Hitting probabilities for Brownian motion on Riemannian manifolds

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1 Introduction and preliminaries

Let M be a complete non-compact Riemannian manifold. Let $(X_t, \mathbb{P}_x)$ be the Brownian motion on M, that is, the stochastic process generated by the Laplace-Beltrami operator Δ. Let also $p(t,x,y)$ be the heat kernel on M, that is, the minimal positive fundamental solution of the heat equation $\partial_t u = \Delta u$ on $(0,\infty) \times M$. Then $p(t,x,y)$ is also the transition density of X_t, which means that for any Borel set $A \subset M$,

$$\mathbb{P}_x(X_t \in A) = \int_A p(t,x,y)dy$$

where dy denotes the Riemannian measure.

Considerable efforts have been made to obtain upper and lower estimates of the heat kernel $p(t,x,y)$. See, for instance, [3], [8], [18], [25], [30], [32] and the references therein. The aim of this paper is to estimate the hitting probability function

$$\psi_K(t,x) := \mathbb{P}_x(\exists\, s \in [0,t]\,,\ X_s \in K)$$

where $K \subset M$ is a fixed compact set. In words, $\psi_K(t,x)$ is the probability that Brownian motion started at x hits K by time t. Our goal is to obtain precise estimates on ψ_K for all $t > 0$ and x outside a neighborhood of K, hence avoiding the somewhat different question of the behavior of ψ_K near the boundary of K. In the context of Riemannian manifolds, this natural question has been considered only in a handful of papers including [2], [4]. We were led to study ψ_K in our attempt to develop sharp heat kernel estimates on manifolds with more than one end. Indeed, the proof of the heat kernel estimates announced in [20] depends in a crucial way on the results of the present paper (see [21]). In this context, it turns out to be important to estimate also the time derivative $\partial_t \psi_K(t,x)$ which is a positive function.

We develop a general approach which allows to obtain estimates of ψ_K in terms of the heat kernel $p(t,x,y)$ or closely related objects such as the Dirichlet heat kernel $p_U(t,x,y)$ of some open set U. In the case when X_t is transient, that is, M is non-parabolic, we show that the behavior of $\psi_K(t,x)$, away from K, is comparable to that of

$$\int_0^t p(s,x,y)ds,$$

where y is a reference point on ∂K. If $(X_t)_{t>0}$ is recurrent, that is, M is parabolic, we obtain similar estimates through

$$\int_0^t p_U(s,x,y)ds$$

where U is a certain region slightly larger than $\Omega := M \setminus K$. We also show that $\partial_t \psi_K(t,x)$ is comparable to $p_\Omega(t,x,y)$ where y stays at a certain distance from ∂K. For precise statements, see Theorems 3.4, 3.7, 3.10 and Corollaries 3.13, 3.14.

Using the known results concerning the heat kernel $p(t,x,y)$ and the results of [23] on $p_U(t,x,y)$, we obtain in Theorems 4.5 and 4.11 some specific bounds on ψ_K for important classes of manifolds, including manifolds of non-negative Ricci curvature. Some examples are presented in Section 5. Consider, for instance, the case $M = \mathbb{R}^2$ and K being the unit ball centered at the origin. Then our results imply the following estimates, for $|x|$ large enough:

(i) If $0 < t < 2\,|x|^2$ then

$$\frac{c}{\log|x|} \exp\left(-C\frac{|x|^2}{t}\right) \le \psi_K(t,x) \le \frac{C}{\log|x|} \exp\left(-c\frac{|x|^2}{t}\right),$$

for some positive constants C, c.

(*ii*) If $t \geq 2|x|^2$ then

$$\psi_K(t,x) \simeq \frac{\log\sqrt{t} - \log|x|}{\log\sqrt{t}},$$

and

$$\partial_t \psi_K \simeq \frac{\log|x|}{t(\log t)^2}.$$

Here the relation $f \simeq g$ means that the ratio f/g is bounded by positive constants from above and below, for a specified range of the variables.

We develop these results below in the somewhat more general framework of weighted manifolds, possibly with a non-trivial boundary. We now explain this framework in detail.

Weighted manifolds. Let M be a Riemannian manifold of dimension N, possibly with a boundary which will be then denoted by δM. (Note that δM is a part of M so that all points on δM are interior points of M as a topological space.) The Riemannian metric g_{ij} induces the geodesic distance $d(x,y)$ between points $x, y \in M$.

Given a smooth positive function σ on M, let μ be the measure on M given by $d\mu(x) = \sigma(x)dx$ where dx is the Riemannian measure. Similarly, let μ' be the measure with the density σ with respect to the Riemannian measure of codimension 1 on any smooth hypersurface, in particular, on δM. The pair (M,μ) is called *a weighted manifold*, and it will serve as the underlying space in this paper.

The differential operators. For any smooth function f on M, denote by ∇f its *gradient*, that is, the vector field given by

$$(\nabla f)^i = \sum_{j=1}^N g^{ij}\frac{\partial f}{\partial x_j},$$

where g^{ij} are the entries of the inverse of the metric tensor g_{ij}. A weighted manifold possesses *the divergence* div_μ defined by

$$\operatorname{div}_\mu F := \frac{1}{\sigma\sqrt{g}}\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sigma\sqrt{g}F^i\right),$$

where F is a smooth vector field and $g := \det\|g_{ij}\|$. If $\sigma \equiv 1$ then div_μ is the Riemannian divergence $\operatorname{div} F$.

The Laplace operator Δ_μ of (M,μ) is the second order differential operator defined by

$$\Delta_\mu f := \operatorname{div}_\mu(\nabla f) = \sigma^{-1}\operatorname{div}(\sigma\nabla f).$$

We say that a smooth function f on (M,μ) is *harmonic* if $\Delta_\mu f = 0$ in $M \setminus \delta M$ and $\frac{\partial}{\partial \mathbf{n}} f = 0$ on δM where $\mathbf{n}$ is the inward unit normal vector field on δM.

Boundaries and integration by parts. For any set $\Omega \subset M$, set $\delta\Omega = \delta M \cap \Omega$. If Ω is open then Ω can be itself considered as a manifold with boundary $\delta\Omega$. Let $\partial\Omega$ be the topological boundary of Ω in M. When $\delta M = \emptyset$, we say that a set $\Omega \subset M$ has smooth boundary if $\partial\Omega$ is a smooth submanifold (without boundary) of co-dimension 1. In general, we have a more complicated definition of smooth boundary which takes into account $\delta\Omega$ as well as possible intersection of $\partial\Omega$ with δM.

Definition 1.1 We say that a set $\Omega \subset M$ has smooth boundary if each component Γ of $\partial\Omega$ satisfies one of the following two conditions *(see Fig. 1)*:

(i) either Γ is a smooth submanifold in M of co-dimension 1 whose boundary $\delta\Gamma$ lies on δM, and Γ is transversal to δM at $\delta\Gamma$ *(including the case $\delta\Gamma = \emptyset$)*;

(ii) or Γ lies in δM and Γ has smooth boundary as a subset of δM.

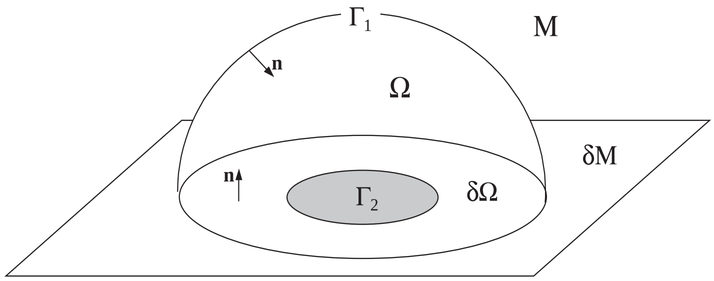

Figure 1 The boundary $\partial\Omega$ consists of two components Γ_1 and Γ_2 satisfying (i) and (ii) respectively.

Assume that Ω is an open set with smooth boundary, and let $\mathbf{n}$ be the inward normal unit vector field on $\partial\Omega$ and $\delta\Omega$. Then, for sufficiently regular functions f, g, we have the integration-by-parts formulas

$$\int_{\Omega} g\Delta_\mu f \, d\mu = -\int_{\Omega} (\nabla f, \nabla g) \, d\mu - \int_{\partial\Omega\cup\delta\Omega} g\frac{\partial f}{\partial \mathbf{n}} d\mu', \tag{1.1}$$

and

$$\int_{\Omega} g\Delta_\mu f \, d\mu = \int_{\Omega} f\Delta_\mu g \, d\mu + \int_{\partial\Omega\cup\delta\Omega} \left(f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}\right) d\mu' . \tag{1.2}$$

In the absence of $\delta\Omega$, the standard regularity condition sufficient for (1.1) and (1.2) is

$$f, g \in C^2(\Omega) \cap C^1(\overline{\Omega}). \tag{1.3}$$

In general, if

$$f, g \in \mathcal{R}(\Omega) := C^2(\Omega \setminus \delta\Omega) \cap C^1\left(\overline{\Omega} \setminus (\partial\Omega \cap \overline{\delta\Omega})\right) \cap L^\infty(\Omega), \tag{1.4}$$

then (1.1) and (1.2) hold. The regularity class $\mathcal{R}(\Omega)$ coincides with (1.3) if $\delta\Omega$ is empty. When $\delta\Omega$ is non-empty then the proof of (1.1) and (1.2) follows from [13, Prop. 2]. The point is that the intersection $\partial\Omega \cap \overline{\delta\Omega}$ has co-dimension 2 and hence does not affect the validity of (1.1) and (1.2) provided f and g are bounded.

Let us observe that if $\Omega \subset M$ is a precompact open set with smooth boundary then the (unique) weak solution f to the boundary value problem

$$\begin{cases} \Delta_\mu f = 0 \\ f|_{\partial\Omega} = f_0 \\ \left.\frac{\partial f}{\partial \mathbf{n}}\right|_{\delta\Omega} = 0 \end{cases}$$

belongs to $\mathcal{R}(\Omega)$ provided $f_0 \in C^1(\partial\Omega)$.

