3.4 and 3.5. In Section 4 we prove some Lemmas, in particular containing the aforementioned computation (1.10) in the general case. In Section 5-8 we prove the above mentioned theorems. In Section 9 we give some examples.

Acknowledgement. The authors are grateful to Alexander Bendikov, Haim Brezis, and Wolfhard Hansen for stimulating conversations on the subject of this paper.

The second author would like to thank the Mathematics Department at Bielefeld University for its hospitality and support.

2. Weighted manifolds

Let M be a smooth Riemannian manifold with the Riemannian metric tensor $g = (g_{ij})$. The associated Laplace-Beltrami operator $\mathcal{L}_0$ acts on C^2 functions u on M and is given in any chart $x_1, ..., x_n$ by the formula

$$\mathcal{L}_0 u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_{x_i} \left(\sqrt{\det g} g^{ij} \partial_{x_j} u \right)$$

where $\det g$ is the determinant of the matrix $g = (g_{ij})$, and $\left(g^{ij}\right)$ is the inverse matrix of (g_{ij}). The Riemannian measure m_0 is given in the same chart by

$$dm_0 = \sqrt{\det g} dx_1 ... dx_n,$$

so that $\mathcal{L}_0$ is symmetric with respect to m_0. Using the gradient operator ∇ defined by

$$(\nabla u)^i = \sum_{j=1}^{n} g^{ij} \partial_{x_j} u$$

and the divergence div on vector fields F^i

$$\operatorname{div} F = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{n} \partial_{x_i} \left(\sqrt{\det g} F^i \right),$$

one represents $\mathcal{L}_0$ in the form

$$\mathcal{L}_0 = \operatorname{div} \circ \nabla.$$

Let ω be a smooth positive function on M and consider the measure m on M given by

$$dm = \omega dm_0.$$

The couple (M, m) is called a weighted manifold or a manifold with density, and ω in this context is called a weight. The following operator $\mathcal{L}$

$$\mathcal{L} u := \frac{1}{\omega} \operatorname{div} (\omega \nabla u) = \frac{1}{\omega \sqrt{\det g}} \sum_{i,j=1}^{n} \partial_{x_i} \left(\omega \sqrt{\det g} g^{ij} \partial_{x_j} u \right) \tag{2.1}$$

acting on C^2 functions u on M, is called the (weighted) Laplace operator of (M, m). It is easy to see that $\mathcal{L}$ is symmetric with respect to measure m.

Of course, for $\omega = 1$ we have $\mathcal{L} = \mathcal{L}_0$. For a general weight ω, define the weighted divergence by

$$\operatorname{div}_\omega = \frac{1}{\omega} \circ \operatorname{div} \circ \omega$$

and obtain

$$\mathcal{L} = \operatorname{div}_\omega \circ \nabla.$$

Note that ∇ remains the Riemannian gradient and does not depend on the weight ω.

It is easy to show that the weighted Laplace operator $\mathcal{L}$ satisfies the same product and chain rules as the classical Laplace operator (cf. [19, Section 3.6]). Namely, for two C^2 functions u, v on M we have

$$\mathcal{L}(uv) = u\mathcal{L}v + 2\langle \nabla u, \nabla v\rangle + v\mathcal{L}u \tag{2.2}$$

where $\langle \nabla u, \nabla v\rangle$ is the inner product of the Riemannian gradients, which is independent of the weight ω. Also, for any C^2 function ϕ defined on $u(M)$ we have

$$\mathcal{L}\phi(u) = \phi'(u)\mathcal{L}u + \phi''(u)|\nabla u|^2. \tag{2.3}$$

As an example, consider in an open set $\Omega \subseteq \mathbb{R}^n$ the following operator

$$Lu = b(x)\sum_{i,j=1}^n \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}u\right), \tag{2.4}$$

where b, a_{ij} are smooth functions, $b > 0$ and $a_{ij} = a_{ji}$. Assume that L is elliptic, that is, the matrix $(a_{ij}(x))$ is positive definite for any x (the uniform ellipticity is not assumed). Then L coincides with the weighted Laplace operator $\mathcal{L}$ of $\mathbb{R}^n$ with the Riemannian metric g and weight ω given by

$$\left(g^{ij}\right) = b\,(a_{ij}), \qquad \omega = b^{\frac{n}{2}-1}\sqrt{\det a},$$

where $a = \left(a_{i_j}\right)$. Indeed, it follows that

$$\det g = \det(g_{ij}) = \frac{1}{b^n \det a},$$

and substitution into (2.1) yields

$$\begin{aligned}\mathcal{L}u &= \frac{\sqrt{b^n\det a}}{b^{\frac{n}{2}-1}\sqrt{\det a}}\sum_{i,j=1}^n \partial_{x_i}\left(b^{\frac{n}{2}-1}\sqrt{\det a}\frac{1}{\sqrt{b^n\det a}}ba^{ij}\partial_{x_j}u\right)\\ &= b\sum_{i,j=1}^n \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}u\right) = Lu.\end{aligned}$$

The measure m associated with $\mathcal{L}$ is given by

$$dm = \omega\sqrt{\det g} = b^{\frac{n}{2}-1}\sqrt{\det a}\frac{1}{\sqrt{b^n\det a}} = \frac{1}{b}dx, \tag{2.5}$$

where dx is Lebesgue measure.

Therefore, all the results that we obtain for a general weighted manifold (M, m), apply to the operator (2.4) in a domain of $\mathbb{R}^n$ with the measure m from (2.5). In particular, if $b \equiv 1$ as was assumed in the Introduction, then L is given by (1.2) and m is Lebesgue measure.

3. Statements of the main results

For any open connected set $\Omega \subseteq M$ denote by $G^\Omega(x,y)$ the infimum of all positive fundamental solutions of $\mathcal{L}$ in Ω. The following dichotomy is true: either $G^\Omega(x,y) \equiv \infty$ or $G^\Omega(x,y) < \infty$ for all $x \neq y$. In the latter case we say that G^Ω is finite. If G^Ω is finite then G^Ω is the symmetric positive Green function of $\mathcal{L}$ in Ω (see [18] and [19, Ch.13]). If Ω is relatively compact then G^Ω is finite and satisfies the Dirichlet boundary condition on the regular part of $\partial\Omega$.

If G^{Ω} is finite then, for any function $f \in L^1_{loc}(\Omega, m)$, set

$$G^{\Omega} f(x) = \int_{\Omega} G^{\Omega}(x, y) f(y) \, dm(y),$$

where in the case $f \geq 0$ the integral is understood in the sense of Lebesgue; for a signed f the integral is understood as follows:

$$G^{\Omega} f(x) = G^{\Omega} f_+(x) - G^{\Omega} f_-(x)$$

(where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$), assuming that at least one of the values $G^{\Omega} f_+(x)$, $G^{\Omega} f_-(x)$ is finite. In this case we say that $G^{\Omega} f(x)$ is well-defined.

Note that if $f \geq 0$ in Ω and $f > 0$ on a set of positive measure then $G^{\Omega} f > 0$ in Ω.

If Ω is relatively compact then $G^{\Omega}(x, \cdot) \in L^1(\Omega)$, which implies that $G^{\Omega} f$ is finite for any $f \in L^{\infty}(\Omega)$. For arbitrary Ω it is still true that $G^{\Omega}(x, \cdot) \in L^1_{\mathrm{loc}}(\Omega)$ for every $x \in \Omega$.

Denote by $\partial_{\infty} M$ the infinity point of the one-point compactification of M (see for example [19, Sec. 5.4.3]). For any open subset $\Omega \subseteq M$ denote by $\partial_{\infty}\Omega$ the union of $\partial\Omega$ and $\partial_{\infty} M$, if Ω is not relatively compact, and set $\partial_{\infty}\Omega = \partial\Omega$ if Ω is relatively compact.

Definition. For a function u defined in $\Omega \subseteq M$ let us write

$$\lim_{y \to \partial_{\infty}\Omega} u(y) = 0, \tag{3.1}$$

if $\lim_{k \to \infty} u(y_k) = 0$ for any sequence $\{y_k\}$ in Ω that converges to a point of $\partial_{\infty}\Omega$; the latter means, that either $\{y_k\}$ converges to a point on $\partial\Omega$ or diverges to $\partial_{\infty} M$. In the same way we understand similar equalities and inequalities involving $\limsup$ and $\liminf$.

For example, if Ω is relatively compact, then (3.1) means that $\lim u(y_k) = 0$ for any sequence $\{y_k\}$ converging to a point on $\partial\Omega$. If $\Omega = M$ then $\partial\Omega = \emptyset$ and (3.1) means that $\lim u(y_k) = 0$ for any sequence $y_k \to \partial_{\infty} M$, that is, for any sequence $\{y_k\}$ that leaves any compact subset of M. In particular, for $M = \mathbb{R}^n$ (3.1) is equivalent to $u(y) \to 0$ as $|y| \to 0$.

We will use the notation

$$\chi_u(x) = \begin{cases} 1, & u(x) > 0, \\ 0, & u(x) \leq 0. \end{cases}$$

Theorem 3.1. *Let M be an arbitrary weighted manifold, and let $\Omega \subseteq M$ be a connected open subset of M with a finite Green function G^{Ω}. Suppose $V, f \in C(\Omega)$ and assume $f \geq 0$, $f \not\equiv 0$ in Ω. Let $u \in C^2(\Omega)$ satisfy*

$$\text{in the case } q > 0: \quad -\mathcal{L}u + Vu^q \geq f \quad \text{in } \Omega, \quad u \geq 0, \tag{3.2}$$

or,

$$\text{in the case } q < 0: \quad \begin{cases} -\mathcal{L}u + Vu^q \leq f \quad \text{in } \Omega, \\ \lim_{y \to \partial_{\infty}\Omega} u(y) = 0, \end{cases} \quad u > 0. \tag{3.3}$$

Set $h = G^{\Omega} f$ and assume that $h < \infty$ in Ω. Assume also that $G^{\Omega}(h^q V)(x)$ (respectively $G^{\Omega}(\chi_u h^q V)(x)$ in the case $0 < q < 1$) is well-defined for all $x \in \Omega$. Then the following statements hold for all $x \in \Omega$.

(i) If $q = 1$, then

$$u(x) \geq h(x) e^{-\frac{1}{h(x)} G^{\Omega}(hV)(x)}. \tag{3.4}$$

(ii) If $q > 1$, then necessarily

$$-(q-1) G^{\Omega}(h^q V)(x) < h(x), \tag{3.5}$$

and the following estimate holds:

$$u(x) \geq \frac{h(x)}{\left[1+(q-1)\frac{G^\Omega(h^qV)(x)}{h(x)}\right]^{\frac{1}{q-1}}}. \tag{3.6}$$

(iii) *If* $0<q<1$, *then*

$$u(x) \geq h(x)\left[1-(1-q)\frac{G^\Omega(\chi_u h^qV)(x)}{h(x)}\right]_+^{\frac{1}{1-q}}. \tag{3.7}$$

(iv) *If* $q<0$ *then necessarily* (3.5) *holds, and*

$$u(x) \leq h(x)\left[1-(1-q)\frac{G^\Omega(h^qV)(x)}{h(x)}\right]^{\frac{1}{1-q}}. \tag{3.8}$$

Note that the condition $f \not\equiv 0$ implies $h>0$ in Ω. Note also that without loss of generality the open set Ω in Theorem 3.1 can be taken to be M. However, we have preferred the present formulation for the sake of convenience in applications.

Remark. In the case $q \geq 1$, it follows from (3.4) and (3.6) that the condition

$$G^\Omega\left(h^qV\right)(x) < +\infty$$

implies $u(x)>0$. Moreover, if for some $C>0$ and all $x\in\Omega$

$$G^\Omega\left(h^qV\right)(x) \leq Ch(x),$$

then $u \geq ch$ in Ω with some constant $c=c(C)>0$.

