HARNACK INEQUALITY FOR THE HEAT EQUATION ON NON-COMPACT RIEMANNIAN MANIFOLDS

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Abstract

The classical Harnack inequality states that any positive harmonic function defined in a ball $B(z,2R) \subset \mathbb{R}^n$ (of the radius $2R$ and centered at a point z) satisfies the following inequality

$$\sup_{B(z,R)} u \le P \inf_{B(z,R)} u$$

with the constant P depending on the dimension n only. An analogous inequality holds for the heat equation in a cylinder $C = B(z,2R) \times (0,4R^2)$. Namely, if $u(x,t)$ is a positive solution in the cylinder C, then

$$\sup_{C_1} u \le P \inf_{C_2} u$$

where $C_1 = B(z,R) \times (R^2, 2R^2), C_2 = B(z,R) \times (3R^2, 4R^2)$ and P depends only on n.

Let us consider a complete non-compact Riemannian manifold M and the Laplacian Δ on M related to the Riemannian metric. One may ask which geometric properties of the Riemannian manifold ensure the analogous Harnack inequalities for the Laplace and the heat equations to hold. It should be emphasized that we are interested only in a uniform Harnack inequality i.e. the constant P must be independent of the radius R and of the centre z of the geodesic ball in question.

Bombieri and Giusti [1] and Yau [7] analysed Moser's proofs of Harnack inequalities in the Euclidean case. In fact they have found out that Moser's proof can be carried over to the manifold M provided the following geometric properties hold:

(a) (doubling volume property) for any couple of concentric geodesic balls $B(x,R), B(x,2R)$ of radii R and $2R$

$$\mu B(x,2R) \le A \mu B(x,R),$$

where the constant A is the same for all balls.

(b) (Poincaré inequality) for any ball $B(x,R)$ and any smooth function f in the ball

$$\int_{B(x,R)} |\nabla f|^2 \ge \frac{a}{R^2} \inf_{\xi \in \mathbb{R}} \int_{B(x,R)} (f-\xi)^2$$

where $a > 0$ is a positive constant independent of x, R, f;

(c) (Sobolev inequality) for any smooth compactly supported function f on M

$$\int_M |\nabla f| \ge b \Big(\int_M |f|^{\frac{n}{n-1}} \Big)^{\frac{n-1}{n}}$$

where $b > 0$ is a positive constant independent of f.

Our main result states, that the parabolic Harnack inequality can be deduced from (a) and (b) only. What is more, the hypothesis (b) can be replaced by a weaker one. The precise statement is as follows.

Theorem 1 *Suppose that the Riemannian manifold M satisfies the hypothesis (a) and the following one*

(b') for some constant $N > 1$ and $a > 0$, for any geodesic ball $B(x, R) \subset M$ and for any smooth function f defined in the ball $B(x, NR)$ the following inequality holds:

$$\int_{B(x,NR)} |\nabla f|^2 \geq \frac{a}{R^2} \inf_{\xi \in \mathbb{R}} \int_{B(x,R)} (f - \xi)^2,$$

then the uniform parabolic Harnack inequality is valid with the constant P depending on A, a, N.

Of course, the elliptic Harnack inequality holds under these conditions too.

It turns out that the hypotheses of the theorem 1 are not only sufficient but necessary as well.

Theorem $1'$ *Suppose that the Harnack inequality for the heat equation is known to be valid in any ball with the same constant P, then the doubling volume property (a) and the weak Poincaré inequality are true with the constant A, a, N depending only on P.*

My paper [2] contains the proof of the Theorem 1 and the first part of the Theorem $1'$, namely, the implication "Harnack inequality $\Longrightarrow$(a)". An entire proof of the Theorem $1'$ as well as an independent proof of the Theorem 1 was given by Saloff-Coste [5] .

Since (a) and (b') are preserved by a quasi-isometric transformation of a manifold then we get the following consequence of the Theorems 1, $1'$.

Corollary *The uniform parabolic Harnack inequality is a quasi-isometry invariant.*

It is still unknown whether the elliptic Harnack inequality is stable under a quasi-isometry. At least the condition (a) is not necessary for the elliptic Harnack inequality. Let us note for comparison that the Liouville property (i.e. the absence of a non-trivial bounded harmonic function defined all over the manifold) which is a consequence of the elliptic Harnack inequality, is not stable under a quasi-isometric transformation as was shown by T.Lyons [L4] .

The idea behind the proof of the Theorem 1 is based upon the fact that the Sobolev inequality (c) can be replaced by an isoperimetric inequality for the first Dirichlet eigenvalue (of Faber-Krahn type) when obtaining a mean-value type property . We use the following two theorems instead of Moser's iteration arguments to avoid applying of the Sobolev inequality.

Theorem 2 *Under hypotheses (a) and (b') the following property holds:*

(c') for any ball $B(z, R)$ for any region $\Omega \Subset B(z, R)$ and for any $f \in C_c^\infty(\Omega)$

$$\int_\Omega |\nabla f|^2 \geq \frac{b}{R^2} \left(\frac{\mu B(z, R)}{\Omega} \right)^\beta \int_\Omega f^2$$

where $b, \beta > 0$ are positive constants, depending on a, A, N.

Another formulation of this inequality is that the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet boundary value problem in the region Ω has a lower bound

$$\lambda_1(\Omega) \geq \frac{b}{R^2} \left(\frac{\mu B(z, R)}{\Omega} \right)^\beta .$$

We apply this theorem together with the following one.

Theorem 3 *Suppose that in some ball $B(z,R)$ the condition (c') is valid with some constants b,β, then for any solution $u(x,t)$ of the heat equation in the cylinder $C = B(z,R)\times(0,R^2)$ the inequality holds:*

$$u(z,R^2)^2 \le \frac{\mathrm{const}_{b,\beta}}{\mu C}\int_C u(x,t)^2 dxdt$$

Hence, conditions (a) and (b') imply together a mean-value type property of solutions to the heat equation.

In conclusion, we propose some graphic geometric condition implying (a) and (b'). Let Γ_q^x be a homothety (moving points along the shortest geodesics) centered at a point $x\in M$ with a coefficient $q\in(0,1)$.

Theorem 4 *Suppose that for any $q\in(\frac{1}{2},1)$, for any $x\in M$ and for any open bounded set $\Omega\subset M$ the image $\Omega_q\equiv\Gamma_q^x(\Omega)$ is measurable and has the volume at least as large as $c\mu\Omega$ (i.e. $\mu\Omega_q\ge c\mu\Omega$) where $c>0$ is independent of x,q,Ω, then the conditions (a) and (b') hold with the constants A,a,N depending only on c.*

It can be proved with ease that this "homothety property" holds on a Riemannian manifold with a non-negative Ricci curvature. Hence, the parabolic Harnack inequality is also valid on such a manifold. Earlier the Harnack inequality on the positively curved manifolds was proved by Yau [6] in the elliptic case and by Yau and Li [3] in the parabolic one.

Note that the geodesics in the definition of the homothety can be replaced by a more general family of curves satisfying the conditions

(1) any two distinct points of a manifold are connected by one of the curves of the family;
(2) any segment of any curve belongs to the family too;
(3) the length of any curve is controlled above by the distance between its ends up to a constant multiple.

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