The heat kernel. Let $C_0^\infty(M)$ denote the set of smooth functions on M with compact support (functions from $C_0^\infty(M)$ do not necessarily vanish on δM). The operator Δ_μ with initial domain $C_0^\infty(M)$ is essentially self-adjoint in $L^2(M,\mu)$ and non-positive definite. It gives rise to the heat semigroup $P_t = e^{t\Delta_\mu}$ which has a positive smooth symmetric kernel $p(t,x,y)$ called the *heat kernel* of (M,μ). Alternatively, the heat kernel can be defined as the minimal positive solution $u(t,x) = p(t,x,y)$ of the Cauchy problem on $M \times (0,+\infty)$

$$\begin{cases} \partial_t u = \Delta_\mu u \\ u|_{t=0} = \delta_y \\ \frac{\partial u}{\partial \mathbf{n}}\big|_{\delta M} = 0 \end{cases} \tag{1.5}$$

(see [5], [10], [28]). The heat kernel satisfies the following properties:

- the semigroup identity

$$p(t,x,y) = \int_M p(s,x,z)p(t-s,z,y)d\mu(z), \tag{1.6}$$

  for all $0 < s < t$ and $x, y \in M$;

- the total mass inequality

$$\int_M p(t,x,y)d\mu(y) \le 1. \tag{1.7}$$

The operator Δ_μ generates a diffusion process $(X_t)_{t\ge 0}$ on M (reflected at δM) which will be called *the Brownian motion* on (M,μ). Denote by $\mathbb{P}_x$ the law of X_t given $X_0 = x \in M$ and by $\mathbb{E}_x$ the corresponding expectation. The heat kernel p is equal to the transition density for X_t with respect to measure μ, that is, for any Borel set $A \subset M$,

$$\mathbb{P}_x (X_t \in A) = \int_A p(t,x,y)d\mu(y).$$

As any open set $\Omega \subset M$ can be regarded as a manifold with boundary $\delta\Omega$, all the constructions above can be repeated for Ω yielding the heat semigroup P_t^Ω with the kernel $p_\Omega(t,x,y)$, which is called the *Dirichlet heat kernel* of Ω. We extend $p_\Omega(t,x,y)$ to all $x, y \in M$ by setting it to 0 if x or y is outside Ω. Then p_Ω vanishes and is continuous at regular points of the boundary $\partial\Omega$, and satisfies the Neumann boundary condition on $\delta\Omega$.

Observe that p_Ω increases with Ω, a fact which follows from the parabolic comparison principle. Let $\{\mathcal{E}_k\}$ be an *exhaustion* of M, that is an increasing sequence of precompact open sets $\mathcal{E}_k \subset M$ with smooth boundaries $\partial\mathcal{E}_k$ such that $\cup_k \mathcal{E}_k = M$. Then the sequence $\{p_{\mathcal{E}_k}\}$ of the corresponding heat kernels increases and converges to the global heat kernel p (see [10]).

Green function. The *Green function* of (M,μ) is defined by

$$G(x,y) = \int_0^\infty p(t,x,y)d\mu(y). \tag{1.8}$$

Equivalently, $G(x,y)$ can be defined as the infimum of all positive fundamental solutions of the operator Δ_μ with the Neumann condition on δM. It is known that either $G(x,y) \equiv \infty$ or $G(x,y) < \infty$ for all $x \ne y$.

Similarly, one defines $G_\Omega(x,y)$ for any open set $\Omega \subset M$. If Ω is precompact and $M \setminus \overline{\Omega}$ is non-empty then G_Ω is the fundamental solution of Δ_μ with the Dirichlet condition on $\partial\Omega$ and the Neumann condition on $\delta\Omega$. In this case $G_\Omega(x,y) < \infty$ for all $x \ne y$. If M is non-compact and $\{\mathcal{E}_k\}$ is an exhaustion of M then the sequence $\{G_{\mathcal{E}_k}\}$ increases and converges to G as $k \to \infty$.

Capacity. Given a non-empty closed set F and an open set Ω on M such that $F \subset \Omega$, define the *capacity* $\operatorname{cap}(F, \Omega)$ of the capacitor (F, Ω) as

$$\operatorname{cap}(F,\Omega) := \inf_{\substack{\phi \in Lip_0(\Omega) \\ \phi|_F = 1}} \int_\Omega |\nabla \phi|^2 \, d\mu. \tag{1.9}$$

Here $Lip_0(\Omega)$ is the class of all Lipschitz functions compactly supported in $\overline{\Omega}$. Note that $Lip_0(\Omega)$ can be replaced by $C_0^\infty(\Omega)$ without changing the value of the capacity. Various properties of capacity can be found in [27, Sect. 2.2.1].

Assume that Ω is precompact, ∂F and $\partial \Omega$ are non-empty, and consider the following boundary value problem in $\Omega \setminus F$

$$\begin{cases} \Delta_\mu \varphi = 0 \\ \varphi|_{\partial\Omega} = 0 \\ \varphi|_{\partial F} = 1 \\ \frac{\partial \varphi}{\partial \mathbf{n}}\Big|_{\delta(\Omega \setminus F)} = 0. \end{cases} \tag{1.10}$$

The unique (Perron) solution φ of this problem is called *the equilibrium potential* of the capacitor (F, Ω). In general, the equilibrium potential does not necessarily belong to the class of test functions in the definition of capacity. However, one always has

$$\operatorname{cap}(F,\Omega) = \int_{\Omega \setminus F} |\nabla \varphi|^2 \, d\mu. \tag{1.11}$$

Moreover, if U is a precompact open set with smooth boundary such that $K \subset U \subset \Omega$ then

$$\operatorname{cap}(F,\Omega) = \int_{\partial U} \frac{\partial \varphi}{\partial \mathbf{n}} d\mu', \tag{1.12}$$

where $\mathbf{n}$ is the inward unit normal vector field on ∂U. If Ω and F have smooth boundaries then $\varphi \in \mathcal{R}(\Omega \setminus F)$, and (1.12) follows from (1.1). In particular, in this case we have $\operatorname{cap}(F, \Omega) > 0$.

The equilibrium potential φ is defined by (1.10) as a function in $\Omega \setminus K$. Let us extend φ by 1 in $\overset{o}{K}$ and set $\varphi(x) = \liminf_{y \to x} \varphi(y)$ for $x \in \partial K$. Then φ becomes a lower semicontinuous superharmonic function in Ω. Similarly, we extend φ by 0 outside $\overline{\Omega}$.

If $\Omega = M$ then we write $\operatorname{cap}(F)$ for $\operatorname{cap}(F, M)$. Given an open subset $\Omega \subset M$ and a closed set $K \subset \Omega$, define $\operatorname{cap}_\Omega(K)$ as the capacity of K in the manifold Ω. From the definition, it easily follows that

$$\operatorname{cap}_\Omega(K) = \operatorname{cap}(K, \Omega).$$

Parabolicity. We say that (M, μ) is *parabolic* if $G(x, y) \equiv \infty$, and *non-parabolic* otherwise. For example, $\mathbb{R}^N$ is parabolic if and only if $N \leq 2$. It is well known that the following properties are equivalent:

- The weighted manifold (M, μ) is parabolic.
- The Brownian motion X_t on (M, μ) is recurrent.
- For any compact set $F \subset M$, $\operatorname{cap}(F) = 0$.
- For some compact set $F \subset M$ with non-empty interior, $\operatorname{cap}(F) = 0$.
- Any positive superharmonic function on (M, μ) is constant.

See, for example, [12], [19], [31].

2 Basic properties of hitting probabilities

2.1 Definition of hitting probabilities

For any closed subset $K \subset M$, denote by τ_K the first time the Brownian motion X_t visits K, that is

$$\tau_K = \inf\{t \geq 0 : X_t \in K\}.$$

Since X_t has continuous paths and K is closed, τ_K is a stopping time (see e.g., [24, Ch. 1.]) Let us set

$$\psi_K(t,x) := \mathbb{P}_x(\tau_K \leq t). \tag{2.1}$$

In other words, $\psi_K(t,x)$ is the probability that the Brownian motion hits K by time t. Observe that $\psi_K(t,x)$ is an increasing function in t, is bounded by 1, and $\psi_K(x,t) = 1$ if $x \in K$.

We also define

$$\psi_K(x) := \lim_{t\to\infty} \psi_K(t,x) = \mathbb{P}_x(\tau_K < \infty), \tag{2.2}$$

which is the probability that the Brownian motion ever hits K. Clearly, $0 \leq \psi_K(x) \leq 1$ on M and $\psi_K(x) = 1$ on K. Note that the parabolicity of (M,μ) is equivalent to the fact that $\psi_K(x) \equiv 1$ for any/some compact K with non-empty interior.

Let us consider also a regularized version of ψ_k defined by

$$\widehat{\psi}_K(x) := \mathbb{P}_x\left(0 < \tau_K < \infty\right).$$

It is obvious from (2.1) that $\widehat{\psi}_K(x) \leq \psi_K(x)$. Both functions $\widehat{\psi}_K(x)$ and $\psi_K(x)$ are harmonic in $\Omega := M \setminus K$ and coincide in Ω. Also, they are equal to 1 in the interior of K. On ∂K, the functions $\widehat{\psi}_K(x)$ and $\psi_K(x)$ may differ but it is known that

$$\mu\left\{x \in M : \psi_K(x) \neq \widehat{\psi}_K(x)\right\} = 0 \tag{2.3}$$

(see [7], [11]).

We will frequently consider the difference

$$\psi_K(x) - \psi_K(t,x) = \mathbb{P}_x\left(t < \tau_K < \infty\right).$$

Clearly, $\psi_K(x) - \psi_K(t,x)$ is the probability that Brownian motion ever hits K, and does it for the first time *after* time t. There is the following crucial relation between $\psi_K(x)$ and $\psi_K(t,x)$.

Lemma 2.1 *For an arbitrary closed set $K \subset M$, we have for all $t > 0$ and $x \in M$*

$$\psi_K(x) - \psi_K(t,x) = P_t^\Omega \psi_K(x), \tag{2.4}$$

where $\Omega := M \setminus K$.

Proof. If $x \in K$ then the both sides of (2.4) vanish. Assume that $x \in \Omega$ and consider the function

$$P_t^\Omega \widehat{\psi}_K(x) = \int_M p_\Omega(t,x,y)\widehat{\psi}_K(y)d\mu(y).$$

Clearly, $p_\Omega(t,x,y)d\mu(y)$ is the law of X_t started at x and conditioned not to hit $\partial\Omega$ (and hence K) by time t. Since $\widehat{\psi}_K(y)$ is the probability that the Brownian motion hits K at some positive time started at y, the Markov property implies that $P_t^\Omega \widehat{\psi}_K(x)$ is the probability that the Brownian motion hits K, but does it after time t. Hence, we obtain

$$\psi_K(x) - \psi_K(t,x) = P_t^\Omega \widehat{\psi}_K(x) = P_t^\Omega \psi_K(x),$$

where the last equality holds for $t > 0$ due to (2.3). ■

Corollary 2.2 *The function $\psi_K(t,x)$ satisfies in $\Omega \times (0,+\infty)$ the heat equation*

$$\partial_t \psi_K = \Delta_\mu \psi_K$$

and the Neumann condition $\frac{\partial}{\partial \mathbf{n}}\psi_K = 0$ on $\delta\Omega$.