In the case $0<q<1$ the function u can vanish in Ω, but the estimate of u cannot depend on the values of V on the set $\{u=0\}$. This explains the appearance of the factor χ_u and the subscript $+$ on the right-hand side of (3.7).

In the case $q<0$, the boundary condition $\lim_{y\to\partial_\infty\Omega} u(y)=0$ is needed as without this condition, for positive V, the function $u+C$ would also be a solution to (3.3) for any $C>0$, so that u could not admit any upper bound.

Remark. The lower estimates of Theorem 3.1 $(i),(ii),(iii)$ remain valid even if the expression $G^\Omega(h^qV)$ is not well-defined in the above sense, provided it is understood as follows

$$G^\Omega(h^qV)(x) := \liminf_{n\to\infty}\int_{\Omega_n} G^{\Omega_n}(x,y)h^q(y)V(y)dy, \tag{3.9}$$

where $\{\Omega_n\}$ is any exhaustion of Ω by relatively compact subsets with smooth boundaries. The same is true for the upper estimate of (iv) where one can use $\limsup$ in place of $\liminf$.

In the case $q=1$ and $h=G^\Omega f$, this means

$$G^\Omega(hV)(x) = G_2^\Omega f(x) = \liminf_{n\to\infty}\int_{\Omega_n} G_2^{\Omega_n}(x,y)f(y)dy, \quad x\in\Omega, \tag{3.10}$$

where G_2^Ω stands for the second iteration of the Green kernel with respect to $V(y)dy$:

$$G_2^\Omega(x,y) = \int_\Omega G^\Omega(x,z)G^\Omega(z,y)V(z)dz, \quad x,y\in\Omega. \tag{3.11}$$

In some cases $G_2^\Omega(x,y)$ in (3.11) can be understood as an improper integral. (See Example 1 in Section 9 below.)

Remark. Suppose $q>1$ in Theorem 3.1. The necessary condition (3.5) for the existence of a positive solution of (3.2) in the case $V\leq 0$ was proved in [27], without the sharp constant $\frac{1}{q-1}$, but for general quasi-metric kernels, including a wide variety of differential and integral operators. It was also shown in [27] that the stronger condition

$$-G^{\Omega}(h^q V)(x) \le \left(1-\frac{1}{q}\right)^q \frac{1}{q-1}\, h(x), \quad x \in \Omega, \tag{3.12}$$

is sufficient for the existence of a solution u such that

$$h \le u \le C(q)\, h$$

Brezis and Cabré [5] subsequently proved the necessity of (3.5) with the sharp constant $\frac{1}{q-1}$ in the case of $\mathcal{L} = \Delta$ in bounded domains of $\mathbb{R}^n$ (see also Theorem 3.5 below).

In the proof of Theorem 3.1, we use Theorem 3.2 below that deals with relatively compact sets $\Omega \subset M$. Fix a function $h \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ such that

$$h > 0 \ \text{ in } \ \Omega \ \text{ and } \ -\mathcal{L}h \ge 0 \ \text{ in } \ \Omega. \tag{3.13}$$

Consider in Ω the following boundary value inequalities:

$$\begin{cases} -\mathcal{L}u + Vu^q \ge -\mathcal{L}h & \text{in } \Omega \\ u \ge h & \text{on } \partial\Omega \\ u \ge 0 & \text{in } \Omega \end{cases} \qquad \text{in the case } q > 0, \tag{3.14}$$

and

$$\begin{cases} -\mathcal{L}u + Vu^q \le -\mathcal{L}h & \text{in } \Omega \\ u \le h & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad \text{in the case } q < 0, \tag{3.15}$$

where $V \in C(\Omega)$ and $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$. In the next theorem we compare u and h as follows.

Theorem 3.2. *Let (M, m) be an arbitrary weighted manifold, and let $\Omega \subset M$ be a relatively compact connected open subset of M. Let a function $h \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy (3.13).*

Let $V \in C(\Omega)$ and suppose that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is a solution to either (3.14) or (3.15). Assume also that $G^{\Omega}(h^q V)(x)$ (respectively $G^{\Omega}(\chi_u h^q V)(x)$ in the case $0 < q < 1$) is well-defined for all $x \in \Omega$. Then statements (i)-(iv) of Theorem 3.1 hold.

Remark. In the linear case $q = 1$, we obtain a simple proof of the well-known lower estimate of solutions to the Schrödinger equation:

$$u(x) \ge h(x) e^{-\frac{1}{h(x)} G^{\Omega}(hV)(x)}, \quad \text{for all } \ x \in \Omega. \tag{3.16}$$

This estimate in the special case $h = 1$ is usually deduced via the Feynman-Kac formalism (see [2], [8]) using Jensen's inequality. In the case $V \ge 0$, alternative proofs based on potential theory methods in a very general setting are given in [18], [20]. In the case $V \le 0$, a similar lower estimate and a matching upper estimate (but without sharp constants) are obtained in [14], [15] for general quasi-metric kernels.

An interesting special case is when h is the solution of the Dirichlet problem:

$$\begin{cases} -\mathcal{L}h = 1 & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

In other words, $h(x) = \mathbb{E}_x[\tau_\Omega]$, where $\tau_\Omega = \inf\{t \colon X_t \notin \Omega\}$ is the first exit time from Ω of the (rescaled) Brownian motion X_t, and $x \in \Omega$ is a starting point. For bounded $C^{1,1}$ domains, $h(x) \simeq d_\Omega(x)$, where

$$d_\Omega(x) = \operatorname{dist}(x, \partial\Omega). \tag{3.18}$$

This gives sharp estimates:

$$u(x) \geq c\, d_\Omega(x)\, e^{-\frac{c}{d_\Omega(x)} G^\Omega(d_\Omega V)(x)}, \quad \text{for all} \;\; x \in \Omega. \tag{3.19}$$

if $q = 1$, as well as the corresponding estimates for other values of q.

For bounded Lipschitz domains with sufficiently small Lipschitz constant (less than $(n-1)^{1/2}$, which is sharp), it is known that (see [4])

$$h(x) \simeq \rho(x) = \min(1, G^\Omega(x, x_0)),$$

where x_0 is a fixed pole in Ω, and so (3.19) holds with ρ in place of d_Ω. The corresponding estimates hold for other values of $q \in \mathbb{R}$ as well.

Returning again to the case of an arbitrary (not necessarily relatively compact) Ω, in the next theorem we give estimates of solutions u of (3.2)-(3.3) with $f = 0$. They are applicable to the so-called gauge ($q = 1$), "large" solutions ($q > 1$), or "ground state" solutions ($-\infty < q < 1$) to the corresponding equations in unbounded domains in $\mathbb{R}^n$ or non-compact manifolds.

Theorem 3.3. *Let M be an arbitrary weighted manifold, and let $\Omega \subseteq M$ be an open connected set with a finite Green function G^Ω. Suppose $V \in C(\Omega)$. Let $u \in C^2(\Omega)$ satisfy either the inequality*

$$-\mathcal{L}u + V\, u^q \geq 0, \quad u \geq 0 \text{ in } \Omega, \quad \text{if } q > 0, \tag{3.20}$$

or

$$-\mathcal{L}u + V\, u^q \leq 0, \quad u > 0 \text{ in } \Omega, \;\text{if } q < 0. \tag{3.21}$$

Assume also that $G^\Omega V(x)$ (respectively $G^\Omega(\chi_u V)(x)$ in the case $0 < q < 1$) is well-defined for all $x \in \Omega$. Then the following statements hold for all $x \in \Omega$.

(i) If $q = 1$ and

$$\liminf_{y \to \partial_\infty \Omega} u(y) \geq 1 \tag{3.22}$$

then

$$u(x) \geq e^{-\; G^\Omega V(x)}. \tag{3.23}$$

(ii) If $q > 1$ and

$$\lim_{y \to \partial_\infty \Omega} u(y) = +\infty, \tag{3.24}$$

then necessarily $G^\Omega V(x) > 0$, and

$$u(x) \geq \left[(q-1)\, G^\Omega V(x)\right]^{-\frac{1}{q-1}}. \tag{3.25}$$

(iii) If $0 < q < 1$, then

$$u(x) \geq \left[-(1-q)\, G^\Omega(\chi_u V)(x)\right]_+^{\frac{1}{1-q}}. \tag{3.26}$$

(iv) If $q < 0$, and

$$\lim_{y \to \partial_\infty \Omega} u(y) = 0, \tag{3.27}$$

then necessarily $G^\Omega V(x) < 0$, and

$$u(x) \leq \left[-(1-q)\, G^\Omega V(x)\right]^{\frac{1}{1-q}}. \tag{3.28}$$

In the next theorem we provide criteria for the existence of positive solutions for the equation

$$-\mathcal{L}u + u^q V = f \quad \text{in } \Omega \tag{3.29}$$

under some additional assumptions and give two-sided pointwise estimates for these solutions.

Theorem 3.4. *Let M be a weighted manifold and $\Omega \subset M$ be a connected relatively compact open set with smooth boundary. Let $f \geq 0$ and V be locally Hölder continuous functions in Ω and in addition $f \in C\left(\overline{\Omega}\right)$. Set $h = G^{\Omega} f$. Then the following statements hold.*

(i) For $q > 1$ and $V \leq 0$, suppose that for all $x \in \Omega$

$$- G^{\Omega}\left(h^q V\right)(x) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{q-1}\, h(x). \tag{3.30}$$

Then (3.29) has a nonnegative solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, and it satisfies for all $x \in \Omega$

$$\frac{h(x)}{\left[1 + (q-1)\frac{G^{\Omega}(h^q V)(x)}{h(x)}\right]^{\frac{1}{q-1}}} \leq u(x) \leq \frac{q}{q-1}\, h(x). \tag{3.31}$$

(ii) For $q < 0$ and $V \geq 0$, suppose that for all $x \in \Omega$

$$G^{\Omega}(h^q V)(x) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{1-q}\, h(x). \tag{3.32}$$

Then (3.29) has a nonnegative solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, and it satisfies for all $x \in \Omega$

$$\frac{1}{1 - \frac{1}{q}}\, h(x) \leq u(x) \leq \left[1 - (1-q)\frac{G^{\Omega}(h^q V)(x)}{h(x)}\right]^{\frac{1}{1-q}} h(x). \tag{3.33}$$

Note that the terms in square brackets in both (3.31) and (3.33) are positive and < 1; it follows that in both cases (i) and (ii) $u \simeq h$ in Ω. Since $h(x) \simeq d_{\Omega}(x) := \operatorname{dist}(x, \partial\Omega)$, we obtain $u(x) \simeq d_{\Omega}(x)$.

In the next theorem we give an abstract version of Theorem 3.4 that provides an existence result together with pointwise estimates of solutions u for the following integral equation with $q \in \mathbb{R} \setminus \{0\}$:

$$u(x) + \int_{\Omega} K(x,y)\, u(y)^q\, V(y)\, dm(y) = h(x) \quad dm - \text{a.e. in } \Omega. \tag{3.34}$$

Here (Ω, m) is a measure space with σ-finite nonnegative measure m, $0 < u < \infty$ dm-a.e., and $K : \Omega \times \Omega \to \bar{\mathbb{R}}_+ \cup \{+\infty\}$ is a nonnegative measurable kernel.

The coefficient V is assumed to be a measurable function in Ω with a definite sign (either $V \geq 0$, or $V \leq 0$). In fact, we can use $d\omega$ in place of $V\, dm$, with an arbitrary σ-finite measure ω (either nonnegative, or nonpositive) in Ω, where $0 < u < +\infty$ $d\omega$-a.e., and the integral equation holds $d\omega$-a.e.

For a nonnegative Borel measure μ in Ω, we will write

$$K\mu(x) = \int_{\Omega} K(x,y)\, d\mu(y),$$

and $Kf(x) = K(f dm)(x)$ for a nonnegative measurable function f.

