Hardy inequality and heat semigroup estimates for Riemannian manifolds with singular data

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2 June 2011

Abstract

Upper bounds are obtained for the heat content of an open set D in a geodesically complete Riemannian manifold M with Dirichlet boundary condition on ∂D, and non-negative initial condition. We show that these upper bounds are close to being sharp if (i) the Dirichlet-Laplace-Beltrami operator acting in $L^2(D)$ satisfies a strong Hardy inequality with weight δ^2, (ii) the initial temperature distribution, and the specific heat of D are given by $\delta^{-\alpha}$ and $\delta^{-\beta}$ respectively, where δ is the distance to ∂D, and $1 < \alpha < 2, 1 < \beta < 2$.

Mathematics Subject Classification (2000): 58J32; 58J35; 35K20.

Keywords: Hardy inequality, heat content, singular data.

[*]Partially supported by Project MTM2009-07756 (Spain)

[†]Partially supported by SFB701 (Germany)

[‡]Supported by National Science Foundation Grant PHY-0757791

1 Introduction

Let D be a smooth, connected, m- dimensional Riemannian manifold and let Δ be the Laplace-Beltrami operator on D. It is well known (see [11], [14]) that the heat equation

$$\Delta u = \frac{\partial u}{\partial t}, \quad x \in D, \quad t > 0, \tag{1}$$

has a unique minimal positive fundamental solution $p(x, y; t)$ where $x \in D$, $y \in D$, $t > 0$. This solution, the Dirichlet heat kernel for D, is symmetric in x, y, strictly positive, jointly smooth in $x, y \in D$ and $t > 0$, and it satisfies the semigroup property

$$p(x, y; s + t) = \int_D p(x, z; s)p(z, y; t)dz, \tag{2}$$

for all $x, y \in D$ and $t, s > 0$, where dz is the Riemannian measure on D. Equation (1) with the initial condition

$$u(x; 0^+) = \psi(x), \quad x \in D, \tag{3}$$

has a solution

$$u_\psi(x; t) = \int_D p(x, y; t)\psi(y)dy, \tag{4}$$

for any function ψ on D from a variety of function spaces like $C_b(D)$ or $L^p(D)$, $1 \le p < \infty$. Note that $u_\psi \in C_b(D)$ if $\psi \in C_b(D)$ or that $u_\psi \in L^p(D)$ if $\psi \in L^p(D)$. Initial condition (3) is understood in the sense that $u_\psi(\cdot; t) \to \psi(\cdot)$ as $t \to 0^+$, where the convergence is appropriate for the function space of initial conditions. For example, if $\psi \in C_b(D)$ then the convergence is locally uniform, or if $\psi \in L^p(D)$, $1 \le p < \infty$ then the convergence is in the norm of $L^p(D)$. In general, (4) is not the unique solution of (1)-(3). However, it has the following distinguished property: if $\psi \ge 0$ then u_ψ is the minimal non-negative solution of that problem (and if ψ is signed then $u_\psi = u_{\psi_+} - u_{\psi_-}$). If D is an open subset of another Riemannian manifold M and if the boundary ∂D of D in M is smooth then the minimality property of u_ψ implies that, for any $t > 0$,

$$\lim_{x \to \partial D} u_\psi(x; t) = 0. \tag{5}$$

If ∂D is non-smooth then (5) can still be understood in a weak sense. Expression (4) makes sense for any non-negative measurable function ψ on D, provided the value $+\infty$ is allowed for u_ψ. It is known that if $u_\psi \in L^1_{loc}(D \times \mathbb{R}_+)$ then u_ψ is a smooth function in $D \times \mathbb{R}_+$ and it solves (1) (see p. 201 in [14]). For any two non-negative measurable functions ψ_1, ψ_2 on D, we define for $t > 0$

$$Q_{\psi_1,\psi_2}(t) = \iint_{D\times D} p(x, y; t)\psi_1(x)\psi_2(y)dxdy. \tag{6}$$

Using the properties of the Dirichlet heat kernel we have for $0 < s < t$

$$Q_{\psi_1,\psi_2}(t) = \int_D u_{\psi_1}(x; s)u_{\psi_2}(x; t - s)dx. \tag{7}$$

Assuming that D is an open subset of a complete Riemannian manifold M, $Q_{\psi_1,\psi_2}(t)$ has the following physical interpretation: it is the amount of heat in

D at time t if D has initial temperature distribution ψ_1, and a specific heat ψ_2, while the ∂D is kept at fixed temperature 0.

This function has been subject of a thorough investigation. Its asymptotic behavior for small t is well understood if D has compact closure with C^∞ boundary, and both ψ_1 and ψ_2 are C^∞ on the closure $\overline{D}$ of D. In that case $Q_{\psi_1,\psi_2}(t)$ has an asymptotic series in $t^{1/2}$, and its coefficients are computable in terms of local geometric invariants [2, 12]. No such series are known if D is unbounded, or if either the initial data or ∂D are non-smooth.

In this paper we will obtain upper bounds for the heat content $Q_{\psi_1,\psi_2}(t)$ under quite general assumptions on D and on ψ_1 and ψ_2.

We are particularly interested in the situation where D is a open subset of another manifold M, and where $\psi_1(x)$ and $\psi_2(x)$ blow up as $x \to \partial D$. In order to guarantee finite heat content for $t > 0$, sufficient cooling at ∂D needs to take place. This will be guaranteed by a condition on D, that is formulated in terms of a Hardy inequality. Note that in this setting $Q_{\psi_1,\psi_2}(t)$ may be unbounded as $t \to 0^+$, and one of the interesting points of this study is to obtain the rate of convergence of $Q_{\psi_1,\psi_2}(t)$ to $+\infty$ as $t \to 0^+$.