Proof. By (2.4), since both $\psi_K(x)$ and $P_t^\Omega \psi_K(x)$ satisfy these conditions, so does $\psi_K(t,x)$. ■

Remark 2.3 If we assume that the process X_t is stochastically complete, that is, $P_t 1 \equiv 1$, then we have also

$$P_t^\Omega 1(x) = 1 - \psi_K(t,x). \tag{2.5}$$

Indeed, the left hand side of (2.5) is the probability that the Brownian motion with the killing boundary condition on $\partial\Omega$ stays in Ω until time t. This is equal to the probability that the global Brownian motion on M does not hit $\partial\Omega$ up to the time t, which coincides with the right hand side of (2.5).

2.2 Equilibrium measure

If (M,μ) is non-parabolic and $K \subset M$ is any compact set then the function $\psi_K(x)$ has the following representation

$$\psi_K(x) = \int_K G(x,y)\, de_K(y), \quad \forall\, x \in M, \tag{2.6}$$

where e_K is *the equilibrium measure* of K (see [6]). We will only use the properties of e_K that it is a Radon measure supported by ∂K, it satisfies (2.6) and

$$e_K(K) = \operatorname{cap}(K). \tag{2.7}$$

If K has smooth boundary then the measure e_K is given by

$$de_K = -\frac{\partial \psi_K}{\partial \mathbf{n}} d\mu', \tag{2.8}$$

where $\mathbf{n}$ is the normal vector field on ∂K inward with respect to $\Omega = M \setminus K$. Let us outline the proof of (2.8). Suppose f is harmonic in Ω and g satisfies in $\Omega \setminus \delta\Omega$ the equation $\Delta_\mu g = -\delta_x$ (where $x \in \Omega$) and the Neumann condition on $\delta\Omega$. If the integration by parts formula (1.2) can be applied then it yields

$$f(x) = \int_{\partial\Omega} \left(\frac{\partial g}{\partial \mathbf{n}} f - \frac{\partial f}{\partial \mathbf{n}} g \right) d\mu'. \tag{2.9}$$

Taking here $f = \psi_K$ and $g = G(x,\cdot)$ and observing that $f \equiv 1$ on $\partial\Omega$, we obtain

$$\psi_K(x) = \int_{\partial\Omega} \frac{\partial G(x,\cdot)}{\partial \mathbf{n}} d\mu' - \int_{\partial\Omega} G(x,\cdot) \frac{\partial \psi_K}{\partial \mathbf{n}} d\mu'. \tag{2.10}$$

This would imply (2.6) with e_K defined by (2.8) if we show that the first integral in (2.10) vanishes. Indeed, by Definition 1.1 of smooth boundary, each component Γ of $\partial\Omega$ is either a smooth hypersurface in M transversal to δM or Γ lies on δM. In the first case, Γ bounds a precompact open set $K_0 \subset K$ so that

$$\int_\Gamma \frac{\partial G(x,\cdot)}{\partial \mathbf{n}} d\mu' = -\int_{K_0} \Delta_\mu G(x,\cdot)\, d\mu = 0,$$

since $G(x,\cdot)$ is harmonic inside K. In the second case, $\Gamma \subset \delta M$ so that $\frac{\partial G}{\partial \mathbf{n}} = 0$ on Γ.

However, for the functions f and g as above the integration by parts is illegal because Ω is not precompact. To complete the proof, one must exhaust M by precompact regions and use the corresponding approximations for ψ_K and G (as in the proof of Lemma 2.5 below). Passage to the limit is possible by the local regularity of solutions of elliptic equations up to the boundary.

The following lemma will be used to obtain lower bounds for $\psi_K(x,t)$ (see Lemma 3.9).

Lemma 2.4 *Let (M,μ) be non-parabolic, K be a compact subset of M. Set $\Omega = M \setminus K$. Then, for all $t > 0$ and $x \in M$,*

$$\psi_K(x) - \psi_K(x,t) = \int_0^\infty \int_\Omega \int_K p_\Omega(t,x,y)p(s,y,z)de_K(z)d\mu(y)ds. \tag{2.11}$$

The proof immediately follows from (2.4), (2.6), (1.8).

2.3 The time derivative

The following lemma will be used to obtain upper bounds for ψ_K and its time derivative.

Lemma 2.5 *Let $K \subset M$ be a compact set with non-empty smooth boundary. Set $\Omega := M \setminus K$. Then, for all $t > 0$ and $x \in \Omega$, we have*

$$\partial_t \psi_K(t,x) = \int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}} p_\Omega(t,x,\cdot)d\mu', \tag{2.12}$$

where $\mathbf{n}$ is the inward normal unit vector field at $\partial\Omega$.

Proof. Denote for simplicity $\psi_K = \psi$. The informal line of reasoning showing (2.12) runs as follows:

$$\begin{aligned}
\partial_t \psi(t,x) &= -\partial_t P_t^\Omega \psi(x) \\
&= -\int_\Omega \partial_t p_\Omega(t,x,\cdot)\psi \, d\mu \\
&= -\int_\Omega \Delta_\mu p_\Omega(t,x,\cdot)\psi \, d\mu \\
&= -\int_\Omega p_\Omega(t,x,\cdot)\Delta_\mu \psi \, d\mu \\
&\quad + \int_{\partial\Omega\cup\delta\Omega} \left[\frac{\partial}{\partial \mathbf{n}} p_\Omega(t,x,\cdot)\psi - p_\Omega(t,x,\cdot)\frac{\partial \psi}{\partial \mathbf{n}}\right] d\mu' \qquad (2.13) \\
&= \int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}} p_\Omega(t,x,\cdot)d\mu',
\end{aligned}$$

where we have applied (2.4), integration by parts as in (1.2), and the conditions

$$\Delta_\mu \psi = 0, \quad \psi|_{\partial\Omega} = 1, \quad p_\Omega|_{\partial\Omega} = 0, \quad \frac{\partial}{\partial \mathbf{n}} p_\Omega = \frac{\partial}{\partial \mathbf{n}} \psi = 0 \text{ on } \delta\Omega.$$

However, the integration by parts is a priori illegal since Ω is not precompact. To make this argument rigorous, we have to approximate Ω by precompact sets and then pass to the limit.

Let $\{\mathcal{E}_k\}$ be an *exhaustion* of M. By this we mean that each $\mathcal{E}_k$ is a precompact open set with smooth boundary $\partial\mathcal{E}_k$; also, we assume that $\mathcal{E}_k$ increase to M as $k \to \infty$. In addition we may assume that each $\mathcal{E}_k$ contains K, and set $\Omega_k = \Omega \cap \mathcal{E}_k = \mathcal{E}_k \setminus K$. We can consider $\mathcal{E}_k$ itself as a manifold, instead of M, and perform the computations above for this manifold. Indeed, consider on $\mathcal{E}_k$ the corresponding heat kernel $p_{\Omega_k}(t,x,y)$ and the hitting probabilities $\psi_k(t,x)$ and $\psi_k(x)$. All these functions vanish on $\partial\mathcal{E}_k$ and satisfy the Neumann boundary condition on $\delta\mathcal{E}_k$. Integration-by-parts is justified in Ω_k so that the computation above yields

$$\partial_t\psi_k(t,x) = \int_{\partial\Omega} \frac{\partial}{\partial\mathbf{n}} p_{\Omega_k}(t,x,y)d\mu'(y). \tag{2.14}$$

We are left to pass to the limit as $k \to \infty$. It is known (see [3, Lemma 3, p.187]) that for all $x, y \in \Omega$ and $t > 0$

$$\psi_k(x) \nearrow \psi(x) \text{ and } p_{\Omega_k}(t,x,y) \nearrow p_\Omega(t,x,y)$$

whence we conclude by (2.4) that

$$\psi_k(t,x) \nearrow \psi(t,x)$$

(in fact, monotonicity of $\psi_k(t,x)$ in k is obvious; what we need from (2.4) is the convergence). By local properties of parabolic equations, we obtain that

$$\partial_t\psi_k(t,x) \to \partial_t\psi(t,x) \tag{2.15}$$

for all $x \in \Omega$ and $t > 0$. In other words, the left hand side of (2.14) converges to that of (2.12) as $k \to \infty$. Since $p_{\Omega_k} = 0$ on $\partial\Omega$ and p_{Ω_k} is non-negative and increases in k, the normal derivative $\frac{\partial}{\partial\mathbf{n}} p_{\Omega_k}$ on $\partial\Omega$ is non-negative and also increases in k. Local estimates of solutions to the heat equation up to the boundary imply

$$\frac{\partial}{\partial\mathbf{n}} p_{\Omega_k} \nearrow \frac{\partial}{\partial\mathbf{n}} p_\Omega \quad \text{on } \partial\Omega.$$

By the monotone convergence theorem, we conclude that the right hand side of (2.14) convergence to that of (2.12), which finishes the proof. ■

Remark 2.6 Integrating (2.12) in t from 0 to ∞, we obtain

$$\psi_K(x) = \int_{\partial\Omega} \frac{\partial G_\Omega(x,\cdot)}{\partial\mathbf{n}} d\mu'. \tag{2.16}$$

Alternatively, (2.16) can be deduced from (2.9) taking there $f = \psi_K$ and $g = G_\Omega(x,\cdot)$, which however, also requires an approximation argument in the spirit of the proof above.

3 General estimates of hitting probabilities

Throughout this section, (M,μ) is a weighted manifold, $K \subset M$ is a compact set, $\Omega := M \setminus K$, and K' is a precompact open neighborhood of K. The main results are Theorems 3.4, 3.7, and 3.10 providing estimates for $\psi_K(t,x)$.

3.1 Estimates based on the equilibrium potential

Lemma 3.1 *Assume that both K and K' have non-empty smooth boundaries. Then, for any function $\varphi \in \mathcal{R}(K'\backslash K)$ such that*

$$\varphi|_{\partial K} = 1, \quad \varphi|_{\partial K'} = 0, \quad \left.\frac{\partial\varphi}{\partial\mathbf{n}}\right|_{\delta(K'\backslash K)} = 0, \tag{3.1}$$

we have, for all $x \in \Omega$ and $t > 0$,

$$\partial_t \psi_K(t,x) = \int_{K'\setminus K} p_\Omega(t,x,\cdot)\Delta_\mu\left(\varphi^2\right) d\mu - \int_{K'\setminus K} \partial_t p_\Omega(t,x,\cdot)\varphi^2\, d\mu. \tag{3.2}$$

Remark 3.2 Since $p_\Omega(t,x,y)$ and $\partial_t\psi_K(t,x)$ vanish if $x \notin \Omega$, (3.2) is, in fact, satisfied for all $x \in M$.