Theorem 3.5. *Let (Ω, m) be a measure space with σ-finite measure m, and let K be a nonnegative kernel on $\Omega \times \Omega$. Let h be a measurable function such that*

$$0 < h < +\infty \quad dm\text{−a.e. in } \Omega. \tag{3.35}$$

Let V be a measurable function in Ω. Then the following statements hold.

(i) For $q > 1$, and $V \leq 0$, suppose that the following condition holds,

$$- K(h^q V)(x) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{q-1}\, h(x) \quad dm\text{−a.e. in } \Omega. \tag{3.36}$$

Then (3.34) *has a minimal positive solution* u*, and it satisfies*

$$h(x) \leq u(x) \leq \frac{q}{q-1}\, h(x) \quad \text{in } \Omega. \tag{3.37}$$

(ii) *For* $q<0$ *and* $V \geq 0$*, suppose that the following condition holds,*

$$K(h^q V)(x) \leq \left(1-\frac{1}{q}\right)^q \frac{1}{1-q}\, h(x) \quad dm-\text{a.e. in } \Omega. \tag{3.38}$$

Then (3.34) *has a maximal positive solution* u*, and it satisfies*

$$\frac{1}{1-\frac{1}{q}}\, h(x) \leq u(x) \leq h(x) \quad dm-\text{a.e.} \quad \text{in } \Omega. \tag{3.39}$$

Remark. Statement (i) of Theorem 3.5 is essentially known, and we include it here only for the sake of completeness. It holds under a less restrictive assumption

$$-K(H^q V)(x) \leq \left(1-\frac{1}{q}\right)^{q^2} \frac{1}{(q-1)^q}\, H(x) \quad dm\text{–a.e. in } \Omega, \tag{3.40}$$

where $H = -K(h^q V)$; in this case, $u \simeq h + H$ (see [27]).

4. Some auxiliary material

In this section we prove some lemmas needed for the proofs of Theorems 3.1, 3.2. Everywhere M stands for an arbitrary weighted manifold.

Lemma 4.1. *Let* v, h *be* C^2*-functions in* $\Omega \subseteq M$*, and* ϕ *be a* C^2*-function on an interval* $I \subset \mathbb{R}$ *such that* $v(\Omega) \subset I$*. Then the following identity is true:*

$$\mathcal{L}(h\phi(v)) = \phi'(v)\mathcal{L}(hv) + \phi''(v)|\nabla v|^2 h + \big(\phi(v) - v\phi'(v)\big)\mathcal{L}h. \tag{4.1}$$

Consequently, if $\phi' \neq 0$ *then*

$$-\mathcal{L}(hv) = -\frac{\mathcal{L}(h\phi(v))}{\phi'(v)} + \frac{\phi''(v)}{\phi'(v)}|\nabla v|^2 h + \left(\frac{\phi(v)}{\phi'(v)} - v\right)\mathcal{L}h. \tag{4.2}$$

Proof. For functions $u \in C^2(\Omega)$, consider the following operator

$$\widetilde{\mathcal{L}}u = \frac{1}{h^2}\operatorname{div}_\omega(h^2\nabla u) = \frac{1}{\omega h^2}\operatorname{div}(\omega h^2 \nabla u),$$

that is, the weighted Laplace operator of the weighted manifold $\left(\Omega, h^2 dm\right) = \left(\Omega, \omega h^2 dm_0\right)$. Using the product rule for $\operatorname{div}_\omega$, we obtain

$$\widetilde{\mathcal{L}}u = \mathcal{L}u + 2\langle \frac{\nabla h}{h}, \nabla u\rangle.$$

On the other hand, by the product rule (2.2) for $\mathcal{L}$ we have

$$\mathcal{L}(hu) = h\mathcal{L}u + 2\langle \nabla h, \nabla u\rangle + u\mathcal{L}h,$$

which implies the identity

$$\mathcal{L}(hu) = h\widetilde{\mathcal{L}}u + u\mathcal{L}h. \tag{4.3}$$

Using (4.3) with $u = \phi(v)$ and applying the chain rule (2.3) for $\widetilde{\mathcal{L}}$, we obtain

$$\begin{aligned}
\mathcal{L}(h\phi(v)) &= h\widetilde{\mathcal{L}}\phi(v) + \phi(v)\mathcal{L}h \\
&= h\left(\phi'(v)\widetilde{\mathcal{L}}v + \phi''(v)|\nabla v|^2\right) + \phi(v)\mathcal{L}h \\
&= \phi'(v)(h\widetilde{\mathcal{L}}v + v\mathcal{L}h) + \phi''(v)|\nabla v|^2 h + \big(\phi(v) - v\phi'(v)\big)\mathcal{L}h \\
&= \phi'(v)\mathcal{L}(hv) + \phi''(v)|\nabla v|^2 h + \big(\phi(v) - v\phi'(v)\big)\mathcal{L}h,
\end{aligned}$$

which proves (4.1). Then (4.2) follows immediately from (4.1). $\square$

Lemma 4.2. *Let ϕ be a C^2 function on an interval $I \subset \mathbb{R}$ such that $\phi > 0$ and $\phi' > 0$ in I. For two functions $v, h \in C^2(\Omega)$, $h > 0$, set*

$$u = h\phi(v)$$

assuming that $\phi(v)$ is well-defined, that is, $v(\Omega) \subset I$.

If the function u satisfies the inequality

$$-\mathcal{L}u + Vu^q \geq -\mathcal{L}h \tag{4.4}$$

in Ω, where $V \in C(\Omega)$, $q \in \mathbb{R} \setminus \{0\}$, then the function v satisfies in Ω the inequality

$$-\mathcal{L}(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)} \geq \left(\frac{\phi(v) - 1}{\phi'(v)} - v \right) \mathcal{L}h + \frac{\phi''(v)}{\phi'(v)} |\nabla v|^2 h. \tag{4.5}$$

If instead u satisfies

$$-\mathcal{L}u + Vu^q \leq -\mathcal{L}h, \tag{4.6}$$

then (4.5) holds with $\leq$ instead of $\geq$.

Proof. It follows from $u = h\phi(v)$ and (4.4) that

$$\mathcal{L}(h\phi(v)) \leq h^q V \phi(u)^q + \mathcal{L}h. \tag{4.7}$$

Substituting this into (4.2) we obtain

$$-\mathcal{L}(hv) \geq -\frac{h^q V \phi(u)^q + \mathcal{L}h}{\phi'(v)} + \frac{\phi''(v)}{\phi'(v)} |\nabla v|^2 h + \left(\frac{\phi(v)}{\phi'(v)} - v \right) \mathcal{L}h,$$

whence (4.5) follows. The second claim is proved in the same way. $\square$

Lemma 4.3. *Under the hypotheses of Lemma 4.2, assume in addition that $\mathcal{L}h \leq 0$ in Ω and $0 \in I$. If in I*

$$\phi(0) = 1, \quad \phi' > 0, \quad \phi'' \geq 0, \tag{4.8}$$

then the function v satisfies the following differential inequality in Ω

$$-\mathcal{L}(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)} \geq 0. \tag{4.9}$$

If instead of (4.8) we have

$$\phi(0) = 1, \quad \phi' > 0, \quad \phi'' \leq 0, \tag{4.10}$$

then v satisfies in Ω

$$-\mathcal{L}(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)} \leq 0. \tag{4.11}$$

Proof. Consider the case (4.8). By the mean value theorem, for any $v \in I$ there exists $\xi \in [0, v]$ such that

$$\frac{\phi(v) - 1}{v} = \frac{\phi(v) - \phi(0)}{v} = \phi'(\xi).$$

By the convexity of ϕ we obtain $\phi'(\xi) \leq \phi'(v)$ provided $v > 0$, that is

$$\frac{\phi(v) - 1}{v} \leq \phi'(v) \quad \text{for } v > 0,$$

and the opposite inequality in the case $v < 0$. It follows that, for all $v \in I$,

$$\frac{\phi(v) - 1}{\phi'(v)} - v \leq 0.$$

Substituting into (4.5) and using also $\mathcal{L}h \leq 0$ and (4.8), we obtain (4.9). The proof in the case (4.10) is similar. $\square$

Remark. Note that in the case $\mathcal{L}h \equiv 0$ the condition $\phi(0) = 1$ in (4.8) and (4.10) is not required as in this case the term

$$\left(\frac{\phi(v)-1}{\phi'(v)} - v\right)\mathcal{L}h$$

vanishes identically.

Lemma 4.4. *Suppose Ω is an open subset of M and F is a l.s.c. $\mathcal{L}$-superharmonic function in Ω. Suppose $F = F_1 + F_2$, where*

$$\liminf_{x\to\partial_\infty\Omega} F_1(x) \geq 0 \quad \textit{and} \quad F_2 \geq -P, \tag{4.12}$$

where $P = G^\Omega\mu$ is a Green potential of a positive measure μ in Ω so that $P \not\equiv +\infty$ on every component of Ω. Then $F \geq 0$ in Ω.

Proof. Indeed, the function $F + P$ is obviously superharmonic, and $F + P \geq F_1$. Hence $\liminf_{x\to\partial_\infty\Omega}(F+P)(x) \geq 0$, and by the standard form of the maximum principle $F+P \geq 0$ on Ω (cf. [3], [19, Sec. 5.4.3]). Hence F is a superharmonic majorant of $-P$, whose least superharmonic majorant must be zero (with the same proof as in the classical case [3, Theorem 4.2.6]), which yields $F \geq 0$. □

The following version of the maximum principle will be frequently used.

Lemma 4.5. *Let Ω be an open subset of M and let $v \in C^2(\Omega)$ satisfy*

$$\begin{cases} -\mathcal{L}v \geq f & \textit{in } \Omega, \\ \liminf_{x\to\partial_\infty\Omega} v(x) \geq 0, & \end{cases}$$

where $f \in C(\Omega)$ such that $G^\Omega f$ is well defined in Ω. Then for all $x \in \Omega$

$$v(x) \geq G^\Omega f(x). \tag{4.13}$$

Proof. If $G^\Omega f_- = +\infty$ then (4.13) is trivially satisfied. Hence, assume in the sequel that $G^\Omega f_- < \infty$. Let us approximate f from below by a sequence $\{f_n\}$ of C^1 functions in Ω such that $f_n \uparrow f$ as $n \to \infty$ and $G^\Omega f_n^- < \infty$ (where $f_n^\pm := (f_n)_\pm$). Moreover, we can also assume that f_n^+ is compactly supported in Ω.

Fix n and consider in Ω two functions

$$F_1 = v + G^\Omega f_n^- \quad \text{and} \quad F_2 = -G^\Omega f_n^+.$$

The hypotheses (4.12) of Lemma 4.4 are obviously satisfied. The function

$$F = v + G^\Omega f_n^- - G^\Omega f_n^+$$

is superharmonic in Ω since

$$-\mathcal{L}F = -\mathcal{L}v + f_n^- - f_n^+ = f - f_n \geq 0.$$

By Lemma 4.4 we conclude that $F \geq 0$ in Ω and, hence,

$$v \geq G^\Omega f_n^+ - G^\Omega f_n^-.$$

Letting $n \to \infty$ and using the convergence theorems we obtain (4.13) □

5. Proof of Theorem 3.2

We start the proof with a particular case of Theorem 3.2 where the idea of the proof is most transparent and not buried in technical complications.

Proof of Theorem 3.2 in the special case $h > 0$, $u > 0$ in $\overline{\Omega}$, and $V \in C(\overline{\Omega})$. In this case the function $G^{\Omega}(h^q V)(x)$ is finite for all $x \in \Omega$.

Choose a function ϕ (to be used in Lemma 4.3) to solve the initial value problem

$$\phi'(s) = \phi(s)^q, \quad \phi(0) = 1. \tag{5.1}$$

For $q = 1$ this gives

$$\phi(s) = e^s, \quad s \in \mathbb{R}, \tag{5.2}$$

while for $q \neq 1$ we obtain

$$\phi(s) = [(1-q)s + 1]^{\frac{1}{1-q}}, \quad s \in I_q, \tag{5.3}$$

where the domain I_q of ϕ is given by:

$$I_q = \begin{cases} (-\infty, \frac{1}{q-1}) & \text{if } q > 1, \\ (-\infty, +\infty) & \text{if } q = 1, \\ (-\frac{1}{1-q}, +\infty) & \text{if } q < 1 \end{cases} \tag{5.4}$$

(see Fig. 1).

