Hardy's Inequality and Green Function on Metric Measure Spaces

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Abstract

We prove an abstract form of Hardy's inequality for local and non-local regular Dirichlet forms on metric measure spaces, using the Green operator of the Dirichlet form in question. Under additional assumptions such as the volume doubling, the reverse volume doubling, and certain natural estimates of the Green function, we obtain the "classical" form of Hardy's inequality containing distance to a reference point or set.

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1 Introduction

1.1 A historical overview: Hardy's inequality on $\mathbb{R}^n$ and manifolds

The classical Hardy's inequality was first proved by Hardy [48] in order to find an elementary proof of a double series inequality of Hilbert. For its prehistory development (in both discrete and continuous forms) over the decade 1906-1928, we refer the interested readers to [57]. A modern form of Hardy's inequality in $\mathbb{R}^n$, $n > 2$, is as follows (cf. [49]):

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^2}\,dx \le \int_{\mathbb{R}^n}|\nabla f(x)|^2\,dx \quad \text{for all } f\in C_c^1(\mathbb{R}^n), \tag{1.1}$$

where $C_c^1(\mathbb{R}^n)$ denotes the class of continuously differentiable functions on $\mathbb{R}^n$ with compact support. Hardy's inequality has found numerous applications in various areas of mathematics such as partial differential equations, geometric analysis, probability theory and etc. We refer the reader to [3, 21, 58, 64, 21] and the references therein for more information about Hardy's inequality in Euclidean spaces and related historical reviews.

Generalizations of (1.1) to Riemannian manifolds can be found in [1, 14, 20, 34, 56]. Let M be a Riemannian manifold, Δ be the Laplace-Beltrami operator on M, and μ be the Riemannian measure. Then, for any positive superharmonic function ϕ on M, the following version of Hardy's inequality is true:

$$\int_M \frac{-\Delta\phi}{\phi} f^2\,d\mu \le \int_M |\nabla f|^2\,d\mu \quad \text{for all } f\in C_c^2(M). \tag{1.2}$$

The following short proof of (1.2) was given in [33, Section 4.4] and [34, p. 258] (see also [42]). Consider the weighted manifold $(M,\tilde{\mu})$ with $d\tilde{\mu} = \phi^2\,d\mu$. An easy calculation shows that the weighted Laplacian

$$\Delta_{\tilde{\mu}} u := \phi^{-2}\operatorname{div}(\phi^2\nabla u)$$

satisfies the following identities: the product rule

$$-\phi\,\Delta_{\tilde{\mu}}(\phi^{-1}f) = -\Delta f + \frac{\Delta\phi}{\phi} f \tag{1.3}$$

and the Green formula

$$-\int_M u\,\Delta_{\tilde{\mu}}u\,d\tilde{\mu} = \int_M |\nabla u|^2 d\tilde{\mu} \ge 0 \ \text{ for all } u\in C_c^2(M). \tag{1.4}$$

Applying (1.4) with $u = \phi^{-1}f$ and using (1.3), we obtain (1.2).

Note that (1.2) is sharp in the sense that it recovers (1.1) when $M = \mathbb{R}^n$, $n > 2$, because, for the function $\phi(x) = |x|^{-\frac{n-2}{2}}$, we have

$$\frac{-\Delta\phi(x)}{\phi(x)} = \frac{(n-2)^2}{4}\frac{1}{|x|^2}.$$

1.2 Abstract Hardy's inequality on metric measure spaces

Motivated by (1.1) and (1.2), our main aim in this paper is to establish Hardy's inequality on *metric measure spaces* (M, d, μ), including manifolds and fractal spaces. We say that (M, d, μ) is a metric measure space if (M, d) is a separable metric space such that all the metric balls in M are precompact, and μ is a Radon measure on M with full support. We assume that a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is defined on $L^2(M, \mu)$. Then, instead of the energy integral $\int_M |\nabla f|^2\, d\mu$ in (1.2) we use $\mathcal{E}(f, f)$, and instead of the function ϕ we use a function Gh where G is the Green operator of $(\mathcal{E}, \mathcal{F})$. Hence, for a certain class of positive functions h on M, (1.2) transforms to

$$\int_M \frac{h}{Gh} f^2\, d\mu \le \mathcal{E}(f, f) \text{ for all } f \in \mathcal{F}. \tag{1.5}$$

Hardy's inequality in the form (1.5) is proved in this paper in Theorem 3.1 for strongly local regular Dirichlet forms and in Theorem 4.5 – for general (non-local) regular Dirichlet forms.

Given a Radon measure ν on M, one can ask under which conditions the following even more general form of Hardy's inequality is valid:

$$\int_M f^2\, d\nu \le \mathcal{E}(f, f).$$

This question was studied in [11, 27, 65] where the answer was given in terms of a certain testing inequality expressed via the Dirichlet form and the measure ν. Our versions of Hardy's inequality in Theorems 3.1 and 4.5 are much more explicit and do not follow from the results of [11, 27, 65].

1.3 "Classical" versions of Hardy's inequalities

Let us describe some applications and consequences of Theorems 3.1 and 4.5. We use the following notation

$$V(x, r) = \mu(B(x, r)) \text{ and } V(x, y) = \mu(B(x, d(x, y)))$$

for all $x, y \in M$ and $r > 0$, and consider the following conditions.

▷ *Volume doubling condition* (**VD**): there exists $C_D \in (1, \infty)$ such that

$$V(x, 2r) \le C_D V(x, r) \text{ for all } x \in M \text{ and } r > 0. \tag{VD}$$

Condition (**VD**) is equivalent to

$$\frac{V(x, R)}{V(x, r)} \le C\left(\frac{R}{r}\right)^{\alpha_+} \text{ for all } x \in M \text{ and } 0 < r \le R < \infty,$$

for some positive C, α_+. The exponent α_+ is called the *upper volume dimension* of (M, d, μ).

▷ *Reverse volume doubling condition* (**RVD**): there exists $c > 0$ such that

$$\frac{V(x, R)}{V(x, r)} \ge c\left(\frac{R}{r}\right)^{\alpha_-} \text{ for all } x \in M \text{ and } 0 < r \le R < \infty. \tag{RVD}$$

The exponent α_- is called the *lower volume dimension* of (M, d, μ).

▷ *Condition* $(\mathbf{G})_\beta$ with $\beta > 0$: the Dirichlet form $(\mathcal{E}, \mathcal{F})$ admits a Green function $G(x, y)$, which is jointly continuous in $M \times M \setminus$ diag and satisfies the estimate

$$G(x, y) \simeq \frac{d(x, y)^\beta}{V(x, y)} \quad \text{for all distinct } x, y \in M, \tag{$\mathbf{G}$}_\beta$$

where diag $= \{(x, y) \in M \times M : \ x = y\}$.

Let us remark that both (**VD**) and (**RVD**) are satisfied if (M, d, μ) is Ahlfors α-regular, that is, if

$$V(x, r) \simeq r^\alpha$$

for all $r > 0$ and $x \in M$. In this case $\alpha = \alpha_+ = \alpha_-$ is the Hausdorff dimension of (M, d), and the measure μ is comparable with the Hausdorff measure H^α (cf. [39]).

The parameter β from $(\mathbf{G})_\beta$ is called the *walk dimension* of $(\mathcal{E}, \mathcal{F})$. The reason of this terminology will be clear from Example 2.5 below as β is the exponent of the space/time scaling for the Markov process associated with $(\mathcal{E}, \mathcal{F})$.

Under the hypotheses (**VD**) , (**RVD**) and $(\mathbf{G})_\beta$ with $\beta < \alpha_-$, we apply Theorem 4.5 and establish in Theorem 5.6 a "classical" form of Hardy's inequality: for all $x_o \in M$ and $f \in \mathcal{F}$,

$$\int_M \frac{f(x)^2}{d(x_o, x)^\beta} \, d\mu(x) \leq C\mathcal{E}(f, f). \tag{1.6}$$

Note that $\mathbb{R}^n$ satisfies the hypotheses of Theorem 5.6 provided $n > 2$ and $\beta = 2$. Theorem 5.6 applies also on many fractals spaces where $(\mathbf{G})_\beta$ with $\beta > 2$ is satisfied – see Example 5.12 in Section 5.3.

Due to the fact that the condition $(\mathbf{G})_\beta$ contains implicit constants, the constant C in (1.6) can not be determined explicitly. However, in specific settings like Euclidean spaces or graphs, one can obtain explicit constants in Hardy's inequality by applying directly (1.5) for suitable functions h as in Theorem 5.1 and Example 5.11.

Let us emphasize that in Theorem 5.6 the Dirichlet form does not have to be local. In the case when the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is strongly local and $\beta = 2$, we apply Theorem 3.1 and obtain in Corollary 5.8 the estimate (1.6) under the weaker hypotheses when $(\mathbf{G})_2$ is replaced by the upper bound

$$G(x, y) \leq C \frac{d(x, y)^2}{V(x, y)}. \tag{$\mathbf{G}_\leq$}_2$$

1.4 Weighted Hardy's inequality

For further applications of Theorems 3.1 and 4.5 (or, Corollary 5.8 and Theorem 5.6), we obtain *weighted* Hardy's inequalities for strongly local Dirichlet forms. It is known that in $\mathbb{R}^n$ with $n > 2$ also the following weighted Hardy inequality holds:

$$\frac{(n - \sigma - 2)^2}{4} \int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^{\sigma+2}} dx \leq \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{|x|^\sigma} dx \tag{1.7}$$

for any $\sigma \in [0, n - 2)$ and $f \in C_c^\infty(\mathbb{R}^n)$ (see, for example, [60, p. 657, (7)], [12, Corollary 4] or [22, Theorem 13]). Under some mild conditions of the strongly Dirichlet form $(\mathcal{E}, \mathcal{F})$ (see Proposition 7.1), any weight function $w : M \to (0, \infty]$ that is continuous and locally integrable can induce a strongly local regular Dirichlet form $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ on $L^2(M, \mu_w)$, where $d\mu_w = w \, d\mu$. Applying Theorem 3.1 for $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$, we obtain in in Corollary 7.2 an abstract version of the weighted Hardy inequality.

From that we deduce in Proposition 7.3 the following new type of Hardy's inequality: for any convex domain $\Omega \subset \mathbb{R}^n$, for any $\sigma \in (0, 1)$ and for all $f \in Lip_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^2 \, d(x, \partial\Omega)^\sigma} \, dx \leq C \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{d(x, \partial\Omega)^\sigma} \, dx,$$

where the constant C depends only on n and σ and hence, is independent of Ω. Although for bounded convex domains there are already various weighted Hardy's inequalities (see [25, 59, 2, 62]), they do not cover Proposition 7.3 (see Remark 7.4 for more details).

In Theorem 7.12 we have developed a systematic way for obtaining weighted Hardy's inequality in the form

$$\int_M \frac{f(x)^2}{d(x,x_o)^2} w(x)\, d\mu(x) \le C \int_M w\, d\Gamma(f,f), \tag{1.8}$$

where $\Gamma(f,f)$ is the energy measure of the Dirichlet form and the weight w is determined by the distance function to a certain closed null set Σ in M (see Definition 7.9).

In Proposition 7.15, we apply Theorem 7.12 in the case when Σ is a closed subset of a hyperplane in $\mathbb{R}^n$ and obtain that, for all $f \in W^{1,2} \cap C_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^2\, d(x,\Sigma)^\sigma} dx \le C \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{d(x,\Sigma)^\sigma} dx$$

where the range of σ is determined by the Assouad dimension of Σ (cf. (7.31), (7.32) and (7.36)). For instance, in Example 7.17 Σ is subspace of $\mathbb{R}^n$ and in Example 7.19 Σ is a Sierpinski carpet.

Our weighted Hardy inequality (1.8) seems to be entirely new in the setting of Dirichlet forms, and its proof is quite involved. We use a weighted Dirichlet form $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$, where $\mathcal{E}^{(w)}(f,f)$ is defined by the right hand side of (1.8), and prove that $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ satisfies the condition $(\mathbf{G})_2$. The latter is highly non-trivial because by hypothesis we know $(\mathbf{G})_2$ only for $(\mathcal{E}, \mathcal{F})$, and the weight function w should not be bounded or separated from zero. To explain the strategy of the proof, consider the following conditions:

▷ *Upper bound estimate* $(\mathbf{UE})_\beta$*:* the heat kernel $p_t(x,y)$ of $(\mathcal{E}, \mathcal{F})$ exists, is Hölder continuous in $x,y \in M$, and satisfies

$$p_t(x,y) \le \frac{C}{V(x,t^{1/\beta})} \exp\left\{-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right\} \tag{$\mathbf{UE}$}_\beta$$

for all $x,y \in M$ and all $t \in (0,\infty)$, where C and c are positive constants.

▷ *Near-diagonal lower bound estimate* $(\mathbf{NLE})_\beta$*:* the heat kernel $p_t(x,y)$ exists, is Hölder continuous in $x,y \in M$, and satisfies

$$p_t(x,y) \ge \frac{C^{-1}}{V(x,t^{1/\beta})} \quad \text{when } d(x,y) < \epsilon t^{1/\beta} \tag{$\mathbf{NLE}$}_\beta$$

for all $x,y \in M$ and all $t \in (0,\infty)$, where C and ϵ are positive constants.

We use the following two highly nontrivial results:

▷ the equivalence

$$(\mathbf{G})_\beta \Leftrightarrow (\mathbf{UE})_\beta + (\mathbf{NLE})_\beta, \tag{1.9}$$

established in Theorem 6.1;

▷ the stability of $(\mathbf{UE})_2 + (\mathbf{NLE})_2$ under certain non-uniform changes of weight in the Dirichlet form (see [42], [70, Theorem 1.0.1]).

Combining these results, we deduce $(\mathbf{G})_2$ for $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ from $(\mathbf{G})_2$ for $(\mathcal{E}, \mathcal{F})$.

1.5 Organization of the paper

This paper is organized as follows.

In Section 2 we describe our basic setup and recall some basic facts about Dirichlet forms, their Green functions and heat kernels.

In Section 3 we prove Hardy's inequality for strongly local regular Dirichlet forms (Theorem 3.1).

In Section 4 we prove Hardy's inequality for general (non-local) regular Dirichlet forms (Theorem 4.5).

In Section 5 we apply Theorem 4.5 to obtain Hardy's inequality in the explicit form (1.6) in various settings. In particular, we obtain in Theorem 5.1 a discrete version of Hardy's inequality on $\mathbb{Z}^n$. We prove also the aforementioned Theorem 5.6 and Corollary 5.8 as well as Theorem 5.10 containing Hardy's inequality for a subordinated Dirichlet form.

In Section 6 we prove the equivalence (1.9) for strongly local Dirichlet forms (Theorem 6.1). This equivalence is interesting on its own merit, but we need it for the proof of Theorem 7.12 as it was explained above. Previously (1.9) was known in the setting of random walks on graphs – see [45]. Different ways of characterization of the heat kernel upper and lower estimates have been considered in many papers; see for example, [8, 45, 46, 32, 36, 37, 40, 38] and references therein. In particular, it was proved in [37] that $(\mathbf{UE})_\beta$ and $(\mathbf{NLE})_\beta$ are equivalent to certain estimates of the restricted Green functions G^B in balls B provided G^B are jointly continuous off the diagonal. However, we do not apply this result here since the proof of joint continuity of G^B would have required at least as much work as a direct proof of (1.9).

The main ingredients of the proof of Theorem 6.1 are the mean exit time estimate $(\mathbf{E})_\beta$ and the elliptic Harnack inequality $(\mathbf{H})$ that are explained in Section 6. Our strategy for the proof of $(\mathbf{H})$ is based on the argument in [37, Lemma 8.2], but a crucial point here is to gain upper and lower bounds for a positive harmonic function via an integral of the Green function with respect to a certain Riesz measure (see the proof of Proposition 6.8).

In Section 7, we prove weighted Hardy's inequalities of Corollary 7.2 and Theorem 7.12, and give explicit examples using the distance function to the boundary of a convex set, a single point or a non-empty closed subset of a hyperplane (Propositions 7.3, 7.13, and 7.15).

Notation. Throughout the paper we use the following notation.

For any $p \in [1, \infty]$ and any open set $\Omega \subset M$, denote as usual by $L^p(\Omega, \mu)$ or $L^p(\Omega)$ the real-valued Lebesgue space on Ω. When $\Omega = M$ we write $L^p = L^p(M, \mu)$. We use $(\cdot, \cdot)$ to denote the inner product in L^2. Set

$$L^p_{\mathrm{loc}} = \{f : \ f \in L^p(\Omega) \text{ for any precompact open set } \Omega \subset M\}.$$

For any set $E \subset M$, $\overline{E}$ denotes the closure of E, and $E^c = M \setminus E$.

For any function $f : M \to \mathbb{R}$, its support $\operatorname{supp} f$ is the complement of the largest open set where $f = 0$ μ-a.e..

For any open set $\Omega \subset M$, $C(\Omega)$ is the space of all continuous functions on Ω with sup-norm, and $C_c(\Omega)$ is the subspace of $C(\Omega)$ consisting of functions with compact supports. In the case $\Omega = M$ we write $C = C(M)$ and $C_c = C_c(M)$.

The letters C and c are used to denote positive constants that are independent of the variables in question, but may vary at each occurrence.

The relation $u \lesssim v$ (resp., $u \gtrsim v$) between functions u and v means that $u \leq Cv$ (resp., $u \geq Cv$) for a positive constant C and for a specified range of the variables. We write $u \simeq v$ if $u \gtrsim v \gtrsim u$.

2 Basic setup

2.1 Volume doubling

Let (M, d, μ) be a metric measure space such that all the metric balls in M are precompact and μ is a Radon measure on M with full support. It is known that if (M, d) is connected and (**VD**) holds then

$$(\mathbf{RVD}) \Leftrightarrow \operatorname{diam} M = \infty \Leftrightarrow \mu(M) = \infty;$$

see [38, Corollary 5.3], [30, Theorem 1.1] or [18, Propositions 2.1 and 2.2]. Clearly, if both (**VD**) and (**RVD**) are satisfied then $0 < \alpha_- \leq \alpha_+$. If (M, d, μ) satisfies (**RVD**) then $\mu(\{x\}) = 0$ for all $x \in M$, so that (M, d, μ) is non-atomic.

The conditions (**VD**) and (**RVD**) are known to hold on many families of metric measure spaces. For example, (**VD**) and (**RVD**) are satisfied for the Euclidean space $\mathbb{R}^n$, convex unbounded domains in $\mathbb{R}^n$, Riemannian manifolds of non-negative Ricci curvature, nilpotent Lie groups, and on many fractal-like spaces; see [5, 7, 17, 18, 32, 37, 38, 45, 54, 67, 72].

2.2 Dirichlet forms

Let (M, d, μ) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on L^2, that is, $\mathcal{E}$ is a symmetric, non-negative definite, closed, Markovian bilinear form in L^2 with domain $\mathcal{F}$ that is a dense subspace of L^2. The domain $\mathcal{F}$ is a Hilbert space endowed with the following norm:

$$\|u\|_{\mathcal{F}}^2 = \mathcal{E}(u, u) + \|u\|_{L^2}^2.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_c$ is dense both in $\mathcal{F}$ (with respect to the norm $\|\cdot\|_{\mathcal{F}}$) and in C_c (with respect to the supremum norm). For more information of Dirichlet forms, we refer the reader to [28].

Definition 2.1. For any open set $\Omega \subset M$ and a set $A \Subset \Omega$, a *cutoff function* ϕ of the pair (A, Ω) is a function $\phi \in \mathcal{F} \cap C_c(\Omega)$ such that $0 \leq \phi \leq 1$ in M and $\phi = 1$ in an open neighborhood of $\overline{A}$.

It is known that if $(\mathcal{E}, \mathcal{F})$ is regular then, for any open set $\Omega \subset M$ and any $A \Subset \Omega$, there exists always a cutoff function of (A, Ω); see [28, p.27].

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(u, v) = 0$ for any two functions $u, v \in \mathcal{F}$ with compact supports such that $u = \text{const}$ in some open neighborhood of $\operatorname{supp} v$.

Any Dirichlet form $(\mathcal{E}, \mathcal{F})$ has the generator – a non-negative definite self-adjoint operator $\mathcal{L}$ in L^2 such that $\operatorname{dom}(\mathcal{L}) \subset \mathcal{F}$ and

$$\mathcal{E}(u, v) = (\mathcal{L}u, v) \quad \text{for all } u \in \operatorname{dom}(\mathcal{L}) \text{ and } v \in \mathcal{F}.$$

For any $t \geq 0$ set $P_t = e^{-t\mathcal{L}}$ so that P_t is a bounded, self-adjoint, positivity preserving operator in L^2. The family $\{P_t\}_{t \geq 0}$ is called the *heat semigroup* of $(\mathcal{E}, \mathcal{F})$. If P_t for $t > 0$ has an integral kernel then the latter is called the heat kernel and is denoted by $p_t(x, y)$ so that for all $f \in L^2$ and $t > 0$,

$$P_t f(x) = \int_M p_t(x, y) f(y) \, d\mu(y) \quad \text{for } \mu\text{-a.a. } x \in M.$$

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. For any non-empty open set $\Omega \subset M$, define $\mathcal{F}(\Omega)$ as the closure of $\mathcal{F} \cap C_c(\Omega)$ in $\mathcal{F}$. Then $\mathcal{F}(\Omega)$ is dense in $L^2(\Omega)$ and $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet from in $L^2(\Omega)$, that is called the part of $(\mathcal{E}, \mathcal{F})$ on Ω. Denote by $\mathcal{L}^\Omega$ the generator of $(\mathcal{E}, \mathcal{F}(\Omega))$ and by $\{P_t^\Omega\}$ the corresponding heat semigroup. It is known that, for any $0 \leq f \in L^2(\Omega)$ and $t \geq 0$,

$$P_t^\Omega f \leq P_t f.$$

Set also

$$\lambda_{\min}(\Omega) = \inf \operatorname{spec} \mathcal{L}^{\Omega}.$$

It is known that

$$\lambda_{\min}(\Omega) = \inf_{u \in \mathcal{F}(\Omega) \setminus \{0\}} \frac{\mathcal{E}(u,u)}{\|u\|_{L^2}^2} = \inf_{u \in (\mathcal{F} \cap C_c(\Omega)) \setminus \{0\}} \frac{\mathcal{E}(u,u)}{\|u\|_{L^2}^2} \tag{2.1}$$

2.3 Green function

The positivity preserving property of the heat semigroups allows to extend $P_t f$ from $f \in L^2$ to all non-negative measurable functions u on M (of course, the value $+\infty$ for $P_t f$ is allowed in this case). It is easy to verify that the semigroup property $P_{t+s} f = P_t (P_s f)$ holds also in this extended setting.

Define the *Green operator* G for all non-negative measurable functions f on M by

$$Gf = \int_0^\infty P_t f \, dt.$$

Of course, the value $+\infty$ is allowed for Gf.

A function $G(x,y)$ on $M \times M$ is called the *Green function* (or the Green kernel) if it takes values in $[0,+\infty]$, is jointly measurable, non-negative, and satisfies for any non-negative f the identity

$$Gf(x) = \int_M G(x,y) f(y) \, d\mu(y) \text{ for } \mu\text{-a.a. } x \in M.$$

For instance, if the heat semigroup $\{P_t\}$ has the heat kernel $p_t(x,y)$ then

$$G(x,y) = \int_0^\infty p_t(x,y) \, dt$$

(although the integral here may diverge). Note that the Green function is always symmetric in x,y which follows from the symmetry of P_t.

Let Ω be a non-empty open subset of M. Denote by P_t^Ω the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$ and by G^Ω the Green operator. It is known that, for any non-negative f,

$$0 \le P_t^\Omega f \le P_t f \text{ for all } t > 0,$$

whence also $0 \le G^\Omega f \le Gf$.

Remark 2.2. Assume that $\lambda_{\min}(\Omega) > 0$. Then the operator $\mathcal{L}^\Omega$ has a bounded inverse in $L^2(\Omega)$, and $(\mathcal{L}^\Omega)^{-1} = G^\Omega|_{L^2(\Omega)}$. In this case G^Ω has the following property: for any $f \in L^2(\Omega)$, we have $G^\Omega f \in \mathcal{F}(\Omega)$ and

$$\mathcal{E}(G^\Omega f, \phi) = (f, \phi) \quad \text{for all } \phi \in \mathcal{F}(\Omega); \tag{2.2}$$

see [37, Lemma 5.1].