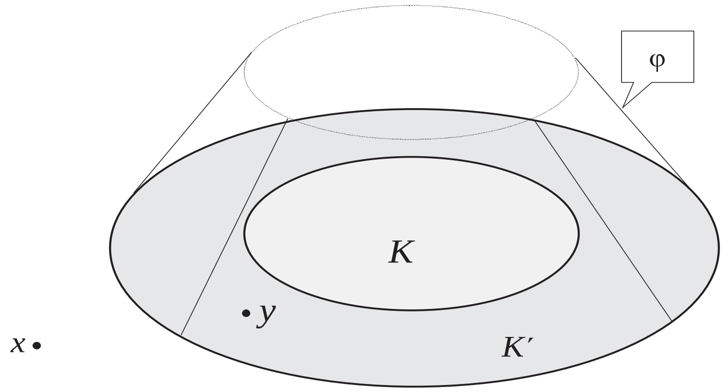

Figure 2 Sets K, K' and function ϕ

Proof. Let us denote for simplicity $u(t,y) := p_\Omega(t,x,y)$ and let $\mathbf{n}$ be the inward normal vector field on the boundary of $K' \setminus K$. By Lemma 2.5, we have

$$\partial_t\psi_K(t,x) = \int_{\partial K} \frac{\partial u}{\partial \mathbf{n}} d\mu'. \tag{3.3}$$

The function u satisfies the heat equation

$$\partial_t u = \Delta_\mu u. \tag{3.4}$$

Multiply (3.4) by φ^2 and integrating over $K'\setminus K$, we obtain

$$\int_{K'\setminus K} \varphi^2 \partial_t u\, d\mu = \int_{K'\setminus K} \varphi^2 \Delta_\mu u\, d\mu = \int_{K'\setminus K} u\Delta_\mu \varphi^2 d\mu - \int_{\partial(K'\setminus K)} \frac{\partial u}{\partial \mathbf{n}} \varphi^2 d\mu' + \int_{\partial(K'\setminus K)} \frac{\partial \varphi^2}{\partial \mathbf{n}} u\, d\mu'.$$

Note that the terms containing integration over $\delta(K' \setminus K)$ vanish because both u and φ satisfy the Neumann condition on $\delta(K' \setminus K)$. Since $u|_{\partial K} = 0$, $\varphi|_{\partial K} = 1$, $\varphi|_{\partial K'} = 0$ and

$$\left.\frac{\partial\varphi^2}{\partial \mathbf{n}}\right|_{\partial K'} = 2\varphi \left.\frac{\partial \varphi}{\partial \mathbf{n}}\right|_{\partial K'} = 0,$$

we obtain

$$\int_{\partial K} \frac{\partial u}{\partial \mathbf{n}} d\mu' = \int_{K'\setminus K} u\Delta_\mu \varphi^2 d\mu - \int_{K'\setminus K} \varphi^2 \partial_t u\, d\mu\,,$$

which together with (3.3) implies (3.2). ■

Corollary 3.3 *Let K and K' have non-empty smooth boundaries. Let φ be the equilibrium potential of capacitor (K,K'). Then, for all $x \in \Omega$ and $t > 0$,*

$$\partial_t\psi_K(t,x) = 2\int_{K'\setminus K} p_\Omega(t,x,\cdot)\left|\nabla\varphi\right|^2 d\mu - \int_{K'\setminus K} \partial_t p_\Omega(t,x,\cdot)\varphi^2\, d\mu. \tag{3.5}$$

Proof. Since $\Delta_\mu \varphi = 0$ in Ω, we obtain

$$\Delta_\mu(\varphi^2) = \sigma^{-1}\mathrm{div}\left(\sigma\nabla\varphi^2\right) = 2\varphi\sigma^{-1}\mathrm{div}\left(\sigma\nabla\varphi\right) + 2\left|\nabla\varphi\right|^2 = 2\varphi\Delta_\mu\varphi + 2\left|\nabla\varphi\right|^2 = 2\left|\nabla\varphi\right|^2.$$

Substituting into (3.2) and using (3.1), we obtain (3.5). ■

Theorem 3.4 *Let K and K' have non-empty boundaries. Then, for all $x \in \Omega$ and $t > 0$,*

$$\partial_t \psi_K(t,x) \leq 2\mathrm{cap}(K,K') \sup_{y\in K'\setminus K} p_\Omega(t,x,y) + \mu(K'\setminus K) \sup_{y\in K'\setminus K} \left|\partial_t p_\Omega(t,x,y)\right|. \tag{3.6}$$

If in addition K'' is a compact set such that $K \subset K'' \subset K'$ then, for all $x \in \Omega$ and $t > 0$,

$$\partial_t \psi_K(t,x) \geq 2m\,\mathrm{cap}(K,K') \inf_{y\in K'\setminus K''} p_\Omega(t,x,y) - \mu(K'\setminus K) \sup_{y\in K'\setminus K} \left|\partial_t p_\Omega(t,x,y)\right|, \tag{3.7}$$

where $m = \inf_{K''}\varphi$ and φ is the equilibrium potential of capacitor (K,K').

Remark 3.5 If $\mathrm{cap}(K,K') > 0$ then the constant m in (3.7) is positive. Indeed, if $m = \inf_{K''}\varphi = 0$ then $\varphi(x) = 0$ for some $x \in K'$. Since K' is connected, the strong minimum principle for superharmonic functions implies $\varphi \equiv 0$ in K'. However, this contradicts $\mathrm{cap}(K,K') > 0$.

We precede the proof of Theorem 3.4 by the following lemma.

Lemma 3.6 *Under the hypotheses of Theorem 3.4 and assuming that K and K' have smooth boundaries, we have*

$$\int_{K'\setminus K''} \left|\nabla\varphi\right|^2 d\mu \geq (\inf_{K''}\varphi)\,\mathrm{cap}(K,K'). \tag{3.8}$$

Proof. If $m := \inf_{K''}\varphi = 0$ then (3.8) holds trivially. Assuming $m > 0$, consider the sets

$$U_\lambda = \{x \in M : \varphi(x) > \lambda\}.$$

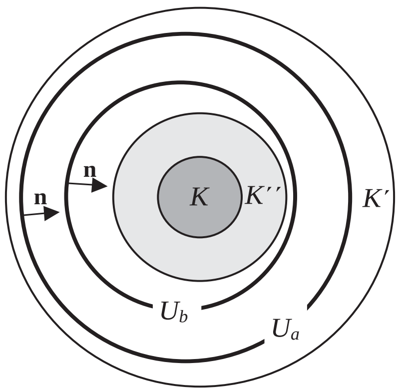

Figure 3 Sets U_a and U_b

As follows from Sard's theorem, for almost all $0 < \lambda < 1$ the set U_λ has a smooth boundary. Taking $0 < a < b < m$ so that ∂U_a and ∂U_b are smooth, we have $K'' \subset U_b \subset U_a \subset K'$ (see Fig. 3) and

$$\int\limits_{K'\setminus K''} \left|\nabla\varphi\right|^2 d\mu \geq \int\limits_{U_a\setminus U_b} \left|\nabla\varphi\right|^2 d\mu = \int\limits_{\partial(U_a\setminus U_b)} \varphi\frac{\partial\varphi}{\partial\mathbf{n}} d\mu' = \int\limits_{\partial U_b} \varphi\frac{\partial\varphi}{\partial\mathbf{n}} d\mu' - \int\limits_{\partial U_a} \varphi\frac{\partial\varphi}{\partial\mathbf{n}} d\mu' = (b-a)\,\mathrm{cap}(K,K'),$$

where we have applied (1.1), (1.10), and (1.12). Letting $a \downarrow 0$ and $b \uparrow m$ we obtain (3.8). ∎

Proof of Theorem 3.4. Let $\{K_n\}$ be a decreasing sequence of compact sets with non-empty smooth boundaries such that $\bigcap_n K_n = K$ and $\{K'_n\}$ be an increasing sequence of open sets with non-empty smooth boundaries such that $\bigcup_n K'_n = K'$. Denote by φ_n the equilibrium potential of (K_n, K'_n). Since

$$\int_{K'_n \setminus K_n} |\nabla \varphi_n|^2 \, d\mu = \mathrm{cap}(K_n, K'_n),$$

the identity (3.5) implies

$$\partial_t \psi_{K_n}(t,x) \le 2\mathrm{cap}(K_n, K'_n) \sup_{y \in K'_n \setminus K_n} p_{\Omega_n}(t,x,y) - \partial_t P_t^{\Omega_n} \varphi_n^2(x) \tag{3.9}$$

Clearly, as $n \to \infty$,

$$\psi_{K_n}(t,x) \searrow \psi_K(t,x) \quad \text{and} \quad p_{\Omega_n}(t,x,y) \nearrow p_\Omega(t,x,y),$$

(cf. the discussion in Section 2.3). In particular, we have also $\partial_t \psi_{K_n}(t,x) \longrightarrow \partial_t \psi_K(t,x)$ as ψ_{K_n} solves the heat equation. Also, φ_n converges to φ locally uniformly in $K' \setminus K$, which together with (1.12) yields

$$\mathrm{cap}(K_n, K'_n) \longrightarrow \mathrm{cap}(K, K')$$

(see also [27, 2.2.1 (iii)-(iv)]). Since $0 \le \varphi_n \le 1$ and $p_{\Omega_n} \le p_\Omega$, the dominated convergence theorem yields

$$P_t^{\Omega_n} \varphi_n^2(x) \longrightarrow P_t^\Omega \varphi^2(x),$$

for all $x \in \Omega$, $t > 0$. Since $P_t^{\Omega_n} \varphi_n^2(x)$ solves the heat equation, this implies also the convergence of the time derivatives. Hence, passing to the limit in (3.9) and applying

$$\left| \partial_t P_t^\Omega \varphi^2(x) \right| \le \int_{K' \setminus K} |\partial_t p_\Omega(t,x,y)| \, d\mu(y) \le \mu(K' \setminus K) \sup_{y \in K' \setminus K} |\partial_t p(t,x,y)| \,, \tag{3.10}$$

we obtain (3.6).