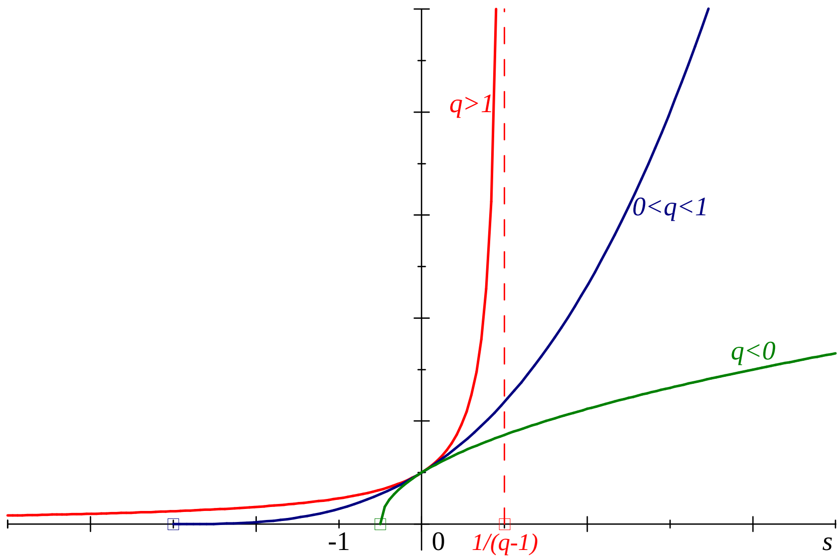

Figure 1. Examples of the function ϕ in three cases $q > 1$, $0 < q < 1$, $q < 0$. The boxed points have the abscissa $\frac{1}{q-1}$.

Note that in all cases $\phi(I_q) = (0, \infty)$. Also we have

$$\phi'(s) = [(1-q)s + 1]^{\frac{q}{1-q}}, \quad \phi''(s) = q[(1-q)s + 1]^{\frac{2q-1}{1-q}}. \tag{5.5}$$

In particular, $\phi' > 0$ in I_q, whereas $\phi'' > 0$ for $q > 0$ and $\phi'' < 0$ for $q < 0$. Consequently, the inverse function ϕ^{-1} is well-defined on $(0, \infty)$.

In the case $0 < q < 1$ it will be convenient for us to extend the domain of ϕ to all $s \leq -\frac{1}{1-q}$ by setting $\phi(s) = 0$ so that in this case we have for all $s \in (-\infty, \infty)$

$$\phi(s) = [(1-q)s + 1]_+^{\frac{1}{1-q}}. \tag{5.6}$$

Observe that all the estimates (3.4), (3.6), (3.7) that we need to prove in the case $q > 0$ can be written in the unified form

$$\frac{u(x)}{h(x)} \geq \phi\left(-\frac{1}{h(x)}G^\Omega(h^q V)(x)\right), \tag{5.7}$$

for all $x \in \Omega$. Similarly, estimate (3.8) in the case $q < 0$ is equivalent to the opposite inequality

$$\frac{u(x)}{h(x)} \leq \phi\left(-\frac{1}{h(x)}G^\Omega(h^q V)(x)\right). \tag{5.8}$$

Since by hypothesis the functions h and u are positive in $\overline{\Omega}$, the function

$$v = \phi^{-1}\left(\frac{u}{h}\right) \tag{5.9}$$

is well-defined in $\overline{\Omega}$ and belongs to the class $C^2(\Omega) \cap C\left(\overline{\Omega}\right)$.

Consider first the case $q > 0$. In this case we will deduce (5.7) from the following inequality for v:

$$v(x) \geq -\frac{1}{h(x)}G^\Omega(h^q V)(x), \tag{5.10}$$

for all $x \in \Omega$. Indeed, if (5.10) holds then applying ϕ to both sides of (5.10) and observing that $\phi(v) = \frac{u}{h}$, we obtain (5.7). However, we should first verify that the both sides of (5.10) are in the domain of ϕ. In the cases $q = 1$ and $0 < q < 1$ the (extended) domain of ϕ is $(-\infty, +\infty)$, so that there is no problem. In the case $q > 1$ we have $v(x) \in I_q = (-\infty, \frac{1}{q-1})$ by (5.9), which implies that the right hand side of (5.10), being bounded by $v(x)$, is also in I_q. This argument also shows that in Ω

$$\frac{1}{q-1} > -\frac{1}{h(x)}G^\Omega(h^q V)(x),$$

which proves (3.5).

To prove (5.10) observe that the function $u = h\phi(v)$ satisfies

$$-\mathcal{L}u + Vu^q \geq -\mathcal{L}h \geq 0$$

in Ω as required by Lemma 4.3. In the case $q > 0$ the function ϕ satisfies (4.8), and we obtain by inequality (4.9) of Lemma 4.3 and by (5.1) that in Ω

$$-\mathcal{L}(hv) + h^q V \geq 0. \tag{5.11}$$

Since $u \geq h$ on $\partial\Omega$, it follows that on $\partial\Omega$

$$hv = h\phi^{-1}\left(\frac{u}{h}\right) \geq h\phi^{-1}(1) = 0.$$

Since hv satisfies (5.11) and the boundary condition $hv \geq 0$ on $\partial\Omega$, we obtain by the maximum principle that in Ω

$$hv \geq -G^\Omega(h^q V), \tag{5.12}$$

which is equivalent to (5.10).

Consider now the case $q < 0$. Then we have

$$-\mathcal{L}u + Vu^q \leq -\mathcal{L}h$$

and, hence, obtain by inequality (4.11) of Lemma 4.3 and (5.1) that in Ω,

$$-\mathcal{L}(hv) + h^q V \leq 0. \tag{5.13}$$

In this case we have $u \leq h$ on $\partial\Omega$, which implies $hv \leq 0$ on $\partial\Omega$. Using (5.13) with this boundary condition, we obtain that in Ω

$$hv \leq -G^{\Omega}(h^q V)$$

and, hence,

$$v \leq -\frac{1}{h} G^{\Omega}(h^q V). \tag{5.14}$$

Since $v(x) \in I_q = (-\frac{1}{1-q}, +\infty)$, it follows that both sides of (5.14) belong to I_q. Consequently, we have

$$-\frac{1}{1-q} < -\frac{1}{h} G^{\Omega}(h^q V),$$

which proves (3.5). Applying ϕ to both sides of (5.14), we obtain (5.8) and, hence, (3.8). □

Proof of Theorem 3.2 in the general case. We will use the same function ϕ as defined above by (5.2)-(5.3), but it will be convenient to extend the domain I_q of ϕ to the endpoints of the interval I_q by taking the limits of ϕ at the endpoints. The extended domain of ϕ is therefore the interval

$$\overline{I}_q := \begin{cases} [-\infty, \frac{1}{q-1}] & \text{if } q > 1, \\ [-\infty, +\infty] & \text{if } q = 1, \\ [-\frac{1}{1-q}, +\infty] & \text{if } q < 1. \end{cases}$$

Moreover, in the case $0 < q < 1$ we extend $\phi(s)$ further to all $s \in [-\infty, +\infty]$ by using (5.6).

With these extensions the required estimates (3.4), (3.6) and (3.7) in the case $q > 0$ can be written in the unified form (5.7), and the estimate (3.8) – in the form (5.8).

Consider first the case $q > 0$. For any $\varepsilon > 0$, set

$$u_\varepsilon = u + \varepsilon$$

and define the function v_ε in Ω via

$$v_\varepsilon = \phi^{-1}\left(\frac{u_\varepsilon}{h}\right),$$

where ϕ is the same as above. Since u_ε and h are positive in Ω, the function v_ε is well-defined in Ω and belongs to $C^2(\Omega)$. Note also that $v_\varepsilon(\Omega) \subset I_q$.

Applying identity (4.2) to functions $h, v_\varepsilon \in C^2(\Omega)$, we obtain

$$-\mathcal{L}(hv_\varepsilon) = -\frac{\mathcal{L}(h\phi(v_\varepsilon))}{\phi'(v)} + \frac{\phi''(v_\varepsilon)}{\phi'(v_\varepsilon)} |\nabla v_\varepsilon|^2 h + \left(\frac{\phi(v_\varepsilon)}{\phi'(v_\varepsilon)} - v_\varepsilon\right) \mathcal{L}h.$$

Since

$$-\mathcal{L}(h\phi(v_\varepsilon)) = -\mathcal{L}u_\varepsilon = -\mathcal{L}u,$$

it follows

$$-\mathcal{L}(hv_\varepsilon) = \frac{-\mathcal{L}u}{\phi'(v_\varepsilon)} + \frac{\phi''(v_\varepsilon)}{\phi'(v_\varepsilon)} |\nabla v_\varepsilon|^2 h + \left(\frac{\phi(v_\varepsilon)}{\phi'(v_\varepsilon)} - v_\varepsilon\right) \mathcal{L}h. \tag{5.15}$$

Observe also that by (5.1)

$$\phi'(v_\varepsilon) = \phi(v_\varepsilon)^q = \left(\frac{u_\varepsilon}{h}\right)^q. \tag{5.16}$$

Since $q > 0$, we have by (3.14)

$$-\mathcal{L}u \geq -Vu^q - \mathcal{L}h.$$

Substituting this and (5.16) into (5.15), we obtain

$$-\mathcal{L}(hv_\varepsilon) \geq -h^q \left(\frac{u}{u_\varepsilon}\right)^q V + \frac{\phi''(v_\varepsilon)}{\phi'(v_\varepsilon)}|\nabla v_\varepsilon|^2 h + \left(\frac{\phi(v_\varepsilon)-1}{\phi'(v_\varepsilon)} - v_\varepsilon\right)\mathcal{L}h.$$

Since ϕ satisfies (4.8) and, hence, the last two terms on the right-hand side of the preceding inequality are nonnegative (cf. the proof of Lemma 4.3), we arrive at

$$-\mathcal{L}(hv_\varepsilon) \geq -h^q \left(\frac{u}{u_\varepsilon}\right)^q V \ \text{ in } \Omega. \tag{5.17}$$

In the case $q \neq 1$, $q > 0$ we have by (5.3)

$$\phi^{-1}(s) = \frac{s^{1-q}-1}{1-q}, \quad s > 0,$$

and, hence, in Ω

$$hv_\varepsilon = h\phi^{-1}\left(\frac{u_\varepsilon}{h}\right) = \frac{1}{1-q}\left(h^q u_\varepsilon^{1-q} - h\right).$$

It follows that, for all $y \in \partial\Omega$,

$$\lim_{x\to y,\, x\in\Omega} h(x)v_\varepsilon(x) = \frac{1}{1-q}\left(h^q(y)u_\varepsilon(y)^{1-q} - h(y)\right) \geq 0,$$

since $u_\varepsilon(y) \geq h(y) + \varepsilon > h(y)$.