The following two-sided estimate $(\mathbf{G})_\beta$ for the Green function $G(x,y)$ are fundamental for us to derive Hardy's inequalities.

Definition 2.3. Given $\beta > 0$, we say that condition $(\mathbf{G})_\beta$ is satisfied if the Green function $G(x,y)$ exists, is jointly continuous in $(M \times M) \setminus \operatorname{diag}$, and

$$G(x,y) \simeq \frac{d(x,y)^\beta}{V(x,y)} \text{ for all distinct } x,y \in M. \tag{$(\mathbf{G})_\beta$}$$

Note that the estimate $(\mathbf{G})_\beta$ can be obtained from certain heat kernel bounds as follows.

Lemma 2.4. *Assume that* (M,d,μ) *satisfies* (**VD**) *and that the heat kernel of* $(\mathcal{E},\mathcal{F})$ *exists. If the heat kernel* p_t *satisfies for all* $t>0$ *and* μ*-a.a.* $x,y\in M$ *the following inequality*

$$p_t(x,y)\lesssim \frac{1}{V(x,t^{1/\beta})}\wedge\frac{1}{V(x,y)}, \tag{2.3}$$

then

$$G(x,y)\lesssim\int_{d(x,y)}^{\infty}\frac{r^{\beta-1}}{V(x,r)}dr, \tag{2.4}$$

for μ*-a.a.* $x,y\in M$*. If, for any* $t>0$ *and* μ*-a.a.* $x,y\in M$*,*

$$p_t(x,y)\gtrsim\frac{1}{V(x,t^{1/\beta})}\quad\text{provided }\ d(x,y)^{\beta}\le t, \tag{2.5}$$

then

$$G(x,y)\gtrsim\int_{d(x,y)}^{\infty}\frac{r^{\beta-1}}{V(x,r)}dr \tag{2.6}$$

for μ*-a.a.* $x,y\in M$*.*

Consequently, if both (2.3) *and* (2.5) *are satisfied then*

$$G(x,y)\simeq\int_{d(x,y)}^{\infty}\frac{r^{\beta-1}}{V(x,r)}dr, \tag{2.7}$$

for μ*-a.a.* $x,y\in M$*. Furthermore, if in addition* (**RVD**) *holds with* $\alpha_->\beta$ *then the Green function satisfies* $(\mathbf{G})_\beta$*.*

Proof. Set for simplicity $\rho=d(x,y)$. It follows from (2.5) that

$$G(x,y)\gtrsim\int_{\rho^{\beta}}^{\infty}\frac{dt}{V(x,t^{1/\beta})}\simeq\int_{\rho}^{\infty}\frac{\beta r^{\beta-1}dr}{V(x,r)},$$

which proves (2.6). It follows from (2.3) that

$$G(x,y)\lesssim\int_{\rho^{\beta}}^{\infty}\frac{dt}{V(x,t^{1/\beta})}+\int_{0}^{\rho^{\beta}}\frac{dt}{V(x,\rho)}\simeq\int_{\rho}^{\infty}\frac{\beta r^{\beta-1}dr}{V(x,r)}+\frac{\rho^{\beta}}{V(x,\rho)}.$$

It remains to observe that, by (**VD**),

$$\int_{\rho}^{\infty}\frac{r^{\beta-1}dr}{V(x,r)}\ge\int_{\rho}^{2\rho}\frac{r^{\beta-1}dr}{V(x,r)}\ge\frac{\rho^{\beta}}{V(x,2\rho)}\gtrsim\frac{\rho^{\beta}}{V(x,\rho)}, \tag{2.8}$$

whence (2.4) follows. Combining (2.6) and (2.4) gives (2.7).

If (**RVD**) is satisfied with $\alpha_->\beta$ then

$$\begin{aligned}\int_{\rho}^{\infty}\frac{r^{\beta-1}}{V(x,r)}dr&=\frac{\rho^{\beta}}{V(x,\rho)}\int_{\rho}^{\infty}\frac{V(x,\rho)\,r^{\beta}}{V(x,r)\rho^{\beta}}\frac{dr}{r}\\&\lesssim\frac{\rho^{\beta}}{V(x,\rho)}\int_{\rho}^{\infty}\left(\frac{\rho}{r}\right)^{\alpha_-}\left(\frac{r}{\rho}\right)^{\beta}\frac{dr}{r}\\&\simeq\frac{\rho^{\beta}}{V(x,\rho)}\int_{1}^{\infty}s^{-(\alpha_--\beta)}\frac{ds}{s}\\&\lesssim\frac{\rho^{\beta}}{V(x,\rho)},\end{aligned}$$

which together with (2.7) and (2.8) implies

$$G(x,y)\simeq\int_{\rho}^{\infty}\frac{r^{\beta-1}}{V(x,r)}dr\simeq\frac{\rho^{\beta}}{V(x,\rho)},$$

that is $(\mathbf{G})_\beta$. □

Example 2.5. Assume that the heat kernel $p_t(x,y)$ on (M,d,μ) exists and satisfies the following sub-Gaussian estimate: for all $t>0$ and μ-a.a. $x,y\in M$,

$$p_t(x,y) \asymp \frac{C}{V(x,t^{1/\beta})}\exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right), \tag{2.9}$$

where $\beta>1$ is the *walk dimension* and the symbol $\asymp$ means that both inequalities with $\le$ and $\ge$ are satisfied but with different values of positive constants C and c. For example, (2.9) is satisfied with $\beta=2$ on any Riemannian manifold of non-negative Ricci curvature (see [61]) as well as with $\beta>2$ on many fractal spaces (see Example 5.12 below).

Clearly, (2.9) implies both (2.3) and (2.5). Indeed, (2.3) and (2.5) are trivial in the case $t\ge d(x,y)^\beta$, while in the case $t<d(x,y)^\beta$ we have, setting $r=d(x,y)$,

$$V(x,r)\,p_t(x,y) \lesssim \frac{V(x,r)}{V(x,t^{1/\beta})}\exp\left(-c\left(\frac{r}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \lesssim \left(\frac{r}{t^{1/\beta}}\right)^{\alpha_+}\exp\left(-c\left(\frac{r}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right)\le C$$

so that

$$p_t(x,y)\le \frac{C}{V(x,r)},$$

which proves (2.3).

Example 2.6. For certain jump processes on fractal spaces the heat kernel satisfies the following stable-like estimate

$$p_t(x,y)\simeq \frac{1}{V(x,t^{1/\beta})}\wedge\frac{t}{V(x,y)\,d(x,y)^\beta}; \tag{2.10}$$

see [16]. For example, if $V(x,r)\simeq r^\alpha$ then (2.10) becomes

$$p_t(x,y)\simeq \frac{1}{t^{\alpha/\beta}}\wedge\frac{t}{d(x,y)^{\alpha+\beta}}\simeq\frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}.$$

This estimate is satisfied with $\alpha=n$ for a symmetric stable process in $\mathbb{R}^n$ of index β.

If $t\ge d(x,y)^\beta$ then (2.10) becomes

$$p_t(x,y)\simeq\frac{1}{V(x,t^{1/\beta})},$$

while in the case $t<d(x,y)^\beta$ inequality (2.10) implies

$$p_t(x,y)\simeq\frac{t}{V(x,y)\,d(x,y)^\beta}\le\frac{1}{V(x,y)}.$$

Hence, in both cases the estimates (2.3) and (2.5) are satisfied, and by Lemma 2.4 the Green function satisfies $(\mathbf{G})_\beta$.

3 Hardy's inequality for strongly local regular Dirichlet forms

In the setting of strongly local regular Dirichlet forms, in order to prove an abstract version of Hardy's inequality, we adopt the method of changing measures explained in introduction. The following theorem is the main result of this section.

Theorem 3.1. *Let $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form in $L^2(M,\mu)$. Assume that $\lambda_{\min}(\Omega) > 0$ for all precompact open sets $\Omega \subset M$. Let h be a non-negative measurable function on M such that*

$$G(h \wedge a) \in L^\infty_{\mathrm{loc}} \tag{3.1}$$

for any positive constant a. Then, for any $f \in \mathcal{F}$,

$$\int_M \frac{h}{Gh} f^2\, d\mu \le \mathcal{E}(f, f). \tag{3.2}$$

Remark 3.2. If h and Gh vanish simultaneously at some points then at these points we set $\frac{h}{Gh} = 0$.

Before the proof, let us recall some necessary notions from the theory of strongly local Dirichlet forms. According to [28, Section 3.2] or [15, Section 4.3], for any $u \in \mathcal{F} \cap L^\infty$, there exists a unique positive Radon measure $\Gamma(u,u)$ on M such that

$$\int_M f\, d\Gamma(u,u) = \mathcal{E}(uf, u) - \frac{1}{2}\mathcal{E}(u^2, f) \quad \text{for all } f \in \mathcal{F} \cap C_c.$$

This measure $\Gamma(u,u)$ is called the *energy measure* of u. For any $u, v \in \mathcal{F} \cap L^\infty$, define a signed energy measure $\Gamma(u,v)$ by

$$\int_M f\, d\Gamma(u,v) = \frac{1}{2}\Big(\mathcal{E}(uf, v) + \mathcal{E}(u, vf) - \mathcal{E}(uv, f)\Big) \quad \text{for all } f \in \mathcal{F} \cap C_c.$$

Note that $\Gamma(u,v)$ is symmetric and bilinear, and it can be extended to all $u, v \in \mathcal{F}$. It is known that

$$\mathcal{E}(u,v) = \int_M d\Gamma(u,v) \quad \text{for all } u, v \in \mathcal{F}; \tag{3.3}$$

see, for example, [13] or [28, Lemma 3.2.3].

Let $\mathcal{F}_{\mathrm{loc}}$ be the space of all μ-measurable functions u on M satisfying the following property: for every precompact open subset $\Omega \subset M$ there exists a function $u' \in \mathcal{F}$ such that $u = u'$ μ-a.e. on Ω. The locality of $(\mathcal{E},\mathcal{F})$ allows to extend $\mathcal{E}(u,v)$ to all $u \in \mathcal{F}_{\mathrm{loc}}$ and $v \in \mathcal{F}_c$, where $\mathcal{F}_c$ denotes a subspace of $\mathcal{F}$ consisting of functions with compact support. Indeed, there exists $u' \in \mathcal{F}$ such that $u = u'$ in a neighborhood of $\operatorname{supp} v$, and $\mathcal{E}(u', v)$ is obviously independent of the choice of u', so that we set $\mathcal{E}(u,v) := \mathcal{E}(u',v)$. It follows that the identity (3.3) holds also for $u \in \mathcal{F}_{\mathrm{loc}}$ and $v \in \mathcal{F}_c$.

It is known that the space $\mathcal{F} \cap L^\infty$ is closed under multiplication of functions; see, for example, [28, Theorem 1.4.2(ii)]. This implies that $\mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ is closed under multiplication[1].

For strongly local Dirichlet forms, $\Gamma(u,v)$ can be extended to all $u, v \in \mathcal{F}_{\mathrm{loc}}$; see [15, Theorem 4.3.11] and [68, p.189]. Moreover, by [68, p.190], we know that $\Gamma(u,v)$ satisfies the following Leibniz product rule

$$d\Gamma(uv, w) = u\, d\Gamma(v,w) + v\, d\Gamma(u,w) \quad \text{for all } u, v \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \text{ and } w \in \mathcal{F}_{\mathrm{loc}}. \tag{3.4}$$

The following lemma is a key ingredient for the proof of Theorem 3.1.

Lemma 3.3. *Let $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(M,\mu)$. If ϕ is a positive measurable function on M such that*

$$\text{both } \phi \text{ and } \phi^{-1} \text{ belong to } \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}, \tag{3.5}$$

[1]Indeed, if $f, g \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ then, for any precompact open set Ω, there exist $f', g' \in \mathcal{F}$ such that $f = f'$ and $g = g'$ in Ω. Both f' and g' can be chosen to be bounded on M because otherwise f' can be replaced by $(f' \wedge C) \vee (-C)$ for any $C > \|f\|_{L^\infty(\Omega)}$, and the same is valid for g'. Hence, $f'g' \in \mathcal{F} \cap L^\infty$. Since $fg = f'g'$ in Ω, we conclude that $fg \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$.

then

$$\mathcal{E}(f,f)-\mathcal{E}(\phi,\phi^{-1}f^2)=\int_M \phi^2\,d\Gamma(\phi^{-1}f,\phi^{-1}f)\geq 0 \quad \textit{for all} \quad f\in\mathcal{F}_c\cap L^\infty. \tag{3.6}$$

Consequently, we have

$$\mathcal{E}(\phi,\phi^{-1}f^2)\leq\mathcal{E}(f,f) \quad \textit{for all} \quad f\in\mathcal{F}_c\cap L^\infty. \tag{3.7}$$

Proof. Since $\phi^{-1}\in\mathcal{F}_{\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}$ and both functions f and f^2 lie in $\mathcal{F}_c\cap L^\infty$, we obtain

$$\phi^{-1}f \;\text{ and }\phi^{-1}f^2\in\mathcal{F}_c\cap L^\infty \tag{3.8}$$

(indeed, both $\phi^{-1}f$ and $\phi^{-1}f^2$ belong to $\mathcal{F}_{\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}$ and have compact supports). By (3.3) we have

$$\mathcal{E}(f,f)-\mathcal{E}(\phi,\phi^{-1}f^2)=\int_M d\Gamma(f,f)-\int_M d\Gamma(\phi,\phi^{-1}f^2).$$

Applying the Leibniz rule (3.4), we obtain

$$\begin{aligned}
d\Gamma(f,f)-d\Gamma(\phi,\phi^{-1}f^2)&=d\Gamma\left(\left(\phi^{-1}f\right)\phi,f\right)-d\Gamma\left(\phi,\left(\phi^{-1}f\right)f\right)\\
&=\left(\phi^{-1}f\,d\Gamma(\phi,f)+\phi\,d\Gamma(\phi^{-1}f,f)\right)-\left(\phi^{-1}f\,d\Gamma(\phi,f)+f\,d\Gamma(\phi,\phi^{-1}f)\right)\\
&=\phi\,d\Gamma(\phi^{-1}f,\phi\phi^{-1}f)-f\,d\Gamma(\phi,\phi^{-1}f)\\
&=\left(\phi^2\,d\Gamma(\phi^{-1}f,\phi^{-1}f)+f\,d\Gamma(\phi^{-1}f,\phi)\right)-f\,d\Gamma(\phi,\phi^{-1}f)\\
&=\phi^2\,d\Gamma(\phi^{-1}f,\phi^{-1}f),
\end{aligned}$$

whence it follows that

$$\mathcal{E}(f,f)-\mathcal{E}(\phi,\phi^{-1}f^2)=\int_M \phi^2\,d\Gamma(\phi^{-1}f,\phi^{-1}f)\geq 0.$$

This proves (3.6) and, hence, (3.7). □

Remark 3.4. If in addition to (3.5) assume that $\phi\in\mathrm{dom}(\mathcal{L})$ then

$$\mathcal{E}(\phi,\phi^{-1}f^2)=\left(\mathcal{L}\phi,\phi^{-1}f^2\right)=\int_M\frac{\mathcal{L}\phi}{\phi}f^2\,d\mu.$$

Hence, (3.7) becomes

$$\int_M\frac{\mathcal{L}\phi}{\phi}f^2\,d\mu\leq\mathcal{E}(f,f),$$

which coincides with (1.2) when (M,d,μ) is a Riemannian manifold and $\mathcal{L}=-\Delta$. Moreover, by the Leibniz rule (3.4), one can verify that the identity (3.6) coincides with [71, (3.2)] provided that f satisfies additional conditions in terms of ϕ and the generalized Laplace operator (that is defined based on the energy measure).

Proof of Theorem 3.1. It suffices to prove (3.2) for all $f\in\mathcal{F}\cap C_c$ since for any $f\in\mathcal{F}$ there exists a sequence $\{f_n\}$ from $\mathcal{F}\cap C_c$ converging to f in $\mathcal{F}$. Applying (3.2) to each f_n, passing to the limit as $n\to\infty$ and using Fatou's lemma in the left hand side, we obtain (3.2) for f.

Hence, we assume further that $f\in\mathcal{F}\cap C_c$. Let Ω be a precompact open subset of M containing $\operatorname{supp} f$ so that $f\in\mathcal{F}(\Omega)$. Let a,ε be positive constants. Set

$$h_a=h\wedge a$$

and consider in Ω the function

$$\phi=G^\Omega h_a+\varepsilon.$$

By (3.1), ϕ is bounded in Ω. Since $\lambda_{\min}(\Omega) > 0$ and $h_a \in L^2(\Omega)$, we have $G^{\Omega}h_a \in \mathcal{F}(\Omega)$ and, hence, $\phi \in \mathcal{F}_{\text{loc}}(\Omega)$. Since $\phi \geq \varepsilon$, it follows that $\phi^{-1} \in \mathcal{F}_{\text{loc}} \cap L^{\infty}(\Omega)$ (indeed, $\phi^{-1} = F \circ \phi$ where $F(t) := \varepsilon^{-1} \wedge t^{-1}$ is Lipschitz; see [28, Theorem 1.4.2(v)]). Therefore, ϕ satisfies the hypotheses of Lemma 3.3 in Ω, and we conclude that

$$\mathcal{E}(\phi, \phi^{-1} f^2) \leq \mathcal{E}(f, f).$$

By (3.8) we have $\phi^{-1} f^2 \in \mathcal{F}_c(\Omega)$, and by (2.2) and the strong locality

$$\begin{aligned}\mathcal{E}(\phi, \phi^{-1} f^2) &= \mathcal{E}(G^{\Omega}h_a + \varepsilon, \phi^{-1} f^2) = \mathcal{E}(G^{\Omega}h_a, \phi^{-1} f^2) \\ &= \left(h_a, \phi^{-1} f^2\right) = \int_{\Omega} \frac{h_a}{G^{\Omega}h_a + \varepsilon} f^2 d\mu \\ &\geq \int_{\Omega} \frac{h_a}{Gh + \varepsilon} f^2 d\mu\end{aligned}$$

whence

$$\int_{\Omega} \frac{h_a}{Gh + \varepsilon} f^2 d\mu \leq \mathcal{E}(f, f).$$

Letting $a \to \infty$, $\varepsilon \to 0$, and $\Omega \to M$, we obtain (3.2). □

A non-negative measurable function u on M is called *excessive* if $P_t u \leq u$ for all $t \geq 0$. Consequently, if u is excessive, then $P_t u \leq P_s u$ for all $t \geq s \geq 0$.

Corollary 3.5. *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(M, \mu)$. Assume that $\lambda_{\min}(\Omega) > 0$ for all precompact open sets $\Omega \subset M$. Let $u \in L^{\infty}_{\text{loc}}$ be a positive excessive function on M. Then, for any $f \in \mathcal{F}$,*

$$-\int_M f^2 \partial_t \log(P_t u)\, d\mu \leq \mathcal{E}(f, f). \tag{3.9}$$

Proof. Fix $t > 0$ and set

$$h = -\partial_t P_t u$$

so that h is a non-negative measurable function on M. We have

$$\begin{aligned}Gh &= \int_0^{\infty} P_s h\, ds = -\int_0^{\infty} P_s(\partial_t P_t u)\, ds \\ &= -\int_0^{\infty} \partial_t(P_{t+s}u)\, ds = -\int_0^{\infty} \partial_s(P_{t+s}u)\, ds \\ &= -\int_t^{\infty} \partial_s(P_s u)\, ds \leq P_t u.\end{aligned}$$

Hence,

$$Gh \leq P_t u \leq u$$

which implies that $Gh \in L^{\infty}_{\text{loc}}$. By Theorem 3.1 we conclude that

$$\int_M \frac{h}{Gh} f^2 d\mu \leq \mathcal{E}(f, f).$$

Observing that

$$\frac{h}{Gh} \geq \frac{-\partial_t P_t u}{P_t u} = -\partial_t \log(P_t u),$$

we obtain (3.9). □

4 Hardy's inequality for regular Dirichlet forms

In this section, we prove an analogue of Theorem 3.1 for general (non-local) regular Dirichlet forms. The main result is Theorem 4.5 below.

4.1 Extended Dirichlet forms

Given a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on L^2, denote by $\mathcal{F}_e$ the family of all μ-measurable functions u on M such that u is finite μ-a.e. on M and there exists a sequence $\{u_n\}\subset\mathcal{F}$ such that

$$\lim_{n\to\infty} u_n = u \quad \mu\text{-a.e. on } M \quad \text{and} \quad \lim_{n,m\to\infty}\mathcal{E}(u_n-u_m,u_n-u_m)=0.$$

For any $u\in\mathcal{F}_e$, by [28, Theorem 1.5.2(i)], the limit

$$\mathcal{E}(u,u)=\lim_{n\to\infty}\mathcal{E}(u_n,u_n)$$

exists and does not depend on the choice of the sequence $\{u_n\}$. Moreover, by [28, Theorem 1.5.2(iii)],

$$\mathcal{F}=\mathcal{F}_e\cap L^2.$$

The pair $(\mathcal{E},\mathcal{F}_e)$ is called an *extended Dirichlet form.*

As was discussed in the previous section, both $\mathcal{F}\cap L^\infty$ and $\mathcal{F}_{\text{loc}}\cap L^\infty_{\text{loc}}$ are closed under multiplication of functions. The following lemma extends this property to $\mathcal{F}_e$.

Lemma 4.1. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on the metric measure space (M,d,μ). Then, for any $u\in\mathcal{F}_e\cap L^\infty_{\text{loc}}$ and any $\psi\in\mathcal{F}_c\cap L^\infty$, we have*

$$u\psi\in\mathcal{F}\cap L^\infty. \tag{4.1}$$

Consequently,

$$\mathcal{F}_e\cap L^\infty_{\text{loc}}\subset\mathcal{F}_{\text{loc}}. \tag{4.2}$$

Proof. Let us first show that (4.1) implies (4.2). Indeed, given a function $u\in\mathcal{F}_e\cap L^\infty_{\text{loc}}$ and a precompact open subset $\Omega\subset M$, we need to find a function $g\in\mathcal{F}$ such that $u=g$ μ-a.e. on Ω. Let ψ be a cutoff function of Ω in M. By (4.1) we have $g:=u\psi\in\mathcal{F}$. Since $g=u$ in Ω, we obtain (4.2).

Now let us prove (4.1). We use the following result [28, (1.3.18) and (1.4.8)]: for any Borel measurable function f on M,

$$f\in\mathcal{F}\Leftrightarrow f\in L^2 \text{ and } \lim_{\tau\to\infty}\mathcal{E}^{(\tau)}(f,f)<\infty, \tag{4.3}$$

where

$$\mathcal{E}^{(\tau)}(f,f)=\frac{\tau}{2}\int_{M\times M}(f(x)-f(y))^2\,d\sigma_\tau(x,y)+\tau\int_M f^2 s_\tau\,d\mu \tag{4.4}$$

for some positive symmetric Radon measure $\sigma_\tau(\cdot,\,\cdot)$ on $M\times M$ satisfying $\sigma_\tau(M,\,E)\le\mu(E)$ for any Borel measurable set E, and s_τ is a function such that $0\le s_\tau\le 1$ on M. It is also known that $\mathcal{E}^{(\tau)}(f,f)$ is non-decreasing as $\tau\to\infty$ so that the limit in (4.3) always exists, finite or infinite. Moreover, by [28, Theorem 1.5.2(i)-(ii)] if $f\in\mathcal{F}_e$ then

$$\lim_{\tau\to\infty}\mathcal{E}^{(\tau)}(f,f)=\mathcal{E}(f,f)<\infty.$$

Let $u\in\mathcal{F}_e\cap L^\infty_{\text{loc}}$ and $\psi\in\mathcal{F}_c\cap L^\infty$. Without loss of generality we can assume that u and ψ are Borel measurable. Clearly, we have $u\psi\in L^\infty\cap L^2$ so that, by (4.3), in order to prove that $u\psi\in\mathcal{F}$, it suffices to verify that

$$\lim_{\tau\to\infty}\mathcal{E}^{(\tau)}(u\psi,u\psi)<\infty.$$

Without loss of generality, we can assume that $\|\psi\|_{L^\infty} = 1$. The set $\{x \in M : |\psi(x)| > 1\}$ is a Borel set of μ-measure zero. Modifying ψ on this set by setting $\psi = 0$ we can assume without loss of generality that

$$|\psi(x)| \le 1 \quad \text{for all } x \in M.$$

Let Ω be a precompact open set containing $\operatorname{supp}\psi$. Similarly, after modifying ψ on a Borel set of μ-measure zero, we can assume that $\psi(x) = 0$ for all $x \in \Omega^c$.