Given a positive measurable function h on a manifold D, we say that the Dirichlet Laplacian acting in $L^2(D)$ satisfies a strong Hardy inequality with weight h if, for all $w \in C_c^\infty(D)$,

$$\int_D |\nabla w|^2 \geq \int_D \frac{w^2}{h} \,. \tag{8}$$

Here, and in what follows, we put $\int_D f = \int_D f(x)dx$ if this does not cause confusion. We also put $|D| = \int_D 1$, and $\|f\|_p = \left(\int_D |f|^p\right)^{1/p}$. A typical example of a Hardy inequality is when D is an open subset of another manifold M, and

$$h(x) = c^2\delta(x)^2, \tag{9}$$

where $c \geq 2$ is a constant, δ is the distance to the boundary,

$$\delta(x) = \min\{d(x,y) : y \in \partial D\},$$

and $d(x,y)$ is the geodesic distance from x to y on M. Both the validity and applications of Hardy inequalities with weight (9) have been investigated extensively [1], [7], [9], [10], [11], [4]. For example, inequality (8) holds with weight (9) with $c = 4$ if D is simply connected with non-empty boundary in $\mathbb{R}^2$, with $c = 2$ if D is convex in $\mathbb{R}^m$, and for some $c \geq 2$ if D is bounded with smooth boundary in $\mathbb{R}^m$.

In [3] it was shown that if D has finite measure and satisfies the Hardy inequality with weight h, and if ψ is a non-negative measurable function on D, such that, for some $q > 1$,

$$\|\psi h^{1/q}\|_{q/(q-1)} < \infty, \tag{10}$$

then, for all $t > 0$,

$$Q_{\psi,1}(t) \leq \left(\frac{q^2}{4(q-1)}\right)^{1/q} \|\psi h^{1/q}\|_{q/(q-1)} \|1 - u_1(\cdot;t)\|_1^{1/q} t^{-1/q}, \tag{11}$$

where $Q_{1,1}$ is defined by (6) for $\psi_1 = \psi_2 = 1$, that is,

$$Q_{1,1}(t) = \int_D u_1\,(x;t)\,dx = \iint_{D\times D} p(x,y;t)dxdy.$$

A similar estimate holds for arbitrary open sets $D \subset \mathbb{R}^m$, satisfying the Hardy inequality with weight h. If ψ is a non-negative measurable function on D such that, for some $q > 1$,

$$\|\max\{\psi, 1\}h^{1/q}\|_{q/(q-1)} < \infty, \tag{12}$$

then, for all $t > 0$,

$$Q_{\psi,1}(t) \le a(q)\|\psi h^{1/q}\|_{q/(q-1)}\|h^{1/(q(q-1))}\|_q t^{-1/(q-1)}, \tag{13}$$

where

$$a(q) = 4^{-1/q}\left(\frac{q}{q-1}\right)^{(2q-1)/(q(q-1))}. \tag{14}$$

Below we give a sufficient condition for the finiteness of $Q_{\psi_1,\psi_2}(t)$ for all $t > 0$, and reduce the problem of finding upper bounds for $Q_{\psi_1,\psi_2}(t)$ to the case $\psi_1 = \psi_2$.

Theorem 1. *Let ψ_1 and ψ_2 be non-negative and Borel measurable on a manifold D.*

(i) If $Q_{\psi_i,\psi_i}(t) < \infty, i = 1, 2$, for all $t > 0$, then $Q_{\psi_1,\psi_2}(t) < \infty$ for all $t > 0$, and

$$Q_{\psi_1,\psi_2}(t) \le \left(Q_{\psi_1,\psi_1}(t)Q_{\psi_2,\psi_2}(t)\right)^{1/2},\ t > 0. \tag{15}$$

(ii) If $Q_{\psi_i,1}(t) < \infty, i = 1, 2$, for all $t > 0$, and if

$$c_t := \sup_{x\in D} p(x, x; t) < \infty, \quad t > 0, \tag{16}$$

then

$$Q_{\psi_1,\psi_2}(t) \le c_{t/3}Q_{\psi_1,1}(t/3)Q_{\psi_2,1}(t/3) < \infty,\ t > 0.$$

Our main results are the following three theorems, in which we assume that D is a Riemannian manifold that satisfies the Hardy inequality with some weight h, and ψ is a non-negative measurable function on D. In particular we do not assume any smoothness conditions on ∂D, nor do we assume that D has finite measure or that D is bounded.

Theorem 2. *If $|D| < \infty$, and if there exists $1 < q \le 2$ such that*

$$\|\psi h^{1/q}\|_{q/(q-1)} < \infty, \tag{17}$$

then, for all $t > 0$,

$$Q_{\psi,\psi}(t) \le \frac{q^{(4-q)/q}}{(2(q-1))^{2/q}}\|\psi h^{1/q}\|^2_{q/(q-1)}\|1 - u_1\,(\cdot;t)\|_1^{(2-q)/q}t^{-2/q}. \tag{18}$$

Theorem 3. *If $1 < q \le 2$ is such that* (17) *holds and that*

$$\|h^{1/q}\|_{q/(q-1)} < \infty,$$

then

$$Q_{\psi,\psi}(t) \le b(q)\ \|\psi h^{1/q}\|^2_{q/(q-1)}\ \|h^{1/q}\|^{(2-q)/(q-1)}_{q/(q-1)} t^{-1/(q-1)},\ t > 0, \tag{19}$$

where

$$b(q) = 2^{(4-3q)/(q(q-1))} \left(\frac{q}{q-1}\right)^{(q^2-4q+2)/(q(1-q))}. \tag{20}$$

Theorem 4. *If $0 \le r \le 2$, and $1 < q \le 2$ are such that*

$$||\psi^r||_q < \infty,$$

and

$$\|\psi^{2-r} h^{1/q}\|_{(q-1)/q} < \infty,$$

then

$$Q_{\psi,\psi}(t) \le \left(\frac{q}{4(q-1)}\right)^{1/q} \|\psi^r\|_q \|\psi^{2-r} h^{1/q}\|_{q/(q-1)} t^{-1/q},\ t > 0. \tag{21}$$

In Theorem 5 in Section 3 we use the bounds of Theorems 2 and 4 together with (15) to obtain an upper bound for the heat content of D, when D satisfies a Hardy inequality with weight (9), and $\psi_1(x) = \delta(x)^{-\alpha}$ and $\psi_2(x) = \delta(x)^{-\beta}$, where $\alpha, \beta \in (1, 2)$. Even though the bounds in e.g. 2 and 4 look very different, both of them are needed to cover the maximal range of α and β in Theorem 5.