For $q = 1$ we have $\phi^{-1}(s) = \ln s$ and, hence, in Ω

$$hv_\varepsilon = h\ln\left(\frac{u_\varepsilon}{h}\right). \tag{5.18}$$

For any $y \in \partial\Omega$ such that $h(y) > 0$, we obtain

$$\lim_{x\to y,\, x\in\Omega} h(x)v_\varepsilon(x) = h(y)\ln\left(\frac{u_\varepsilon(y)}{h(y)}\right) > 0,$$

and if $h(y) = 0$, then, using $u_\varepsilon \geq \varepsilon$, we obtain from (5.18)

$$\lim_{x\to y,\, x\in\Omega} h(x)v_\varepsilon(x) = 0. \tag{5.19}$$

Hence, in the case $q > 0$, we can extend hv_ε by continuity to $\overline{\Omega}$ so that $hv_\varepsilon \in C(\overline{\Omega}) \cap C^2(\Omega)$ and

$$hv_\varepsilon \geq 0 \ \text{ on } \partial\Omega.$$

Note that $h^q \left(\frac{u}{u_\varepsilon}\right)^q V \in C(\Omega)$ and $G^\Omega\left(h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right)$ is well-defined in Ω, since

$$G^\Omega\left(h^q\left(\frac{u}{u_\varepsilon}\right)^q V_\pm\right) \leq G^\Omega\left(h^q V_\pm\right),$$

and $G^\Omega(h^q V)$ is well-defined by hypothesis. Hence, by the maximum principle of Lemma 4.5, we conclude from (5.17) and (5.19) that

$$hv_\varepsilon \geq -G^\Omega\left(h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right)$$

and, hence,

$$v_\varepsilon \geq -\frac{1}{h}G^\Omega\left(h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right) \ \text{ in } \Omega. \tag{5.20}$$

Assume now $q \geq 1$. Assume also that $G^\Omega(h^q V_+) \not\equiv +\infty$ in Ω, because otherwise, (3.4), (3.5) and (3.6) are trivially satisfied, and so there is nothing to prove. Let us first show that under these assumptions $u > 0$ in Ω. Observe that if $G^\Omega(h^q V_+) \not\equiv +\infty$ in Ω, then $G^\Omega(h^q V_+)(x) < +\infty$ for every $x \in \Omega$. Indeed, for an open set $\Omega' \Subset \Omega$ with smooth boundary, fix a function $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ in Ω'. Then the function $G^{\Omega}\left(h^q V_+\right) - G^{\Omega}\left(\eta h^q V_+\right)$ is harmonic in Ω', and $G^{\Omega}\left(\eta h^q V_+\right)$ is bounded in Ω since $\eta h^q V_+ \in C(\overline{\Omega})$. Consequently, $G^{\Omega}\left(h^q V_+\right)$ is finite in Ω', and hence in Ω.

It follows from (5.20) and $u \leq u_\varepsilon$, that

$$v_\varepsilon \geq -\frac{1}{h} G^{\Omega}\left(h^q V_+\right). \tag{5.21}$$

Since the value $v_\varepsilon = \phi^{-1}\left(\frac{u_\varepsilon}{h}\right)$ belongs to I_q and the value of the right hand side of (5.21) lies in $[-\infty, 0]$, which, in the present case $q \geq 1$, is contained in $\overline{I}_q$, we can apply ϕ to both sides of this inequality and obtain

$$u_\varepsilon \geq h\phi\left(-\frac{G^{\Omega}\left(h^q V_+\right)}{h}\right). \tag{5.22}$$

Letting $\varepsilon \to 0$ we obtain

$$u \geq h\phi\left(-\frac{G^{\Omega}\left(h^q V_+\right)}{h}\right) \quad \text{in } \Omega.$$

Since $G^{\Omega}\left(h^q V_+\right) < \infty$, it follows that $u > 0$ in Ω as was claimed.

Let us return to (5.20). Since $v_\varepsilon \in I_q$ and, hence, the right hand side of (5.20) lies in $\overline{I}_q$, we can apply ϕ to the both sides of this inequality and obtain

$$u_\varepsilon \geq h\phi\left(-\frac{G^{\Omega}\left(h^q \left(\frac{u}{u_\varepsilon}\right)^q V\right)}{h}\right) \quad \text{in } \Omega, \tag{5.23}$$

The positivity of u in Ω implies $\frac{u}{u_\varepsilon} \uparrow 1$ in Ω as $\varepsilon \to 0$, whence by the monotone convergence theorem,

$$G^{\Omega}\left(h^q \left(\frac{u}{u_\varepsilon}\right)^q V\right) \to G^{\Omega}\left(h^q V\right) \quad \text{as } \varepsilon \to 0 \tag{5.24}$$

pointwise in Ω. In particular, we have, for any $x \in \Omega$,

$$-\frac{G^{\Omega}\left(h^q V\right)(x)}{h(x)} \in \overline{I}_q. \tag{5.25}$$

Letting $\varepsilon \to 0$ in (5.23), we deduce, for $q \geq 1$,

$$u \geq h\phi\left(-\frac{G^{\Omega}\left(h^q V\right)}{h}\right) \quad \text{in } \Omega,$$

which proves (3.4) and (3.6). In the case $q > 1$, it follows that

$$\phi\left(-\frac{G^{\Omega}\left(h^q V\right)}{h}\right) \leq \frac{u}{h} < \infty$$

and, hence,

$$-\frac{G^{\Omega}\left(h^q V\right)}{h} < \frac{1}{q-1},$$

which proves (3.5).

Assume now $0 < q < 1$. We employ the same argument up to (5.20). The extended function ϕ is defined in this case on $[-\infty, +\infty]$ by (5.6). Applying ϕ to the both sides of (5.20) we obtain

$$u_\varepsilon \geq h\phi\left(-\frac{1}{h} G^{\Omega}\left(h^q \left(\frac{u}{u_\varepsilon}\right)^q V\right)\right). \tag{5.26}$$

In this case u can actually vanish inside Ω. Letting $\varepsilon \to 0$, we see that $\frac{u}{u_\varepsilon}(x) \uparrow 1$ if $u(x) > 0$ and $\frac{u}{u_\varepsilon} = 0$ if $u(x) = 0$, that is

$$\frac{u}{u_\varepsilon} \uparrow \chi_u \text{ pointwise in } \Omega.$$

Passing to the limit in (5.26) as $\varepsilon \to 0$ and using the monotone convergence theorem gives

$$u \geq h\phi\left(-\frac{1}{h}G^\Omega\left(\chi_u h^q V\right)\right) \text{ in } \Omega. \tag{5.27}$$

which is equivalent to (3.7).

Consider the last case $q < 0$. We define for any $\varepsilon > 0$ the function v_ε in a slightly different way as follows:

$$v_\varepsilon = \phi^{-1}\left(\frac{u}{h_\varepsilon}\right),$$

where $h_\varepsilon = h + \varepsilon$. Since $\frac{u}{h_\varepsilon} > 0$ in Ω, we obtain $v_\varepsilon \in C^2(\Omega)$. The function

$$\phi^{-1}(s) = \frac{s^{1-q} - 1}{1-q}, \tag{5.28}$$

initially defined for $s > 0$, extends continuously to $s = 0$ by setting $\phi^{-1}(0) = -\frac{1}{1-q}$. Since $\frac{u}{h_\varepsilon}$ is continuous and nonnegative in $\overline{\Omega}$, we obtain $v_\varepsilon \in C\left(\overline{\Omega}\right)$. Moreover, since on the boundary $\partial\Omega$ we have $u \leq h < h_\varepsilon$, it follows that $v_\varepsilon \leq \phi^{-1}(1) = 0$ and, hence,

$$h_\varepsilon v_\varepsilon \leq 0 \text{ on } \partial\Omega. \tag{5.29}$$

Since $\mathcal{L}h_\varepsilon \leq 0$ and $u = h_\varepsilon \phi(v_\varepsilon)$ satisfies by (3.15)

$$-\mathcal{L}u + Vu^q \leq -\mathcal{L}h_\varepsilon,$$

we obtain by inequality (4.11) of Lemma 4.3 and (5.1) that

$$-\mathcal{L}(h_\varepsilon v_\varepsilon) + h_\varepsilon^q V \leq 0 \text{ in } \Omega. \tag{5.30}$$

Since $q < 0$ and

$$G^\Omega(h_\varepsilon^q V_\pm) \leq G^\Omega(h^q V_\pm),$$

it follows that $G^\Omega(h_\varepsilon^q V)$ is well-defined. Hence, we obtain from (5.30) and (5.29) by the maximum principle of Lemma 4.5, that

$$h_\varepsilon v_\varepsilon \leq -G^\Omega(h_\varepsilon^q V) \quad \text{in } \Omega,$$

that is,

$$v_\varepsilon \leq -\frac{G^\Omega(h_\varepsilon^q V)}{h_\varepsilon} \text{ in } \Omega. \tag{5.31}$$

Since $v_\varepsilon(\Omega) \subset I_q = (-\frac{1}{1-q}, \infty)$, it follows that

$$-\frac{G^\Omega(h_\varepsilon^q V)}{h_\varepsilon} \in (-\frac{1}{1-q}, +\infty] \subset \overline{I}_q. \tag{5.32}$$

Applying ϕ to both sides of (5.31), we obtain

$$\phi(v_\varepsilon) \leq \phi\left(-\frac{G^\Omega(h_\varepsilon^q V)}{h_\varepsilon}\right) \quad \text{in } \Omega,$$

which is equivalent to

$$u \leq h_\varepsilon \left[1 - (1-q)\frac{G^\Omega(h_\varepsilon^q V)}{h_\varepsilon}\right]^{\frac{1}{1-q}} \quad \text{in } \Omega$$

and, hence, to

$$u \leq h_\varepsilon \left[1-(1-q)\,\frac{G^\Omega(h_\varepsilon^q V_+)}{h_\varepsilon}+(1-q)\,\frac{G^\Omega(h_\varepsilon^q V_-)}{h_\varepsilon}\right]^{\frac{1}{1-q}}. \tag{5.33}$$

Note that the expression in the square brackets here belongs to $(0,+\infty]$ by (5.32). In particular, we have $G^\Omega(h_\varepsilon^q V_+)<\infty$. Since $0<h<h_\varepsilon$ in Ω and $q<0$, we see that in Ω

$$\frac{G^\Omega(h_\varepsilon^q V_-)}{h_\varepsilon} \leq \frac{G^\Omega(h^q V_-)}{h}. \tag{5.34}$$

Since $h_\varepsilon^q \uparrow h^q$ as $\varepsilon \to 0$, we obtain by the monotone convergence theorem, that

$$G^\Omega(h_\varepsilon^q V_+) \to G^\Omega(h^q V_+) \ \text{ pointwise in } \Omega. \tag{5.35}$$

Since by hypothesis $G^\Omega(h^q V)$ is well-defined, we obtain as $\varepsilon\to 0$ from (5.33), (5.34) and (5.35) that

$$u \leq h\left[1-(1-q)\,\frac{G^\Omega(h^q V)}{h}\right]^{\frac{1}{1-q}} \ \text{ in } \ \Omega.$$

By construction the expression in the square brackets here belongs to $[0,+\infty]$. Since by hypothesis $u>0$ in Ω, we obtain that this expression cannot vanish, which proves (3.5) in this case. □

6. Proof of Theorem 3.1

Consider first the case $q>0$. By hypothesis, the function f is continuous and nonnegative in Ω. In the proof we need f to be locally Hölder continuous because in this case the function $G^U f$ is of the class C^2 for any relatively compact domain $U\subset\Omega$.

Let us approximate a given continuous function f in Ω from below by a sequence $\{f_k\}_{k=1}^\infty$ of C^1 functions f_k so that

$$f_k \uparrow f \ \text{ as } \ k\to\infty \tag{6.1}$$

pointwise. Replacing each f_k by $(f_k)_+$, we obtain a sequence $\{f_k\}$ of nonnegative locally Lipschitz functions satisfying (6.1).

Set $h_k = G^\Omega f_k$ and observe that $h_k \leq h<\infty$ and $h_k\uparrow h$ pointwise in Ω as $k\to\infty$. Since

$$G^\Omega\left(h_k^q V_\pm\right) \leq G^\Omega\left(h^q V_\pm\right),$$

we see that one of the values $G^\Omega\left(h_k^q V_\pm\right)$ is finite and, hence, $G^\Omega\left(h_k^q V\right)$ is well-defined. Since

$$G^\Omega\left(h_k^q V_\pm\right) \to G^\Omega\left(h^q V_\pm\right),$$

we obtain that

$$G^\Omega\left(h_k^q V\right) \to G^\Omega\left(h^q V\right) \tag{6.2}$$

pointwise in Ω. The same is true for $G^\Omega\left(\chi_u h_k^q V\right)$ in the case (iii).