Without loss of generality, we can also assume that $\|u\|_{L^\infty(\Omega)} = 1$. Modifying u on the Borel null set $\{x \in \Omega : u(x) > 1\}$, we can assume that

$$|u(x)| \le 1 \quad \text{for all } x \in \Omega.$$

Let us verify that, for all $x, y \in M$,

$$|u(x)\psi(x) - u(y)\psi(y)| \le |\psi(x) - \psi(y)| + |u(x) - u(y)|. \tag{4.5}$$

Indeed, if $x, y \in \Omega$ then

$$\begin{aligned} |u(x)\psi(x) - u(y)\psi(y)| &\le |u(x)|\,|\psi(x) - \psi(y)| + |\psi(y)|\,|u(x) - u(y)| \\ &\le |\psi(x) - \psi(y)| + |u(x) - u(y)|. \end{aligned}$$

If $x \in \Omega^c$ and $y \in \Omega$ then $\psi(x) = 0$ and

$$|u(x)\psi(x) - u(y)\psi(y)| = |u(y)|\,|\psi(y)| = |u(y)|\,|\psi(x) - \psi(y)| \le |\psi(x) - \psi(y)|,$$

and if $x, y \in \Omega^c$ then $|u(x)\psi(x) - u(y)\psi(y)| = 0$.

It follows from (4.5) that

$$\begin{aligned} \int_{M\times M} ((u\psi)(x) - (u\psi)(y))^2 \, d\sigma_\tau(x, y) \le 2 &\int_{M\times M} (\psi(x) - \psi(y))^2 \, d\sigma_\tau(x, y) \\ +2 &\int_{M\times M} (u(x) - u(y))^2 \, d\sigma_\tau(x, y). \end{aligned}$$

Since $|u\psi| \le |u|$, we have also

$$\int_M (u\psi)^2 s_\tau \, d\mu \le \int_M u^2 s_\tau \, d\mu.$$

From this and (4.4), it follows that

$$\mathcal{E}^{(\tau)}(u\psi, u\psi) \le 2\mathcal{E}^{(\tau)}(\psi, \psi) + 2\mathcal{E}^{(\tau)}(u, u)$$

and, hence,

$$\lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(u\psi, u\psi) \le 2 \lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(\psi, \psi) + 2 \lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(u, u) < \infty,$$

which finishes the proof. □

4.2 Transience of Dirichlet forms

According to [28, Section 1.5], a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *transient* if there exists a bounded μ-measurable function g that is strictly positive μ-a.e. on M and such that

$$\int_M |u| g \, d\mu \le \sqrt{\mathcal{E}(u, u)} \quad \text{for all } u \in \mathcal{F}.$$

By [28, Lemma 1.5.5], if $(\mathcal{E},\mathcal{F})$ is transient then $\mathcal{E}(u,v)$ is an inner product in $\mathcal{F}_e$ and $\mathcal{F}_e$ with this inner product is a Hilbert space. By [28, Theorem 1.5.4], if $(\mathcal{E},\mathcal{F})$ is transient, then, for any non-negative μ-measurable function f on M satisfying

$$\int_M fGf\,d\mu < \infty,$$

we have that $Gf \in \mathcal{F}_e$ and

$$\mathcal{E}(Gf,\phi) = \int_M f\phi\,d\mu \;\text{ for all } \phi \in \mathcal{F}_e. \tag{4.6}$$

As it follows from [28, Lemma 1.5.1], in order to show that $(\mathcal{E},\mathcal{F})$ is transient, it suffices to find a μ-a.e. strictly positive function $g \in L^1$ such that

$$Gg(x) < \infty \quad \text{for } \mu\text{-a.a. } x \in M. \tag{4.7}$$

Lemma 4.2. *If the Green function $G(x,y)$ exists and belongs to $L^1_{\text{loc}}(M\times M)$ then $(\mathcal{E},\mathcal{F})$ is transient.*

Proof. It suffices to construct a strictly positive function $g \in L^1$ such that

$$Gg \in L^1_{\text{loc}},$$

which will imply (4.7). Observe first that if A and B are precompact subsets of M then

$$\int_B G1_A d\mu = \int_B\left(\int_A G(x,y)\,d\mu(y)\right)d\mu(x) = \|G\|_{L^1(B\times A)} < \infty. \tag{4.8}$$

Fix a point $x_o \in M$, set $B_k = B(x_o, 2^k)$,

$$A_0 = B_0, \quad A_k = B_k \setminus B_{k-1} \text{ for } k \geq 1,$$

and define g by

$$g = \sum_{k=0}^{\infty} c_k 1_{A_k},$$

where $\{c_k\}_{k=0}^{\infty}$ is sequence of positive reals yet to be determined. Clearly, $g > 0$ on M. By (4.8) we have, for all indices k,n,

$$\int_{B_n} G1_{A_k} d\mu = \|G\|_{L^1(B_n\times A_k)}$$

and, hence,

$$\int_{B_n} Gg d\mu = \sum_{k=0}^{\infty} c_k \|G\|_{L^1(B_n\times A_k)}. \tag{4.9}$$

Choose c_k for all $k = 0,1,...$ so that $c_k \|G\|_{L^1(B_k\times A_k)} \leq 2^{-k}$. Then the series in (4.9) converges for any n, whence $Gg \in L^1_{\text{loc}}$ follows. □

Corollary 4.3. *If (M,d,μ) satisfies* (**VD**) *and, for some $\beta > 0$.*

$$G(x,y) \lesssim \frac{d(x,y)^\beta}{V(x,y)} \quad \text{for } \mu\text{-a.a. distinct } x,y \in M,$$

then $(\mathcal{E},\mathcal{F})$ is transient. In particular, (**VD**) + $(\mathbf{G})_\beta$ *imply the transience.*

Proof. Indeed, by **(VD)**, we have, for all $x \in M$ and $R \in (0, \infty)$,

$$\begin{aligned}\int_{B(x,R)} \frac{d(x,y)^\beta}{V(x,y)}\, d\mu(y) &= \sum_{j=0}^{\infty} \int_{B(x,2^{-j}R)\setminus B\left(x,2^{-(j+1)}R\right)} \frac{d(x,y)^\beta}{V(x,y)}\, d\mu(y) \\ &\le \sum_{j=0}^{\infty} (2^{-j}R)^\beta \frac{V(x,2^{-j}R)}{V(x,2^{-(j+1)}R)} \\ &\le C_D \sum_{j=0}^{\infty} (2^{-j}R)^\beta \\ &\simeq R^\beta. \end{aligned} \tag{4.10}$$

Now, for any ball $B(x_o, R)$, we obtain, using (4.10),

$$\begin{aligned}\int_{B(x_o,R)} \int_{B(x_o,R)} G(x,y)\, d\mu(y)\, d\mu(x) &\lesssim \int_{B(x_o,R)} \left(\int_{B(x_o,R)} \frac{d(x,y)^\beta}{V(x,y)}\, d\mu(y) \right) d\mu(x) \\ &\le \int_{B(x_o,R)} \left(\int_{B(x,2R)} \frac{d(x,y)^\beta}{V(x,y)}\, d\mu(y) \right) d\mu(x) \\ &\lesssim \int_{B(x_o,R)} R^\beta d\mu(x) < \infty, \end{aligned}$$

which implies $G \in L^1_{\text{loc}}(M \times M)$. Hence, $(\mathcal{E}, \mathcal{F})$ is transient by Lemma 4.2. □

4.3 Admissible functions and Hardy's inequality

Definition 4.4. Let G be the Green operator of a Dirichlet form. A positive μ-measurable function h on M is called (μ, G)-admissible if it satisfies the following three conditions:

(*i*) $Gh \in L^\infty_{\text{loc}}$;

(*ii*) $(Gh)^{-1} \in L^\infty_{\text{loc}}$;

(*iii*) $\int_M hGh\, d\mu < \infty$.

The next theorem is our main result about Hardy's inequality for general regular Dirichlet forms.

Theorem 4.5. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on (M, d, μ) and G be its Green operator. If h is a (μ, G)-admissible function on M, then the following Hardy's inequality holds:*

$$\int_M \frac{h}{Gh} f^2 d\mu \le \mathcal{E}(f, f) \quad \textit{for all } f \in \mathcal{F}. \tag{4.11}$$

Remark 4.6. If $(\mathcal{E}, \mathcal{F})$ is strongly local then Theorem 3.1 gives the same Hardy's inequality (4.11) under a weaker hypothesis (3.1) instead of (μ, G)-admissibility.

Proof. Due to the regularity of the Dirichlet form $(\mathcal{E}, \mathcal{F})$, it suffices to show (4.11) for all $f \in \mathcal{F} \cap C_c$ (see the proof of Theorem 3.1).

Let us first verify that if a (μ, G)-admissible function h exists then $(\mathcal{E}, \mathcal{F})$ is transient. Indeed, it suffices to construct a positive function $g \in L^1$ such that $g \le h$ (then (4.7) is satisfied by $Gh \in L^\infty_{\text{loc}}$). Indeed, define a sequence $\{A_k\}_{k=0}^\infty$ of subsets of M as in Lemma 4.2, choose positive c_k so that

$$c_k \mu(A_k) \le 2^{-k},$$

and set

$$g(x) = \min\{c_k, h(x)\} \quad \text{if } x \in A_k.$$

Clearly, $0 < g \le h$ and

$$\int_{A_k} g\, d\mu \le c_k \mu(A_k) \le 2^{-k}$$

whence $g \in L^1$ follows.

By [28, Theorem 1.5.4], the condition (*iii*) of Definition 4.4 and the transience of $(\mathcal{E}, \mathcal{F})$ imply that

$$w := Gh \in \mathcal{F}_e. \tag{4.12}$$

The condition (*i*) of Definition 4.4, that is, $w \in L^\infty_{\mathrm{loc}}$, and (4.12) imply by Lemma 4.1 that

$$w \in \mathcal{F}_{\mathrm{loc}}.$$

By condition (*ii*) of Definition 4.4, for any ball $B \subset M$ there is $\varepsilon > 0$ such that $w \ge \varepsilon$ in B. By using [28, Theorem 1.4.2(v)], we conclude that $w^{-1} \in \mathcal{F}_{\mathrm{loc}}$ (indeed, we have $w^{-1} = F \circ w$, where $F(t) := \varepsilon^{-1} \wedge t^{-1}$ is a Lipschitz function). Hence, $w^{-1} \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$. It follows that, for any $f \in \mathcal{F} \cap C_c$,

$$w^{-1} f^2 \in \mathcal{F} \subset \mathcal{F}_e. \tag{4.13}$$

By the transience of $(\mathcal{E}, \mathcal{F})$ and (4.6), we obtain

$$\int_M \frac{h}{Gh} f^2\, d\mu = \int_M h\left(w^{-1} f^2\right) d\mu = \mathcal{E}(Gh, w^{-1} f^2) = \mathcal{E}(w, w^{-1} f^2).$$

Hence, the proof of (4.11) amounts to verifying that

$$\mathcal{E}(w, w^{-1} f^2) \le \mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F} \cap C_c. \tag{4.14}$$

According to [28, Lemma 4.5.4, Theorem 4.5.2] and [28, Theorem 7.2.1], a regular Dirichlet form $\mathcal{E}$ admits a Beurling-Deny and LeJan decomposition: for all $u, v \in \mathcal{F}_e$,

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \int_{M \times M} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\, dJ(x, y) + \int_M \tilde{u}(x)\tilde{v}(x)\, dk(x), \tag{4.15}$$

where $\mathcal{E}^{(c)}$ is a strongly local symmetric form with domain $\mathcal{F}_e$, $\tilde{u}$ and $\tilde{v}$ denote quasi continuous versions of u and v, J is a symmetric positive Radon measure on $(M \times M) \setminus \mathrm{diag}$ (the jumping measure) and k is a positive Radon measure on M (the killing measure).

Let now w be a quasi continuous version of Gh. Then $w^{-1} f^2$ and $w^{-1} f$ are also quasi continuous. By (4.12), (4.13) and (4.15), we have

$$\begin{aligned} \mathcal{E}(w, w^{-1} f^2) = \mathcal{E}^{(c)}(w, w^{-1} f^2) & \\ + \int_{(M \times M)\setminus \mathrm{diag}} (w(x) - w(y))\Big(w(x)^{-1} f(x)^2 - w(y)^{-1} f(y)^2\Big)\, dJ(x, y) & \\ + \int_M w(x) w(x)^{-1} f(x)^2\, dk(x). & \end{aligned} \tag{4.16}$$

By $f \in \mathcal{F} \cap C_c$ and (4.15), we have

$$\mathcal{E}(f, f) = \mathcal{E}^{(c)}(f, f) + \int_{(M \times M)\setminus \mathrm{diag}} (f(x) - f(y))^2\, dJ(x, y) + \int_M f(x)^2 dk(x)\,. \tag{4.17}$$

In order to prove (4.14), we compare the corresponding terms in the right hand sides of (4.16) and (4.17). Clearly, the third terms in the the right hand sides of (4.16) and (4.17) are equal to each other. Since both w and w^{-1} are in $\mathcal{F}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}$, the argument in Lemma 3.3 shows that

$$\mathcal{E}^{(c)}(w, w^{-1} f^2) \le \mathcal{E}^{(c)}(f, f).$$

Finally, in order to compare the middle terms, observe that, for all $x, y \in M$,

$$\begin{aligned}
&(w(x) - w(y))\Big(w(x)^{-1} f(x)^2 - w(y)^{-1} f(y)^2\Big) \\
&\quad = f(x)^2 + f(y)^2 - w(x)w(y)^{-1} f(y)^2 - w(y)w(x)^{-1} f(x)^2 \\
&\quad = \Big(f(x) - f(y)\Big)^2 + 2 f(x) f(y) - w(x)w(y)\Big(w(y)^{-1} f(y)\Big)^2 - w(y)w(x)\Big(w(x)^{-1} f(x)\Big)^2 \\
&\quad = \Big(f(x) - f(y)\Big)^2 + w(x)w(y)\Big[2w(x)^{-1} f(x) w(y)^{-1} f(y) - \Big(w(y)^{-1} f(y)\Big)^2 - \Big(w(x)^{-1} f(x)\Big)^2\Big] \\
&\quad = \Big(f(x) - f(y)\Big)^2 - w(x)w(y)\Big(w(x)^{-1} f(x) - w(y)^{-1} f(y)\Big)^2 \\
&\quad \le \Big(f(x) - f(y)\Big)^2.
\end{aligned}$$

This proves (4.14) and, hence, (4.11). □

Remark 4.7. As we see from the proof, the positivity of the function h was used only in the first part in order to prove that $(\mathcal{E}, \mathcal{F})$ is transient. If it is known a priori that $(\mathcal{E}, \mathcal{F})$ is transient then we can allow h to be non-positive provided all the conditions (i)-(iii) of Definition 4.4 are satisfied.

We conclude this section with the following corollary.

Corollary 4.8. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on (M, d, μ), and $\mathcal{L}$ be its generator. If a positive function $\phi \in \mathrm{dom}\,(\mathcal{L})$ satisfies $\phi, \phi^{-1} \in L^{\infty}_{\mathrm{loc}}$ and $\int_M \phi\, \mathcal{L}\phi\, d\mu < \infty$, then*

$$\int_M \frac{\mathcal{L}\phi}{\phi} f^2\, d\mu \le \mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F}. \tag{4.18}$$

Proof. Indeed, applying Theorem 4.5 with $h = \mathcal{L}\phi$ and observing that $\phi = Gh$, we obtain (4.18) from (4.11). □

5 Some "classical" versions of Hardy's inequality

In this section, we mainly apply Theorem 4.5 to obtain various versions of Hardy's inequality on metric measure spaces, which are generalizations of classical/discrete/fractional Hardy's inequality.

5.1 Discrete Hardy's inequality

We show here how Theorem 4.5 yields a discrete Hardy's inequality in $\mathbb{Z}^n$, where $n \in \mathbb{N}$. For any $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, we set

$$\|k\| = |k_1| + \cdots + |k_n|$$

and define the graph structure in $\mathbb{Z}^n$ as follows: for $k, m \in \mathbb{Z}^n$ we say that k and m are neighbors and write $k \sim m$ if $\|k - m\| = 1$.

Define for all $s \ge 1$ the function

$$\omega(s) = \sum_{i=1}^{\infty} \binom{4i}{2i} \frac{1}{2^{4i-1}(4i-1)} \frac{1}{s^{2i}} = \frac{1}{4s^2} + \frac{5}{64s^4} + \frac{21}{512s^6} + \cdots.$$

Denote

$$\Gamma = \{k = (k_1, \ldots, k_n) \in \mathbb{Z}^n : \; k_i = 0 \text{ for some } i = 1, ..., n\}.$$

Theorem 5.1. *For any function* $f : \mathbb{Z}^n \to \mathbb{R}$ *such that* $f \in l^2(\mathbb{Z}^n)$ *and* $f|_\Gamma = 0$, *the following discrete Hardy's inequality holds:*

$$2n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \omega(\|k\|) f(k)^2 \le \sum_{\{k,m \in \mathbb{Z}^n : m \sim k\}} |f(m) - f(k)|^2. \tag{5.1}$$

Since $\omega(s) \ge \frac{1}{4s^2}$, the inequality (5.1) implies

$$\frac{n}{2} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{f(k)^2}{\|k\|^2} \le \sum_{\{k,m \in \mathbb{Z}^n : m \sim k\}} |f(m) - f(k)|^2.$$

If $n = 1$ and a function $f : \mathbb{Z} \to \mathbb{R}$ vanishes for $k \le 0$, we derive from (5.1) that

$$\sum_{k=1}^{\infty} \omega(k) f(k)^2 \le \sum_{k=1}^{\infty} (f(k) - f(k-1))^2 . \tag{5.2}$$

This inequality was proved in [52, 53] and shown there to be optimal. Of course, (5.2) implies the classical discrete Hardy's inequality

$$\frac{1}{4} \sum_{k=1}^{\infty} \frac{f(k)^2}{k^2} \le \sum_{k=1}^{\infty} (f(k) - f(k-1))^2 ,$$

where the constant $1/4$ is the best possible; see [49, p. 239].

Let us compare (5.1) with the result of [53, Theorems 0.2 and 7.2] that says the following: if $n \ge 3$ then, for any finitely supported function φ on $\mathbb{Z}^n$,

$$2 \sum_{k \in \mathbb{Z}^n \setminus \{0\}} w(\|k\|) \varphi(k)^2 \le \sum_{\{k,m \in \mathbb{Z}^n : m \sim k\}} |\varphi(m) - \varphi(k)|^2, \tag{5.3}$$

where w is an optimal Hardy weight that has the following asymptotic behaviour:

$$w(s) = \frac{(n-2)^2}{4s^2} + O\left(\frac{1}{s^3}\right) \text{ as } s \to \infty.$$

The corresponding weight in (5.1) is

$$n\omega(s) = \frac{n}{4s^2} + O\left(\frac{1}{s^3}\right) \text{ as } s \to \infty.$$

For $n \ge 5$ the weight w is obviously better, for $n = 4$ the weights are equivalent: $w(s) \sim n\omega(s)$ as $s \to \infty$, while for $n = 3$ the weight $n\omega(s)$ is better than $w(s)$ by a factor 3. This is not surprising because the class of functions f in (5.1) has a restriction $f|_\Gamma = 0$ while functions φ in (5.3) must only be finitely supported. For the same reason, (5.1) holds also for $n = 1, 2$ while for (5.3) $n \ge 3$ is required.

Proof of Theorem 5.1. Define the distance on $\mathbb{Z}^n$ by $d(k,m) = \|k-m\|$ and let μ be the degree measure, that is, $\mu(k) = 2n$ for all $k \in \mathbb{Z}^n$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $\mathbb{Z}^n$ is given by

$$\mathcal{E}(f,f) = \frac{1}{2} \sum_{\{k.m \in \mathbb{Z}^n : m \sim k\}} |f(m) - f(k)|^2,$$

where $\mathcal{F} = l^2(\mathbb{Z}^n)$. The discrete Laplacian Δ is defined on all functions $f : \mathbb{Z}^n \to \mathbb{R}$ by

$$\Delta f(k) = \frac{1}{2n} \sum_{m \sim k} (f(m) - f(k)), \quad k \in \mathbb{Z}^n.$$

It is known that the generator $\mathcal{L}$ of $(\mathcal{E}, \mathcal{F})$ coincides with $-\Delta|_{l^2}$; see [51].

Consider the set $\Omega = \mathbb{Z}^n \setminus \Gamma$ and the function space

$$\mathcal{F}(\Omega) = \{f \in \mathcal{F} : f|_\Gamma = 0\}$$

so that $(\mathcal{E}, \mathcal{F}(\Omega))$ is the part of $(\mathcal{E}, \mathcal{F})$ on Ω. For any $N \in \mathbb{N}$, consider the following function on $\mathbb{Z}^n$:

$$\phi_N(k) = \begin{cases} \|k\|^{\frac{1}{2}} = (|k_1| + \cdots + |k_n|)^{\frac{1}{2}} & \text{if } 0 \le \|k\| \le N \\ N^{\frac{1}{2}} & \text{if } \|k\| > N. \end{cases}$$

Clearly, if $\|k\| > N$ then

$$\Delta\phi_N(k) = 0.$$

For any $k \in \Omega$ with $0 < \|k\| \le N - 1$, there exist n vertices $m \sim k$ satisfying $\phi_N(m) = (\|k\| + 1)^{\frac{1}{2}}$, and another n vertices $m \sim k$ satisfying $\phi_N(m) = (\|k\| - 1)^{\frac{1}{2}}$, which implies that

$$\begin{aligned} -\frac{\Delta\phi_N(k)}{\phi_N(k)} &= \frac{1}{2n} \sum_{m \sim k} \frac{\phi_N(k) - \phi_N(m)}{\phi_N(k)} \\ &= \frac{2\|k\|^{\frac{1}{2}} - (\|k\| + 1)^{\frac{1}{2}} - (\|k\| - 1)^{\frac{1}{2}}}{2\|k\|^{\frac{1}{2}}} \\ &= \frac{1}{2}\left(2 - \left(1 + \frac{1}{\|k\|}\right)^{\frac{1}{2}} - \left(1 - \frac{1}{\|k\|}\right)^{\frac{1}{2}}\right). \end{aligned}$$

Using the Taylor expansions of the functions $t \mapsto (1 + t)^{\frac{1}{2}}$ and $t \mapsto (1 - t)^{\frac{1}{2}}$ that converge in $[-1, 1]$, we obtain

$$\begin{aligned} 2 - (1+t)^{\frac{1}{2}} - (1-t)^{\frac{1}{2}} &= 2 - \sum_{j=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - j + 1)}{j!} t^j \\ &\quad - \sum_{j=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - j + 1)}{j!} (-t)^j \\ &= -2 \sum_{i=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - 2i + 1)}{(2i)!} t^{2i} \\ &= 2 \sum_{i=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (4i - 3)}{2^{2i}(2i)!} t^{2i} \\ &= \sum_{i=1}^{\infty} \binom{4i}{2i} \frac{t^{2i}}{2^{4i-1}(4i - 1)} = \omega\left(\frac{1}{t}\right) \end{aligned}$$

It follows that

$$-\frac{\Delta\phi_N(k)}{\phi_N(k)} = \frac{1}{2}\omega(\|k\|) \quad \text{for all } k \in \Omega \text{ with } 0 < \|k\| \le N - 1.$$

If $k \in \Omega$ and $\|k\| = N$, then there exist n vertices $m \sim k$ satisfying $\phi_N(m) = (\|k\| - 1)^{\frac{1}{2}} = (N - 1)^{\frac{1}{2}}$, and another n vertices $m \sim k$ satisfying $\phi_N(m) = N^{\frac{1}{2}}$, which implies that

$$-\frac{\Delta\phi_N(k)}{\phi_N(k)} = \frac{1}{2}\left(\frac{N^{\frac{1}{2}} - (N - 1)^{\frac{1}{2}}}{N^{\frac{1}{2}}}\right).$$

Hence, we obtain that, for all $k \in \Omega$,

$$-\frac{\Delta\phi_N(k)}{\phi_N(k)} = \eta_N(k) := \frac{1}{2}\begin{cases}\omega(\|k\|) & \text{if } 0 < \|k\| \le N-1\\ \frac{N^{\frac{1}{2}}-(N-1)^{\frac{1}{2}}}{N^{\frac{1}{2}}} & \text{if } \|k\| = N\\ 0 & \text{if } \|k\| \ge N.\end{cases} \tag{5.4}$$

Set $h_N = \phi_N \eta_N$ so that

$$-\Delta\phi_N = h_N \text{ in } \Omega. \tag{5.5}$$

Note that $\phi_N \ge 0$ and $h_N \ge 0$ in Ω. In particular, the function ϕ_N is non-negative and superharmonic in Ω (let us mention that outside Ω it may happen that $-\Delta\phi_N < 0$, for example, $-\Delta\phi_N(0) < 0$). Since ϕ_N is non-constant, it follows that that $(\mathcal{E}, \mathcal{F}(\Omega))$ is transient. In particular, the Green function G^Ω exists. It follows from (5.5) by the comparison principle that

$$\phi_N \ge G^\Omega h_N \text{ in } \Omega. \tag{5.6}$$

It is easy to see that the function $h = h_N$ satisfies in Ω all the conditions (i)-(iii) of Definition 4.4. Indeed, (i) holds by (5.6), (ii) holds because $G^\Omega h_N > 0$ by the strong minimum principle for superharmonic functions on graphs, and (iii) holds because h_N has finite support.