Theorem 2 has a curious consequence. We claim that if a manifold D has finite measure $|D|$, and is stochastically complete then no Hardy inequality holds on D (which confirms the philosophy that the Hardy inequality corresponds to cooling that comes from the boundary). Indeed, stochastic completeness means that $u_1 \equiv 1$. In this case, $\|1 - u_1(\cdot; t)\|_1 = 0$ so that we obtain from (18) that $Q_{\psi,\psi}(t) = 0$ whenever function ψ satisfies the condition (17) for some $q \in (1, 2)$. However, if h is finite then it is easy to construct a non-trivial function ψ that satisfies (17): choose any measurable set S with finite positive measure such that h is bounded on S, and let $\psi = 1_S$. Then (17) holds with any $q > 1$ while $Q_{\psi,\psi}(t) > 0$ so that we obtain contradiction. Of course, without the finiteness of $|D|$, the Hardy inequality may hold on stochastically complete manifolds like $\mathbb{R}^m \setminus \{0\}$.

This paper is organized as follows. In Section 2 we will prove Theorems 1, 2, 3 and 4. In Section 3 we will state and prove Theorem 5. Finally in Section 4 we obtain very refined asymptotics in the special case of the ball in $\mathbb{R}^3$ with $\psi_1(x) = \delta(x)^{-\alpha}, \alpha < 2, \psi_2(x) = \delta(x)^{-\beta}, \beta < 2$, and $\alpha + \beta > 3$ (Theorem 7). This special case shows that the bound obtained in Theorem 5 is close to being sharp. Moreover it suggests formulae for the first few terms in the asymptotic series of a compact Riemannian manifold D with the singular data above.

2 Proofs of Theorems 1, 2, 3 and 4

Proof of Theorem 1. In both parts, it suffices to prove the claims for non-negative functions ψ_1, ψ_2 from $L^2(D)$. Arbitrary non-negative measurable functions ψ_1, ψ_2 can be approximated by monotone increasing sequences of non-negative functions from $L^2(D)$, whence the both claims follow by the monotone convergence theorem.

To prove part (i) we use symmetry and the semigroup property, and obtain by (7) for $s = t/2$ that

$$\begin{aligned} Q_{\psi_1,\psi_2}(t) &= \int_D u_{\psi_1}(x;t/2)u_{\psi_2}(x;t/2)dx \\ &\le \left(\int_D u^2_{\psi_1}(x;t/2)dx\right)^{1/2}\left(\int_D u^2_{\psi_2}(x;t/2)dx\right)^{1/2} \\ &= \left(Q_{\psi_1,\psi_1}(t)Q_{\psi_2,\psi_2}(t)\right)^{1/2}. \end{aligned}$$

It follows from (2) and (16) that

$$p(x,y;t) \le (p(x,x;t)p(y,y;t))^{1/2} \le c_t. \tag{22}$$

To prove part (ii) we have by (22) that

$$\begin{aligned} p(x,y;t) &= \iint_{D\times D} p(x,z_1;t/3)p(z_1,z_2;t/3)p(z_2,y;t/3)dz_1dz_2 \\ &\le c_{t/3}u_1(x;t/3)u_1(y;t/3). \end{aligned} \tag{23}$$

This together with definition (6) completes the proof. □

For the proofs of Theorems 2, 3, 4, choose a sequence $\{D_n\}$ that consists of precompact open subsets of D with smooth boundaries such that $\overline{D}_n \subset D_{n+1}$ and $\bigcup_n D_n = D$. Obviously, Hardy inequality (8) remains true in any D_n with the same weight h, because $C_c^\infty(D_n) \subset C_c^\infty(D)$. Moreover, we claim that (8) holds for any function $w \in C(\overline{D}_n) \cap C^1(D_n)$ that satisfies the boundary condition $w|_{\partial D_n} = 0$. Indeed, if $\int_{D_n} |\nabla w|^2 = \infty$ then (8) is trivially satisfied. If $\int_{D_n} |\nabla w|^2 < \infty$ then w belongs to the Sobolev space $W^{1,2}(D_n)$. Extend function w to D_{n+1} by setting $w = 0$ in $D_{n+1} \setminus \overline{D}_n$. Due to the boundary condition $w|_{\partial D_n} = 0$, we obtain that $w_n \in W^{1,2}(D_{n+1})$. Since w is compactly supported in D_{n+1}, it follows that $w \in W_0^{1,2}(D_{n+1})$ where $W_0^{1,2}(\Omega)$ is the closure $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$. Since the Hardy inequality (8) holds for functions from $C_c^\infty(D_{n+1})$, passing to the limit in $W^{1,2}(D_{n+1})$ and using Fatou's lemma, we obtain that w also satisfies (8).