Since $f_k\leq f$, we obtain that u satisfies $-\mathcal{L}u+Vu^q\geq f_k$ in Ω. Therefore, if statements $(i),(ii),(iii)$ are already proved for locally Lipschitz functions f, then we obtain the corresponding lower bounds (3.4), (3.6), (3.7) of u with h_k in place of h. Letting $k\to\infty$ and using (6.2), we obtain the same estimates of u via h as claimed.

In the case (ii) we still need to prove (3.5) for h assuming that it is true with h_k in place of h. Passing to the limit as $k\to\infty$, we obtain a non-strict inequality

$$-(q-1)G^\Omega(h^q V)(x) \leq h(x). \tag{6.3}$$

However, estimate (3.6) implies that the expression in the square brackets in (3.6) cannot vanish, which yields a strict inequality in (6.3), that is, (3.5).

Continuing the proof in the case $q>0$, we can assume now that f is locally Hölder (even Lipschitz) continuous. Let $\{\Omega_n\}_{n=1}^{\infty}$ be an exhaustion of Ω by relatively compact, connected, open sets $\Omega_n \Subset \Omega$ with smooth boundaries. Set $h_n = G^{\Omega_n} f$. Since f is locally Hölder continuous and $\partial\Omega_n$ is regular, we have $h_n \in C^2(\Omega_n) \cap C\left(\overline{\Omega}_n\right)$ and

$$\begin{cases} -\mathcal{L}h_n = f & \text{in } \Omega_n, \\ h_n = 0 & \text{on } \partial\Omega_n. \end{cases}$$

We can always take n large enough so that $f \not\equiv 0$ in Ω_n and, hence, $0 < h_n < \infty$ in Ω_n.

Observe that by the monotone convergence theorem

$$h_n \uparrow h := G^{\Omega} f \quad \text{as } n \to \infty.$$

Fix a point $x \in \Omega$ and let n be so large that $x \in \Omega_n$. Since u satisfies (3.2) in Ω, it follows that

$$\begin{cases} -\mathcal{L}u + Vu^q \geq f = -\mathcal{L}h_n & \text{in } \Omega_n, \\ u \geq 0 = h_n & \text{on } \partial\Omega_n. \end{cases}$$

Applying Theorem 3.2 in Ω_n we obtain

$$u(x) \geq \begin{cases} h_n(x) e^{-\frac{G^{\Omega_n}(h_n V)(x)}{h_n(x)}}, & \text{if } q=1, \\ h_n(x)\left[1 + (q-1)\frac{G^{\Omega_n}(h_n^q V)(x)}{h_n(x)}\right]^{-\frac{1}{q-1}}, & \text{if } q>1, \\ h_n(x)\left[1 + (q-1)\frac{G^{\Omega_n}(\chi_n h_n^q V)(x)}{h_n(x)}\right]_+^{-\frac{1}{q-1}}, & \text{if } 0<q<1, \end{cases} \tag{6.4}$$

where $\chi_n := \chi_{u|_{\Omega_n}}$. Since $h_n^q \uparrow h^q$ as $n \to \infty$, we obtain by the monotone convergence theorem,

$$\lim_{n\to\infty} G^{\Omega_n}(h_n^q V_\pm)(x) = G^{\Omega}(h^q V_\pm)(x) \tag{6.5}$$

(and a similar identity for the term with $\chi_n h_n^q V$). Passing to the limit in (6.4) as $n \to \infty$, we arrive at

$$u(x) \geq \begin{cases} h(x) e^{-\frac{G^{\Omega}(hV)(x)}{h(x)}}, & \text{if } q=1, \\ h(x)\left[1 + (q-1)\frac{G^{\Omega}(h^q V)(x)}{h(x)}\right]^{-\frac{1}{q-1}}, & \text{if } q>1, \\ h(x)\left[1 + (q-1)\frac{G^{\Omega}(\chi_u h^q V)(x)}{h(x)}\right]_+^{-\frac{1}{q-1}}, & \text{if } 0<q<1, \end{cases} \tag{6.6}$$

which proves estimates (3.6), (3.7), (3.8).

In the case $q>1$ the expression in square brackets in (6.6) is non-negative as the limit of that of (6.4). However, since the exponent $-\frac{1}{q-1}$ is in this case negative and $\frac{u(x)}{h(x)} < \infty$, it actually has to be positive, which proves (3.5).

Consider now the case $q<0$. In this case we approximate f from above by a sequence of C^1 functions f_k such that $f_k \downarrow f$ and set $h_k = G^{\Omega} f_k$. The function f_1 should be chosen so close to f that $h_1 < \infty$. Then $h_k \downarrow h$ pointwise in Ω, and, since $q<0$, we have $h_k^q \uparrow h^q$ as $k \to \infty$. The same argument as in the case $q>0$ shows that $G^{\Omega}\left(h_k^q V\right)$ is well-defined and (6.2) holds. Since $f_k \geq f$, the function u satisfies in Ω the inequality $-\mathcal{L}u + Vu^q \leq f_k$. If (iv) is already proved for locally Hölder continuous f, then we conclude that (3.8) holds with h_k instead of h. Letting $k \to \infty$, we complete the proof (condition (3.5) is proved in the same way as in the case $q>0$).

Hence, we assume in what follows that f is locally Hölder continuous. In this case the proof goes the same way as in Theorem 3.2. Observe first that $G^{\Omega} f \in C^2(\Omega)$. Indeed, for any relatively compact open set $\Omega' \subset \Omega$ with smooth boundary it is known that $G^{\Omega'} f \in C^2(\Omega')$. Since the difference $G^{\Omega} f - G^{\Omega'} f$ is harmonic in Ω', it follows that it is smooth in Ω', which implies that $G^{\Omega}f \in C^2(\Omega')$. By exhausting Ω with relatively compact open subsets, we obtain $G^{\Omega}f \in C^2(\Omega)$ as claimed.

For any $\varepsilon > 0$ set $h_\varepsilon = \varepsilon + G^{\Omega}f$, so that $-\mathcal{L}h_\varepsilon = f$. Since $u, h_\varepsilon > 0$ in Ω, the function $v_\varepsilon = \phi^{-1}\left(\frac{u}{h_\varepsilon}\right)$ belongs to $C^2(\Omega)$ and, similarly to the proof of Theorem 3.2 (cf. (5.30)), we obtain the following inequality in Ω

$$-\mathcal{L}(h_\varepsilon v_\varepsilon) + h_\varepsilon^q V \le 0.$$

Note that in this case we have by (5.28)

$$h_\varepsilon v_\varepsilon = h_\varepsilon \phi^{-1}\left(\frac{u}{h_\varepsilon}\right) = h_\varepsilon^q \frac{u^{1-q} - h_\varepsilon^{1-q}}{1-q}.$$

Using the boundary condition in (3.3) and $h_\varepsilon \ge \varepsilon$, we obtain

$$\limsup_{y \to \partial_\infty \Omega} (h_\varepsilon v_\varepsilon)(y) \le 0.$$

Applying Lemma 4.5 to $-h_\varepsilon v_\varepsilon$ we obtain

$$-h_\varepsilon v_\varepsilon \ge G^{\Omega}(h_\varepsilon^q V).$$

Letting $\varepsilon \to 0$ and arguing as in the proof of Theorem 3.2, we finish the proof.

Remark. Note that (6.4) implies immediately the lower bounds of Theorem 3.1(i), (ii), (iii) by passing to the limit as $n \to \infty$, provided we use a relaxed definition of the expression $G^{\Omega}(h^q V)$ given by (3.9). A similar observation holds also for the upper estimate of (iv).

7. Proof of Theorem 3.3

The proof is similar to that of Theorem 3.2, but simpler. Let $\{\Omega_n\}$ be an exhaustion of Ω as above.

Assume first $q \ge 1$ and define for any n a function $h_n \in C^2(\Omega_n) \cap C\left(\overline{\Omega}\right)$ as the solution of

$$\begin{cases} \mathcal{L}h_n = 0 & \text{in } \Omega_n, \\ h_n = u & \text{on } \partial\Omega_n. \end{cases}$$

In cases $(i), (ii)$, we have $h_n > 0$ in Ω_n for large enough n by (3.22) and (3.24) respectively. By Theorem 3.2 it follows that $u(x) > 0$ for all $x \in \Omega_n$. Consequently, $u(x) > 0$ for all $x \in \Omega$.

In the case $q = 1$, set $h \equiv 1$, $v = \ln u$. As in the proof of Theorem 3.2 (cf. (5.11)), we obtain

$$-\mathcal{L}v + V \ge 0.$$

Since by (3.22) we have $\liminf_{y \to \partial_\infty \Omega} v(y) \ge 0$, we conclude by Lemma 4.5 that

$$\ln u(x) = v(x) \ge -G^{\Omega}V(x), \tag{7.1}$$

which proves (3.23).

In the case $q > 1$, we set $\nu_n = \inf_{\partial\Omega_n} u$, where by (3.24) we can assume $\lim_{n\to\infty} \nu_n = +\infty$. Then by Theorem 3.2 with $h \equiv \nu_n$, we obtain in Ω_n

$$\begin{aligned} u &\ge \nu_n \left[1 + (q-1)\nu_n^{q-1} G^{\Omega_n} V\right]^{-\frac{1}{q-1}} \\ &= \left[\nu_n^{-(q-1)} + (q-1) G^{\Omega_n} V\right]^{-\frac{1}{q-1}} \end{aligned} \tag{7.2}$$

where

$$-(q-1) G^{\Omega_n} V < \nu_n^{-(q-1)} \quad \text{in } \Omega_n. \tag{7.3}$$

It follows from (7.3) that $G^{\Omega}V_{-}(x) \neq +\infty$, since otherwise both $G^{\Omega}V_{\pm}(x) = +\infty$. Hence, by letting $n \to +\infty$ in (7.3), we see that $G^{\Omega}V(x) \geq 0$, and consequently by the monotone convergence theorem (7.2) yields

$$u(x) \geq \left[(q-1)G^{\Omega}V(x)\right]^{-\frac{1}{q-1}}.$$

Since $u(x) < \infty$, we actually have a strict inequality $G^{\Omega}V(x) > 0$.

Consider now the case $0 < q < 1$. We set

$$\phi(v) = [(1-q)v]^{\frac{1}{1-q}}, \quad v \in I_q = (0, +\infty).$$

Then clearly

$$\phi'(v) = [(1-q)v]^{\frac{q}{1-q}} > 0, \quad \phi''(v) = q\,[(1-q)v]^{\frac{2q-1}{1-q}} > 0,$$

and (5.1) holds. For a sequence $\varepsilon_n \downarrow 0$, we set $u_n = u + \varepsilon_n$, and define v_n by

$$v_n = \phi^{-1}(u_n), \quad n = 1, 2, \ldots.$$

Using Lemma 4.3 in the case $h \equiv 1$ so that $\mathcal{L}h = 0$ (in this case the condition $\phi(0) = 1$ in (4.8) is not required, see Remark after the proof of Lemma 4.3), we obtain as in the proof of Theorem 3.2,

$$-\mathcal{L}v_n + \left(\frac{u}{u_n}\right)^q V \geq 0.$$

Since $v_n > 0$ on $\partial\Omega_n$, it follows from the maximum principle

$$v_n \geq -G^{\Omega_n}\left(\left(\frac{u}{u_n}\right)^q V\right) \quad \text{in } \Omega_n. \tag{7.4}$$

As $n \to \infty$ we obtain $v_n \to \phi^{-1}(u)$, and

$$\lim_{n\to\infty} G^{\Omega_n}\left(\left(\frac{u}{u_n}\right)^q V_{\pm}\right) = G^{\Omega}\left(\chi_{\Omega^+}V_{\pm}\right)$$

by the monotone convergence theorem. Passing to the limit in (7.4) as $n \to \infty$ gives

$$\phi^{-1}(u) \geq -G^{\Omega}\left(\chi_{\Omega^+}V\right),$$

which is equivalent to (3.25).