By Remark 4.7, we can apply Theorem 4.5 with $h = h_N$ and conclude that, for all $f \in \mathcal{F}(\Omega)$,

$$\int_\Omega \frac{h_N}{G^\Omega h_N} f^2\, d\mu \le \mathcal{E}(f, f). \tag{5.7}$$

The left-hand side here can be estimated by (5.6) and (5.4) as follows:

$$\int_\Omega \frac{h_N}{G^\Omega h_N} f^2\, d\mu \ge \int_\Omega \frac{h_N}{\phi_N} f^2\, d\mu = \int_\Omega \eta_N f^2\, d\mu \ge \sum_{0<\|k\|<N} \frac{1}{2}\omega(\|k\|)\, f(k)^2 2n.$$

Combining with (5.7) and letting $N \to \infty$, we obtain (5.1). □

5.2 Hardy's inequality and distance function

In this subsection we obtain an explicit form of Hardy's inequality under the hypotheses **(VD)**, **(RVD)** and $(\mathbf{G})_\beta$. For that, we construct explicitly (μ, G)-admissible functions that can be used in Theorem 4.5. The main result is stated in Theorem 5.6 below.

Let us begin with the following Selberg-type integral formula on (M, d, μ).

Lemma 5.2. *Assume that (M, d, μ) satisfies* **(VD)** *and* **(RVD)** *with lower volume dimension α_-. If β and ε are positive reals such that $\beta + \varepsilon < \alpha_-$, then the following estimate*

$$\int_M \frac{d(x,z)^\beta}{V(x,z)} \frac{d(z,y)^\varepsilon}{V(z,y)}\, d\mu(z) \simeq \frac{d(x,y)^{\beta+\varepsilon}}{V(x,y)} \tag{5.8}$$

holds uniformly for all distinct $x, y \in M$.

Proof. By condition **(RVD)**, there exists a large constant $K > 2$ such that for all $x \in M$ and $R > 0$,

$$\frac{V(x, KR)}{V(x, R)} \ge 2. \tag{5.9}$$

Set $r = d(x, y)$. In order to prove the lower bound in (5.8), observe first that

$$d(x,z) < \frac{r}{2} \Rightarrow d(y,z) \simeq r, \tag{5.10}$$

whence

$$\int_M \frac{d(x,z)^\beta}{V(x,z)}\frac{d(z,y)^\varepsilon}{V(z,y)}\,d\mu(z) \geq \int_{B(x,r/2)\setminus B(x,r/2K)} \frac{d(x,z)^\beta}{V(x,z)}\frac{d(y,z)^\varepsilon}{V(y,z)}\,d\mu(z)$$
$$\simeq \frac{r^{\beta+\varepsilon}}{V(x,r)^2}\left(V(x,r/2) - V(x,r/2K)\right).$$

Using further (5.9), we obtain

$$\int_M \frac{d(x,z)^\beta}{V(x,z)}\frac{d(z,y)^\varepsilon}{V(z,y)}\,d\mu(z) \gtrsim \frac{r^{\beta+\varepsilon}}{V(x,r)^2}\frac{1}{2}V(x,r/2) \simeq \frac{r^{\beta+\varepsilon}}{V(x,r)}.$$

Before we prove the upper bound in (5.8), observe that, by (4.10), for any $\sigma \in (0,\infty)$ and $R \in (0,\infty)$,

$$\int_{B(x,R)} \frac{d(x,z)^\sigma}{V(x,z)}\,d\mu(z) \lesssim R^\sigma. \tag{5.11}$$

Let us prove that, if $0 < \theta < \alpha_-$, then

$$\int_{B(x,R)^c} \frac{d(x,z)^\theta}{V(x,z)^2}\,d\mu(z) \lesssim \frac{R^\theta}{V(x,R)} \tag{5.12}$$

uniformly in $x \in M$ and $R \in (0,\infty)$. Indeed, applying (**RVD**) and $\theta < \alpha_-$, we obtain

$$\begin{aligned}\int_{B(x,R)^c} \frac{d(x,z)^\theta}{V(x,z)^2}\,d\mu(z) &\leq \sum_{j=0}^\infty \int_{B\left(x,2^{j+1}R\right)\setminus B\left(x,2^jR\right)} \frac{d(x,z)^\theta}{V(x,z)^2}\,d\mu(z)\\ &\lesssim \sum_{j=0}^\infty \frac{(2^{j+1}R)^\theta}{V(x,2^jR)}\\ &\lesssim \frac{R^\theta}{V(x,R)}\sum_{j=0}^\infty 2^{j\theta}\frac{V(x,R)}{V(x,2^jR)}\\ &\lesssim \frac{R^\theta}{V(x,R)}\sum_{j=0}^\infty 2^{j(\theta-\alpha_-)} \simeq \frac{R^\theta}{V(x,R)},\end{aligned}$$

which proves (5.12).

Now, we use (5.11) and (5.12) to verify the upper bound in (5.8). Using (5.10) and (5.11), we obtain

$$\int_{B(x,r/2)} \frac{d(x,z)^\beta}{V(x,z)}\frac{d(z,y)^\varepsilon}{V(z,y)}\,d\mu(z) \simeq \frac{r^\varepsilon}{V(x,r)}\int_{B(x,r/2)} \frac{d(x,z)^\beta}{V(x,z)}\,d\mu(z) \lesssim \frac{r^{\beta+\varepsilon}}{V(x,r)}. \tag{5.13}$$

Similarly, if $r/2 \leq d(z,x) < 2r$, then

$$d(z,y) \leq d(z,x) + d(x,y) < 3r \quad \text{and} \quad V(x,z) \simeq V(x,r),$$

which, together with (5.11) implies

$$\int_{B(x.2r)\setminus B(x,r/2)} \frac{d(x,z)^\beta}{V(x,z)}\frac{d(z,y)^\varepsilon}{V(z,y)}\,d\mu(z) \simeq \frac{r^\beta}{V(x,r)}\int_{B(y,3r)} \frac{d(z,y)^\varepsilon}{V(z,y)}\,d\mu(z) \lesssim \frac{r^{\beta+\varepsilon}}{V(x,r)}. \tag{5.14}$$

For any $z \in M$ satisfying $d(z,x) \geq 2r$, we have by (**VD**) that

$$d(z,y) \simeq d(x,z) \;\text{ and }\; V(z,y) \simeq V(x,z),$$

which yields by (5.12) and $\beta+\varepsilon<\alpha_-$ that

$$\int_{B(x,2r)^c} \frac{d(x,z)^\beta}{V(x,z)}\frac{d(z,y)^\varepsilon}{V(z,y)}\,d\mu(z) \simeq \int_{B(x,2r)^c} \frac{d(x,z)^{\beta+\varepsilon}}{V(x,z)^2}\,d\mu(z) \lesssim \frac{r^{\beta+\varepsilon}}{V(x,r)}. \tag{5.15}$$

Adding up (5.13), (5.14) and (5.15), we conclude that

$$\int_M \frac{d(x,z)^\beta}{V(x,z)}\frac{d(z,y)^\varepsilon}{V(z,y)}\,d\mu(z) \lesssim \frac{r^{\beta+\varepsilon}}{V(x,r)},$$

which finishes the proof of (5.8). □

Remark 5.3. The Selberg integral formula [66, p. 118, (6)] in $\mathbb{R}^n$ says that, if a_1,a_2 are positive reals satisfying $a_1+a_2>n$, then for all distinct $x,y\in\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |x-z|^{-a_1}|z-y|^{-a_2}\,dz = C_{n,a_1,a_2}|x-y|^{n-a_1-a_2}, \tag{5.16}$$

where

$$C_{n,a_1,a_2} = \pi^{\frac{n}{2}}\frac{\Gamma(\frac{n-a_1}{2})\Gamma(\frac{n-a_2}{2})\Gamma(\frac{a_1+a_2-n}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{2n-a_1-a_2}{2})}.$$

The inequality (5.8) can be regarded as a generalization of the identity (5.16).

We use Lemma 5.2 in order to construct a function h that is admissible in the sense of Definition 4.4.

Lemma 5.4. *Assume that (M,d,μ) satisfies* **(VD)** *and* **(RVD)** *with lower volume dimension α_-. Let β and ε be positive reals such that $\beta+\varepsilon<\alpha_-$ and let the Green function $G(x,y)$ satisfy $(\mathbf{G})_\beta$. Fix an arbitrary point $x_o\in M$, a real $\rho>0$ and define*

$$h(x) = \begin{cases} \frac{\rho^\varepsilon}{V(x_o,\rho)} & \text{if } d(x_o,x)<\rho \\ \frac{d(x_o,x)^\varepsilon}{V(x_o,x)} & \text{if } d(x_o,x)\ge\rho. \end{cases} \tag{5.17}$$

Then, the Green potential of h satisfies

$$\inf_{B(x_o,R)} Gh > 0 \qquad \text{for all } R\in(0,\infty) \tag{5.18}$$

and

$$Gh(x) \le C\begin{cases} \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)} & \text{if } d(x_o,x)<2\rho \\ \frac{d(x_o,x)^{\beta+\varepsilon}}{V(x_o,x)} & \text{if } d(x_o,x)\ge 2\rho \end{cases} \tag{5.19}$$

where C is a positive constant independent of x,x_o and ρ.

Proof. The inequality (5.18) follows from $\inf_{B(x_o,R)} h>0$ and

$$\inf_{x,y\in B(x_o,R)} G(x,y) \simeq \inf_{x,y\in B(x_o,R)} \frac{d(x,y)^\beta}{V(x,y)} \gtrsim \frac{R^\beta}{V(x_o,R)} > 0. \tag{5.20}$$

Indeed, for any $x,y\in B(x_o,R)$, setting $r=d(x,y)$, we obtain that $r<2R$ and

$$\frac{R^\beta}{V(x_o,R)}\Big/\frac{r^\beta}{V(x,r)} = \frac{V(x,r)}{V(x_o,R)}\left(\frac{R}{r}\right)^\beta \simeq \frac{V(x,r)}{V(x,2R)}\left(\frac{R}{r}\right)^\beta \lesssim \left(\frac{r}{2R}\right)^{\alpha_-}\left(\frac{R}{r}\right)^\beta = \left(\frac{r}{R}\right)^{\alpha_--\beta} \lesssim 1,$$

which proves (5.20) and, hence, (5.18).

In order to prove (5.19), we apply $(\mathbf{G})_\beta$, (5.17) and split the integral in the definition of Gh into two parts as follows:

$$\begin{aligned} Gh(x) &\simeq \int_M \frac{d(x,y)^\beta}{V(x,y)} h(y)\, d\mu(y) \\ &\simeq \int_{B(x_o,\rho)} \frac{d(x,y)^\beta}{V(x,y)} \frac{\rho^\varepsilon}{V(x_o,\rho)}\, d\mu(y) + \int_{B(x_o,\rho)^c} \frac{d(x,y)^\beta}{V(x,y)} \frac{d(x_o,y)^\varepsilon}{V(x_o,y)}\, d\mu(y) \\ &=: I_1 + I_2. \end{aligned} \tag{5.21}$$

Set $r = d(x_o, x)$. We estimate I_1 and I_2 in (5.21) by considering two cases: $r \geq 2\rho$ and $r < 2\rho$.

Case $r \geq 2\rho$. If $y \in B(x_o, \rho)$ then

$$d(x,y) \leq d(x_o, x) + d(x_o, y) < r + \rho < 2r$$

and

$$d(x,y) \geq d(x_o, x) - d(x_o, y) > r - \rho > r/2$$

so that

$$d(x,y) \simeq r \text{ and } V(x,y) \simeq V(x_o, r).$$

It follows that

$$I_1 \simeq \int_{B(x_o,\rho)} \frac{r^\beta}{V(x_o,r)} \frac{\rho^\varepsilon}{V(x_o,\rho)} d\mu(y) \simeq \frac{r^\beta \rho^\varepsilon}{V(x_o,r)} \lesssim \frac{r^{\beta+\varepsilon}}{V(x_o,r)}.$$

By Lemma 5.2, we have

$$I_2 \lesssim \frac{r^{\beta+\varepsilon}}{V(x_o,r)}.$$

Combining the last two estimates and (5.21), we obtain

$$Gh(x) \lesssim \frac{r^{\beta+\varepsilon}}{V(x_o,r)} \text{ provided } r \geq 2\rho.$$

Case $r < 2\rho$. In this case, applying (5.11) gives

$$I_1 \simeq \frac{\rho^\varepsilon}{V(x_o,\rho)} \int_{B(x_o,\rho)} \frac{d(x,y)^\beta}{V(x,y)}\, d\mu(y) \lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}.$$

So, it remains to estimate I_2. By **(VD)** and (5.11) we obtain

$$\begin{aligned} \int_{B(x_o,4\rho)\setminus B(x_o,\rho)} \frac{d(x,y)^\beta}{V(x,y)} \frac{d(x_o,y)^\varepsilon}{V(x_o,y)}\, d\mu(y) &\simeq \frac{\rho^\varepsilon}{V(x_o,\rho)} \int_{B(x_o,4\rho)\setminus B(x_o,\rho)} \frac{d(x,y)^\beta}{V(x,y)}\, d\mu(y) \\ &\lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}. \end{aligned} \tag{5.22}$$

If $y \in B(x_o, 4\rho)^c$, then $r < \frac{1}{2} d(x_o, y)$ and

$$d(x,y) \leq d(x_o, y) + d(x_o, x) < 2d(x_o, y)$$

and

$$d(x,y) \geq d(x_o, y) - d(x_o, x) > \frac{1}{2} d(x_o, y),$$

whence

$$V(x,y) \simeq V(x_o, y).$$

Using also (5.12), we obtain

$$\int_{B(x_o,4\rho)^c} \frac{d(x,y)^\beta}{V(x,y)} \frac{d(x_o,y)^\varepsilon}{V(x_o,y)} d\mu(y) \simeq \int_{B(x_o,4\rho)^c} \frac{d(x_o,y)^{\beta+\varepsilon}}{V(x_o,y)^2} d\mu(y) \lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}. \tag{5.23}$$

Combining (5.22) and (5.23) yields

$$I_2 \lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}.$$

Substituting the estimates of I_1 and I_2 into (5.21), we obtain

$$Gh(x) \lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)} \ \text{ provided } \ r < 2\rho,$$

which finishes the proof of (5.19). □

Corollary 5.5. *Under the hypotheses of Lemma 5.4, assume that $\beta + 2\varepsilon < \alpha_-$. Then the function h in (5.17) is (μ, G)-admissible.*

Proof. Note that (5.18) and (5.19) imply that h satisfies the conditions (i) and (ii) of Definition 4.4. Let us verify the remaining condition (iii) in Definition 4.4. By (5.19), (5.11), (5.12) and $\beta + 2\varepsilon < \alpha_-$, we obtain

$$\begin{aligned}
\int_M h\, Gh\, d\mu &= \left(\int_{B(x_o,\rho)} + \int_{B(x_o,2\rho)\setminus B(x_o,\rho)} + \int_{B(x_o,2\rho)^c}\right) h\, Gh\, d\mu \\
&\lesssim \int_{B(x_o,\rho)} \frac{\rho^\varepsilon}{V(x_o,\rho)} \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)} d\mu(x) \\
&\quad + \int_{B(x_o,2\rho)\setminus B(x_o,\rho)} \frac{d(x_o,x)^\varepsilon}{V(x_o,x)} \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)} d\mu(x) \\
&\quad + \int_{B(x_o,2\rho)^c} \frac{d(x_o,x)^\varepsilon}{V(x_o,x)} \frac{d(x_o,x)^{\beta+\varepsilon}}{V(x_o,x)} d\mu(x) \\
&\lesssim \frac{\rho^{\beta+2\varepsilon}}{V(x_o,\rho)} < \infty,
\end{aligned}$$

which finishes the proof. □

Applying Theorem 4.5 with the admissible function h as in (5.17), we derive Hardy's inequality (1.6).

Theorem 5.6. *Assume that (M, d, μ) satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension α_-. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on M that satisfies $(\mathbf{G})_\beta$ with $0 < \beta < \alpha_-$. Then there exists a positive constant C depending only on the constants in the hypotheses, such that, for all $x_o \in M$ and $f \in \mathcal{F}$,*

$$\int_M \frac{f(x)^2}{d(x_o,x)^\beta} d\mu(x) \le C\mathcal{E}(f,f). \tag{5.24}$$

Proof. Choose a number ε such that $0 < 2\varepsilon < \alpha_- - \beta$. For this ε and $\rho \in (0,\infty)$, we define the function h as in (5.17) and adopt all other notation from Lemma 5.4. By Corollary 5.5, h is (μ, G)-admissible. By Theorem 4.5 we conclude that, for all $f \in \mathcal{F}$,

$$\int_M f^2 \frac{h}{Gh} d\mu \le \mathcal{E}(f,f). \tag{5.25}$$

Applying (5.17) and (5.19), we obtain

$$\int_M f^2 \frac{h}{Gh}\, d\mu \geq \int_{B(x_o,2\rho)^c} f^2 \frac{h}{Gh}\, d\mu \gtrsim \int_{B(x_o,2\rho)^c} \frac{f(x)^2}{d(x_o,x)^\beta}\, d\mu(x)$$

with implicit constant independent of x_o and ρ. Substituting the last estimate into (5.25) and letting $\rho \to 0$, we obtain

$$\int_M \frac{f(x)^2}{d(x_o,x)^\beta}\, d\mu(x) = \lim_{\rho\to 0} \int_{B(x_o,2\rho)^c} \frac{f(x)^2}{d(x_o,x)^\beta}\, d\mu(x) \lesssim \mathcal{E}(f,f),$$

which concludes the proof. □

As an example of application, we apply Theorem 5.6 to deduce the following estimate of $\lambda_{\min}(\Omega)$.

Corollary 5.7. *Under the assumptions of Theorem 5.6, for any non-empty open bounded $\Omega \subset M$, we have*

$$\lambda_{\min}(\Omega) \gtrsim \left(\operatorname{diam}(\Omega)\right)^{-\beta}. \tag{5.26}$$

Proof. Set $D = \operatorname{diam}\Omega$, fix a point $x_o \in \Omega$ and let $u \in \mathcal{F} \cap C_c(\Omega)$. We have $\operatorname{supp} u \subset \Omega$ and

$$\|u\|_{L^2}^2 = \int_{B(x_o,D)} |u(x)|^2\, d\mu(x) \leq \int_M \left(\frac{D}{d(x,x_o)}\right)^\beta |u(x)|^2\, d\mu(x).$$

By Theorem 5.6, we have

$$\int_M \frac{u(x)^2}{d(x_o,x)^\beta}\, d\mu(x) \lesssim \mathcal{E}(u,u).$$

Combining the last two inequalities yields

$$\|u\|_{L^2}^2 \lesssim D^\beta \mathcal{E}(u,u),$$

which implies (5.26) by (2.1). □

If $(\mathcal{E},\mathcal{F})$ is strongly local then the proof of Theorem 5.6 simplifies as in this case we can apply Theorem 3.1 instead of Theorem 4.5 and, hence, do not need Corollary 5.5.

Corollary 5.8. *Assume that (M,d,μ) satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension $\alpha_- > 2$. Let $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form in $L^2(M,\mu)$ such that the Green function $G(x,y)$ is jointly continuous in $M \times M \setminus \operatorname{diag}$ and*

$$G(x,y) \leq C\frac{d(x,y)^2}{V(x,y)} \quad \textit{for all distinct } x,y \in M. \tag{$\mathbf{G}_\leq$)$_2$}$$

Assume that $\lambda_{\min}(\Omega) > 0$ for all precompact open sets $\Omega \subset M$. Then there exists a positive constant C such that for all $x_o \in M$ and $f \in \mathcal{F}$,

$$\int_M \frac{f(x)^2}{d(x_o,x)^2}\, d\mu(x) \leq C\mathcal{E}(f,f). \tag{5.27}$$

Proof. Following the lines in Lemma 5.4 (take $\beta = 2$ therein), we obtain by $(\mathbf{G}_\leq)_2$ that the function h in (5.17) still satisfies (5.19), so that

$$\frac{h(x)}{Gh(x)} \gtrsim d(x_o,x)^2 \quad \text{if} \quad d(x_o,x) \geq 2\rho$$

with implicit constants independent of $x_o \in M$ and $\rho \in (0,\infty)$. For the function h in (5.17), it is obvious that $(\mathbf{G}_{\le})_\mathbf{2}$ implies

$$G(h \wedge a) \in L^\infty_{\mathrm{loc}}$$

for any positive constant a. Thus, by (3.2) in Theorem 3.1, we obtain

$$\int_M \frac{f(x)^2}{d(x_o,x)^2}\,d\mu(x) \lesssim \mathcal{E}(f,f) \quad \text{for all } \ f \in \mathcal{F}.$$

□

Remark 5.9. Let M be a complete non-compact Riemannian manifold, d the geodesic distance and μ the Riemannian volume. Let $(\mathcal{E},\mathcal{F})$ be the canonical Dirichlet form on M that is,

$$\mathcal{E}(f,f) = \int_M |\nabla f|^2\,d\mu, \tag{5.28}$$

where $f \in \mathcal{F} = W^{1,2}(M)$. Assume that M satisfies the *relative Faber-Krahn inequality*: for all balls $B(x,R)$ in M and for all open sets $\Omega \subset B(x,R)$,

$$\lambda_{\min}(\Omega) \ge \frac{c}{R^2}\left(\frac{V(x,R)}{\mu(\Omega)}\right)^{\varepsilon}, \tag{5.29}$$

where c, ε are positive constants. By [31, Proposition 5.2], (5.29) implies that the heat kernel on M satisfies the Gaussian upper bound $(\mathbf{UE})_\mathbf{2}$ which further implies $(\mathbf{G}_{\le})_\mathbf{2}$ by Lemma 2.4. Besides, (5.29) implies also (**VD**), and the latter implies (**RVD**) provided that $\operatorname{diam} M = \infty$. Assuming in addition that $\alpha_- > 2$, we can then apply Corollary 5.8 and obtain Hardy's inequality (5.27).

For example, consider a manifold with ends $M = \mathbb{R}^n \# \mathbb{R}^n$ with $n > 2$ where # stands for a connected sum. Note that on $\mathbb{R}^n \# \mathbb{R}^n$ the Faber-Krahn inequality (5.29) holds by [44]. It is easy to see that (**VD**) and (**RVD**) hold on $\mathbb{R}^n \# \mathbb{R}^n$ with $\alpha_- = \alpha_+ = n$. Hence, Corollary 5.8 yields the Hardy's inequality (5.27) on $\mathbb{R}^n \# \mathbb{R}^n$.