To prove (3.7), choose a decreasing sequence $\{K''_n\}$ of compact sets such that $\bigcap_n K''_n = K''$ and $K_n \subset K''_n \subset K'_n$. By (3.5) and (3.8), we obtain

$$\begin{aligned} \partial_t \psi_{K_n}(t,x) &\ge 2 \int_{K'_n \setminus K''_n} |\nabla \varphi_n|^2 \, d\mu \inf_{y \in K'_n \setminus K''_n} p_{\Omega_n}(t,x,y) - \int_{K'_n \setminus K_n} \partial_t p_{\Omega_n}(t,x,\cdot) \varphi_n^2 \, d\mu. \\ &\ge 2(\inf_{K''_n} \varphi_n) \mathrm{cap}(K_n, K'_n) \inf_{y \in K' \setminus K''} p_{\Omega_n}(t,x,y) - \left| \partial_t P_t^{\Omega_n} \varphi_n^2(x) \right|. \end{aligned}$$

Passing to the limit as $n \to \infty$ and using (3.10), we obtain (3.7). ∎

Theorem 3.7 *Assume that* $\mathrm{cap}(K, K') > 0$. *Then we have, for all* $x \notin K'$ *and* $t > 0$,

$$\psi_K(t,x) \le 2\mathrm{cap}(K, K') \int_0^t \sup_{y \in K' \setminus K} p_\Omega(s,x,y) ds \tag{3.11}$$

and

$$\psi_K(x) - \psi_K(t,x) \le 2\mathrm{cap}(K, K') \int_t^\infty \sup_{y \in K' \setminus K} p_\Omega(s,x,y) ds + \mu(K' \setminus K) \sup_{y \in K' \setminus K} p_\Omega(t,x,y). \tag{3.12}$$

Let in addition K' be connected and K'' be a compact set such that $K \subset K'' \subset K'$. Then, for all $x \notin K'$ and $t > 0$,

$$\psi_K(x) - \psi_K(t,x) \geq 2m \operatorname{cap}(K,K') \int_t^\infty \inf_{y \in K' \setminus K''} p_\Omega(s,x,y) ds, \tag{3.13}$$

where $m := \inf_{K''} \varphi > 0$ and φ is the equilibrium potential of capacitor (K,K').

Proof. Assume first that K and K' have smooth boundaries. Integrating (3.5) from 0 to t, we obtain

$$\psi_K(t,x) = 2 \int_0^t \int_{K' \setminus K} p_\Omega(s,x,\cdot) |\nabla \varphi|^2 d\mu ds - \int_{K' \setminus K} p_\Omega(t,x,\cdot) \varphi^2 d\mu$$

where we have used $p_\Omega(0,x,y) = 0$ because $x \neq y$ (indeed, we have $x \notin K'$ and $y \in K'$). Hence,

$$\psi_K(t,x) \leq 2 \int_0^t \int_{K' \setminus K} p_\Omega(s,x,\cdot) |\nabla \varphi|^2 d\mu ds, \tag{3.14}$$

which obviously implies (3.11). Similarly, integrating (3.5) from t to ∞, we obtain

$$\psi_K(x) - \psi_K(t,x) = 2 \int_t^\infty \int_{K' \setminus K} p_\Omega(s,x,\cdot) |\nabla \varphi|^2 d\mu ds + \int_{K' \setminus K} p_\Omega(t,x,\cdot) \varphi^2 d\mu, \tag{3.15}$$

whence the upper bound (3.12) follows. Finally, restricting the first integration in (3.15) to $K' \setminus K''$ and using (3.8), we obtain (3.13). The positivity of m is explained in Remark 3.5.

For general K, K', we use the same approximation procedure as in the previous proof. ■

Remark 3.8 By letting $t \to \infty$ in (3.14), we obtain, for all $x \notin K'$,

$$\psi_K(x) \leq 2 \int_{K' \setminus K} G_\Omega(x,y) |\nabla \varphi|^2 d\mu(y) \leq 2 \operatorname{cap}(K,K') \sup_{y \in K' \setminus K} G_\Omega(x,y). \tag{3.16}$$

Note that $G_\Omega(x,y)$ in (3.16) and $p_\Omega(s,x,y)$ in (3.11), (3.12) can be replaced by $G(x,y)$ and $p(s,x,y)$, respectively, since $p_\Omega \leq p$ and $G_\Omega \leq G$. Let us mention for comparison that (2.6) implies

$$\operatorname{cap}(K) \inf_{y \in \partial K} G(x,y) \leq \psi_K(x) \leq \operatorname{cap}(K) \sup_{y \in \partial K} G(x,y). \tag{3.17}$$

3.2 Estimates based on the equilibrium measure

Lemma 3.9 *Assume (M,μ) is non-parabolic. Then, for all $x \notin K$ and $t > 0$,*

$$\psi_K(t,x) \geq \int_K \int_0^t p(s,x,y) ds \, de_K(y), \tag{3.18}$$

where e_K is the equilibrium measure of K.

Proof. Denote $\Omega := M \setminus K$. Applying Lemma 2.4 and the semi-group identity (1.6), we obtain

$$\begin{aligned}
\psi_K(x) - \psi_K(t,x) &= \int_\Omega \int_K \int_0^\infty p_\Omega(t,x,z) p(s,z,y) ds\, de_K(y) d\mu(z) \\
&\leq \int_K \int_0^\infty \left[\int_\Omega p(t,x,z) p(s,z,y) d\mu(z) \right] ds\, de_K(y) \\
&\leq \int_K \int_0^\infty p(t+s,x,y) ds\, de_K(y) \\
&= \int_K \int_t^\infty p(s,x,y) ds\, de_K(y). \qquad (3.19)
\end{aligned}$$

Hence, by (2.6) and (1.8),

$$\begin{aligned}
\psi_K(t,x) &\geq \int_K \int_0^\infty p(s,x,y) ds\, de_K(y) - \int_K \int_t^\infty p(s,x,y) ds\, de_K(y) \\
&= \int_K \int_0^t p(s,x,y) ds\, de_K(y).
\end{aligned}$$

which was to be proved. ■

Theorem 3.10 *Let (M,μ) be non-parabolic. Then, for all $x \notin K$ and $t > 0$,*

$$\psi_K(t,x) \geq \operatorname{cap}(K) \int_0^t \inf_{y \in \partial K} p(s,x,y) ds \qquad (3.20)$$

and

$$\psi_K(x) - \psi_K(t,x) \leq \operatorname{cap}(K) \int_t^\infty \sup_{y \in \partial K} p(s,x,y) ds. \qquad (3.21)$$

Proof. Indeed, then (3.20) follows from (3.18), (2.7) and the fact that e_K sits on ∂K. Similarly, (3.21) follows from (3.19). Note that (3.20) holds also for a parabolic manifold (M,μ) as in this case the right hand side of (3.20) vanishes due to $\operatorname{cap}(K) = 0$. ■

Estimate (3.20) is trivially true also for parabolic (M,μ) as in this case $\operatorname{cap}(K) = 0$. However, Theorem 3.10 can give in this case a non-trivial lower bound for $\psi_K(t,x)$ as in Corollary 3.11 below. To state it, let us introduce the notion of conductivity. For any two disjoint non-empty sets A and B in M, define the *conductivity* between A and B by

$$\operatorname{cond}(A,B) = \inf_{\substack{\varphi \in Lip(M) \\ \varphi|_A = 1,\, \varphi|_B = 0}} \int_M |\nabla \varphi|^2 \, d\mu.$$

Clearly, $\operatorname{cond}(A,B)$ is symmetric in A, B. Also, each of the sets A, B can be replaced by its boundary. Comparing with the definition (1.9) of capacity we see that if A is compact and D is an open set containing A then

$$\operatorname{cond}(A, M \setminus D) \leq \inf_{\substack{\varphi \in Lip_0(D) \\ \varphi|_A = 1}} \int_M |\nabla \varphi|^2 \, d\mu = \operatorname{cap}(A,D). \qquad (3.22)$$

If in addition D is precompact then equality takes place in (3.22).

Corollary 3.11 *Let $F \subset M$ be a compact set such that ∂K and F are non-empty and disjoint* (see Fig. 4). *Set $U = M \setminus F$. Then, for all $x \in U \setminus K$ and $t > 0$,*

$$\psi_K(t,x) \geq \operatorname{cond}(\partial K, F) \int_0^t \inf_{y \in \partial K} p_U(s,x,y)ds. \tag{3.23}$$

Remark 3.12 Note that $\operatorname{cond}(\partial K, F) > 0$ whenever both K and F have non-empty interior. In this case, (3.23) provides a non-trivial lower bound for $\psi_K(t,x)$ regardless of (M,μ) being parabolic or not. A particularly interesting application for (3.23) is when $F \subset \overset{o}{K}$. In this case we have

$$\operatorname{cond}(\partial K, F) = \operatorname{cond}(F, M \setminus K) = \operatorname{cap}(F, \overset{o}{K}) \tag{3.24}$$

so that $\operatorname{cond}(\partial K, F)$ depends only on the intrinsic properties of K.

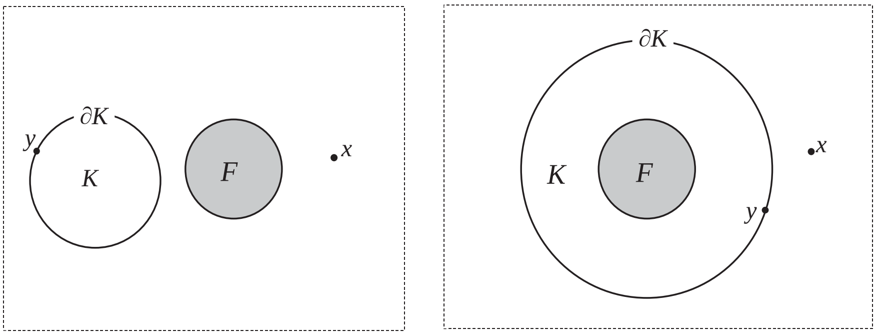

Figure 4 Possible locations of K and F

Proof. Let us apply Theorem 3.10 to estimate from below $\psi_{\partial K, U}(t,x)$ – the hitting probability of the compact set ∂K in manifold (U,μ) (note that $\partial K \subset U$). It is obvious that if $x \notin K$ then

$$\psi_K(t,x) = \psi_{\partial K}(t,x),$$

and if $x \in U \setminus K$ then

$$\psi_{\partial K}(t,x) \geq \psi_{\partial K, U}(t,x).$$

Applying (3.20) to the manifold (U,μ) and the compact ∂K, we obtain

$$\psi_{\partial K, U}(t,x) \geq \operatorname{cap}_U(\partial K) \int_0^t \inf_{y \in \partial K} p_U(s,x,y)ds.$$

Observing that

$$\operatorname{cap}_U(\partial K) = \operatorname{cap}(\partial K, U) \geq \operatorname{cond}(\partial K, F), \tag{3.25}$$

and collecting together all the above estimates, we obtain (3.23). ■

3.3 Two-sided estimates

Here we collect together the estimates of Theorems 3.7 and 3.10.