Finally, let $q < 0$. We argue as in the case $q > 1$, setting $\nu_n = \inf_{\partial\Omega_n} u$ where in view of (3.27) we can assume $\lim_{n\to\infty}\nu_n = 0$. Then by Theorem 3.2 with $h \equiv \nu_n$,

$$u(x) \leq \left[\nu_n^{1-q} - (1-q)G^{\Omega_n}V(x)\right]^{\frac{1}{1-q}} \quad \text{in } \Omega_n, \tag{7.5}$$

where

$$(1-q)\,G^{\Omega_n}V(x) < \nu_n^{1-q} \quad \text{in } \Omega_n.$$

It follows as in the case $q > 1$ that $GV_{+}(x) \neq +\infty$, and $GV_{+}(x) \leq GV_{-}(x)$. Letting $n \to +\infty$ in (7.5), we deduce (3.28), which yields the strict inequality $GV_{+}(x) < GV_{-}(x)$, since $u(x) > 0$.

8. Proof of Theorems 3.4 and 3.5

Proof of Theorem 3.4. We prove only statement (*ii*) (for $q < 0$) since statement (*i*) (for $q > 1$) is proved in a similar but simpler way. We use the method of sub- and super-solutions, understood in the classical sense: if there exist $\underline{u}$, $\overline{u} \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $0 < \underline{u} \leq \overline{u}$ in Ω, $\underline{u} = \overline{u} = 0$ on $\partial\Omega$, and

$$-\mathcal{L}\underline{u} + V\underline{u}^q \leq f, \qquad -\mathcal{L}\overline{u} + V\overline{u}^q \geq f \quad \text{in } \Omega,$$

then there exists a solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ to (3.29) such that $\underline{u} \le u \le \overline{u}$. (See [13], Theorem 1.2.3, in the case $M = \mathbb{R}^n$ and $\mathcal{L} = \Delta$; the same proof which relies on standard interior regularity estimates works in the general case.)

Clearly, setting $\overline{u} = h = G^\Omega f \in C(\overline{\Omega}) \cap C^2(\Omega)$ gives a supersolution since $V \ge 0$, and consequently

$$-\mathcal{L}\overline{u} + V\overline{u}^q \ge -\mathcal{L}\overline{u} = f, \quad \overline{u} = 0 \text{ on } \partial\Omega.$$

The main problem is to find a subsolution which we define by

$$\underline{u} = h - \lambda^q \, G^\Omega(h^q V),$$

where $\lambda > 0$ is a constant to be determined later. Using (3.32) we see that $\underline{u} > 0$ provided

$$\left(1 - \frac{1}{q}\right)^q \frac{1}{1-q} < \lambda^{-q}. \tag{8.1}$$

Under the assumptions imposed on f it follows that $h \in C(\overline{\Omega}) \cap C^2(\Omega)$. We need to show that $\underline{u} \in C(\overline{\Omega}) \cap C^2(\Omega)$. As in the proof of Theorem 3.1 (iv), let Ω' be an arbitrary relatively compact subset of Ω with smooth boundary. Then $G^\Omega(h^qV) - G^{\Omega'}(h^qV)$ is a harmonic function in Ω'. Since $h > 0$ in Ω', it follows that $h^qV \in C(\overline{\Omega})$ and is locally Hölder-continuous. Hence, $G^{\Omega'}(h^qV) \in C^2(\Omega')$, and consequently $G^\Omega(h^qV) \in C^2(\Omega')$ as well. To show that $G^\Omega(h^qV) \in C(\overline{\Omega})$, notice that h vanishes continuously on $\partial\Omega$. Using (3.32), we deduce that the same is true for $G^\Omega(h^qV)$.

It remains to show that $-\mathcal{L}\underline{u} + V\underline{u}^q \le f$. Since $q < 0$ and hence $\underline{u}^q \ge h^q$, it follows

$$-\mathcal{L}\underline{u} + V\underline{u}^q = f - \lambda^q \, h^q V + \underline{u}^q V \le f,$$

provided

$$\lambda \, h \le \underline{u} = h - \lambda^q \, G^\Omega(h^q V),$$

or equivalently,

$$G^\Omega(h^qV) \le \lambda^{-q}(1-\lambda)h.$$

Optimizing over all $\lambda \in (0,1)$, we obtain that the maximum of the right-hand side is obtained for $\lambda = \frac{1}{1-\frac{1}{q}}$, which coincides with condition (3.32),

$$G^\Omega(h^qV) \le \left(1 - \frac{1}{q}\right)^q \frac{1}{1-q} \, h.$$

Notice that (8.1) obviously holds with this choice of λ as well. Thus, $\underline{u}$ is a classical subsolution which is positive in Ω, and $\underline{u} \le \overline{u}$ as desired. Consequently, there exists a classical solution u such that $\underline{u} \le u \le \overline{u}$. Moreover,

$$u \ge \underline{u} = h - \lambda^q G^\Omega(h^qV) = h - \left(1 - \frac{1}{q}\right)^{-q} G^\Omega(h^qV) \ge \frac{1}{1-\frac{1}{q}} \, h,$$

which proves the lower bound for u in (3.33).

The upper bound was obtained above in Theorem 3.1(iv). □

Proof of Theorem 3.5. The case $q > 1$, $V \le 0$ is considered in [27] and [5] (see also [31]), so we give a proof only in the case $q < 0$, $V \ge 0$. Let us assume that

$$K(h^qV)(x) \le a \, h(x) \quad dm - \text{a.e. in } \Omega, \tag{8.2}$$

for some constant $a > 0$, where h satisfies (3.35).

Set $u_0 = h$, and construct a sequence of consecutive iterations u_k by

$$u_{k+1} + K(u_k^q V) = h, \quad k = 0, 1, 2, \ldots.$$

Clearly, by (8.2),

$$(1-a)h(x) \le u_1(x) = h(x) - K(h^qV)(x) \le h(x) = u_0(x).$$

We set $b_0 = 1$, $b_1 = 1 - a$, and continue the argument by induction. Suppose that for some $k = 1, 2, \ldots$

$$b_k\, h(x) \le u_k(x) \le u_{k-1}(x) \quad \text{in } \Omega. \tag{8.3}$$

Since $q < 0$ and $V \ge 0$, we deduce using estimates (8.2) and (8.3),

$$(1 - a\, b_k^q)\, h(x) \le h(x) - b_k^q\, K(h^q V)(x) \le h(x) - K(u_k^q V)(x) = u_{k+1}(x).$$

On the other hand,

$$u_{k+1}(x) = h(x) - K(u_k^q V)(x) \le h(x) - K(u_{k-1}^q V)(x) = u_k(x).$$

Hence,

$$b_{k+1}\, h(x) \le u_{k+1}(x) \le u_k(x), \quad \text{where } b_{k+1} = 1 - a\, b_k^q.$$

We need to pick $a > 0$ small enough, so that $b_k \downarrow b$, where $b > 0$, and $b = 1 - a\, b^q$.

In other words, we are solving the equation

$$\frac{1-x}{a} = x^q \tag{8.4}$$

by consecutive iterations $b_{k+1} = 1 - a b_k^q$ starting from the initial value $b_0 = 1$. Clearly, this equation has a solution $0 < x < 1$ if and only if $0 < a \le a_*$, where $y = \frac{1-x}{a_*}$ is the tangent line to the convex curve $y = x^q$. Here the optimal value a_* is found by equating the derivatives, and solving the system of equations

$$x_*^q = \frac{1 - x_*}{a}, \quad q x_*^{q-1} = -\frac{1}{a_*},$$

which gives

$$a_* = \left(1 - \frac{1}{q}\right)^q \frac{1}{1-q}, \quad x_* = \frac{1}{1 - \frac{1}{q}}.$$

Letting $a = a_*$, we see that by the convexity of $y = x^q$, (8.4) has a unique solution $x_* = \frac{1}{1-\frac{1}{q}}$, and by induction, $x_* < b_{k+1} < b_k < 1$, so that

$$b_k \downarrow b = x_* = \frac{1}{1 - \frac{1}{q}} > 0.$$

From this it follows that (8.3) holds for all $k = 1, 2, \ldots$. Passing to the limit as $k \to \infty$, and using the monotone convergence theorem shows that $u = \lim_{k\to\infty} u_k$ is a solution of (3.34) such that

$$b\, h(x) \le u(x) \le u_0(x) = h(x).$$

Moreover, it is easy to see by construction that u is a maximal solution, that is, if $\tilde{u}$ is another nonnegative solution to (3.34), then $\tilde{u} \le u_k$ for every $k = 0, 1, 2, \ldots$, and consequently $\tilde{u} \le u$ in Ω. □

9. Examples

In this section, we consider several examples which demonstrate various phenomena that may affect behavior of solutions to the equations considered above. In the liner case $q = 1$ (Schrödinger equations), many examples concerning possible behavior of Green's functions on domains and manifolds for $V \ge 0$ are given in [20]; the case $V \le 0$ is considered in [14] and [15] (see also [8], [21], [29], [30]). In the superlinear case for $q > 1$ and $V \ge 0$ we refer to [5] and [27] for existence results as well as pointwise estimates of solutions, and many examples. The case $q > 1$ and $V \le 0$ (equations with absorption) is studied in [28]. In the sublinear case $0 < q < 1$, existence of bounded positive solutions, along with uniqueness, and pointwise estimates of bounded solutions on $\mathbb{R}^n$ were obtained in [6]. Recently, sharp existence results and matching two-sided estimates for weak positive solutions (not necessarily bounded) in $\mathbb{R}^n$ were given in [9]; see also [10] for a characterization of finite energy solutions.

Here we give an example involving a rapidly oscillating V in the case $q = 1$, and also illustrate various phenomena with regards to pointwise behavior of solutions in the less studied case $q < 0$, for both $V \geq 0$ and $V < 0$. (Related results for $q < 0$ where obtained in [4], [13], [16] and [17].)

Example 1. We consider first the linear case $q = 1$ in Theorem 3.1:

$$-u'' + V\,u = f \quad \text{in } \Omega, \tag{9.1}$$

for $\Omega = (0, 1)$, $M = \mathbb{R}^1$. Let $f = 1$, and $h = Gf = \frac{1}{2}x(1-x)$. The corresponding Green function is $G(x, y) = \min(x(1-y), y(1-x))$.

We start with a positive solution with zero boundary values to (9.1),

$$u(x) = x(1-x)\Big(1 + x\sin\Big(\frac{\pi}{x^\alpha}\Big)\Big), \qquad x \in (0,1), \quad \alpha > 0. \tag{9.2}$$

Then

$$u'(x) = x(1-x)\Big(1 + x\sin\Big(\frac{\pi}{x^\alpha}\Big)\Big), \qquad x \in (0,1), \quad \alpha > 0. \tag{9.3}$$

The corresponding $V = \frac{u''+1}{u}$ is found from (9.1),

$$V = V_1 + V_2 + V_3, \tag{9.4}$$

where

$$\begin{aligned}
V_1(x) &= -\frac{\alpha^2\pi^2 x^{-2\alpha-1}\sin(\frac{\pi}{x^\alpha})}{1 + x\sin(\frac{\pi}{x^\alpha})},\\
V_2(x) &= \frac{\alpha(\alpha-1)(1-2x)\pi x^{-\alpha-1}\cos(\frac{\pi}{x^\alpha}) - \alpha\pi(1-2x)x^{-\alpha}\cos(\frac{\pi}{x^\alpha})}{1 + x\sin(\frac{\pi}{x^\alpha})},\\
V_3(x) &= \frac{(1-2x)\sin(\frac{\pi}{x^\alpha})}{1 + x\sin(\frac{\pi}{x^\alpha})} - \frac{2\sin(\frac{\pi}{x^\alpha})}{(1-x)(1 + x\sin(\frac{\pi}{x^\alpha}))}.
\end{aligned}$$

Thus, V has a highly oscillatory behavior at the end-point $x = 0$, where V_1 is the leading term. Nevertheless, due to the cancellation phenomenon, we have $u \simeq h$.