Note that $(\mathbf{G})_\mathbf{2}$ does not hold on $\mathbb{R}^n \# \mathbb{R}^n$ (cf. [43]), so that Theorem 5.6 is not applicable in the case $M = \mathbb{R}^n \# \mathbb{R}^n$. For further results on manifolds with ends see also [41].

5.3 Subordinated Green function and fractional Hardy's inequality

For any $\delta \in (0,1)$ the operator $\mathcal{L}^\delta$ generates the *subordinated* heat semigroup $\{e^{-t\mathcal{L}^\delta}\}_{t\ge 0}$ and the associated Dirichlet form $(\mathcal{E}^{(\delta)},\mathcal{F}^{(\delta)})$. It is well known that (cf. [73] and [32, Section 5.4])

$$e^{-t\mathcal{L}^\delta} = \int_0^\infty \eta_t^{(\delta)}(s) e^{-s\mathcal{L}}\,ds \quad \text{for all } t \ge 0,$$

where $\{\eta_t^{(\delta)}(s)\}_{t\ge 0}$ is a family of non-negative continuous functions on $[0,\infty)$ that is called a *subordinator*. Moreover, if $(\mathcal{E},\mathcal{F})$ is regular, then $(\mathcal{E}^{(\delta)},\mathcal{F}^{(\delta)})$ is also regular; see [63, Proposition 3.1]. If $\{e^{-t\mathcal{L}}\}_{t\ge 0}$ has the heat kernel $p_t(x,y)$ then $\{e^{-t\mathcal{L}^\delta}\}_{t\ge 0}$ has the heat kernel

$$p_t^{(\delta)}(x,y) = \int_0^\infty \eta_t^{(\delta)}(s) p_s(x,y)\,ds \quad \text{for all } x,y \in M.$$

Using the following identity from [63, (6)]

$$\int_0^\infty \eta_t^{(\delta)}(s)\,dt = \frac{s^{\delta-1}}{\Gamma(\delta)} \quad \text{for all } s > 0,$$

we obtain the following expression for the subordinated Green function $G^{(\delta)}$:

$$G^{(\delta)}(x,y) = \int_0^\infty p_t^{(\delta)}(x,y)\,dt = \int_0^\infty \int_0^\infty \eta_t^{(\delta)}(s) p_s(x,y)\,ds\,dt = c_\delta \int_0^\infty s^{\delta-1} p_s(x,y)\,ds. \tag{5.30}$$

Theorem 5.10. *Assume that* (M, d, μ) *satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension* α_-. *Let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirichlet form on* M. *Assume that the heat kernel of* $(\mathcal{E}, \mathcal{F})$ *exists and satisfies* (2.3) *and* (2.5) *for some* $\beta \in (0, \alpha_-)$. *Then, for any* $\delta \in (0,1)$, *the subordinated Green kernel* $G^{(\delta)}$ *satisfies*

$$G^{(\delta)}(x,y) \simeq \frac{d(x,y)^{\delta\beta}}{V(x,y)} \quad \text{for distinct } x, y \in M. \tag{$(\mathbf{G}^{(\delta)})_\beta$}$$

Consequently, there exists a constant $C > 0$ *such that, for all* $f \in \mathcal{F}^{(\delta)}$,

$$\int_M \frac{f(x)^2}{d(x_o, x)^{\beta\delta}}\, d\mu(x) \le C\mathcal{E}^{(\delta)}(f, f). \tag{5.31}$$

Proof. The inequality (5.31) follows directly from Theorem 5.6 and $(\mathbf{G}^{(\delta)})_\beta$. Let us verify that the subordinated Green kernel $G^{(\delta)}$ satisfies $(\mathbf{G}^{(\delta)})_\beta$. By (5.30), (2.5) and (**VD**), we obtain the lower bound of $G^{(\delta)}$:

$$G^{(\delta)}(x,y) \ge c_\delta \int_{d(x,y)^\beta}^{2d(x,y)^\beta} s^{\delta-1} p_s(x,y)\, ds \gtrsim \int_{d(x,y)^\beta}^{2d(x,y)^\beta} \frac{s^{\delta-1}}{V(x, s^{1/\beta})}\, ds \simeq \frac{d(x,y)^{\delta\beta}}{V(x,y)}.$$

Recall that, by Lemma 2.4, (2.3) and (2.5) imply $(\mathbf{G})_\beta$. Applying (5.30), (2.3) and $(\mathbf{G})_\beta$, we obtain the upper bound of $G^{(\delta)}$:

$$\begin{aligned} G^{(\delta)}(x,y) &= c_\delta \left(\int_0^{d(x,y)^\beta} + \int_{d(x,y)^\beta}^\infty \right) s^{\delta-1} p_s(x,y)\, ds \\ &\lesssim \int_0^{d(x,y)^\beta} \frac{s^{\delta-1}}{V(x,y)}\, ds + d(x,y)^{\beta(\delta-1)} \int_{d(x,y)^\beta}^\infty p_s(x,y)\, ds \\ &\lesssim \frac{d(x,y)^{\delta\beta}}{V(x,y)} + d(x,y)^{\beta(\delta-1)} G(x,y) \simeq \frac{d(x,y)^{\delta\beta}}{V(x,y)}, \end{aligned}$$

which finishes the proof. □

Example 5.11. In $\mathbb{R}^n$ $(n \ge 3)$ the following fractional version of Hardy's inequality is known: if $\delta \in (0,1)$ then

$$c_{n,\delta} \int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^{2\delta}}\, dx \le \int_{\mathbb{R}^n} \left| (-\Delta)^{\frac{\delta}{2}} f(x) \right|^2\, dx \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \tag{5.32}$$

where the constant $c_{n,\delta} := \left(\frac{2^\delta \Gamma(\frac{n+2\delta}{4})}{\Gamma(\frac{n-2\delta}{4})} \right)^2$ is the best possible (see [10, p. 1873, Corollary 1]). Observe that (1.1) can be viewed as the limiting case of (5.32) as $\delta \to 1$:

$$c_{n,1} \int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^2}\, dx \le \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \tag{5.33}$$

where $c_{n,1} = \left(\frac{2\Gamma(\frac{n+2}{4})}{\Gamma(\frac{n-2}{4})} \right)^2 = \left(\frac{n-2}{2} \right)^2$ is also best possible.

Consider in $\mathbb{R}^n$ $(n \ge 3)$ the Dirichlet form $(\mathcal{E}, \mathcal{F})$ where

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^n} |\nabla f|^2\, dx \tag{5.34}$$

and

$$f \in \mathcal{F} = W^{1,2} = \{ f \in L^2(\mathbb{R}^n) : \nabla f \in L^2(\mathbb{R}^n) \}. \tag{5.35}$$

The generator of $(\mathcal{E},\mathcal{F})$ is the Laplacian $-\Delta = -\sum_{j=1}^{n}\partial_{x_j}^2$, the heat kernel $\{p_t\}_{t>0}$ of the heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ is the Gauss-Weierstrass function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{4t}\right),$$

and the Green function is given by

$$G(x,y) = \int_0^\infty p_t(x,y)\,dt = \frac{\Gamma(\frac{n-2}{2})}{4\pi^{n/2}}|x-y|^{2-n}. \tag{5.36}$$

For the subordinated Dirichlet form $(\mathcal{E}^{(\delta)},\mathcal{F}^{(\delta)})$ we have

$$\mathcal{F}^{(\delta)} = \left\{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^2}{|x-y|^{n+2\delta}}\,dx\,dy < \infty\right\}$$

and

$$\mathcal{E}^{(\delta)}(f,f) = \left((-\Delta)^\delta f, f\right) = \int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{\delta}{2}}f(x)\right|^2 dx \quad \text{for } f \in \mathcal{F}^{(\delta)};$$

see [32, Theorem 5.2]. Hence, by Theorem 5.10 (resp., Theorem 5.6) with $\beta = 2$ we obtain (5.32) (resp., (5.33)) with *some* constant $c_{n,\delta} > 0$.

Let us show how Theorem 4.5 yields (5.32) and (5.33) with the sharp constant $c_{n,\delta}$. To unify the notation, denote by $(\mathcal{E}^{(1)},\mathcal{F}^{(1)})$ the Dirichlet form $(\mathcal{E},\mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are as in (5.34) and (5.35). Moreover, denote by $G^{(1)}$ the Green function G in (5.36). From [66, p. 117] it follows that

$$G^{(\delta)}(x,y) = \frac{\Gamma(\frac{n-2\delta}{2})}{4^\delta\pi^{n/2}\Gamma(\delta)}|x-y|^{2\delta-n}.$$

Applying Theorem 4.5 to $(\mathcal{E}^{(\delta)},\mathcal{F}^{(\delta)})$ and $G^{(\delta)}$, we have Hardy's inequality (4.11), where we choose the admissible function h to be

$$h_r(x) = \begin{cases} r^{\epsilon-n}, & |x| \leq r \\ |x|^{\epsilon-n}, & |x| > r, \end{cases}$$

where $r > 0$ and $0 < \epsilon < n-2\delta$. Now let in (4.11) $r \to 0$. By the Selberg integral formula in (5.16) (see also [66, p. 118, (6)]), we obtain

$$\lim_{r\to 0}\frac{h_r(x)}{G^{(\delta)}h_r(x)} = \frac{|x|^{\epsilon-n}}{\frac{\Gamma(\frac{n-2\delta}{2})}{4^\delta\pi^{n/2}\Gamma(\delta)}\int_{\mathbb{R}^n}|x-y|^{2\delta-n}|y|^{\epsilon-n}\,dy} = \frac{2^{2\delta}\Gamma(\frac{2\delta+\epsilon}{2})\Gamma(\frac{n-\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{n-2\delta-\epsilon}{2})}\frac{1}{|x|^{2\delta}}.$$

Taking here $\epsilon = \frac{n-2\delta}{2}$, we obtain

$$\lim_{r\to 0}\frac{h_r(x)}{G^{(\delta)}h_r(x)} = \left(\frac{2^\delta\Gamma(\frac{n+2\delta}{4})}{\Gamma(\frac{n-2\delta}{4})}\right)^2\frac{1}{|x|^{2\delta}} = \frac{c_{n,\delta}}{|x|^{2\delta}},$$

which implies (5.32) and (5.33).

Example 5.12. Let us show how Theorem 5.6 can be applied on fractal spaces. Most fractals can be regarded as a metric measure space (M,d,μ) that is α-regular for some $\alpha > 0$. On large families of fractals it was possible to construct a strongly local Dirichlet form $(\mathcal{E},\mathcal{F})$ that is self-similar with respect to the fractal structure and, moreover, the corresponding heat kernel satisfies the following sub-Gaussian bounds

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{5.37}$$

with some $\beta \geq 2$. For the Sierpinski gasket this was done by Barlow and Perkins [9], for p.c.f. fractals by Kigami [54] and for generalized Sierpinski carpets – by Barlow and Bass [7] (see also [5, 32, 55, 67] for the further development of this subject). Moreover, it follows from [6] that, for any pair of reals α, β satisfying

$$2 \leq \beta \leq \alpha + 1,$$

there exists an α-regular metric measure space (M, d, μ) and a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mu)$ such that the heat kernel of $(\mathcal{E}, \mathcal{F})$ satisfies (5.37).

If $\alpha > \beta$ in (5.37) then, integrating this estimate in t, we obtain by Lemma 2.4 the following estimate for the Green function:

$$G(x, y) \simeq d(x, y)^{\beta - \alpha}$$

that is equivalent to $(\mathbf{G})_\beta$. Hence, by Theorem 5.6 we obtain Hardy's inequality (5.24). In this setting the parameters α, β can take arbitrary values within the restriction

$$2 \leq \beta < \alpha.$$

Fix now some $\delta \in (0, 1)$ and consider the subordinated Dirichlet form $\left(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)}\right)$ on $L^2(M, \mu)$. Then the heat kernel $p_t^{(\delta)}$ of $\left(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)}\right)$ satisfies the following estimate

$$p_t^{(\delta)}(x, y) \simeq \frac{1}{t^{\alpha/\beta'}} \left(1 + \frac{d(x, y)}{t^{1/\beta'}}\right)^{-(\alpha + \beta')}$$

where $\beta' = \beta\delta$ (see [32] or [35]). If δ is small enough so that $\alpha > \beta'$ then, integrating this estimate in t, we obtain the following estimate for the Green function:

$$G^{(\delta)}(x, y) \simeq d(x, y)^{\beta' - \alpha}.$$

Hence, by Theorem 5.6, we obtain the following Hardy's inequality

$$\int_M \frac{f(x)^2}{d(x_o, x)^{\beta'}} \, d\mu(x) \leq C \mathcal{E}^{(\delta)}(f, f).$$

Note that α and β' can take here arbitrary values with the only restriction

$$0 < \beta' < \alpha.$$

The Sierpinski gasket and carpet satisfy (5.37) but with $\beta > \alpha$. A bounded Sierpinski carper is shown on Fig. 1. In this case we have $\alpha = \frac{\log 8}{\log 3}$ and $\beta \approx 2.09 > \alpha$. Nevertheless, we still have Hardy's inequality for $\mathcal{E}^{(\delta)}$ provided $\delta < \alpha/\beta$.

In order to get explicit examples with $\beta < \alpha$, consider a *generalized Sierpinski carpet* $SC(a, b, k)$ constructed in [7]. Here a, b, k are integers such that $k \geq 2$, $a > b \geq 1$ and $a = b \bmod 2$. Divide the unit cube $[0, 1]^k \subset \mathbb{R}^k$ into a^k equal cubes of side a^{-1} and take out the central block of b^k such cubes. Then repeat this procedures with each of the remaining cubes of side a^{-1}, etc. In the end one obtains a compact subset of $[0, 1]^k$ that is called $SC(a, b, k)$. For example, the Sierpinski carpet on Fig. 1 is $SC(3, 1, 2)$.

We need an unbounded version of $SC(a, b, k)$ that is obtained by gluing together countably many appropriately scaled compact versions. The resulting unbounded fractal is α-regular with

$$\alpha = \frac{\log N}{\log a},$$

where $N = a^k - b^k$, and admits a strongly local Dirichlet form with the heat kernel bound (5.37) for some $\beta > 2$. The exact value of the walk dimension β is unknown but it is known that

$$\beta < \frac{\log(Ns)}{\log a},$$

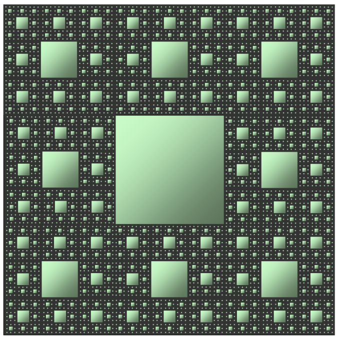

Figure 1: The bounded Sierpinski carpet.

where $s = \frac{a}{a^{k-1}-b^{k-1}}$. Clearly, if $s < 1$ then we obtain the desired condition $\beta < \alpha$. In particular, we have $s < 1$ provided $k \geq 3$ because in this case

$$a^{k-1} - b^{k-1} > (a-b)\, a^{k-2} = a\,(a-b)\, a^{k-3} \geq a.$$

Therefore, Theorem 5.6 applies on any generalized Sierpinski carpet $SC\,(a, b, k)$ with $k \geq 3$.

6 Green functions and heat kernels

The main goal of this section is to show the equivalence between the Green function estimate $(\mathbf{G})_\beta$ and the upper/lower bound of the heat kernel. This equivalence will be used in Section 7 in order to obtain a weighted Hardy's inequality.

6.1 Statement of Theorem 6.1

The following theorem is the main result of this section.

Theorem 6.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on the metric measure space (M, d, μ) that satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension α_-. Then, for any $0 < \beta < \alpha_-$, the following two statements are equivalent:*

(i) the Green function $G(x, y)$ exists, is jointly continuous off-diagonal, and satisfies $(\mathbf{G})_\beta$;

(ii) the heat kernel $p_t(x, y)$ exists, is Hölder continuous in $x, y \in M$, and satisfies the following upper bound estimate

$$p_t(x, y) \leq \frac{C}{V(x, t^{1/\beta})} \exp\left\{-c\left(\frac{d(x, y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right\} \qquad (\mathbf{UE})_\beta$$

as well as the near-diagonal lower bound estimate

$$p_t(x, y) \geq \frac{C^{-1}}{V(x, t^{1/\beta})} \quad \text{when } d(x, y) < \epsilon t^{1/\beta} \qquad (\mathbf{NLE})_\beta$$

for all $x, y \in M$ and all $t \in (0, \infty)$, where C and c, ϵ are positive constants.

Combining Theorems 6.1 and 5.10, we have the following fractional version of Hardy's inequality for strongly local Dirichlet forms.

Corollary 6.2. *Assume that* (M, d, μ) *satisfies* **(VD)** *and* **(RVD)** *with lower volume dimension* α_-. *Let* $(\mathcal{E}, \mathcal{F})$ *be a strongly local regular Dirichlet form on* M *and satisfies* $(\mathbf{G})_\beta$ *for some* $\beta \in (0, \alpha_-)$. *Then, given any* $\delta \in (0, 1)$, *the subordinated Green kernel* $G^{(\delta)}$ *satisfies* $(\mathbf{G}^{(\delta)})_\beta$. *Moreover, there exists a constant* $C > 0$ *such that, for all* $f \in \mathcal{F}^{(\delta)}$,

$$\int_M \frac{f(x)^2}{d(x_o, x)^{\beta\delta}}\, d\mu(x) \leq C\mathcal{E}^{(\delta)}(f, f).$$

6.2 Overview of the proof of Theorem 6.1

The detailed proof of Theorem 6.1 is presented in the subsections below. Here we give an overview of the proof. In Section 6.3 we prove the implication $(ii) \Rightarrow (i)$. The estimates and the continuity of the Green functions follow from similar properties of the heat kernel upon integration in time.

The proof of the implication $(i) \Rightarrow (ii)$ is much more involved. For that we need the following definitions.

Definition 6.3. Let $\Omega \subset M$ be an open subset. A function $u \in \mathcal{F}$ is said to be *harmonic* in Ω if

$$\mathcal{E}(u, \phi) = 0 \quad \text{for all } \phi \in \mathcal{F}(\Omega).$$

A function $u \in \mathcal{F}$ is said to be *superharmonic* (resp., *subharmonic*) in Ω if

$$\mathcal{E}(u, \phi) \geq 0 \quad (\text{ resp., } \mathcal{E}(u, \phi) \leq 0) \ \text{ for all } 0 \leq \phi \in \mathcal{F}(\Omega).$$

Definition 6.4. We say that the *elliptic Harnack inequality* **(H)** holds if there exist constants $C \in (1, \infty)$ and $\delta \in (0, 1)$ such that, for any ball $B \subset M$ and for any function $u \in \mathcal{F}$ that is harmonic and non-negative in B,

$$\operatorname*{esssup}_{x \in \delta B} u(x) \leq C \operatorname*{essinf}_{x \in \delta B} u(x). \tag{\textbf{H}}$$

Definition 6.5. We say that the mean exit time estimate $(\mathbf{E})_\beta$ holds if there exist constants $C \in (1, \infty)$ and $\delta \in (0, 1)$ such that, for any ball $B \subset M$ of radius $r > 0$, the restricted Green operator G^B exists and satisfies

$$C^{-1} r^\beta \leq \operatorname*{essinf}_{x \in \delta B} G^B 1(x) \leq \operatorname*{esssup}_{x \in B} G^B 1(x) \leq C r^\beta. \tag{$(\mathbf{E})_\beta$}$$

It is known that $(\mathbf{UE})_\beta + (\mathbf{NLE})_\beta \Leftrightarrow (\mathbf{E})_\beta + (\mathbf{H})$; see [46, Theorem 7.4]. We show in Sections 6.5 and 6.6 that $(\mathbf{G})_\beta \Rightarrow (\mathbf{E})_\beta$ and $(\mathbf{G})_\beta \Rightarrow (\mathbf{H})$, thus yielding $(i) \Rightarrow (ii)$.

6.3 Proof of $(\mathbf{UE})_\beta + (\mathbf{NLE})_\beta \Rightarrow (\mathbf{G})_\beta$

Proof of Theorem 6.1 $(ii) \Rightarrow (i)$. Since the heat kernel is Hölder continuous in $x, y \in M$, the Green function can be then defined pointwise by the identity

$$G(x, y) = \int_0^\infty p_t(x, y)\, dt. \tag{6.1}$$

The estimate $(\mathbf{G})_\beta$ of the Green function has been already proved in Lemma 2.4; see also Example 2.5.

Let us now prove the continuity of $G(x, y)$ off-diagonal. By (6.1) we have

$$|G(x, y) - G(x_o, y)| \leq \int_0^\infty |p_t(x, y) - p_t(x_o, y)|\, dt. \tag{6.2}$$

Next, we will use the following elementary estimate: if $0 \le a < 1$ then for all $x \in M$ and $R \in (0, \infty)$,

$$\int_0^\infty t^{-a} \left(\frac{1}{V(x, t^{1/\beta})} \wedge \frac{1}{V(x,R)} \right) dt \lesssim \frac{R^{(1-a)\beta}}{V(x,R)}. \tag{6.3}$$

Indeed, using (**RVD**) and $a\beta + \alpha_- > \beta$, we obtain

$$\begin{aligned}\int_{R^\beta}^\infty t^{-a} \frac{1}{V(x,t^{1/\beta})} dt &= \frac{1}{V(x,R)} \int_{R^\beta}^\infty t^{-a} \frac{V(x,R)}{V(x,t^{1/\beta})} dt \\ &\lesssim \frac{1}{V(x,R)} \int_{R^\beta}^\infty t^{-a} \left(\frac{R}{t^{1/\beta}} \right)^{\alpha_-} dt \\ &\simeq \frac{1}{V(x,R)} \int_0^1 \left(\frac{s}{R} \right)^{a\beta} s^{\alpha_-} \beta R^\beta s^{-(\beta+1)} ds \\ &\simeq \frac{\beta R^{(1-a)\beta}}{V(x,R)} \int_0^1 s^{a\beta+\alpha_- -\beta-1} ds \simeq \frac{R^{(1-a)\beta}}{V(x,R)}.\end{aligned}$$

By $a < 1$ we have also

$$\int_0^{R^\beta} t^{-a} \frac{1}{V(x,R)} dt \simeq \frac{R^{(1-a)\beta}}{V(x,R)},$$

whence (6.3) follows.