Assume for a moment that the statements of the theorems have been proved in each domain D_n. Then one can take the limit in (18), (19), (21) as $n \to \infty$, and obtain the statements for D. Indeed, the left hand side of these inequalities is $Q^{D_n}_{\psi,\psi}(t) = \iint_{D_n\times D_n} p_{D_n}(x,y;t)\psi(x)\psi(y)dxdy$, where p_{D_n} is the Dirichlet heat kernel for D_n. This converges to $Q^D_{\psi,\psi}(t)$ as $n \to \infty$. The right hand sides of (18), (19), (21) contain various $L^p(D_n)$-norms that can be estimated from above by the $L^p(D)$-norms. The only exception is the term $\|1 - \int_{D_n} p_{D_n}(\cdot,y;t)dy\|_1$ in (18) that is decreasing as $n \to \infty$. If $|D| < \infty$ then $1 \in L^1(D)$ so that the passage to the limit is justified by the dominated convergence theorem.

Hence, it suffices to prove each of the statements for D_n instead of D. Renaming D_n back to D, we assume in all three proofs that D is a precompact open domain with smooth boundary in M.

Another observation is that all inequalities (18), (19), (21) survive the increasing monotone limits in ψ. So it suffices to prove them when ψ is bounded and has a compact support in D, which will be assumed below. Furthermore, since all the statements of Theorems 2, 3, 4 are homogeneous with respect to ψ, we can assume that $0 \leq \psi \leq 1$. If $\psi \equiv 0$ then there is nothing to prove; hence, we assume that ψ is non-trivial. Then $u_\psi(x;t)$ is smooth and bounded in $\overline{D} \times (0, +\infty)$ and positive in $D \times (0, +\infty)$.

Proof of Theorem 2. Let ν be the outwards normal vector field on ∂D. Using the Green's formula, we obtain

$$\begin{aligned} -\frac{d}{dt}\int_D u_\psi^q &= -q\int_D u_\psi^{q-1}\frac{\partial u_\psi}{\partial t} \\ &= -q\int_D u_\psi^{q-1}\Delta u_\psi \\ &= -q\int_{\partial D} u_\psi^{q-1}\frac{\partial u_\psi}{\partial \nu} + q\int_D \left(\nabla u_\psi^{q-1}, \nabla u_\psi\right) \\ &= q(q-1)\int_D u_\psi^{q-2}\left|\nabla u_\psi\right|^2, \end{aligned} \tag{24}$$

where we have used that $q > 1$ and, hence, $u_\psi^{q-1} = 0$ on ∂D. Observing that $u_\psi^{q/2} \in C\left(\overline{D}\right) \cap C^1(D)$,

$$\left|\nabla u_\psi^{q/2}\right|^2 = \frac{q^2}{4}u_\psi^{q-2}\left|\nabla u_\psi\right|^2,$$

and applying the Hardy inequality (8) to $u^{q/2}$, we obtain that

$$-\frac{d}{dt}\int_D u_\psi^q = \frac{4(q-1)}{q}\int_D |\nabla(u_\psi^{q/2})|^2 \geq \frac{4(q-1)}{q}\int_D \frac{u_\psi^q}{h}. \tag{25}$$

By Hölder's inequality we have that

$$\begin{aligned} Q_{\psi,\psi}(t) &= \int_D u_\psi \psi \\ &\leq \left(\int_D \left(\frac{u_\psi}{h^{1/q}}\right)^q\right)^{1/q}\left(\int \left(\psi h^{1/q}\right)^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \\ &= \left(\int_D \frac{u_\psi^q}{h}\right)^{1/q}\|\psi h^{1/q}\|_{q/(q-1)}. \end{aligned} \tag{26}$$

By (25) and (26) we conclude that

$$-\frac{d}{dt}\int_D u_\psi^q \geq \frac{4(q-1)}{q}\|\psi h^{1/q}\|_{q/(q-1)}^{-q}\left(Q_{\psi,\psi}(t)\right)^q. \tag{27}$$

Note that the function $t \mapsto Q_{\psi,\psi}(t) = \|u_\psi(\cdot;t/2)\|_2^2$ is decreasing in t, which, for example, follows from (24) with $q = 2$. Integrating differential inequality

(27) with respect to t over the interval $[t, 2t]$ gives that

$$\int_D u_\psi^q \geq \frac{4(q-1)}{q} \|\psi h^{1/q}\|_{q/(q-1)}^{-q} \left(Q_{\psi,\psi}(2t)\right)^q t. \tag{28}$$

On the other hand, using $1 < q \leq 2$ and the Hölder inequality, we obtain

$$\int_D u_\psi^q = \int_D u_\psi^{2-q} u_\psi^{2q-2} \leq \left(\int_D u_\psi\right)^{2-q} \left(\int_D u_\psi^2\right)^{q-1}$$

that is,

$$\int_D u_\psi^q \leq \left(Q_{\psi,1}(t)\right)^{2-q} \left(Q_{\psi,\psi}(2t)\right)^{q-1}. \tag{29}$$

Combining (28) and (29) yields

$$Q_{\psi,\psi}(2t) \leq \frac{q}{4(q-1)} ||\psi h^{1/q}||_{q/(q-1)}^q \left(Q_{\psi,1}(t)\right)^{2-q} t^{-1}. \tag{30}$$

Estimating $Q_{\psi,1}$ by (11), we obtain

$$Q_{\psi,\psi}(2t) \leq \frac{q^{(4-q)/q}}{(4(q-1))^{2/q}} \|\psi h^{1/q}\|_{q/(q-1)}^2 \|1 - u_1(\cdot;t)\|_1^{(2-q)/q} t^{-2/q},$$

which completes the proof. □

Proof of Theorem 3. Since $\psi \leq 1$ we have that (12) is satisfied. We obtain by (13) and (30) that

$$Q_{\psi,\psi}(2t) \leq \frac{q}{4(q-1)} a(q)^{2-q} \|\psi h^{1/q}\|_{q/(q-1)}^2 \; \|h^{1/q}\|_{q/(q-1)}^{(2-q)/(q-1)} t^{-1/(q-1)}.$$