For $0 < \alpha < 1$, $G(hV)$ is well-defined, and Theorem 3.1 gives the lower bound

$$u \geq h\,e^{-\frac{G(hV)}{h}}, \tag{9.5}$$

which is sharp since $\frac{G(hV)}{h}$ is a bounded function on Ω. Indeed, it is easy to see that the term $G(hV_3)$ is harmless since hV_3 is bounded in Ω, and hence $G(hV_3) \simeq h$ at the endpoints. To estimate $G(hV_2)$, notice that $|V_2(x)| \leq Cx^{-\alpha-1}$, and consequently by direct estimates

$$G(h|V_2|)(x) = O(x) \quad \text{as } x \to 0^+. \tag{9.6}$$

It remains to notice that due to cancellation, for $0 < \alpha < 1$,

$$G(hV_1)(x) = O(x) \quad \text{as } x \to 0^+ \tag{9.7}$$

as well. This can be verified by looking at the asymptotics of the integrals in the expression

$$G(hV_1)(x) = (1-x)\int_0^x \frac{y^2(1-y)}{2}V_1(y)dy + x\int_x^1 \frac{y(1-y)^2}{2}V_1(y)dy. \tag{9.8}$$

Clearly, $G(hV_1)(x) \simeq 1 - x$ as $x \to 1^-$. For $0 < \alpha < 1$, it is not difficult to see using integration by parts that $G(hV_1)(x) \simeq x$ as $x \to 0^+$; we omit the details here.

If $\alpha = 1$, then $G(hV)$ is not well-defined, and the first term on the right-hand side of (9.8) has to be understood as an improper integral which asymptotically behaves like x as $x \to 0^+$. However, the second term actually has an extra logarithmic factor, so that

$$G(hV) \simeq x \log\left(\frac{1}{x}\right) \quad \text{as } x \to 0^+.$$

This shows that the lower bound $u(x) \ge h\, e^{-\frac{G(hV)}{h}}$ is not sharp in this case.

Example 2. Let $q < 0$, and let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$. Consider inequality (3.3) with $\mathcal{L} = \Delta$, $f \equiv 1$, and

$$V(x) = \frac{\lambda}{d_\Omega(x)^\beta}, \quad x \in \Omega, \quad \lambda > 0, \ \beta > 0,$$

and the corresponding equation

$$-\Delta u + \frac{\lambda}{d_\Omega(x)^\beta} u^q = 1, \quad u > 0 \quad \text{in } \Omega. \tag{9.9}$$

We set

$$h(x) = G^\Omega f(x) \simeq d_\Omega(x), \quad x \in \Omega. \tag{9.10}$$

Theorem 3.1 (iv) gives the following necessary condition,

$$(1-q)\frac{G^\Omega(h^q V)(x)}{h(x)} < 1, \tag{9.11}$$

for the existence of a positive solution u to (3.3) with zero boundary values.

It is easy to see via direct estimates of the Green kernel that, for $\beta \ge 2 + q$, we have $G^\Omega(h^q V) \equiv +\infty$. For $1 + q < \beta < 2 + q$,

$$\frac{G^\Omega(h^q V)(x)}{h(x)} \simeq d_\Omega(x)^{1+q-\beta}, \quad x \in \Omega,$$

For $\beta = 1 + q$, we have

$$\frac{G^\Omega(h^q V)(x)}{h(x)} \simeq \log \frac{A}{d_\Omega(x)}, \quad x \in \Omega,$$

where $A = 2\,\mathrm{diam}(\Omega)$. Hence, for $\beta \ge 1 + q$, condition (9.11) fails, and (3.3) has no positive solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ with zero boundary values. This non-existence result was proved earlier in [13], Theorem 2.1.

In the case $0 < \beta < 1 + q$, direct estimates give

$$(1-q)\frac{G^\Omega(h^q V)(x)}{h(x)} \le c\,\lambda, \tag{9.12}$$

where $c = c(\Omega, q, \beta)$ is a positive constant.

Theorem 3.1 (iv) implies that if (3.3) has a solution u with zero boundary values, then actually (9.12) holds with $c\lambda < 1$, and

$$u(x) \le h(x)\Big[1 - (1-q)\frac{G^\Omega(h^q V)(x)}{h(x)}\Big]^{\frac{1}{1-q}}, \quad x \in \Omega,$$

by estimate (3.8).

Moreover, if (9.12) holds with $c\,\lambda \le \left(1 - \frac{1}{q}\right)^q$, then by Theorem 3.4 there exists a solution $\widetilde{u}$ to (9.9) with zero boundary values which satisfies the lower bound

$$\widetilde{u}(x) \ge \frac{1}{1 - \frac{1}{q}} h(x), \quad x \in \Omega.$$

Hence, $\widetilde{u}(x) \simeq d_\Omega(x)$, and our general upper bound (3.8) is sharp in this case as well.

In Example 4, we will demonstrate that due to a non-uniqueness phenomenon, equations of the type (9.9) may have other solutions which violate the lower bound $u(x) \geq c\, d_\Omega(x)$.

Example 3. Let $q < 0$, and let Ω be a bounded smooth domain in $\mathbb{R}^n$. We consider (3.3) with $f \equiv 1$, and

$$V(x) = -\frac{1}{d_\Omega(x)^\beta}, \qquad \beta > 0,$$

where $d_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$, together with the corresponding equation

$$-\Delta u - \frac{1}{d_\Omega(x)^\beta} u^q = 1, \quad u > 0 \quad \text{in } \Omega. \tag{9.13}$$

As in the previous example, set

$$h(x) = G^\Omega f(x) \simeq d_\Omega(x), \quad x \in \Omega, \tag{9.14}$$

and $A = 2\,\mathrm{diam}(\Omega)$.

Theorem 3.1 (iv) gives the following upper bounds for all positive solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to (3.3) with zero boundary values: for all $x \in \Omega$,

(a) $u(x) \leq C\, d_\Omega(x)$ if $0 < \beta < 1 + q$;

(b) $u(x) \leq C\, d_\Omega(x) \log^{\frac{1}{1-q}}\left(\frac{A}{d_\Omega(x)}\right)$ if $\beta = 1 + q$;

(c) $u(x) \leq C\, d_\Omega(x)^{\frac{2-\beta}{1-q}}$ if $1 + q < \beta < 2 + q$.

The corresponding lower bounds for positive *super-solutions*, not necessarily with zero boundary values, were established in [16], Proposition 2.6 (see also [13], Theorem 3.5): if $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and

$$-\Delta u - \frac{1}{d_\Omega^\beta} u^q \geq 0, \quad u > 0 \quad \text{in } \Omega, \tag{9.15}$$

then, for all $x \in \Omega$,

(a') $u(x) \geq c\, d_\Omega(x)$ if $0 < \beta < 1 + q$;

(b') $u(x) \geq c\, d_\Omega(x) \log^{\frac{1}{1-q}}\left(\frac{A}{d_\Omega(x)}\right)$ if $\beta = 1 + q$;

(c') $u(x) \geq c\, d_\Omega(x)^{\frac{2-\beta}{1-q}}$ if $1 + q < \beta < 2$.

There are no positive solutions u to (9.15) in the case $\beta \geq 2$. For $0 < \beta < 2$, there exists a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ with zero boundary values to equation (9.13) which satisfies both the upper and lower bounds given above.

Thus, our general upper bound (3.8) in Theorem 3.1 (iv) is sharp in all cases, except for $2 + q \leq \beta < 2$, where $G(h^q V) \equiv -\infty$, so that (3.8) becomes trivial.

Example 4. In this example, we encounter the non-uniqueness phenomenon for classical solutions with zero boundary conditions to semilinear equations with negative exponents $q < 0$, where obviously our estimates are not expected to be sharp for all solutions. For simplicity, we consider the one-dimensional case, although similar examples are easy to construct in higher dimensions, with coefficients V depending only on $d_\Omega(x)$.

Consider the following semilinear equation:

$$-u'' + V\, u^q = f \quad \text{in } \Omega, \tag{9.16}$$

for $q < 0$, $\Omega = (-1, 1)$, with zero boundary conditions $u(\pm 1) = 0$. Set $f \equiv 1$ and

$$h = Gf = \frac{1}{2}(1 - x^2).$$

The corresponding Green function is

$$G(x,y) = \min\Big((x+1)(1-y), (y+1)(1-x)\Big).$$

Consider a positive solution with zero boundary values to (9.16) given by

$$u(x) = \lambda\,(1-x^2)^\gamma, \qquad x \in (0,1), \quad \lambda > 0, \quad \gamma > 0. \tag{9.17}$$

Then the corresponding $V = \frac{u''+1}{u^q}$ is found from (9.16),

$$V = V_1 + V_2 + V_3,$$

where

$$V_1(x) = 4\lambda^{1-q}\,\gamma(\gamma-1)(1-x^2)^{\gamma-2-\gamma q},$$
$$V_2(x) = -2\lambda^{1-q}\,\gamma(2\gamma-1)(1-x^2)^{\gamma-1-\gamma q},$$
$$V_3(x) = \lambda^{-q}(1-x^2)^{-\gamma q}.$$

In the case $\gamma = 1$, clearly, $V_1 \equiv 0$, and

$$V(x) = \lambda^{-q}(1-2\lambda)(1-x^2)^{-q}.$$

Then

$$\frac{G(h^q V)(x)}{h(x)} = (2\lambda)^{-q}(1-2\lambda), \quad x \in \Omega.$$

Our estimate (3.8) is sharp in both cases, $V \le 0$ ($\lambda \ge \frac{1}{2}$), and $V \ge 0$ ($0 < \lambda < \frac{1}{2}$):

$$u(x) \le \frac{1-x^2}{2}\Big[1-(1-q)(2\lambda)^{-q}(1-2\lambda)\Big]^{\frac{1}{1-q}},$$

where the constant in square brackets is positive for any choice of $\lambda > 0$, $q < 0$.

In the case $\gamma \ne 1$ the situation is more complicated. Clearly, V_1 is now the most singular term.

For $\gamma > 1$, the behavior of the solution u given by (9.17) at the end-points $x = \pm 1$ is too good to be captured by the upper estimate (3.8); obviously, it is not sharp for this particular u. On the other hand, notice that $V > 0$ if $2\lambda\gamma < 1$; for $\gamma > 1$, it is easy to see by direct estimates that

$$\frac{G(h^q V)(x)}{h(x)} \le C\lambda^{-q}, \quad x \in \Omega. \tag{9.18}$$

Since there exists a positive solution, Theorem 3.1 (*iv*) implies that actually (9.18) holds with $C\lambda^{-q} < \frac{1}{1-q}$.

For $1 < \gamma < \frac{1}{2\lambda}$, which ensures that $V > 0$, every positive solution u with zero boundary values obviously satisfies the upper bound

$$u \le h \quad \text{in } \Omega.$$

Moreover, if (9.18) holds with $C\lambda^{-q} \le \left(1-\frac{1}{q}\right)^q \frac{1}{1-q}$, then by Theorem 3.4 equation (9.16) has a solution $\widetilde{u}$ such that $\widetilde{u} \simeq h$, for which the upper bound (3.8) is indeed sharp.

If $0 < \gamma \le -\frac{q}{1-q}$, then V is too singular at the end-points, so that $G(h^q V) \equiv +\infty$, and (3.8) trivializes.

In the remaining case $-\frac{q}{1-q} < \gamma < 1$, it is easy to see that

$$\frac{G(h^q V)(x)}{h(x)} \simeq (1-x^2)^{-q+\gamma-\gamma q-1}, \quad x \in \Omega,$$

which blows up as $x \to \pm 1$. In this case, (3.8) gives $u(x) \le c(1-x^2)^\gamma$, which is again sharp.

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