Corollary 3.13 *Let (M,μ) be non-parabolic and $K \subset M$ be a compact set such that* $\operatorname{cap}(K) > 0$.

1. *Then, for all $x \notin K'$ and $t > 0$*

$$\operatorname{cap}(K) \int_0^t \inf_{y \in \partial K} p(s,x,y)ds \leq \psi_K(t,x) \leq 2\operatorname{cap}(K,K') \int_0^t \sup_{y \in K' \setminus K} p_\Omega(s,x,y)ds. \tag{3.26}$$

2. *Let K' be connected and K'' be a compact set such that $K \subset K'' \subset K'$. Then, for all $x \notin K'$ and $t > 0$,*

$$c \int_t^\infty \inf_{y \in K' \setminus K''} p_\Omega(s,x,y)ds \le \psi_K(x) - \psi_K(t,x) \le \mathrm{cap}(K) \int_t^\infty \sup_{y \in \partial K} p(s,x,y)ds, \tag{3.27}$$

where $c > 0$ depends on K, K' and K''.

Proof. Indeed, the estimates (3.26) follow from (3.11) and (3.20), and the estimates (3.27) follow from (3.21) and (3.13). ■

Corollary 3.14 *Let (M, μ) be parabolic and let $F \subset \overset{o}{K}$ be a compact set such* $\mathrm{cap}(F, \overset{o}{K}) > 0$. *Set $U = M \setminus F$.*

1. *Then, for all $x \notin K'$ and $t > 0$,*

$$\mathrm{cap}(F, \overset{o}{K}) \int_0^t \inf_{y \in \partial K} p_U(s,x,y)ds \le \psi_K(t,x) \le 2\mathrm{cap}(K, K') \int_0^t \sup_{y \in K' \setminus K} p_\Omega(s,x,y)ds. \tag{3.28}$$

2. *Let K' be connected and K'' be a compact set such that $K \subset K'' \subset K'$. Then, for all $x \notin K'$ and $t > 0$,*

$$c \int_t^\infty \inf_{y \in K' \setminus K''} p_\Omega(s,x,y)ds \le 1 - \psi_K(t,x) \le \mathrm{cap}(F, \overset{o}{K}) \int_t^\infty \sup_{y \in \partial K} p_U(s,x,y)ds, \tag{3.29}$$

where $c > 0$ depends on F, K, K' and K''.

Proof. The upper bound in (3.28) follows from (3.11), and the lower bound in (3.29) follows from (3.13). The other two estimates here follow from the corresponding estimates of Corollary 3.13 when applied to the compact ∂K on the manifold (U, μ), and to the set $K' \setminus F$ instead of K'. Indeed, $\partial K \subset U$, and for $x \notin K$ we have $\psi_K(t,x) = \psi_{\partial K, U}(t,x)$ and $\psi_K(x) = 1$. The fact that $\mathrm{cap}(F) > 0$ implies that (U, μ) is non-parabolic.

We are left to verify that $\mathrm{cap}_U(\partial K) = \mathrm{cap}(F, \overset{o}{K})$. Indeed, we have (cf. (3.24) and (3.25))

$$\mathrm{cap}_U(\partial K) = \mathrm{cap}(\partial K, U) = \mathrm{cond}(\partial K, F) + \mathrm{cap}(K) = \mathrm{cap}(F, \overset{o}{K}),$$

as $\mathrm{cap}(K) = 0$ by the parabolicity of (M, μ). ■

4 Specific estimates of hitting probabilities

In this section, we present estimates of $\partial_t \psi_K(t,x)$ and $\psi_K(t,x)$ which depend on additional assumptions on the heat kernel. The main results are Theorems 4.5 and 4.11.

For any $\delta > 0$ and any set $A \subset M$, let A_δ denote the open δ-neighborhood of A. Throughout the section, we fix $\delta > 0$, a compact set $K \subset M$, and a reference point $o \in K$. Denote $\Omega := M \setminus K$ and $|x| := d(x, o)$.

4.1 Upper estimates I

Proposition 4.1 *Assume that there exists a constant C_0 such that, for all $x \in \Omega$ and for all $t > 0$,*

$$p_\Omega(t,x,x) \le \frac{C_0}{f(x,t)}\,, \tag{4.1}$$

where $f(x,t)$ is a positive function on $M \times (0,+\infty)$ which possesses the following regularity properties:

(i) for any $x \in M$, the function $f(x,t)$ is monotone increasing in t;

(ii) for some $\gamma > 1$ and for all $x \in M$, $0 < t_1 < t_2$,

$$\frac{f(x,\gamma t_1)}{f(x,t_1)} \le C_0 \frac{f(x,\gamma t_2)}{f(x,t_2)}; \tag{4.2}$$

(iii) for some $\alpha > 0$ for all $x, y \in M$, $t > 0$,

$$\frac{f(x,t)}{f(y,t)} \le C_0 \left(1 + \frac{d(x,y)}{\sqrt{t}}\right)^\alpha. \tag{4.3}$$

If $\operatorname{cap}(K, K_\delta) > 0$ then, for all $x \notin K_{2\delta}$, $t > 0$,

$$\psi_K(t,x) \le C \operatorname{cap}(K,K_\delta) \int_0^t \frac{ds}{f(o,\kappa s)} \exp\left(-c\frac{|x|^2}{s}\right), \tag{4.4}$$

$$\psi_K(x) - \psi_K(t,x) \le C \left(\operatorname{cap}(K,K_\delta) \int_t^\infty \frac{ds}{f(o,\kappa s)} + \frac{\mu(K_\delta \setminus K)}{f(o,\kappa t)}\right), \tag{4.5}$$

and

$$\partial_t \psi_K(t,x) \le \frac{C\,\mu(K_\delta \setminus K)}{f(o,\kappa t)\delta^2} \exp\left(-c\frac{|x|^2}{t}\right), \tag{4.6}$$

where $\kappa > 0$ depends on γ, $c > 0$ depends on $(\operatorname{diam} K)/\delta$, and C depends on C_0, α, γ, and $(\operatorname{diam} K)/\delta$.

Example 4.2 Suppose that $p_\Omega(t,x,x) \le C_0 t^{-\alpha/2}$ for all $x \in \Omega$, $t > 0$ and some $\alpha > 0$. In this case, we can set $f(x,t) = t^{\alpha/2}$, which satisfies (4.2) and (4.3) (the latter is trivially satisfied whenever $f(x,t)$ does not depend on x). Then (4.4) and (4.6) yield

$$\psi_K(t,x) \le \frac{C}{|x|^{\alpha-2}} \exp\left(-c\frac{|x|^2}{t}\right) \tag{4.7}$$

and

$$\partial_t \psi_K(t,x) \le \frac{C}{t^{\alpha/2}} \exp\left(-c\frac{|x|^2}{t}\right).$$

If $\alpha > 2$ then (4.5) implies

$$\psi_K(x) - \psi_K(t,x) \le C t^{1-\alpha/2}.$$

Proof of Proposition 4.1. Applying (4.3), we obtain, for all $x \in M$, $s > 0$, $\varepsilon > 0$,

$$\frac{1}{f(x,s)} = \frac{1}{f(o,s)} \frac{f(o,s)}{f(x,s)} \leq \frac{C}{f(o,s)} \left(1 + \frac{|x|^2}{s}\right)^{\alpha/2} \leq \frac{C_\varepsilon}{f(o,s)} \exp\left(\varepsilon \frac{|x|^2}{s}\right). \tag{4.8}$$

By [17, Theorem 3.1], the hypotheses (4.1) and (4.2) imply, for all $x, y \in \Omega$ and $t > 0$,

$$p_\Omega(t,x,y) \leq \frac{4C_0}{\sqrt{f(x,\kappa t) f(y,\kappa t)}} \exp\left(-c\frac{d^2(x,y)}{t}\right),$$

with any $c \in (0, 1/4)$ and some $\kappa = \kappa(c, \gamma) > 0$. Applying here the estimate (4.8) for $s = \kappa t$, we obtain

$$p_\Omega(t,x,y) \leq \frac{C}{f(o,\kappa t)} \exp\left(-c\frac{d^2(x,y)}{t} + \varepsilon \frac{|x|^2 + |y|^2}{t}\right).$$

Choosing ε small enough, we obtain for all $y \in K_\delta \setminus K$ and $x \in M \setminus K_{2\delta}$,

$$p_\Omega(t,x,y) \leq \frac{C}{f(o,\kappa t)} \exp\left(-c\frac{|x|^2}{t}\right). \tag{4.9}$$

Then (4.4) follows from (3.11) and (4.9).

Estimate (4.5) follows from (3.12) and (4.9). To prove (4.6), let us first estimate $|\partial_t p_\Omega(t,x,y)|$. By [17, Corollary 3.3], the hypotheses (4.1) and (4.2) imply, for all $x, y \in \Omega$ and $t > 0$,

$$|\partial_t p_\Omega(t,x,y)| \leq \frac{C}{t\sqrt{f(x,\kappa t) f(y,\kappa t)}} \exp\left(-c\frac{d^2(x,y)}{t}\right). \tag{4.10}$$

Using (4.8) and assuming $y \in K_\delta \setminus K$ and $x \in M \setminus K_{2\delta}$ as above we obtain

$$\begin{aligned} |\partial_t p_\Omega(t,x,y)| &\leq \frac{C}{t f(o,\kappa t)} \exp\left(-c\frac{|x|^2}{t}\right) \\ &= \frac{C}{|x|^2 f(o,\kappa t)} \frac{|x|^2}{t} \exp\left(-c\frac{|x|^2}{t}\right) \\ &\leq \frac{C'}{\delta^2 f(o,\kappa t)} \exp\left(-c'\frac{|x|^2}{t}\right). \end{aligned} \tag{4.11}$$

Substituting (4.9) and (4.11) into (3.6) we obtain, for $x \in M \setminus K_{2\delta}$,

$$\partial_t \psi_K(t,x) \leq \left(\operatorname{cap}(K, K_\delta) + \frac{\mu(K_\delta \backslash K)}{\delta^2}\right) \frac{C}{f(o,\kappa t)} \exp\left(-c\frac{|x|^2}{t}\right).$$

We are left to apply the elementary inequality

$$\operatorname{cap}(K, K_\delta) \leq \frac{\mu(K_\delta \backslash K)}{\delta^2},$$

which follows from the definition of the capacity (1.9) if we use the tent test function. ■