This completes the proof of Theorem 3 since, by (14) and (20),

$$2^{1/(q-1)} \frac{q}{4(q-1)} a(q)^{2-q} = b(q).$$

□

Proof of Theorem 4. By the arithmetic-geometric mean inequality, we have

$$\psi(x)\psi(y) \leq \frac{1}{2} \left(\psi(x)^r \psi(y)^{2-r} + \psi(x)^{2-r} \psi(y)^r\right).$$

By non-negativity and symmetry of the Dirichlet heat kernel

$$Q_{\psi,\psi}(t) \leq \int_D u_{\psi^r} \psi^{2-r}. \tag{31}$$

Next, Hölder's inequality yields

$$\int_D u_{\psi^r} \psi^{2-r} \leq \left(\int_D u_{\psi^r}^q \frac{1}{h}\right)^{1/q} \|\psi^{2-r} h^{1/q}\|_{q/(q-1)}. \tag{32}$$

By (25) we have

$$-\frac{d}{dt} \int_D u_{\psi^r}^q \geq \frac{4(q-1)}{q} \int_D u_{\psi^r}^q \frac{1}{h}. \tag{33}$$

Combining (31), (32), (33) we obtain that

$$(Q_{\psi,\psi}(t))^q \leq -\frac{q}{4(q-1)}\frac{d}{dt}\left(\int_D u^q_{\psi^r}\right)\|\psi^{2-r}h^{1/q}\|^q_{q/(q-1)}.$$

Since the function $t \mapsto Q_{\psi,\psi}(t)$ is decreasing in t, we obtain by integrating the differential inequality (33) with respect to t over the interval $[0,t]$ that

$$t\,(Q_{\psi,\psi}(t))^q \leq \frac{q}{4(q-1)}\left(\int_D \psi^{rq}\right)\|\psi^{2-r}h^{1/q}\|^q_{q/(q-1)},$$

and (21) follows. □

3 Singular initial temperature and singular specific heat

Below we make some further hypothesis on the geometry of D, and obtain an upper bound for the heat content for a wide class of geometries using Theorems 2 and 4, and (15), if the initial temperature distribution and specific heat are given by $\delta^{-\alpha}, 1<\alpha<2$, and $\delta^{-\beta}, 1<\beta<2$ respectively.

Theorem 5. *Let D be an open set in a smooth complete m-dimensional Riemannian manifold M, and suppose that*

i. The Ricci curvature on M is non-negative.

ii. For $x\in D$,

$$\psi_\alpha(x)=\delta(x)^{-\alpha}.$$

iii. D has finite inradius i.e. $\rho_D=\sup\{\delta(x):x\in D\}<\infty$.

iv. There exist constants $\kappa_D<\infty, d\in[m-1,m)$ such that

$$\int_{\{x\in D:\delta(x)<\rho\}} 1 \leq \kappa_D\rho^{m-d},\ \ 0<\rho\leq\rho_D. \tag{34}$$

v. The strong Hardy inequality (8) holds with (9) for some $c\geq 2$.

If $1<\alpha<2, 1<\beta<2$, and if $\epsilon>0$ then

$$Q_{\psi_\alpha,\psi_\beta}(t)=O(t^{-\epsilon+(m-d-\alpha-\beta)/2}),\ \ t\to 0. \tag{35}$$

Proof. Note that (iii) and (iv) in Theorem 5 imply that $|D|\leq\kappa_D\rho_D^{m-d}<\infty$. By (15) it suffices to prove (35) in the special case $\alpha=\beta$ with $1<\alpha<2$. In order to estimate $\|1-u_1(\cdot;t)\|_1$ in Theorem 2 we rely on the following lower bound for u_1 (Lemma 5 in [5]).

Lemma 6. *Let M be a smooth, geodesically complete Riemannian manifold with non-negative Ricci curvature, and let D be an open subset of M with boundary ∂D. Then for $x\in D, t>0$*

$$u_1(x;t)\geq 1-2^{(2+m)/2}e^{-\delta(x)^2/(8t)}.$$

To prove (35) we first consider the case

$$(2+m-d)/2<\alpha<2. \tag{36}$$

This set of α's is non-empty since $d\in[m-1,m)$. By (9) we have that

$$\|\psi_\alpha h^{1/q}\|_{q/(q-1)}=c^{2/q}\left(\int_D \delta^{(2-q\alpha)/(q-1)}\right)^{(q-1)/q}. \tag{37}$$

Denote the left hand side of (34) by $\omega_D(\rho)$. Then we can write the right hand side of (37) as

$$c^{2/q}\left(\int_{\mathbb{R}^+}\rho^{(2-q\alpha)/(q-1)}\omega_D(d\rho)\right)^{(q-1)/q}. \tag{38}$$

An integration by parts, using (36) shows that (38) is finite for

$$q<\frac{2-m+d}{\alpha-m+d}. \tag{39}$$

Since α satisfies (36), we have that the right hand side of (39) is in $(1,2)$. We now choose $\epsilon>0$ such that

$$\frac{2-m+d}{\alpha-m+d}(1+\epsilon)^{-1}\in(1,2), \tag{40}$$

and choose q equal to the left hand side of (40). By Lemma 6 and (34) we have that for $t\to 0$

$$\begin{aligned}\|1-u_1(\cdot;t)\|_1&=\int_D(1-u(x;t))dx\\&\le 2^{(m+2)/2}\int_D e^{-\delta^2/(8t)}\\&\le 2^{(m+2)/2}\int_{\mathbb{R}^+}e^{-\rho^2/(8t)}\omega_D(d\rho)\\&=2^{(m+2)/2}e^{-\rho_D^2/(8t)}|D|+2^{(m-2)/2}\kappa_D t^{-1}\int_0^{\rho_D}\rho^{m-d+1}e^{-\rho^2/(8t)}d\rho\\&=O(t^{(m-d)/2}).\end{aligned} \tag{41}$$

By Theorem 2 and (37)-(41) we find that for all α satisfying (36) and all $\epsilon>0$ satisfying (40)

$$Q_{\psi_\alpha,\psi_\alpha}(t)=O(t^{-\epsilon(\alpha-m+d)+(m-d-2\alpha)/2}),\ t\to 0. \tag{42}$$

We conclude that (35) holds for all $\alpha=\beta$ satisfying (36), and all $\epsilon>0$.