For any $x \in M$ and positive t, R, consider the cylinder

$$D((t,x), R) = B(x,R) \times (t - R^\beta, t].$$

It was proved in [8, Corollary 4.2] that $(\mathbf{UE})_\beta + (\mathbf{NLE})_\beta$ imply the following property: there exist $\theta, \delta \in (0,1)$ such that, for any continuous caloric function u in $D((t, x_o), R)$ and for all $x \in B(x_o, \delta R)$

$$|u(t,x) - u(t,x_o)| \lesssim \left(\frac{d(x,x_o)}{R} \right)^\theta \operatorname*{osc}_{(s,z) \in D((t,x_o),R)} u(s,z).$$

Fix $y \in M$ so that $u(t,x) = p_t(x,y)$ is a non-negative continuous caloric function on $M \times (0, \infty)$. Fix also distinct points $x, x_o \in M$ and set $r = d(x, x_o)$. For any $t > T := 2(r/\delta)^\beta$, if we take $R = (t/2)^{1/\beta}$ (this implies that $d(x, x_o) < \delta R$), then the function u is caloric in the cylinder $D((t, x_o), R)$, which implies that

$$|p_t(x,y) - p_t(x_o,y)| \lesssim \left(\frac{r}{R} \right)^\theta \sup_{t/2 \le s \le t} \sup_{z \in B(x_o,R)} p_s(y,z). \tag{6.4}$$

For $s \in [t/2, t]$ we have by $(\mathbf{UE})_\beta$ that

$$p_s(y,z) \lesssim \frac{1}{V(y,t^{1/\beta})} \exp\left(-c \left(\frac{d(y,z)}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right).$$

Since

$$d(y,z) \ge d(y,x_o) - d(x_o,z) \ge d(y,x_o) - R = d(y,x_o) - (t/2)^{1/\beta},$$

it follows that

$$p_s(y,z) \lesssim \frac{1}{V(y,t^{1/\beta})} \exp\left(-c \left(\frac{d(y,x_o)}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right) \lesssim \frac{1}{V(y,t^{1/\beta})} \wedge \frac{1}{V(y,x_o)}.$$

See Example 2.5 for the last formula. Substituting into (6.4), we conclude that, for all $t > T := 2(r/\delta)^\beta$,

$$|p_t(x,y) - p_t(x_o,y)| \lesssim \left(\frac{r}{t^{1/\beta}} \right)^\theta \left(\frac{1}{V(y,t^{1/\beta})} \wedge \frac{1}{V(y,x_o)} \right).$$

Applying (6.3) with $a = \theta/\beta$ (here we may as well assume that θ is small enough satisfying $\theta < \beta$), we obtain

$$\int_T^\infty |p_t(x,y) - p_t(x_o,y)|\, dt \lesssim \int_T^\infty \left(\frac{r}{t^{1/\beta}}\right)^\theta \left(\frac{1}{V(y,t^{1/\beta})} \wedge \frac{1}{V(y,x_o)}\right) dt \lesssim r^\theta \frac{d(x_o,y)^{\beta-\theta}}{V(x_o,y)}.$$

Similarly, we obtain

$$\int_0^T p_t(x,y)\, dt \le \int_0^T \left(\frac{T}{t}\right)^{\theta/\beta} p_t(x,y)\, dt \lesssim \int_0^\infty \left(\frac{r^\beta}{t}\right)^{\theta/\beta} \left(\frac{1}{V(y,t^{1/\beta})} \wedge \frac{1}{V(y,x)}\right) dt \lesssim r^\theta \frac{d(x,y)^{\beta-\theta}}{V(x,y)}$$

and

$$\int_0^T p_t(x_o,y)\, dt \lesssim r^\theta \frac{d(x_o,y)^{\beta-\theta}}{V(x_o,y)}.$$

Substituting the above three estimates into (6.2), we obtain

$$|G(x,y) - G(x_o,y)| \lesssim r^\theta \frac{d(x_o,y)^{\beta-\theta}}{V(x_o,y)} + r^\theta \frac{d(x,y)^{\beta-\theta}}{V(x,y)},$$

which proves the locally uniform Hölder continuity of $G(\cdot,y)$ in $M \setminus \{y\}$ with the Hölder exponent θ. Since $G(x,y)$ is symmetric, this implies a joint continuity of $G(x,y)$ in $(x,y) \in (M \times M) \setminus \mathrm{diag}$. □

6.4 Existence of the restricted Green function

Lemma 6.6. *Let* (**VD**), (**RVD**) *and* $(\mathbf{G})_\beta$ *be satisfied with* $0 < \beta < \alpha_-$. *Then the following are true.*

(i) For any ball $B \subset M$, there exists a non-negative symmetric function $G^B(x,y)$ that is jointly measurable in $x, y \in B$ and satisfies

$$G^B f(x) = \int_B G^B(x,y) f(y)\, d\mu(y) \quad \text{for all } f \in L^2(B) \text{ and } \mu\text{-a.a. } x \in B. \tag{6.5}$$

(ii) There exist constants $\varepsilon \in (0,1)$ and $C > 0$ such that, for any ball B, the restricted Green function $G^B(x,y)$ satisfies

$$G^B(x,y) \le C \frac{d(x,y)^\beta}{V(x,y)} \quad \text{for } \mu\text{-a.a. } x,y \in B \tag{6.6}$$

and

$$G^B(x,y) \ge C^{-1} \frac{d(x,y)^\beta}{V(x,y)} \quad \text{for } \mu\text{-a.a. } x,y \in \varepsilon B. \tag{6.7}$$

Proof. By Corollary 5.7 we have, for any ball $B = B(x_o,R)$,

$$\lambda_{\min}(B) \gtrsim (\mathrm{diam}(B))^{-\beta} > 0.$$

By Remark 2.2, the operator $\mathcal{L}^B$ has a bounded inverse in $L^2(B)$, and the latter is exactly the restricted Green operator G^B. Besides, we have

$$0 \le G^B f \le G f \quad \text{for all } 0 \le f \in L^2(B).$$

Let us now prove the existence of the integral kernel of G^B. For that, we will prove that, for any $0 < \delta < 1$, the operator $G - G^B$ acting from $L^2(\delta B)$ to $L^2(B)$, has an integral kernel. By [38, Lemma 3.3], for the existence of the integral kernel, it suffices to prove that

$$\left\|G - G^B\right\|_{L^2(\delta B) \to L^\infty(B)} < \infty,$$

that is,

$$\left\|Gf - G^B f\right\|_{L^\infty(B)} \lesssim \|f\|_{L^2} \quad \text{for any } 0 \le f \in L^2(\delta B). \tag{6.8}$$

The function $Gf - G^B f$ is harmonic in B. Due to $\lambda_{\min}(B) > 0$, we can apply the maximum principle for harmonic functions (see [37, Lemma 4.1]) and obtain, for any $\lambda \in (\delta, 1)$

$$\begin{aligned} 0 \le \operatorname*{esssup}_{B}\left(Gf - G^B f\right) &\le \operatorname*{esssup}_{B\setminus\lambda B}\left(Gf - G^B f\right) \\ &\le \operatorname*{esssup}_{x\in B\setminus(\lambda B)} Gf(x) \lesssim \sup_{x\in B\setminus(\lambda B)} \int_{\delta B} \frac{d(x,y)^\beta}{V(x,y)} f(y)\, d\mu(y). \end{aligned}$$

Since for all x, y in the above expression

$$(\lambda - \delta) R < d(x,y) < 2R,$$

it follows that

$$\frac{d(x,y)^\beta}{V(x,y)} \le \frac{(2R)^\beta}{V(x,(\lambda-\delta)R)} \lesssim (\lambda-\delta)^{\alpha_+} \frac{R^\beta}{V(x,R)} \lesssim (\lambda-\delta)^{\alpha_+} \frac{R^\beta}{V(x_o,R)}.$$

Therefore, we have

$$\left\|Gf - G^B f\right\|_{L^\infty(B)} \lesssim (\lambda-\delta)^{\alpha_+} \frac{R^\beta}{V(x_o,R)} \|f\|_{L^1}, \tag{6.9}$$

whence (6.8) follows. Hence, the operator $G - G^B$ has an integral kernel, say $K^\delta(x,y)$ that is a non-negative jointly measurable function in $B \times \delta B$.

Clearly, the family $\{K^\delta\}_{\delta\in(0,1)}$ of kernels is consistent in the sense that, for all $0 < \delta' < \delta'' < 1$,

$$K^{\delta'}(x,y) = K^{\delta''}(x,y) \quad \text{for } \mu\text{-a.a. } x \in B \text{ and } y \in \delta' B.$$

Choose a sequence $\delta_k \nearrow 1$ and define in $B \times B$ the kernel

$$K(x,y) = K^{\delta_k}(x,y) \quad \text{for } \mu\text{-a.a. } x \in B \text{ and } y \in \delta_k B.$$

Finally, we define the Green function G^B by

$$G^B(x,y) = G(x,y) - K(x,y).$$

Similarly to the proof of [37, (5.8)], one shows that G^B satisfies (6.5).

Because the operator G^B is positivity preserving, it follows from [38, Lemma 3.2] that

$$G^B(x,y) \ge 0 \quad \text{for } \mu\text{-a.a. } x, y \in B.$$

Moreover, by the symmetry of $\mathcal{E}$, we have, for all $f, g \in \mathcal{F}(B)$,

$$(f, G^B g) = \mathcal{E}(G^B f, G^B g) = \mathcal{E}(G^B g, G^B f) = (g, G^B f),$$

which implies that

$$G^B(x,y) = G^B(y,x) \quad \text{for } \mu\text{-a.a. } x, y \in B.$$

By construction $G^B(x,y) \le G(x,y)$ so that the upper bound (6.6) of $G^B(x,y)$ follows from $(\mathbf{G})_\beta$. In order to prove the lower bound (6.7) of $G^B(x,y)$, it suffices to verify that, for all $0 \le f \in L^2(\varepsilon B)$,

$$\operatorname*{essinf}_{\varepsilon B} G^B f(x) \gtrsim \int_{\varepsilon B} \frac{d(x,y)^\beta}{V(x,y)} f(y)\, d\mu(y),$$

where $\varepsilon > 0$ is yet to be determined. Fix the parameters δ and λ from the previous part of the proof, for example, set $\delta = \frac{1}{2}$ and $\lambda = \frac{3}{4}$. Assuming that $\varepsilon < \frac{1}{2}$, we obtain by (6.9)

$$\left\| Gf - G^B f \right\|_{L^\infty(B)} \le C \frac{R^\beta}{V(x_o, R)} \|f\|_{L^1}$$

so that, for μ-a.a. $x \in \varepsilon B$,

$$G^B f(x) \ge \int_{\varepsilon B} G(x,y) f(y)\, d\mu - C \frac{R^\beta}{V(x_o,R)} \int_{\varepsilon B} f\, d\mu. \tag{6.10}$$

Let us show that the second term in the right hand side of (6.10) is a small fraction of the first one. Since

$$G(x,y) \gtrsim \frac{d(x,y)^\beta}{V(x,y)},$$

so it suffices to verify that, for all $x, y \in \varepsilon B$,

$$\frac{R^\beta}{V(x_o,R)} \le c(\varepsilon) \frac{d(x,y)^\beta}{V(x,y)}, \tag{6.11}$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Indeed, setting $r = d(x,y)$, we obtain

$$\frac{R^\beta}{V(x_o,R)} \Big/ \frac{r^\beta}{V(x,r)} = \frac{V(x,r)}{V(x_o,R)} \left(\frac{R}{r}\right)^\beta \le \frac{V(x,r)}{V(x,R/2)} \left(\frac{R}{r}\right)^\beta \lesssim \left(\frac{r}{R}\right)^{\alpha_-} \left(\frac{R}{r}\right)^\beta = \left(\frac{R}{r}\right)^{\beta-\alpha_-} \lesssim \varepsilon^{\alpha_- - \beta}.$$

Since $\alpha_- > \beta$, this proves (6.11) with $c(\varepsilon) = C\varepsilon^{\alpha_- - \beta}$. It follows that

$$G^B f(x) \ge (1 - Cc(\varepsilon))\, Gf(x)$$

and, hence,

$$G^B(x,y) \ge (1 - Cc(\varepsilon))\, G(x,y) \quad \text{for } \mu\text{-a.a. } x, y \in \varepsilon B. \tag{6.12}$$

By choosing ε small enough we obtain (6.7). □

6.5 $(\mathbf{G})_\beta$ implies $(\mathbf{E})_\beta$

Proposition 6.7. *Let* (**VD**), (**RVD**) *and* $(\mathbf{G})_\beta$ *be satisfied and* $0 < \beta < \alpha_-$. *Then*

$$(\mathbf{G})_\beta \Rightarrow (\mathbf{E})_\beta.$$

Proof. Fix a ball $B = B(x_o, R) \subset M$. Then we obtain from (4.10) that

$$\operatorname*{esssup}_B G^B 1 \le \operatorname*{esssup}_B G 1_B \lesssim R^\beta.$$

Choose $\delta = \varepsilon$ where ε is the constant from (6.7). Then, for μ-a.a. $x \in \delta B$,

$$G^B 1(x) \ge \int_{\delta B} G^B(x,y)\, d\mu(y) \gtrsim \int_{\delta B} \frac{d(x,y)^\beta}{V(x,y)} d\mu(y).$$

Using (6.11), we conclude

$$G^B 1(x) \gtrsim \frac{R^\beta}{V(x_o,R)} V(x_o, \delta R) \gtrsim R^\beta,$$

which finishes the proof of $(\mathbf{E})_\beta$. □

6.6 $(\mathbf{G})_\beta$ implies $(\mathbf{H})$

Proposition 6.8. *Let* $(\mathbf{VD})$, $(\mathbf{RVD})$ *and* $(\mathbf{G})_\beta$ *be satisfied and* $0 < \beta < \alpha_-$. *Then*

$$(\mathbf{G})_\beta \Rightarrow (\mathbf{H}).$$

Proof. If the restricted Green functions G^B are continuous off-diagonal then this was proved in [37, Theorem 3.12 and Lemma 8.2]. Without the continuity of G^B, the key ingredient of the proof – [37, Lemma 6.2(ii)], breaks down[2]. To overcome this difficulty, we have developed here a new approach.

Let $u \in \mathcal{F}$ be non-negative and harmonic in a ball $B = B(x_o, R) \subset M$. We need to prove that

$$\operatorname*{esssup}_{\delta B} u \le C \operatorname*{essinf}_{\delta B} u \tag{6.13}$$

for some constants $C \in (1, \infty)$ and $\delta \in (0, 1)$ independent of B. Without loss of generality, we can assume that $u \in L^\infty$; see [46, p. 1280, Theorem 7.4] for how to remove this additional assumption. Also, by replacing u by u_+, we can assume without loss of generality that $u \ge 0$ on M.

By (6.12), there exists a small $\varepsilon \in (0, \frac{1}{4})$ so that for any ball B

$$\frac{1}{2} G(x, y) \le G^B(x, y) \le G(x, y) \qquad \text{for } \mu\text{-a.a. } x, y \in \varepsilon B. \tag{6.14}$$

Let us fix this ε and use it in what follows. The further proof will be split into three steps.

Step 1. *Riesz measure and a reduced function.* Fix $B = B(x_o, R)$ and consider also the ball

$$B_1 = \frac{\varepsilon}{2} B.$$

By [37, Lemma 6.4], there exists the *reduced function* $\hat{u}$ of u with respect to $(\overline{B}_1, B)$ such that

- $\hat{u} \in \mathcal{F}(B)$;
- $\hat{u} = u$ in $\overline{B}_1$ and $0 \le \hat{u} \le u$ in M;
- $\hat{u}$ is harmonic in $B \setminus \overline{B}_1$ and superharmonic in B.

See Fig. 2 below.

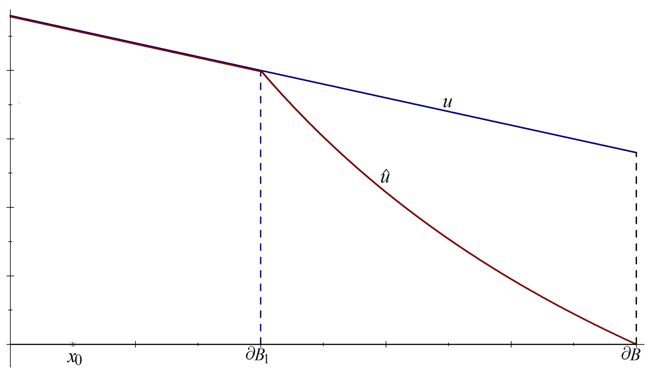

Figure 2: Functions u and $\hat{u}$.

By [37, Lemma 6.2(i)], there exists a regular non-negative Borel measure σ in B such that

$$\int_B \varphi \, d\sigma = \mathcal{E}(\hat{u}, \varphi) \quad \text{for all } \varphi \in \mathcal{F} \cap C_c(B). \tag{6.15}$$

The measure σ is called the *Riesz measure* of the superharmonic function $\hat{u}$. Moreover, the proof of [37, Lemma 6.2(i)] shows that σ does not charge any open set where $\hat{u}$ is harmonic. Since $\hat{u}$ is

[2] Note that a posteriori G^B is still continuous off-diagonal which follows from $(\mathbf{H})$.

harmonic in the both sets B_1 and $B \setminus \overline{B}_1$, we obtain that $\operatorname{supp} \sigma \subset \partial B_1 =: S$. Consequently, the domain of integration in (6.15) can be reduced to S.

Step 2. Let Ω be an open neighborhood of $S = \partial B_1$, such that $\overline{\Omega} \subset B$, for example,

$$\Omega = (1+\tau) B_1 \setminus (1-\tau) \overline{B}_1$$

with a small $\tau \in \left(0, \frac{1}{2}\right)$. Consider also the ball

$$B_2 := \frac{1}{2} B_1 = \frac{\varepsilon}{4} B$$

so that $\overline{B}_2$ and $\overline{\Omega}$ are disjoint; see Fig. 3.

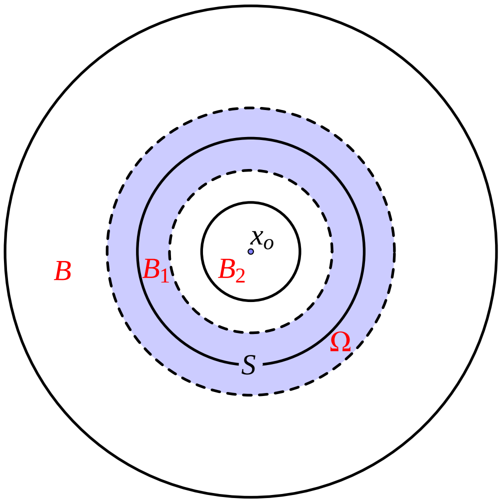

Figure 3: The sets B, B_1, B_2, Ω, S.

Fix a cutoff function ψ of (S, Ω). The aim of this step is to show that, for any function

$$0 \le \phi \in \mathcal{F} \cap C_c(B_2), \tag{6.16}$$

the following inequality holds:

$$\frac{1}{2}\mathcal{E}(\hat{u}, \psi G\phi) \le (u, \phi) \le \mathcal{E}(\hat{u}, \psi G\phi); \tag{6.17}$$

see Fig. 4.

By Remark 2.2, both functions $G^B\phi$ and $(1-\psi)G^B\phi$ belong to $\mathcal{F}(B)$. Since $(1-\psi)G^B\phi$ vanishes in an open neighbourhood of S, we conclude by [37, Proposition A.3] that

$$(1-\psi)G^B\phi \in \mathcal{F}(B \setminus S).$$

Since $\hat{u}$ is harmonic $B \setminus S$ we have

$$\mathcal{E}(\hat{u}, (1-\psi)G^B\phi) = 0. \tag{6.18}$$

Since $u = \hat{u}$ in B_1, ϕ is supported in B_1, and $\hat{u} \in \mathcal{F}(B)$, we obtain, using Remark 2.2 and (6.18) that

$$\begin{aligned}(u, \phi) &= (\hat{u}, \phi) = \mathcal{E}(\hat{u}, G^B\phi)\\ &= \mathcal{E}(\hat{u}, \psi G^B\phi) + \mathcal{E}(\hat{u}, (1-\psi)G^B\phi)\end{aligned}$$

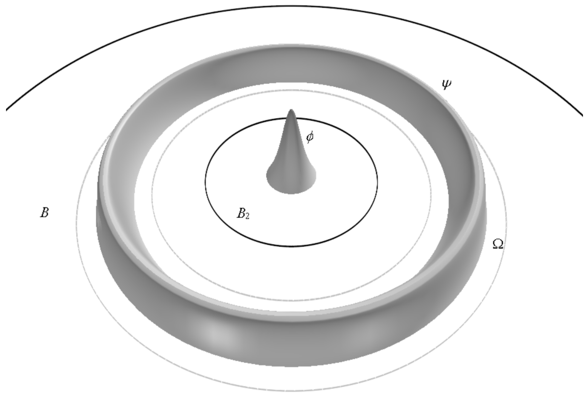

Figure 4: Functions ϕ and ψ.

$$= \mathcal{E}(\hat{u}, \psi G^B \phi). \tag{6.19}$$

By (6.14) we have

$$\frac{1}{2} G\phi \le G^B \phi \le G\phi \quad \mu\text{-a.a. in } \varepsilon B.$$

Since $\operatorname{supp} \psi \subset 2B_1 = \varepsilon B$, it follows that

$$\frac{1}{2} \psi G\phi \le \psi G^B \phi \le \psi G\phi \quad \mu\text{-a.a. in } B.$$

Since both functions $\psi G\phi$ and $\psi G^B \phi$ belong to $\mathcal{F}(B)$ and $\hat{u}$ is superharmonic in B, we obtain

$$\frac{1}{2} \mathcal{E}(\hat{u}, \psi G\phi) \le \mathcal{E}(\hat{u}, \psi G^B \phi) \le \mathcal{E}(\hat{u}, \psi G\phi).$$

This inequality together with (6.19) yields (6.17).

Step 3. Now we can prove the Harnack inequality (6.13). As before, let ψ be a fixed cutoff function of (S, Ω) and ϕ be any function satisfying (6.16). Since $\operatorname{supp} \psi \cap \operatorname{supp} \phi = \emptyset$ and the Green function $G(x, y)$ is jointly continuous off-diagonal, the function $\psi(x) G(x, y) \phi(y)$ is jointly continuous in $(x, y) \in M \times M$. Clearly, we also have $\psi G\phi \in \mathcal{F} \cap C_c(B)$. Applying (6.15) with $\varphi = \psi G\phi$ and the Fubini theorem, we obtain

$$\begin{aligned} \mathcal{E}(\hat{u}, \psi G\phi) &= \int_S \psi(x) G\phi(x)\, d\sigma(x) \\ &= \int_S \psi(x) \left(\int_{B_2} G(x, y) \phi(y)\, d\mu(y) \right) d\sigma(x) \\ &= \int_{B_2} \left(\int_S \psi(x)\, G(x, y)\, d\sigma(x) \right) \phi(y)\, d\mu(y) \\ &= \int_{B_2} \left(\int_S G(x, y)\, d\sigma(x) \right) \phi(y)\, d\mu(y), \end{aligned}$$

where in the last step we have used that $\psi = 1$ on S. Combining with (6.17), we obtain

$$\frac{1}{2} \int_{B_2} \left(\int_S G(x, y)\, d\sigma(x) \right) \phi(y)\, d\mu(y) \le (u, \phi) \le \int_{B_2} \left(\int_S G(x, y)\, d\sigma(x) \right) \phi(y)\, d\mu(y).$$

Since this is true for any non-negative $\phi \in \mathcal{F} \cap C_c(B_2)$ and $\mathcal{F} \cap C_c(B_2)$ is dense in $L^2(B_2)$, we conclude that

$$\frac{1}{2} \int_S G(x, y)\, d\sigma(x) \le u(y) \le \int_S G(x, y)\, d\sigma(x) \quad \text{for } \mu\text{-a.a. } y \in B_2.$$

Since $(\mathbf{G})_\beta$ implies

$$G(x,y) \simeq \frac{R^\beta}{V(x_o,R)} \quad \text{for all } x \in S \text{ and } y \in B_2,$$

we deduce that

$$u(y) \simeq \frac{R^\beta}{V(x_o,R)}\sigma(S) \quad \text{for } \mu\text{-a.a. } y \in B_2.$$

Hence, the Harnack inequality (6.13) holds with $\delta = \frac{1}{4}\varepsilon$. □

7 Weighted Hardy's inequality for strongly local Dirichlet forms

Let (M,d,μ) be a metric measure space and $(\mathcal{E},\mathcal{F})$ be a strongly local Dirichlet form on $L^2(M,\mu)$. The main aim of this section is to obtain a weighted version of Hardy's inequality for strongly local Dirichlet forms.

7.1 Weighted Dirichlet form and weighted Hardy's inequality

For all $x,y \in M$, define

$$d_i(x,y) = \sup\left\{u(x) - u(y) : \ u \in \mathcal{F} \cap C_c, \ d\Gamma(u,u) \le d\mu\right\}.$$

The function $d_i(x,y)$ is called the *intrinsic metric* of $(\mathcal{E},\mathcal{F})$. In general $d_i(x,y)$ is a pseudo-distance.

Let us introduce the following hypotheses (H1)-(H3) that will be used in what follows.

(H1) For any $u \in \mathcal{F}$, the energy measure $\Gamma(u,u)$ is absolutely continuous with respect to μ.

(H2) The intrinsic metric d_i coincides with the original metric d.

(H3) The metric space (M,d) is complete.

It is known that, under these assumptions, the metric space (M,d) is geodesic. Besides, for any non-empty subset E of M, the function $f(x) = d(x,E)$ belongs to $\mathcal{F}_{\text{loc}}$ and satisfies $d\Gamma(f,f) \le d\mu$; see [50].

For example, (H1)-(H3) are satisfied if M is a geodesically complete Riemannian manifold, d is the geodesic distance, μ is the Riemannian measure, and $(\mathcal{E},\mathcal{F})$ is given (5.28).