4.2 Upper estimates II

Let

$$B(x,r) = \{y \in M \mid d(x,y) < r\}$$

denote the geodesic ball of radius r centered at x, and set

$$V(x,r) := \mu(B(x,r)).$$

Consider the following two conditions which in general may be true or not:

$$V(x, 2r) \leq C_0 V(x,r), \quad \text{for all } x \in M,\ r > 0, \tag{4.12}$$

and

$$p(t,x,x) \leq \frac{C_0}{V(x,\sqrt{t})}, \quad \text{for all } x \in M,\ t > 0. \tag{4.13}$$

Obviously, (4.12) and (4.13) are satisfied for $\mathbb{R}^N$. More generally, (4.12) and (4.13) are satisfied if M is a complete Riemannian manifold with non-negative Ricci curvature (see [25]). Other examples are provided by unbounded convex subsets of $\mathbb{R}^N$ regarded as manifolds with boundary. Necessary and sufficient conditions for (4.12) and (4.13) in terms of certain Faber-Krahn type inequality can be found in [15, Proposition 5.2]. If (4.13) holds then the parabolicity of (M,μ) is equivalent to

$$\int^{\infty} \frac{ds}{V(x,\sqrt{s})} = \infty. \tag{4.14}$$

For all $t, r > 0$, define the function

$$H(r,t) := \frac{r^2}{V(o,r)} + \left(\int_{r^2}^{t} \frac{ds}{V(o,\sqrt{s})} \right)_+ . \tag{4.15}$$

Corollary 4.3 *Let (M,μ) be a complete non-compact manifold satisfying (4.12) and (4.13), and let $\operatorname{cap}(K) > 0$. Then, for all $x \notin K_{2\delta}$ and $t > 0$,*

$$\psi_K(t,x) \leq C \operatorname{cap}(K, K_\delta) H(|x|, t) \exp\left(-c \frac{|x|^2}{s} \right), \tag{4.16}$$

$$\psi_K(x) - \psi_K(t,x) \leq C \operatorname{cap}(K) \int_t^{\infty} \frac{ds}{V(o,\sqrt{s})}, \tag{4.17}$$

and

$$\partial_t \psi_K(t,x) \leq \frac{C\, \mu(K_\delta \setminus K)}{V(o,\sqrt{t})\delta^2} \exp\left(-c \frac{|x|^2}{t} \right), \tag{4.18}$$

where $c > 0$ depends on $(\operatorname{diam} K)/\delta$, and C depends on C_0 and $(\operatorname{diam} K)/\delta$.

Proof. We obtain (4.16) and (4.18) from Proposition 4.1. Let us set $f(x,t) = V(x,\sqrt{t})$. Then the hypothesis (4.12) implies both (4.2) and (4.3). Indeed, we have, for all positive t_1 and t_2, by (4.12),

$$\frac{V(x,\sqrt{4t_1})}{V(x,\sqrt{t_1})} \leq C \leq C \frac{V(x,\sqrt{4t_2})}{V(x,\sqrt{t_2})},$$

whence (4.2) follows with $\gamma = 4$.

To show (4.3), let us observe that (4.12) implies, for some $\alpha > 0$,

$$\frac{V(x,R)}{V(x,r)} \leq C\left(\frac{R}{r}\right)^{\alpha}, \tag{4.19}$$

for all $x \in M$ and $R \geq r > 0$. Therefore,

$$\frac{V(x,\sqrt{t})}{V(y,\sqrt{t})} \leq \frac{V(y,\sqrt{t}+d(x,y))}{V(y,\sqrt{t})} \leq C\left(1+\frac{d(x,y)}{\sqrt{t}}\right)^{\alpha}, \tag{4.20}$$

which was to be proved.

Obviously, (4.13) implies (4.1). The estimate (4.4) of Proposition 4.1 gives

$$\psi_K(t,x) \leq C \operatorname{cap}(K,K_\delta) \int_0^t \exp\left(-c\frac{|x|^2}{s}\right)\frac{ds}{V(o,\sqrt{s})}, \tag{4.21}$$

where we have eliminated κ by (4.19). Observing that

$$c'H(r,t)\exp(-\frac{2cr^2}{t}) \leq \int_0^t \frac{\exp\left(-c\frac{r^2}{s}\right)ds}{V(o,\sqrt{s})} \leq C'H(r,t)\exp(-\frac{cr^2}{2t}), \tag{4.22}$$

(where $C', c' > 0$ depend on c and C_0) we obtain (4.16).

Estimate (4.17) follows from inequality (3.21) of Theorem 3.10 using the fact that for all $x \in M \setminus K_\delta$ and $y \in \partial K$,

$$p(t,x,y) \leq \frac{C}{V(o,\sqrt{t})},$$

which is deduced from (4.13) and (4.12) in the same way as (4.9).

Finally, (4.18) follows from (4.6). ■

The next statement provides an upper bound on $\psi_K(t,x)$ using a different approach.

Proposition 4.4 *Let (M,μ) be a complete non-compact manifold satisfying* (4.12) *and* (4.13). *Then, for any $c \in (0,1/4)$, for all $x \in M \setminus K$ and $t > 0$,*

$$\psi_K(t,x) \leq C\exp\left(-c\frac{d^2(x,K)}{t}\right), \tag{4.23}$$

where C depends on C_0 and c.

Proof. We apply the fact that (4.12) and (4.13) imply the following mean value type inequality (see [15, Proposition 5.2 and eq. (3.5)]):

If a function $u(t,y)$ satisfies the heat equation $\partial_t u = \Delta_\mu u$ in cylinder $B(x,r) \times [t/2,t]$ and the Neumann condition $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\delta B(x,r)$ then

$$u^2(t,x) \leq C\left(\left(\frac{r^2}{t}\right)^{\beta} + \frac{t}{r^2}\right)\frac{1}{tV(x,r)}\int_{t/2}^t \int_{B(x,r)} u^2(s,y)d\mu(y)ds, \tag{4.24}$$

with some constants $\beta > 0$ and $C > 0$ depending only on the constants in (4.12) and (4.13).

The following inequality was proved in [16, Theorem 3] [1]:

[1]Inequality (4.25) is an L^2 version of Takeda's inequality [33] - see also [26].

If a function $u(t,y)$, $0\leq u\leq 1$, satisfies the heat equation $\partial_t u=\Delta_\mu u$ in cylinder $A_r\times(0,s]$ (where $A\subset M$ is a precompact set), the Neumann condition $\frac{\partial u}{\partial \mathbf{n}}=0$ on δA_r, and the initial condition $u(0,x)=0$ in A_r, then

$$\int_A u^2(y,s)d\mu(y)\leq \mu(A_r)\left(\frac{r^2}{s}+\frac{s}{r^2}\right)\exp\left(-\frac{r^2}{2s}+1\right). \tag{4.25}$$

Given $x\notin K$, let us set $d=d(x,K)$, $r=(1-\varepsilon)\,d$ and $A=B(x,\varepsilon d)$ where $\varepsilon\in(0,1)$. Then the function $u(t,y)=\psi_K(t,y)$ satisfies both (4.24) and (4.25). Integrating (4.25) in s from $t/2$ to t, we obtain

$$\int_{t/2}^t\int_{B(x,\varepsilon d)}\psi_K^2(s,y)d\mu(y)ds\leq CtV(x,d)\left(\frac{r^2}{t}+\frac{t}{r^2}\right)\exp\left(-\frac{r^2}{2t}\right).$$

Therefore, applying (4.24) in $B(x,\varepsilon d)$ and (4.12), we obtain

$$\psi_K^2(t,x)\leq C\left(\left(\frac{r^2}{t}\right)^{\beta+1}+\left(\frac{t}{r^2}\right)^2\right)\exp\left(-\frac{r^2}{2t}\right). \tag{4.26}$$

If $t\leq r^2$, then (4.26) yields (4.23). If $t>r^2$, then (4.23) follows from $\psi_K\leq 1$. ■

If $x\notin K_{2\delta}$ then (4.23) can also be deduced from (4.16), provided the condition (4.14) holds, that is, (M,μ) is non-parabolic. If (M,μ) is parabolic then we do not know an alternative way of proving (4.23). Moreover, examples show that (4.23) is often sharp for parabolic manifolds.

4.3 Two-sided estimates in the non-parabolic case

In this section we obtain two-sided estimates of $\psi_K(t,x)$ in the case when the heat kernel satisfies the following estimate, for all $x,y\in M$ and $t>0$,

$$\frac{c_1}{V(x,\sqrt{t})}\exp\left(-C_1\frac{d^2(x,y)}{t}\right)\leq p(t,x,y)\leq\frac{C_2}{V(x,\sqrt{t})}\exp\left(-c_2\frac{d^2(x,y)}{t}\right). \tag{4.27}$$

The estimate (4.27) is known to be equivalent to the doubling volume property (4.12) and a certain Poincaré inequality (see [29], [30]). In is known that (4.27) holds in the following settings:

- M is a complete Riemannian manifold of non-negative Ricci curvature, μ is the Riemannian volume (see [25]).
- M is a unbounded convex region in $\mathbb{R}^N$ considered as a manifold with boundary, μ is the Lebesgue measure (see [14]).
- M is a nilpotent Lie groups with left-invariant Riemannian metric, μ is the Haar measure (see [34]).

Many more examples of weighted manifolds where (4.27) holds can be found in [22]. It is known (see [29]) that (4.27) is stable under quasi-isometry of (M,μ).

Note that the hypotheses (4.12) and (4.13) from Section 4.2 imply the upper bound in (4.27) (cf. [17, Theorem 1.1]). On the other hand, (4.27) implies both (4.12) and (4.13). Hence all results of Section 4.2 can be used in the present setting.

Theorem 4.5 *Let (M,μ) be a complete non-compact non-parabolic weighted manifold satisfying* (4.27), *and let* $\operatorname{cap}(K)>0$. *Then the following estimates hold:*

1. *For any $\delta > 0$ and for all $x \notin K_{2\delta}$, $t > 0$,*

$$c\,\mathrm{cap}(K) H(|x|, t) \exp\left(-C\frac{|x|^2}{t}\right) \le \psi_K(t, x) \le C\,\mathrm{cap}(K, K_\delta) H(|x|, t) \exp\left(-c\frac{|x|^2}{t}\right), \tag{4.28}$$

where $c, C > 0$ depend on c_1, c_2, C_1, C_2, and $(\mathrm{diam}\, K)/\delta$.