Next consider the case

$$1<\alpha<(2+m-d)/2. \tag{43}$$

This set of α's is again non-empty since $d\in[m-1,m)$. By (34) we have that

$$\|\psi^r\|_q=\left(\int_{\mathbb{R}^+}\omega_D(d\rho)\rho^{-\alpha rq}\right)^{1/q}<\infty \tag{44}$$

for

$$\alpha r q < m - d, \tag{45}$$

and

$$\|\psi^{2-r} h^{1/q}\|_{q/(q-1)} = \left(\int_{\mathbb{R}+} \omega_D(d\rho) \rho^{(2-\alpha(2-r)q)/(q-1)} \right)^{(q-1)/q} < \infty \tag{46}$$

for

$$\frac{\alpha q(2-r) - 2}{q-1} < m - d. \tag{47}$$

The optimal choice for r is henceforth given by

$$r = 2(\alpha q - 1)\alpha^{-1} q^{-2}. \tag{48}$$

By (43) we also have that $\alpha > 1$. Hence $r \in (0, 2)$. The requirements under (45) and (47) become with this choice of r that

$$q < 2(2\alpha + d - m)^{-1}. \tag{49}$$

Since α satisfies (43), the right hand side of (49) is in $(1, 2)$. We now choose $\epsilon > 0$ such that

$$2((2\alpha + d - m)(1 + 2\epsilon))^{-1} \in (1, 2), \tag{50}$$

and choose q equal to the left hand side of (50). By Theorem 4 and (44)-(49) we find that for all α satisfying (43), and all $\epsilon > 0$ satisfying (50)

$$Q_{\psi_\alpha, \psi_\alpha}(t) = O(t^{-\epsilon(2\alpha - m + d) + (m - d - 2\alpha)/2}), \; t \to 0. \tag{51}$$

We conclude that (35) holds for all $\alpha = \beta$ satisfying (43), and all $\epsilon > 0$.

To prove (35) for the limiting case $\alpha = \beta = (2 + m - d)/2 := \alpha_c$ we note that $Q_{\psi,\phi}(t)$ is monotone on the positive cone of non-negative and measurable ψ and ϕ. Let $\alpha = \alpha_c + \epsilon$ where ϵ is such that $\alpha \in (\alpha_c, 2)$. Since

$$\psi_{\alpha_c} \le \rho_D^{\alpha - \alpha_c} \psi_\alpha.$$

we have by (42) that

$$\begin{aligned} Q_{\psi_{\alpha_c}, \psi_{\alpha_c}}(t) &\le \rho_D^{2(\alpha - \alpha_c)} Q_{\psi_\alpha, \psi_\alpha}(t) \\ &\le \rho_D^{2(\alpha - \alpha_c)} O(t^{-\epsilon(\alpha - m + d) + (m - d - 2\alpha)/2}) \\ &= O(t^{-\epsilon(2 + \epsilon + (d - m)/2) + (m - d - 2\alpha_c)/2}). \end{aligned} \tag{52}$$

We conclude that (35) holds for $\alpha = \beta = \alpha_c$, and all $\epsilon > 0$. □

4 The special case calculation for a ball in $\mathbb{R}^3$

In this section we show by means of an example that the upper bound obtained in Theorem 5 is close to being sharp for $\alpha < 2, \beta < 2, \alpha + \beta > 3$.

Theorem 7. *Let $B_a = \{x \in \mathbb{R}^3 : |x| < a\}$. If $\alpha < 2, \beta < 2, \alpha + \beta > 3, J \in \mathbb{N}$ then there exist coefficients $b_0, b_1, \cdots$ depending on α and β only such that for $t \to 0$*

$$Q_{\psi_\alpha,\psi_\beta}(t) = 4\pi c_{\alpha,\beta} a^2 t^{(1-\alpha-\beta)/2} - 4\pi(c_{\alpha-1,\beta} + c_{\alpha,\beta-1}) a t^{(2-\alpha-\beta)/2}$$
$$+ 4\pi c_{\alpha-1,\beta-1} t^{(3-\alpha-\beta)/2} + \sum_{j=0}^{J} b_j a^{3-j-\alpha-\beta} t^{j/2} + O(t^{(J+1)/2}), \quad (53)$$

where

$$c_{\alpha,\beta} = 2^{-\alpha-\beta}\pi^{-1/2}\Gamma((2-\alpha-\beta)/2)$$
$$\times \int_0^1 (\rho^{-\alpha} + \rho^{-\beta})((1-\rho)^{\alpha+\beta-2} - (1+\rho)^{\alpha+\beta-2}) d\rho, \quad (54)$$

and

$$b_0 = -8\pi((\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3))^{-1},$$
$$b_1 = 0,$$
$$b_2 = 8\pi\alpha\beta((\alpha+\beta+1)(\alpha+\beta)(\alpha+\beta-1))^{-1},$$
$$b_3 = 0. \quad (55)$$

We see that the leading term in (53) jibes with (35) since (9) holds for some $c \geq 2$, and (34) holds with $d = m - 1$.