Let $w : M \to (0,\infty]$ be a continuous, locally integrable function, where "continuous" in this context means that w is continuous on $\{w < \infty\}$ and lower semi-continuous on M. Define a weighted bilinear form $\mathcal{E}^{(w)}$ by

$$\mathcal{E}^{(w)}(u,v) = \int_M w\, d\Gamma(u,v) \ \text{ for all } u,v \in \mathcal{F} \cap C_c$$

and set

$$\mathcal{C}^{(w)} = \left\{u \in \mathcal{F} \cap C_c : \ \mathcal{E}^{(w)}(u,u) < \infty\right\}.$$

We will use the following result from [70, Corollary 6.1.6].

Proposition 7.1. *Let* $(\mathcal{E},\mathcal{F})$ *satisfy* (H1)-(H3) *and let* $w : M \to (0,\infty]$ *be a continuous, locally integrable function. Define*

$$d\mu_w = w d\mu.$$

Then the symmetric bilinear form $(\mathcal{E}^{(w)},\mathcal{C}^{(w)})$ *is closable and its closure* $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$ *is a strongly local regular Dirichlet form on* $L^2(M,\mu_w)$ *that also satisfies* (H1)-(H3).

Combining Proposition 7.1 and Theorem 3.1, we deduce a weighted version of Theorem 3.1.

Corollary 7.2. *Let all the assumptions of Proposition 7.1 be satisfied. Assume that*

$$\lambda_{\min}^{(w)}(\Omega) := \inf_{u \in (\mathcal{F}^{(w)} \cap C_c(\Omega)) \setminus \{0\}} \frac{\mathcal{E}^{(w)}(u,u)}{\|u\|_{L^2(M,\mu_w)}^2} > 0$$

for all precompact open sets $\Omega \subset M$. Let $G^{(w)}$ be the Green function of $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ and h be a non-negative measurable function on M such that

$$G^{(w)}(h \wedge a) \in L^{\infty}_{\mathrm{loc}}$$

for any positive constant a. Then, for any $f \in \mathcal{F}^{(w)}$,

$$\int_M \frac{h}{G^{(w)} h} f^2 w \, d\mu \leq \mathcal{E}^{(w)}(f,f).$$

7.2 Example: Σ is the boundary of a convex domain

In this subsection we apply Corollary 7.2 in order to prove the following statement.

Proposition 7.3. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a non-empty convex domain and let $\sigma \in (0,1)$. Then, for all $f \in Lip_c(\mathbb{R}^n)$, the following inequality holds:*

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^2 \, d(x,\partial\Omega)^{\sigma}} dx \leq C \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{d(x,\partial\Omega)^{\sigma}} dx, \tag{7.1}$$

where the constant C depends only on n and σ (and does not depend on Ω).

Here $Lip_c(\mathbb{R}^n)$ denotes the class of Lipschitz functions in $\mathbb{R}^n$ with compact support. In particular, (7.1) holds for any $f \in Lip_c(\overline{\Omega})$ with $f|_{\partial\Omega} = 0$ as this function extends to that in $Lip_c(\mathbb{R}^n)$ by setting $f = 0$ in $\overline{\Omega}^c$.

Remark 7.4. Let us compare Hardy's inequality (7.1) with previously known results. The following weighted Hardy's inequality was proved in [59, Theorems 1.2 and 3.4]: if $d(x)$ a distance function in $\mathbb{R}^n$ such that, for some real α and $\sigma \neq \alpha - 2$,

$$(\alpha - 2 - \sigma)\left(\Delta d(x) - \frac{\alpha - 1}{d(x)}\right) \geq 0 \ \text{ in } U := \{d(x) > 0\}, \tag{7.2}$$

then, for any $f \in C_c^{\infty}(U)$,

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{d(x)^{\sigma+2}} dx \leq c \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{d(x)^{\sigma}} dx, \tag{7.3}$$

where $c = 4/(\alpha - 2 - \sigma)^2$. For example, if $d(x)$ is the distance to a subspace $\mathbb{R}^l$ of $\mathbb{R}^n$ then

$$\Delta d(x) = \frac{n-l-1}{d(x)},$$

and (7.2) is satisfied with $\alpha = n - l$ and any $\sigma \neq n - l - 2$.

Let $d(x)$ be the distance to $\overline{B}$ where $B = B(0,R)$. Then in $U = \mathbb{R}^n \setminus \overline{B}$ we have $d(x) = |x| - R$ so that

$$\Delta d(x) = \frac{n-1}{|x|} \leq \frac{n-1}{d(x)}.$$

Hence, (7.2) is satisfied with $\alpha = n$ and $\sigma > n - 2$, which yields

$$\int_U \frac{f(x)^2}{(|x| - R)^{\sigma+2}} dx \leq c \int_U \frac{|\nabla f(x)|^2}{(|x| - R)^{\sigma}} dx \tag{7.4}$$

for all $f \in C_c^\infty(U)$. For comparison, our Proposition 7.3 gives in the case $\Omega = B(0,R)$

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^2 \, ||x|-R|^\sigma} dx \le C \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{||x|-R|^\sigma} dx, \tag{7.5}$$

for any $\sigma \in (0,1)$ all $f \in Lip_c(\mathbb{R}^n)$. Of course, neither of (7.4), (7.5) covers the other because the ranges of σ are disjoint. Besides, the range of functions f in (7.5) is wider and includes functions not vanishing on $\overline{B}$.

Let now $d(x)$ be the distance to B^c. In B we have $d(x) = R - |x|$ and

$$\Delta d(x) = -\frac{n-1}{|x|} \le 0$$

so that (7.2) is satisfied with $\alpha = 1$ and $\sigma > -1$. Therefore, (7.3) yields in this case

$$\int_B \frac{f(x)^2}{(R-|x|)^{\sigma+2}} dx \le c \int_B \frac{|\nabla f(x)|^2}{(R-|x|)^\sigma} dx \tag{7.6}$$

for any $f \in C_c^\infty(B)$. Although the range of σ in (7.6) is wider than that in (7.5), still the inequality (7.5) gives a better result for $f \in C_c^\infty(\frac{1}{2}B)$.

We see that the results in [59] do not cover Proposition 7.3 for convex domains and even for balls. Although for bounded convex domains there are already weighted Hardy's inequalities (see [2, 59, 25]), they do not cover Proposition 7.3 either because the Hardy constant in (7.1) does not depend on Ω.

To prove Proposition 7.3, we need several lemmas.

Lemma 7.5. *Let $n \ge 2$ and V be a convex subset of a bounded open set $U \subset \mathbb{R}^n$. Then*

$$H^{n-1}(\partial V) \le H^{n-1}(\partial U), \tag{7.7}$$

where H^{n-1} denotes the $(n-1)$-dimensional Hausdorff measure.

Proof. Let us define a mapping $\Phi : \partial U \to \partial V$ as follows: for any $x \in \partial U$ let $\Phi(x)$ be the nearest to x point of $\overline{V}$. Let us prove that

$$|\Phi(x) - \Phi(y)| \le |x-y|. \tag{7.8}$$

Denote for simplicity $X = \Phi(x)$ and $Y = \Phi(y)$ and first observe that

$$(Y-X)\cdot(X-x) \ge 0. \tag{7.9}$$

Indeed, by the convexity of V, the point $X + t(Y-X)$ lies in $\overline{V}$ for any $t \in (0,1)$, whence

$$|X-x| \le |(X+t(Y-X))-x|,$$

that is,

$$|X-x|^2 \le |(X+t(Y-X))-x|^2 = |X-x|^2 + 2t(Y-X)\cdot(X-x) + t^2|Y-X|^2,$$

or, equivalently,

$$0 \le 2(Y-X)\cdot(X-x) + t|Y-X|^2,$$

which implies (7.9) by letting $t \to 0$. Similarly, we have

$$(X-Y)\cdot(Y-y) \ge 0. \tag{7.10}$$

Adding up (7.9) and (7.10), we obtain

$$(X - Y) \cdot (Y - X + x - y) \geq 0$$

whence

$$|X - Y|^2 \leq (X - Y) \cdot (x - y)$$

and, hence, (7.8). It follows from (7.8) that the mapping Φ reduces Hausdorff measures of all dimensions (cf. [26, p. 75]), whence (7.7) follows. □

Lemma 7.6. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a non-empty convex domain. Then, for any $x_o \in \mathbb{R}^n$ and $R, s \in (0, \infty)$,*

$$|\{x \in B(x_o, R) : d(x, \partial\Omega) < s\}| \leq \omega_n R^{n-1} \min\{2s, n^{-1}R\}, \tag{7.11}$$

where ω_n is the surface area of a unit sphere in $\mathbb{R}^n$.

Dividing by $|B(x_o, R)| = \frac{\omega_n}{n} R^n$, we obtain from (7.11)

$$\frac{|\{x \in B(x_o, R) : d(x, \partial\Omega) < s\}|}{|B(x_o, R)|} \leq \min\left\{2n\frac{s}{R}, 1\right\}.$$

Proof. Note that (7.11) holds trivially if $2ns \geq R$ or if $\{x \in B(x_o, R) : d(x, \partial\Omega) < s\} = \emptyset$. Hence, we assume in what follows that

$$0 < 2ns < R \quad \text{and} \quad \{x \in B(x_o, R) : d(x, \partial\Omega) < s\} \neq \emptyset.$$

Consider the following *signed distance function* $\delta(x)$ to $\partial\Omega$ that is defined by

$$\delta(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega, \\ d(x, \partial\Omega) & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Note that δ is Lipschitz and, hence, differentiable almost everywhere on $\mathbb{R}^n$. It follows from [24, Theorem 5.1.5] that $|\nabla\delta(x)| = 1$ for almost all $x \in \mathbb{R}^n$.

Consider for all $t \in \mathbb{R}$ the set

$$\Omega_t = \{x \in \mathbb{R}^n : \delta(x) < t\}.$$

We claim that Ω_t is a convex set for all $t \in \mathbb{R}$. Indeed, for $t < 0$ this was proved in [47, p. 17, the remark after Fig. 4]. Let us prove the convexity of Ω_t for $t > 0$. Note that, for $t > 0$, we have

$$\Omega_t = \left\{x \in \mathbb{R}^n : d\left(x, \overline{\Omega}\right) < t\right\}.$$

Fix two points $x, y \in \Omega_t$ and prove that the line segment $[x, y]$ is contained in Ω_t. To this end, we choose points $\tilde{x}, \tilde{y} \in \overline{\Omega}$ such that

$$|x - \tilde{x}| < t \quad \text{and} \quad |y - \tilde{y}| < t.$$

Any point $z \in [x, y]$ can be written as $z = \lambda x + (1 - \lambda)y$ for some $\lambda \in [0, 1]$. Since $\tilde{x}, \tilde{y} \in \overline{\Omega}$ and $\overline{\Omega}$ is convex, it follows that

$$\tilde{z} = \lambda\tilde{x} + (1 - \lambda)\tilde{y} \in \overline{\Omega}.$$

Since

$$|z - \tilde{z}| = \left|\left(\lambda x + (1 - \lambda)y\right) - \left(\lambda\tilde{x} + (1 - \lambda)\tilde{y}\right)\right| \leq \lambda\,|x - \tilde{x}| + (1 - \lambda)\,|y - \tilde{y}| < t,$$

we conclude that $z \in \Omega_t$ and, hence, $[x, y] \subset \Omega_t$.

By the coarea formula in [26, p. 112], for any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and any Lebesgue measurable set $A \subset \mathbb{R}^n$,

$$\int_A |\nabla f|\, dx = \int_{\mathbb{R}} H^{n-1}(A \cap \{x \in \mathbb{R}^n : f(x) = t\})\, dt.$$

Applying this formula with $f = \delta$ and using that $|\nabla \delta| = 1$ a.e., we obtain

$$\begin{aligned} |\{x \in B(x_o, R) : d(x, \partial\Omega) < s\}| &= |\{x \in B(x_o, R) : |\delta(x)| < s\}| \\ &= \int_{\{x \in B(x_o,R) : |\delta(x)| < s\}} |\nabla \delta(x)|\, dx \\ &= \int_{-s}^{s} H^{n-1}(\{x \in B(x_o, R) : \delta(x) = t\})\, dt. \end{aligned} \tag{7.12}$$

Clearly, we have

$$\{x \in B(x_o, R) : \delta(x) = t\} = \partial\Omega_t \cap B(x_o, R) \subset \partial(\Omega_t \cap B(x_o, R))$$

and, hence,

$$H^{n-1}(\{x \in B(x_o, R) : \delta(x) = t\}) \le H^{n-1}(\partial(\Omega_t \cap B(x_o, R))). \tag{7.13}$$

See Fig. 5 for the case $t \in (-\infty, 0)$.

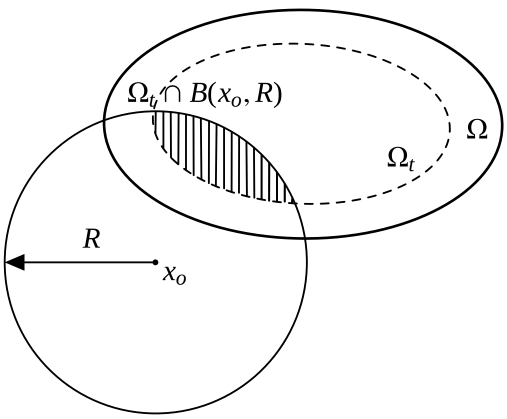

Figure 5: The sets $B(x_o, R)$, Ω and Ω_t for $t \in (-\infty, 0)$.

Since every Ω_t is convex, the set $\Omega_t \cap B(x_o, R)$ is also convex. It follows from (7.7) that

$$H^{n-1}(\partial(\Omega_t \cap B(x_o, R))) \le H^{n-1}(\partial (B(x_o, R))) = \omega_n R^{n-1}. \tag{7.14}$$

Combining (7.12), (7.13) and (7.14), we obtain

$$|\{x \in B(x_o, R) : d(x, \partial\Omega) < s\}| \le 2\omega_n R^{n-1} s,$$

which was to be proved. □

Let v_n denote the volume of a unit ball in $\mathbb{R}^n$, that is, $v_n = \omega_n / n$.

Lemma 7.7. *Let $\Omega \subset R^n$ be a non-empty convex domain. Then, for any $\sigma \in (0,1)$, the weight function*

$$w(x) = d(x, \partial\Omega)^{-\sigma} \quad \text{for all } x \in \mathbb{R}^n$$

satisfies the relation

$$\nu_n r^n (r + d(x_o, \partial\Omega))^{-\sigma} \leq \mu_w(B(x_o, r)) \leq \frac{(6n)^\sigma}{1-\sigma} \nu_n r^n (r + d(x_o, \partial\Omega))^{-\sigma} \tag{7.15}$$

uniformly in $x_o \in \mathbb{R}^n$ and $r > 0$.

Proof. Obviously, for any $y \in B(x_o, r)$, we have

$$d(y, \partial\Omega) \leq d(y, x_o) + d(x_o, \partial\Omega) < r + d(x_o, \partial\Omega),$$

which implies

$$\mu_w(B(x_o, r)) = \int_{B(x_o,r)} d(y, \partial\Omega)^{-\sigma}\, dy \geq (r + d(x_o, \partial\Omega))^{-\sigma} |B(x_o, r)| = \nu_n r^n (r + d(x_o, \partial\Omega))^{-\sigma}.$$

In order to prove the upper bound of $\mu_w(B(x_o, r))$, consider the following two cases.

Case 1: let $d(x_o, \partial\Omega) \geq 2r$. In this case, for any $y \in B(x_o, r)$, we have

$$d(y, \partial\Omega) \geq d(x_o, \partial\Omega) - d(x_o, y) > d(x_o, \partial\Omega)/2,$$

which implies

$$\mu_w(B(x_o, r)) = \int_{B(x_o,r)} d(y, \partial\Omega)^{-\sigma}\, dy \leq \nu_n 2^\sigma r^n d(x_o, \partial\Omega)^{-\sigma} \leq \nu_n 3^\sigma r^n (r + d(x_o, \partial\Omega))^{-\sigma}.$$

Case 2: let $d(x_o, \partial\Omega) < 2r$. By the Fubini theorem and Lemma 7.6, we obtain

$$\begin{aligned}
\mu_w(B(x_o, r)) &= \int_{B(x_o,r)} d(y, \partial\Omega)^{-\sigma}\, dy = \sigma \int_{B(x_o,r)} \left(\int_{d(y,\partial\Omega)}^{\infty} s^{-\sigma-1}\, ds \right) dy \\
&= \sigma \int_0^\infty \left(\int_{\{y \in B(x_o,r):\, d(y,\partial\Omega) < s\}} dy \right) s^{-\sigma-1}\, ds \\
&\leq \sigma r^{n-1} \int_0^\infty \omega_n \min\{2s, n^{-1} r\} s^{-\sigma-1}\, ds \\
&= \omega_n \sigma r^{n-1} \left(2 \int_0^{(2n)^{-1} r} s^{-\sigma}\, ds + n^{-1} r \int_{(2n)^{-1} r}^{\infty} s^{-\sigma-1}\, ds \right) \\
&= \frac{(2n)^\sigma}{1-\sigma} \nu_n r^{n-\sigma} \leq \frac{(6n)^\sigma}{1-\sigma} \nu_n r^n (r + d(x_o, \partial\Omega))^{-\sigma},
\end{aligned}$$

which finishes the proof. □

Now we can prove Proposition 7.3.

Proof of Proposition 7.3. The estimate (7.15) of Lemma 7.7 implies that, for any $x \in \mathbb{R}^n$ and $R \geq r > 0$,

$$c \left(\frac{R}{r} \right)^{n-\sigma} \leq \frac{\mu_w(B(x, R))}{\mu_w(B(x, r))} \leq C \left(\frac{R}{r} \right)^n,$$

where the positive constants c, C depend only on n and σ. Consequently, the metric measure space $(\mathbb{R}^n, \mu_w)$ (with the Euclidean distance) satisfies (**VD**) and (**RVD**) with the upper volume dimension n and the lower volume dimension $n - \sigma > 2$.

The weight function $w(x) = d(x, \partial\Omega)^{-\sigma}$ with $\sigma \in (0,1)$ is locally integrable in $(\mathbb{R}^n, dx)$. Hence, by Proposition 7.1, the quadratic form

$$\mathcal{E}^{(w)}(u,v) = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v)\, w\, dx$$

is a strongly local regular Dirichlet form in $L^2(\mathbb{R}^n, \mu_w)$, and the domain $\mathcal{F}^{(w)}$ of this Dirichlet form has a core

$$C^{(w)} = \left\{ u \in W^{1,2} \cap C_c : \ \mathcal{E}^{(w)}(u,u) < \infty \right\}.$$

Next, observe that the function $w(x) = d(x, \partial\Omega)^{-\sigma}$ belongs to Muckenhoupt weight class A_2, that is,

$$[w]_{A_2} := \sup_{B \text{ ball in } \mathbb{R}^n} \left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1}\, dx \right) < \infty.$$

This follows from [23, Theorem 1.1], but we need also to know that $[w]_{A_2}$ admits an upper bound depending only on n and σ. Indeed, by Lemma 7.7, we have, for any ball $B = B(x_o, r)$,

$$\frac{1}{|B|} \int_B w(x)\, dx = \frac{\mu_w(B)}{|B|} \leq \frac{(6n)^\sigma}{1-\sigma} (r + d(x_o, \partial\Omega))^{-\sigma}$$

while by the triangle inequality

$$\frac{1}{|B|} \int_B w(x)^{-1}\, dx = \frac{1}{|B|} \int_B d(x, \partial\Omega)^\sigma\, dx \leq (r + d(x_o, \partial\Omega))^\sigma.$$

Hence, it follows that

$$\left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1}\, dx \right) \leq \frac{(6n)^\sigma}{1-\sigma}$$

and

$$[w]_{A_2} \leq \frac{(6n)^\sigma}{1-\sigma}.$$

Applying [19, Theorem 1], we obtain that the heat kernel $p_t^{(w)}$ of $e^{-t\mathcal{L}_w}$ satisfies

$$p_t^{(w)}(x,y) \leq \frac{C}{\sqrt{\mu_w(B(x, \sqrt{t}))} \sqrt{\mu_w(B(y, \sqrt{t}))}} \exp\left(-c \frac{|x-y|^2}{t} \right) \tag{7.16}$$

for all $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$, with the positive constants C and c depending only on n and $[w]_{A_2}$, that is, only on n and σ.

By [38, Theorem 2.1], the heat kernel bound (7.16) implies the Faber-Krahn inequality (5.29), whence $\lambda_{\min}^{(w)}(U) > 0$ for any precompact domains U.

Using (7.16) and the reverse volume doubling property of μ_w with the lower volume dimension $n - \sigma > 2$, we obtain by the argument in the proof of Lemma 2.4 that, for distinct $x, y \in \mathbb{R}^n$,

$$G^{(w)}(x,y) \leq C \frac{|x-y|^2}{\mu_w(B(x, |x-y|))}. \tag{7.17}$$

Next, we fix real numbers $\varepsilon \in (0,1)$, $\rho > 0$ and define h as in (5.17), that is,

$$h(x) = \begin{cases} \frac{\rho^\varepsilon}{\mu_w(B(0,\rho))} & \text{if } |x| < \rho \\ \frac{|x|^\varepsilon}{\mu_w(B(0,|x|))} & \text{if } |x| \geq \rho. \end{cases} \tag{7.18}$$

Due to (7.15) and (7.17), we follow the proof of (5.19) and derive that

$$G^{(w)} h(x) \leq C \begin{cases} \frac{\rho^{2+\varepsilon}}{\mu_w(B(0,\rho))} & \text{if } |x| < 2\rho \\ \frac{|x|^{2+\varepsilon}}{\mu_w(B(0,|x|))} & \text{if } |x| \geq 2\rho \end{cases} \tag{7.19}$$

where C is a constant depending only on n and σ.

Hence, we see that all the hypotheses of Corollary 7.2 are satisfied. Therefore, for any $f \in Lip_c(\mathbb{R}^n) \subset W^{1,2} \cap C_c$, we obtain

$$\int_{\mathbb{R}^n} \frac{h}{G^{(w)}h} f^2\, d\mu_w \leq \mathcal{E}^{(w)}(f, f) = \int_{\mathbb{R}^n} |\nabla f(x)|^2 w(x)\, dx.$$

Further, applying (7.18) and (7.19), and letting $\rho \to 0$, we finally obtain

$$\int_{\mathbb{R}^n} \frac{1}{|x|^2} f^2(x) w(x)\, dx \leq C \int_{\mathbb{R}^n} |\nabla f(x)|^2 w(x)\, dx,$$

where the constant C depends only on n and σ, which finishes the proof. □

7.3 Admissible weights

Motivated by [42, 70], we introduce the following definitions. Given a set $\Sigma \subset M$ and $\rho \in (0, 1]$, define for any $x_o \in \Sigma$ and $s \geq 0$ the set

$$\Sigma_\rho(x_o, s) := \{x \in M : \ d(x, x_o) \leq s \text{ and } d(x, \Sigma) \geq \rho s\}\,.$$

Set also

$$\widehat{\Sigma}_\rho(x_o, r) = \bigcup_{0 \leq s \leq r} \Sigma_\rho(x_o, s).$$

Indeed, it is easy to see that

$$\begin{aligned}\widehat{\Sigma}_\rho(x_o, r) &= \{x \in M : \ \rho d(x, x_o) \leq d(x, \Sigma) \leq d(x, x_o) \leq r\}\\ &= \{x \in \overline{B}(x_o, r) : \ \rho d(x, x_o) \leq d(x, \Sigma)\}.\end{aligned}$$

For example, if $\Sigma = \{x_o\}$ then $\Sigma_\rho(x_o, s)$ is the annulus $\overline{B}(x_o, s) \setminus B(x_o, \rho s)$, and $\widehat{\Sigma}_\rho(x_o, r)$ coincides with the closed ball $\overline{B}(x_o, r)$.

Definition 7.8. Let Σ be a non-empty subset of M. Fix $\rho \in (0, 1)$. The set Σ is called *ρ-accessible* if the following conditions are satisfied:

(i) Σ is closed and $\mu(\Sigma) = 0$;

(ii) there exists $\rho' \in (\rho, 1]$ such that, for any $x_o \in \Sigma$ and $s \in (0, \infty)$, the set $\Sigma_{\rho'}(x_o, s)$ is non-empty;

(iii) for any $x_o \in \Sigma$ and $r \in (0, \infty)$, the set $\widehat{\Sigma}_\rho(x_o, r)$ is path connected.