Theorem 7 suggests that for any precompact D with smooth ∂D in M, and for $\alpha < 2, \beta < 2, \alpha + \beta > 3$ and $t \to 0$

$$Q_{\psi_\alpha,\psi_\beta}(t) = c_{\alpha,\beta} \int_{\partial D} t^{(1-\alpha-\beta)/2} - 2^{-1}(c_{\alpha-1,\beta} + c_{\alpha,\beta-1}) \int_{\partial D} L_{gg} t^{(2-\alpha-\beta)/2}$$
$$+ \int_{\partial D} (c_1 L_{gg} L_{hh} + c_2 L_{gh} L_{gh}) t^{(3-\alpha-\beta)/2} + O(1), \quad (56)$$

where c_1 and c_2 are constants depending on α and β only, and which satisfy

$$4c_1 + 2c_2 = c_{\alpha-1,\beta-1},$$

and where L_{gg} is the trace of the second fundamental form on the boundary of ∂D oriented by an inward unit vector field. Since $\int_{\partial B_a} 1 = 4\pi a^2$, $\int_{\partial B_a} L_{gg} = 8\pi a$ and $\int_{\partial B_a} (c_1 L_{gg} L_{hh} + c_2 L_{gh} L_{gh}) = 16\pi c_1 + 8\pi c_2$, we see that (56) holds for the ball in $\mathbb{R}^3$.

The proof of Theorem 7 rests on the following result (pp.237, 367-368 in [8]).

Lemma 8. *Let B_a as in Theorem 7, and let the initial datum be radially symmetric i.e. $\psi_1(x) = f(r)$, where $r = |x|$. Then the solution of (1), (3), (5) is given by*

$$u(x;t) = (4\pi t r^2)^{-1/2} \int_0^a r' f(r') \sum_{n\in\mathbb{Z}} (e^{-(2na-r+r')^2/(4t)} - e^{-(2na+r+r')^2/(4t)}) dr'.$$

To prove Theorem 7 we have by Lemma 8 that

$$Q_{\psi_\alpha,\psi_\beta}(t) =(4\pi/t)^{1/2} \iint_{S_a} rr'(a-r)^{-\alpha}(a-r')^{-\beta} \times \sum_{n\in\mathbb{Z}} (e^{-(2na-r+r')^2/(4t)} - e^{-(2na+r+r')^2/(4t)})drdr', \tag{57}$$

where $S_a = [0,a]\times[0,a]$. Substitution of $a-r=p$ and $a-r'=q$ in (57) gives that

$$Q_{\psi_\alpha,\psi_\beta}(t) = A_0 + A_1 + A_2 + B,$$

where

$$A_0 = (4\pi/t)^{1/2}a^2 \iint_{S_a} p^{-\alpha}q^{-\beta}(e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)})dpdq,$$

$$A_1 = -(4\pi/t)^{1/2}a \iint_{S_a} (p+q)p^{-\alpha}q^{-\beta}(e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)})dpdq,$$

$$A_2 = (4\pi/t)^{1/2} \iint_{S_a} p^{1-\alpha}q^{1-\beta}(e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)})dpdq,$$

and

$$B = (4\pi/t)^{1/2} \iint_{S_a} (a-p)(a-q)p^{-\alpha}q^{-\beta} \sum_{n\ge1} (e^{-(2na+p-q)^2/(4t)} + e^{-(2na+q-p)^2/(4t)} - e^{-(2na+q+p)^2/(4t)} - e^{-(2na-q-p)^2/(4t)})dpdq. \tag{58}$$

We have the following.

Lemma 9. *If* $1<\alpha<2, 1<\beta<2$ *then for* $t\to 0$

$$B = -8\pi^{1/2}3^{-1}a^{-\alpha-\beta}t^{3/2} + O(t^2). \tag{59}$$

Proof. The integrand in (58) can be rewritten as

$$(a-p)(a-q)p^{-\alpha}q^{-\beta}\sum_{n\ge1} e^{-(2na-p-q)^2/(4t)} \times ((e^{(p-2na)q/t} + e^{(q-2na)p/t})(1-e^{-pq/t}) - (1-e^{-2pna/t})(1-e^{-2qna/t})). \tag{60}$$

The contribution from the terms with $n\ge 2$ in (60) is bounded in absolute value by

$$2a^2p^{1-\alpha}q^{1-\beta}t^{-1}\sum_{n\ge2} e^{-a^2(n-1)^2/t}(1+2n^2a^2t^{-1}).$$

After integrating with respect to p and q we see that this term contributes at most $O(e^{-a^2/(2t)})$ to B. Next we will show that the main contribution from the term with $n=1$ in (60) comes from a neighbourhood of the point $(p,q)=(a,a)$. Let

$$C_1(a) = \{(p,q)\in\mathbb{R}^2 : a/3<p<a, a/3<q<a\},$$

and

$$C_2(a) = S_a \setminus C_1(a).$$

On $C_2(a)$ we have that $2a - p - q \geq 2a/3$. Hence the term with $n = 1$ in (60) is bounded on $C_2(a)$ in absolute value by

$$2(a-p)(a-q)p^{1-\alpha}q^{1-\beta}t^{-1}e^{-a^2/(9t)}(1 + 2a^2t^{-1}). \tag{61}$$

Integrating (61) over $C_2(a)$ gives a contribution which is bounded by $O(e^{-a^2/(18t)})$. In order to calculate the contribution from the term with $n = 1$ on $C_1(a)$ we use the expression under (58) instead. First we note that $2a+p-q \geq 2a/3, 2a + q - p \geq 2a/3, 2a + p + q \geq 8a/3$. Hence the first three terms in the summand of (58) with $n = 1$ give after integration over $C_1(a)$ a contribution $O(e^{-a^2/(18t)})$. Putting all this together gives that

$$\begin{aligned} B = &- (4\pi/t)^{1/2} \iint_{C_1(a)} (a-p)(a-q)p^{-\alpha}q^{-\beta} \\ &\times e^{-(2a-q-p)^2/(4t)} dpdq + O(e^{-a^2/(18t)}). \end{aligned}$$

Noting that

$$p^{-\alpha}q^{-\beta} = a^{-\alpha-\beta} + O(a-p) + O(a-q) \tag{62}$$

uniformly in p and q yields after a change of variables that

$$\begin{aligned} B = &- (4\pi/t)^{1/2}a^{-\alpha-\beta} \iint_{S_{a/3}} pqe^{-(p+q)^2/(4t)} \\ &\times (1 + O(p) + O(q))dpdq + O(e^{-a^2/(18t)}), \end{aligned}$$

which agrees with the right hand side of (59). □

By taking higher order terms of the form $(a-p)^{n_1}(a-q)^{n_2}$ in (62) into account one can determine the coefficient $t^{(j+3)/2}, j = 0, 1, 2, \cdots$ in the expansion of B.

To complete the proof of Theorem 7 we rewrite A_0, A_1 and A_2 respectively as follows.

$$\begin{aligned} A_0 =&(4\pi/t)^{1/2}a^2 \left(\int_0^a dp \int_0^p dq + \int_0^a dq \int_0^q dp \right) \\ &\times p^{-\alpha}q^{-\beta}(e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)}) \\ =&(4\pi/t)^{1/2}a^2 \int_0^a p^{1-\alpha-\beta}dp \int_0^1 (\rho^{-\alpha} + \rho^{-\beta}) \\ &\times (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)})d\rho \\ =&4\pi a^2 c_{\alpha,\beta} t^{(1-\alpha-\beta)/2} \\ &- (4\pi/t)^{1/2}a^2 \int_a^\infty p^{1-\alpha-\beta}dp \int_0^1 (\rho^{-\alpha} + \rho^{-\beta}) \\ &\times (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)})d\rho, \end{aligned} \tag{63}$$

$$A_1 = - 4\pi a(c_{\alpha-1,\beta} + c_{\alpha,\beta-1}) t^{(2-\alpha-\beta)/2} + (4\pi/t)^{1/2} a \int_a^\infty p^{2-\alpha-\beta} dp \times \int_0^1 d(\rho^{1-\alpha} + \rho^{-\alpha} + \rho^{1-\beta} + \rho^{-\beta})(e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}) d\rho, \tag{64}$$

and

$$A_2 = 4\pi c_{\alpha-1,\beta-1} t^{(3-\alpha-\beta)/2} - (4\pi/t)^{1/2} \int_a^\infty p^{3-\alpha-\beta} dp \times \int_0^1 d(\rho^{1-\alpha} + \rho^{1-\beta})(e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}) d\rho. \tag{65}$$

The terms to be evaluated in (63), (64) and (65) are all of the form

$$(4\pi/t)^{1/2} a^{2-j} \int_a^\infty p^{1+j-\alpha-\beta} dp \int_0^1 \rho^{-\gamma}(e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}) d\rho, \tag{66}$$

where $j = 0, 1, 2$ respectively. Following arguments similar to the proof of Lemma 9 we see that the contribution of the integral with respect to $\rho \in [0, 1/2)$ in (66) is at most $O(e^{-a^2/(18t)})$. Furthermore

$$(4\pi/t)^{1/2} a^{2-j} \int_a^\infty p^{1+j-\alpha-\beta} dp \int_{1/2}^1 \rho^{-\gamma} e^{-p^2(1+\rho)^2/(4t)} d\rho = O(e^{-a^2/(18t)}). \tag{67}$$

Hence the expression under (66) equals

$$(4\pi/t)^{1/2} a^{2-j} \int_a^\infty p^{1+j-\alpha-\beta} dp \int_{1/2}^1 \rho^{-\gamma} e^{-p^2(1-\rho)^2/(4t)} d\rho + O(e^{-a^2/(18t)}). \tag{68}$$

Expanding $\rho^{-\gamma}$ about $\rho = 1$ we obtain that

$$\begin{aligned} &|\rho^{-\gamma} - 1 - \gamma(1-\rho) - 2^{-1}\gamma(\gamma+1)(1-\rho)^2 \\ &- 6^{-1}\gamma(\gamma+1)(\gamma+2)(1-\rho)^3| \le C(1-\rho)^4, \ 0 \le \rho \le 1/2, \end{aligned} \tag{69}$$

where C depends on γ only. By (69) and (68) we obtain that (66) is equal to

$$\begin{aligned} &2\pi(\alpha+\beta-j-1)^{-1} a^{3-\alpha-\beta} + 4\pi^{1/2}\gamma(\alpha+\beta-j)^{-1} a^{2-\alpha-\beta} t^{1/2} \\ &+ 2\pi\gamma(\gamma+1)(\alpha+\beta-j+1)^{-1} a^{1-\alpha-\beta} t \\ &+ 8\pi^{1/2} 3^{-1}\gamma(\gamma+1)(\gamma+2)(\alpha+\beta-j+2)^{-1} a^{-\alpha-\beta} t^{3/2} + O(t^2). \end{aligned} \tag{70}$$

It remains to compute the coefficients b_0, b_1 and b_2 in Theorem 7. Altogether there are eight terms which contribute to the terms in (70):

$$\begin{array}{lllll} j = 0, & \gamma = \alpha, & \gamma = \beta & & \\ j = 1, & \gamma = \alpha - 1, & \gamma = \beta - 1, & \gamma = \alpha, & \gamma = \beta \\ j = 2, & \gamma = \alpha - 1, & \gamma = \beta - 1\,. & & \end{array}$$

Summing these eight terms yield the expressions for b_0, b_1 and b_2 under (55). To calculate b_3 we have that the above eight $\gamma(\gamma+1)(\gamma+2)$ terms in (70) cancel the contribution from (59). This completes the proof of Theorem 7.

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