For example, if (M, d) is a non-compact complete geodesic space and $\Sigma = \{x_0\}$ then all these conditions are satisfied so that a singleton is ρ-accessible for any $\rho \in (0, 1)$.

Other examples of ρ-accessible sets will be shown in Section 7.4 below.

Definition 7.9. A function $w : M \to (0, \infty]$ is called an *admissible weight* if there exist a set $\Sigma \subset M$ and a function $a : [0, \infty) \to (0, \infty]$ such that

$$w(x) = a(d(x, \Sigma)) \ \text{ for all } x \in M$$

and the following conditions are satisfied:

(i) the set Σ is ρ-accessible for some $\rho \in (0, 1)$;

(*ii*) the function a is continuous, non-increasing, $a(r) < \infty$ for $r > 0$, and there exists a constant $c \in (0,1)$ such that, for any $r > 0$,

$$a(2r) \geq ca\,(r)\,;$$

(*iii*) there exists a positive constant C such that, for any $x_o \in \Sigma$ and any $r > 0$,

$$\mu_w(B(x_o, r)) \leq Ca(r)\mu(B(x_o, r)), \tag{7.20}$$

where $d\mu_w = w\,d\mu$.

It follows that any admissible function w is continuous and locally integrable with respect to μ.

For example, the function $a\,(r) = r^{-\sigma}$ satisfies (*ii*) for any $\sigma > 0$. If $\mu\,(B\,(x_o, r)) \simeq r^{\alpha}$ for all $r > 0$ and $x_o \in \Sigma$ then $a\,(r) = r^{-\sigma}$ satisfies (*iii*) if and only if $0 < \sigma < \alpha$; see [42, Sec. 4.3] and Proposition 7.13 below.

Lemma 7.10. *Assume that the measure μ satisfies* (**VD**). *Let w be an admissible weight as in* Definition 7.9. *Then the measure μ_w also satisfies* (**VD**) *and, for all $x \in M$ and $r > 0$,*

$$\mu_w\,(B\,(x, r)) \simeq \mu\,(B\,(x, r))\,a\,(\xi\,(x) + r) \tag{7.21}$$

where $\xi\,(x) = d\,(x, \Sigma)$.

Proof. The fact that μ_w satisfies (**VD**) was proved in [70, Thm 1.0.1, Prop. 4.2.2]. Note, the condition (*iii*) of Definition 7.8 is not needed for that, while the condition (7.20) is very essential.

In order to prove (7.21), let us first assume that $x \in \Sigma$, that is, $\xi\,(x) = 0$. Then the upper bound in (7.21) follows from (7.20) while the lower bound holds by

$$\mu_w(B(x, r)) = \int_{B(x,r)} a\,(\xi\,(y))\,d\mu\,(y) \geq a\,(r)\,\mu\,(B\,(x, r))$$

because $\xi\,(y) \leq d\,(x, y) \leq r$ and a is monotone decreasing.

Assume now that $\xi\,(x) \geq 2r$. Then, for any $y \in B\,(x, r)$, we have $\xi\,(y) \simeq \xi\,(x)$, whence

$$\mu_w(B(x, r)) = \int_{B(x,r)} a\,(\xi\,(y))\,d\mu\,(y) \simeq a\,(\xi\,(x))\,\mu\,(B\,(x, r)) \simeq a\,(\xi\,(x) + r)\,\mu\,(B\,(x, r))\,.$$

Finally, let $\xi\,(x) < 2r$. Let x' be a point on Σ so that $d\,(x, x') < 2r$. Since the measures μ_w and μ are doubling, we obtain

$$\mu_w\,(B\,(x, r)) \simeq \mu_w\,(B\,(x', r)) \simeq a\,(r)\,\mu\,(B\,(x', r)) \simeq a\,(\xi\,(x) + r)\,\mu\,(B\,(x, r))\,.$$

This proves (7.21) for general $x \in M$.

One can also derive that μ_w satisfies (**VD**) by terms of (7.21) and the monotone decreasing property of the function a. □

The notion of an admissible weight was used in [70, Prop. 4.2.2] to prove the following result.

Theorem 7.11. *Let a strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on (M, d, μ) satisfy* (H1)-(H3) *as well as the uniform parabolic Harnack inequality. Let w be an admissible weight on M. Then the weighted Dirichlet form $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ on (M, d, μ_w) also satisfies the uniform parabolic Harnack inequality.*

We use Theorem 7.11 in order to prove our main result in this section that is the following weighted Hardy's inequality for admissible weights w.

Theorem 7.12. *Assume that* (M,d,μ) *satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension* α_-. *Let* $(\mathcal{E},\mathcal{F})$ *be a strongly local regular Dirichlet form on* $L^2(M,\mu)$ *that satisfies* (H1)-(H3) *as well as* $(\mathbf{G})_2$. *Let* w *be an admissible weight on* M *as in* Definition 7.9, *and assume that the function* $a(r)$ *satisfies for all* $R > r > 0$

$$\frac{a(R)}{a(r)} \gtrsim \left(\frac{R}{r}\right)^{-\sigma} \tag{7.22}$$

for some σ *such that*

$$0 \le \sigma < \alpha_- - 2.$$

Then the Green function $G^{(w)}$ *of* $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$ *satisfies* $(\mathbf{G})_2$ *with respect to* μ_w, *and the following weighted Hardy's inequality holds: for all* $x_o \in M$ *and* $f \in \mathcal{F} \cap C_c$,

$$\int_M \frac{f(x)^2}{d(x,x_o)^2} w(x)\, d\mu(x) \le C \int_M w\, d\Gamma(f,f), \tag{7.23}$$

where the constant C *depends only on the constants in the hypotheses, but is independent of* x_o *and* f.

Proof. By Theorem 6.1, the hypothesis $(\mathbf{G})_2$ implies that the heat kernel p_t of $(\mathcal{E},\mathcal{F})$ satisfies $(\mathbf{UE})_2$ and $(\mathbf{NLE})_2$. Further, by [8, Theorems 3.1 and 3.2] (see also [69]), the conditions $(\mathbf{UE})_2$ and $(\mathbf{NLE})_2$ are equivalent to the parabolic Harnack inequality for $(\mathcal{E},\mathcal{F})$. Since w admissible, we conclude by Theorem 7.11 that the parabolic Harnack inequality for $(\mathcal{E},\mathcal{F})$ implies the parabolic Harnack inequality for $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$. Hence, the heat kernel $p_t^{(w)}$ of $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$ also satisfies the Gaussian estimates $(\mathbf{UE})_2$ and $(\mathbf{NLE})_2$, with respect to the measure μ_w.

Next, we need to make sure that the Green function $G^{(w)}$ of $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$ satisfies $(\mathbf{G})_2$ with respect to the measure μ_w. By Lemma 7.10, the measure μ_w is doubling. By (7.21), (7.22) and (**RVD**) for μ, we obtain, for all $R > r > 0$

$$\frac{\mu_w(B(x,R))}{\mu_w(B(x,r))} \simeq \frac{a(\xi(x)+R)}{a(\xi(x)+r)}\frac{\mu(B(x,R))}{\mu(B(x,r))} \gtrsim \left(\frac{a(\xi(x)+R)}{a(\xi(x)+r)}\right)^{-\sigma}\left(\frac{R}{r}\right)^{\alpha_-} \gtrsim \left(\frac{R}{r}\right)^{\alpha_- - \sigma}$$

so that μ_w satisfies (**RVD**) with lower volume dimension

$$\alpha_-^{(w)} = \alpha_- - \sigma > 2.$$

Applying Lemma 2.4 in the space (M,d,μ_w) we obtain that $G^{(w)}$ satisfies $(\mathbf{G})_2$, with respect to the measure μ_w.

By Theorem 5.6, we conclude that, for any $f \in \mathcal{F}^{(w)}$,

$$\int_M \frac{f(x)^2}{d(x_o,x)^2}\, d\mu(x) \lesssim \mathcal{E}^{(w)}(f,f). \tag{7.24}$$

It remains to verify that (7.23) holds for all $f \in \mathcal{F} \cap C_c$. If the right hand side of (7.23) is ∞, then (7.23) is trivially satisfied. If the right hand side of (7.23) is finite then $f \in C^{(w)} \subset \mathcal{F}^{(w)}$ and

$$\int_M w\, d\Gamma(f,f) = \mathcal{E}^{(w)}(f,f),$$

so that (7.23) follows from (7.24). □

Let us illustrate Theorem 7.12 in the case when Σ is a singleton.

Proposition 7.13. *Assume that* (M,d,μ) *satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension* $\alpha_- \in (2,\infty)$. *Let* $(\mathcal{E},\mathcal{F})$ *be a strongly local regular Dirichlet form on* $L^2(M,\mu)$ *that satisfies* (H1)-(H3) *and admits the Green function* $G(x,y)$ *satisfying* $(\mathbf{G})_2$. *Then, for any* σ *satisfying*

$$0 \le \sigma < \alpha_- - 2,$$

the following weighted Hardy's inequality

$$\int_M \frac{f(x)^2}{d(x_o,x)^{\sigma+2}}\,d\mu(x) \le C\int_M \frac{1}{d(x_o,x)^{\sigma}}\,d\Gamma(f,f) \tag{7.25}$$

holds for all $x_o \in M$ and $f \in \mathcal{F} \cap C_c$, where C depends only on the constants in the hypotheses.

Proof. We will apply Theorem 7.12 with $\Sigma = \{x_o\}$ and the weight

$$w(x) = d(x,x_o)^{-\sigma} \ \text{ for all } x \in M.$$

Let us verify that the weight w is admissible. The conditions (i) and (ii) of Definition 7.9 are obviously satisfied with $a(r) = r^{-\sigma}$. Let us verify the condition (iii) of Definition 7.9. Setting $r = r2^{-k}$, we obtain, using (**RVD**) that

$$\begin{aligned}\mu_w(B(x_o,r)) &= \int_{B(x_o,r)} \frac{d\mu(x)}{d(x_o,x)^{\sigma}} \\ &= \sum_{k=0}^{\infty}\int_{B(x_o,r_k)\setminus B(x_o,r_{k+1})} \frac{d\mu(x)}{d(x_o,x)^{\sigma}} \\ &\le \sum_{k=0}^{\infty} r_{k+1}^{-\sigma}\mu(B(x_o,r_k)) \le C\sum_{k=0}^{\infty} r_k^{-\sigma}\left(\frac{r_k}{r}\right)^{\alpha_-}\mu(B(x_o,r)) \\ &= Cr^{-\sigma}\mu(B(x_o,r))\sum_{k=0}^{\infty}\left(2^{-k}\right)^{\alpha_- -\sigma} = C'r^{-\sigma}\mu(B(x_o,r)),\end{aligned}$$

which proves (7.20). By Theorem 7.12 we obtain (7.23), which is equivalent to (7.25). □

Since $Lip_c(\mathbb{R}^n) \subset W^{1,2}(\mathbb{R}^n) \cap C_c(\mathbb{R}^n)$, it follows from Proposition 7.13 that, for any $f \in Lip_c(\mathbb{R}^n)$ and $0 \le \sigma < n-2$,

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^{\sigma+2}}\,dx \lesssim \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{|x|^{\sigma}}dx,$$

which matches (1.7).

7.4 Example: Σ is a subset of a hyperplane

Here we apply Theorem 7.12 in the case when Σ is a closed subset of a hyperplane in $\mathbb{R}^n$. Let us start with the following observation.

Lemma 7.14. *Let Σ be a non-empty closed subset of a hyperplane in $\mathbb{R}^n$. Then Σ is ρ-accessible for any $\rho \in (0,1)$.*

Proof. Condition (i) of Definition 7.8 is trivially satisfied.

Let us verify that $\Sigma_\rho(x_o,r)$ is non-empty for any $\rho \in (0,1)$, $x_o \in \Sigma$ and $r > 0$, which will imply the condition (ii) of Definition 7.8. Without loss of generality, we can assume that $x_o = 0$ and that Σ is a subset of the hyperplane $\{x_n = 0\}$. Then $\Sigma_\rho(0,r)$ contains the interval $[-r,-\rho r] \cup [\rho r, r]$ on the axis x_n and, hence, is non-empty.

Let us now verify the condition (iii) of Definition 7.8, that is, $\widehat{\Sigma}_\rho(0,r)$ is path connected for any $\rho \in (0,1)$ and $r > 0$. The intersection of $\widehat{\Sigma}_\rho(0,r)$ with the axis x_n is the interval $I_r := [-r,r]$. Let us verify that any point $z \in \widehat{\Sigma}_\rho(0,r)$ can be connected by a continuous path in $\widehat{\Sigma}_\rho(0,r)$ to a point in I_r, which will imply the path connectedness of $\widehat{\Sigma}_\rho(0,r)$.

By the definition of $\widehat{\Sigma}_\rho(0,r)$, we have

$$z \in \widehat{\Sigma}_\rho(0,r) \Leftrightarrow |z| \le r \ \text{ and } \ d(z,\Sigma) \ge \rho|z|.$$

Fix some $z \in \widehat{\Sigma}_\rho(0,r)$ and choose $u, v \in \mathbb{R}^n$ so that u lies in the subspace $\{x_n = 0\}$, v lies on the axis x_n, $|u| = |v| = |z|$, and

$$z = u \sin\phi + v \cos\phi$$

for some $\phi \in [0, \pi/2]$ (see Fig. 6).

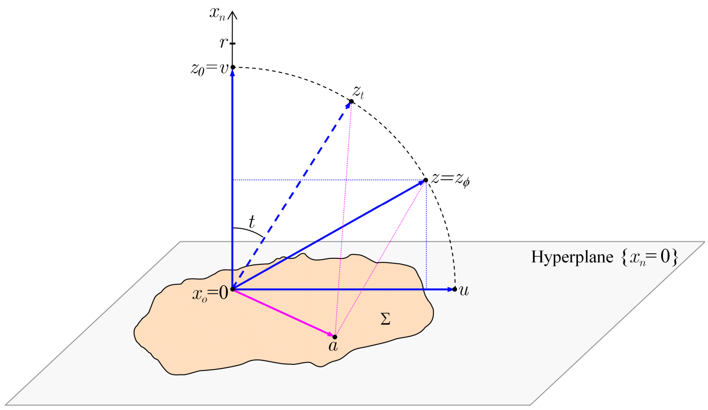

Figure 6: A path in $\widehat{\Sigma}_\rho(0,r)$ that connects $z \in \widehat{\Sigma}_\rho(0,r)$ and $v \in I_r$.

In fact, v is obtained by rotating z towards the axis x_n by an angle ϕ, and u is obtained by rotating z in the opposite direction by the angle $\pi/2 - \phi$. Since $|v| = |z| \le r$, we obtain that $v \in I_r$. For any $t \in [0, \phi]$ set

$$z_t = u \sin t + v \cos t$$

so that $z_0 = v$ and $z_\phi = z$ (see Fig. 6). Let us verify that $z_t \in \widehat{\Sigma}_\rho(0,r)$ for any $t \in [0, \phi]$. Firstly, we have

$$|z_t|^2 = |u|^2 \sin^2 t + 2u \cdot v \sin t \cos t + |v|^2 \cos^2 t = |z|^2 \le r^2$$

so that $|z_t| \le r$. Secondly, we need to verify that $d(z_t, \Sigma) \ge \rho |z_t|$, which is equivalent to

$$d(z_t, a) \ge \rho |z| \quad \text{for all } a \in \Sigma. \tag{7.26}$$

To show (7.26), for any $a \in \Sigma$, we have

$$\begin{aligned} d(z_t, a)^2 &= |u \sin t + v \cos t - a|^2 \\ &= |u|^2 \sin^2 t + 2u \cdot v \sin t \cos t + |v|^2 \cos^2 t - 2a \cdot u \sin t - 2a \cdot v \cos t + |a|^2 \\ &= |z|^2 + |a|^2 - 2a \cdot u \sin t. \end{aligned}$$

If $a \cdot u \le 0$, then the last formula implies $d(z_t, a)^2 \ge |z|^2 \ge \rho^2 |z|^2$. If $a \cdot u > 0$, then

$$d(z_t, a)^2 \ge |z|^2 + |a|^2 - 2a \cdot u \sin\phi = d(z, a)^2 \ge d(z, \Sigma)^2 \ge \rho^2 |z|^2,$$

whence (7.26) follows. Hence, $\{z_t\}$ is a continuous path in $\widehat{\Sigma}_\rho(0,r)$ that connects the points $z = z_\phi$ and $v = z_0 \in I_r$. □

For any set $\Sigma \subset \mathbb{R}^n$ and any $t > 0$ denote

$$\Sigma_t = \{x \in \mathbb{R}^n : d(x, \Sigma) < t\}.$$

Here is our main result in this section.

Proposition 7.15. *Let Σ be a non-empty closed subset of a hyperplane in $\mathbb{R}^n$. Assume that, for any $x_o \in \Sigma$ and $r \geq t > 0$,*

$$\mu\left(\Sigma_t \cap B\left(x_o, r\right)\right) \leq c r^{\alpha} t^{n-\alpha}, \tag{7.27}$$

for some $c > 0$ and $\alpha \in (0, n)$. Then the weight $w(x) = d(x, \Sigma)^{-\sigma}$ is admissible whenever σ satisfying

$$0 \leq \sigma < n - \alpha. \tag{7.28}$$

Consequently, if

$$0 \leq \sigma < n - \max\{\alpha, 2\}, \tag{7.29}$$

then the following weighted Hardy's inequality

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^2 d(x, \Sigma)^{\sigma}} dx \leq C \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{d(x, \Sigma)^{\sigma}} dx \tag{7.30}$$

holds for all $f \in W^{1,2} \cap C_c(\mathbb{R}^n)$, where the constant C depends only on c, n, α and σ.

Proof. By Lemma 7.14, Σ is ρ-accessible for any $\rho \in (0,1)$, so that conditions (i) and (ii) of Definition 7.9 are satisfied with $a(r) = r^{-\sigma}$. Let us verify the condition (iii) of this definition, that is, (7.20).

For any $x_o \in \Sigma$ and $r > 0$, we set $r_k = r 2^{-k}$ and obtain

$$\begin{aligned}
\mu_w\left(B\left(x_o, r\right)\right) &= \int_{B(x_o,r)} \frac{dx}{d(x,\Sigma)^{\sigma}} = \int_{B(x_o,r)\cap\Sigma_r} \frac{dx}{d(x,\Sigma)^{\sigma}} \\
&= \sum_{k=0}^{\infty} \int_{B(x_o,r)\cap(\Sigma_{r_k}\setminus\Sigma_{r_{k+1}})} \frac{dx}{d(x,\Sigma)^{\sigma}} \\
&\leq \sum_{k=0}^{\infty} \mu\left(B\left(x_o, r\right)\cap\Sigma_{r_k}\right) r_{k+1}^{-\sigma} \\
&\leq \sum_{k=0}^{\infty} c r^{\alpha} r_k^{n-\alpha} r_{k+1}^{-\sigma} = \sum_{k=0}^{\infty} 2^{\sigma} c r^{n-\sigma} \left(2^{-k}\right)^{n-\alpha-\sigma} = C r^{-\sigma} |B\left(x_o, r\right)|,
\end{aligned}$$

where $C = C(c, n, \alpha, \sigma)$ and we have used (7.28), that is, $n - \alpha - \sigma > 0$. Hence, (7.20) is verified.

Since $\mathbb{R}^n$ satisfies (**RVD**) with the lower volume dimension n, we see that under the condition (7.29) all the hypotheses of Theorem 7.12 are satisfied, and we obtain (7.23), which is equivalent to (7.30) for $x_o = 0$. □

Remark 7.16. It is easy to see that (7.27) holds provided that Σ satisfies the following condition:

$$\Sigma \cap B\left(x_o, r\right) \text{ can be covered by at most } c\left(\frac{r}{t}\right)^{\alpha} \text{ Euclidean balls of radius } t, \tag{7.31}$$

for some $\alpha, c > 0$ and for all $x_o \in \Sigma$ and $r \geq t > 0$. The infimum of all α for which (7.31) is satisfied is called the *Assouad dimension* of Σ and is denoted by $\dim_A \Sigma$. Hence, the condition (7.31) implies $\dim_A \Sigma \leq \alpha$, and the condition (7.29) can be restated as follows:

$$0 \leq \sigma < n - \max\{\dim_A \Sigma, 2\}. \tag{7.32}$$

For example, if Σ is a Lipschitz curve in $\mathbb{R}^n$ then (7.31) is satisfied with $\alpha = 1$.

Example 7.17. Let Σ be a non-empty closed subset of $\mathbb{R}^l \subset \mathbb{R}^n$ with $2 \leq l < n$. Then (7.31) is satisfied with $\alpha = l$. Hence, Hardy's inequality (7.30) is satisfied provided

$$\sigma < n - l. \tag{7.33}$$

Let $\Sigma = \mathbb{R}^l$ be a subspace of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, set $x = x' + x''$, where $x' \in \mathbb{R}^l$ and $x'' \in \mathbb{R}^{n-l}$. Then $d(x, \Sigma) = |x''|$, and (7.30) becomes

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^2 |x''|^\sigma} dx \le C \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{|x''|^\sigma} dx. \tag{7.34}$$

Let us compare (7.34) with the result of [59, Theorem 3.4] mentioned in Example 7.4: for any real $\sigma \neq n - l - 2$ and any $f \in C_c^\infty(\{|x''| > 0\})$ the following weighted Hardy's inequality holds:

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x''|^{\sigma+2}} dx \le c \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{|x''|^\sigma} dx, \tag{7.35}$$

where $c = 4/(n - l - 2 - \sigma)^2$ (cf. (7.3)). The inequality (7.35) is obviously sharper than (7.34). However, the range of f in (7.35) is smaller, and (7.35) fails for $\sigma = n - l - 2$, while (7.34) is valid for this σ.

There is a number of previously known Hardy's inequalities involving distance to a surface $\Sigma \subset \mathbb{R}^n$ (see, for example, [2, 4, 29, 62]), but none of them works with an arbitrary closed set $\Sigma \subset \mathbb{R}^{n-1}$ as in Proposition 7.15 (see also Example 7.19 below).

Remark 7.18. Assume now that Σ satisfies the following condition:

$$\Sigma \cap B(x_o, r) \text{ can be covered by} \le N \text{ Euclidean balls of radius } r/M \text{ centered at } \Sigma, \tag{7.36}$$

for some $N, M \ge 1$ and for all $x_o \in \Sigma$ and $r > 0$. By iterating this condition, we obtain (7.31) and, hence, (7.27) with

$$\alpha = \frac{\log N}{\log M}.$$

Example 7.19. Let $\Sigma = SC(a, b, k)$ be a generalized Sierpinski carpet from Example 5.12 (bounded or unbounded) that is based on a unit cube in $\mathbb{R}^k$, $k \ge 2$. It suffices to have (7.36) for the constituent cubes of $SC(a, b, k)$ instead of balls $B(x_o, r)$. If Q is such a cube of side $r = a^{-m}$ then $Q \cap \Sigma$ it is covered by N cubes of side $t = a^{-(m+1)}$ where $N = a^k - b^k$. It follows that (7.31) is satisfied with $\alpha = \frac{\log N}{\log a}$ that is the Hausdorff dimension of $SC(a, b, k)$. Therefore, considering $\Sigma = SC(a, b, k)$ as a subset of $\mathbb{R}^n$ with $n > k$, we obtain the Hardy's inequality (7.30) for all

$$0 < \sigma < n - \max\{\alpha, 2\}.$$

Since $\alpha < k$, this range of σ is larger than that of (7.33) whenever $k \ge 3$.

Remark 7.20. In the setting of Proposition 7.15, we have by Lemma 7.10

$$\mu_w(B(x, r)) \simeq r^n (d(x, \Sigma) + r)^{-\sigma}.$$

By Theorem 7.12, the Green function $G^{(w)}(x, y)$ of the Dirichlet form $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ exists and satisfies $(\mathbf{G})_2$ with respect to the measure μ_w, which yields for $r = |x - y|$

$$G^{(w)}(x, y) \simeq \frac{r^2}{\mu_w(B(x, r))} \simeq r^{2-n}(d(x, \Sigma) + r)^{-\sigma}.$$

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