2. *For a large enough δ and for all $x \notin K_{2\delta}$, $t \ge |x|^2$,*

$$c\int_t^\infty \frac{ds}{V(o,\sqrt{s})} \le \psi_K(x) - \psi_K(t,x) \le C\int_t^\infty \frac{ds}{V(o,\sqrt{s})}. \tag{4.29}$$

3. *For a large enough δ and for all $x \notin K_{2\delta}$, $t \ge \delta^2$,*

$$\frac{c}{V(o,\sqrt{t})}\exp\left(-C\frac{|x|^2}{t}\right) \le \partial_t\psi_K(t,x) \le \frac{C}{V(o,\sqrt{t})}\exp\left(-c\frac{|x|^2}{t}\right). \tag{4.30}$$

In (4.29) *and* (4.30) *the constants $c, C > 0$ depend on c_1, c_2, C_1, C_2, and K.*

Remark 4.6 Using the definition (4.15) of the function $H(r,t)$, one can rewrite (4.28) as follows: if $0 < t < 2|x|^2$ then

$$c\frac{\mathrm{cap}(K)\,|x|^2}{V(o,|x|)}\exp\left(-C\frac{|x|^2}{t}\right) \le \psi_K(t,x) \le C\frac{\mathrm{cap}(K,K_\delta)\,|x|^2}{V(o,|x|)}\exp\left(-c\frac{|x|^2}{t}\right) \tag{4.31}$$

and if $t \ge 2|x|^2$ then

$$c\,\mathrm{cap}(K)\int_{|x|^2}^t \frac{ds}{V(o,\sqrt{s})} \le \psi_K(t,x) \le C\,\mathrm{cap}(K,K_\delta)\int_{|x|^2}^t \frac{ds}{V(o,\sqrt{s})}. \tag{4.32}$$

Remark 4.7 Clearly, the estimate (4.29) can be obtained from (4.30) by integrating it from t to ∞, assuming $t \ge |x|^2$. Nevertheless, we give below an independent proof for (4.29), as it is simpler than (4.30).

Proof. Let us observe that for all $x \notin K_{2\delta}$, $y \in K_\delta \setminus K$,

$$\frac{c}{V(o,\sqrt{t})}\exp\left(-C\frac{|x|^2}{t}\right) \le p(t,x,y) \le \frac{C}{V(o,\sqrt{t})}\exp\left(-c\frac{|x|^2}{t}\right). \tag{4.33}$$

Indeed, this follows from (4.27) with swapped x, y, from

$$\frac{c_\varepsilon}{V(o,\sqrt{t})}\exp\left(-\varepsilon\frac{|y|^2}{t}\right) \le \frac{1}{V(y,\sqrt{t})} \le \frac{C_\varepsilon}{V(o,\sqrt{t})}\exp\left(\varepsilon\frac{|y|^2}{t}\right), \tag{4.34}$$

(see (4.20)) and from the fact that $d(x,y)$ is comparable to $|x|$ in the range in question. Given the estimates (4.33), (4.28) follows from (3.26), $p_\Omega \le p$, and (4.22).

The upper bound in (4.29) follows in the same way from that of (3.27) since (4.33) implies

$$p(t,x,y) \le \frac{C}{V(o,\sqrt{t})}$$

(here we do not need neither δ is large nor $t \geq |x|^2$). To prove the lower bound in (4.29), we use again (3.27) and the following lower estimate for p_Ω

$$p_\Omega(t,x,y) \geq cp(Ct,x,y) \geq \frac{c}{V(y,\sqrt{t})} \exp\left(-C\frac{d(x,y)^2}{t}\right), \tag{4.35}$$

which holds for all $t > 0$ provided $|x|$ and $|y|$ are large enough (see [23, Theorem 3.1] – note that the non-parabolicity of (M,μ) is important for (4.35)). Take δ large enough so that (4.35) holds for $|x|, |y| \geq \delta/2$ and $K' := K_\delta$ is connected; set $K'' = K_{\delta/2}$. Then (4.35) implies, for all $y \in K' \setminus K''$ and $x \notin K_{2\delta}$

$$p_\Omega(t,x,y) \geq \frac{c}{V(o,\sqrt{t})} \exp\left(-c\frac{|x|^2}{t}\right). \tag{4.36}$$

Assuming in addition $t \geq |x|^2$, we obtain the lower bound in (4.29) from that of (3.27).

The upper bound in (4.30) follows from (4.18). To prove the lower bound in (4.30), let us recall that by (3.7), for all $x \in \Omega$ and $t > 0$,

$$\partial_t \psi_K(t,x) \geq c \inf_{y \in K' \setminus K''} p_\Omega(t,x,y) - C \sup_{y \in K' \setminus K} |\partial_t p_\Omega(t,x,y)|. \tag{4.37}$$

By $p_\Omega \leq p$, (4.27), (4.10), and (4.20), we obtain, for all $x, y \in \Omega$ and $t > 0$,

$$|\partial_t p_\Omega(t,x,y)| \leq \frac{C}{tV(y,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right). \tag{4.38}$$

For $x \notin K_{2\delta}$ and $y \in K' \setminus K$, this implies

$$|\partial_t p_\Omega(t,x,y)| \leq \frac{C}{tV(o,\sqrt{t})} \exp\left(-c\frac{|x|^2}{t}\right). \tag{4.39}$$

Substituting (4.36) and (4.39) into (4.37) and assuming in addition $t \geq |x|^2$, we obtain

$$\partial_t \psi_K(t,x) \geq \frac{c}{V(o,\sqrt{t})} - \frac{C}{tV(o,\sqrt{t})} \geq \frac{\left(c - C/\delta^2\right)}{V(o,\sqrt{t})} \geq \frac{c/2}{V(o,\sqrt{t})}, \tag{4.40}$$

provided δ is large enough. This proves the lower bound in (4.30) in the range $t \geq |x|^2$.

To obtain the lower bound for $\partial_t \psi_K(t,x)$ in the range $\delta^2 \leq t \leq |x|^2$, observe that $\partial_t \psi_K$ is a non-negative solution of the heat equation in $(0,+\infty) \times \Omega$. Hence, the full range lower bound in (4.30) follows by the standard chaining argument based on the parabolic Harnack inequality that is a consequence of (4.27) (see, for instance, [1], [23, (2.18)] or [22, Theorem 2.7]). ■

Corollary 4.8 *Referring to Theorem* 4.5*, assume in addition that, for some* $c_0 > 0$ *and* $\alpha > 2$*,*

$$\frac{V(o,R)}{V(o,r)} \geq c_0 \left(\frac{R}{r}\right)^\alpha, \quad \forall R \geq r > \delta. \tag{4.41}$$

1. For any $\delta > 0$*, for all* $x \notin K_{2\delta}$ *and* $t > 0$*,*

$$\frac{c\,\mathrm{cap}(K)\,|x|^2}{V(o,|x|)} \exp\left(-C\frac{|x|^2}{t}\right) \leq \psi_K(t,x) \leq \frac{C\,\mathrm{cap}(K,K_\delta)\,|x|^2}{V(o,|x|)} \exp\left(-C\frac{|x|^2}{t}\right), \tag{4.42}$$

where $c, C > 0$ *depend on* c_0*,* c_1*,* c_2*,* C_1*,* C_2*, and* $(\mathrm{diam}\,K)/\delta$*.*

2. For a large enough δ and for all $x \notin K_{2\delta}$ and $t \geq |x|^2$,

$$\frac{c\ |x|^2}{V(o,|x|)} \leq \psi_K(x) - \psi_K(t,x) \leq \frac{C\ |x|^2}{V(o,|x|)}.$$

where $c, C > 0$ depend on c_1, c_2, C_1, C_2 and K.

Proof. The condition (4.41) implies, for all $r > \delta$,

$$\frac{cr^2}{V(o,r)} \leq \int_{r^2}^{\infty} \frac{ds}{V(o,\sqrt{s})} \leq \frac{Cr^2}{V(o,r)}. \tag{4.43}$$

whence we obtain by (4.15)

$$\frac{cr^2}{V(o,r)} \leq H(r,t) \leq \frac{Cr^2}{V(o,r)}.$$

The rest follows by Theorem 4.5. ∎

Example 4.9 To illustrate Corollary 4.8, take $K = B(o,r)$ and $\delta = r$. The result of [23, Lemma 4.3] provides the following bound, for all $R \geq 2r > 0$

$$\frac{1}{2}\int_r^R \frac{sds}{V(o,s)} \leq \operatorname{cap}(B\left(o,r\right), B(o,R))^{-1} \leq C\int_r^R \frac{sds}{V(o,s)}$$

assuming (4.12) and (4.13). Assuming also (4.41), we obtain for such R, r

$$\frac{cr^2}{V(o,r)} \leq \int_r^R \frac{sds}{V(o,s)} \leq \frac{Cr^2}{V(o,r)}.$$

Hence, (4.42) gives

$$\frac{c\,V(o,r)\,|x|^2}{V(o,|x|)r^2}\exp\left(-C\frac{|x|^2}{t}\right) \leq \psi_{B(o,r)}(t,x) \leq \frac{C\,V(o,r)\,|x|^2}{V(o,|x|)r^2}\exp\left(-c\frac{|x|^2}{t}\right),$$

for all $o \in M$, $r, t > 0$ and all $x \in M$ such that $|x| > 4r$.

4.4 Two-sided estimates in the parabolic case

This section describes sharp two sided estimates on ψ_K in the case where the weighted manifold (M,μ) is parabolic and satisfies some additional assumptions. Throughout this section, we also assume that (M,μ) satisfies the two sided heat kernel bounds (4.27). This implies in particular the volume doubling property (4.12).

Given a point $o \in M$, we call the pair (M,o) a *pointed manifold.*

Definition 4.10 We say that a pointed Riemannian manifold (M,o) satisfies the condition (RCA), that is, has *relatively connected annuli*, if there exists $A > 1$ such that, for any $r > A^2$ and all x, y with $|x| = |y| = r$, there exists a continuous path $\gamma : [0,1] \to M$ with $\gamma(0) = x$, $\gamma(1) = y$ whose image is contained in $B(o,Ar) \setminus B(o,r/A)$.

Define a function $h(r)$ for all $r > 0$ by

$$h(r) := 1 + \left(\int_1^r \frac{sds}{V(o,s)}\right)_+ = 1 + \frac{1}{2}\left(\int_1^{r^2} \frac{dt}{V(o,\sqrt{t})}\right)_+. \tag{4.44}$$

Theorem 4.11 *Let (M,μ) be complete, non-compact, parabolic, and satisfy* (4.27) *and* (RCA). *Assume that, for some $\varepsilon > 0$, the set $F := \overline{B(o,\varepsilon)}$ does not intersect δM. Let $\delta > \varepsilon$ be large enough and $B(o,\delta)$ be contained in K.*

1. *For all $x \notin K_{2\delta}$ we have the following: if $0 < t < 2|x|^2$ then*

$$\frac{c|x|^2}{V(o,|x|)h(|x|)} \exp\left(-C\frac{|x|^2}{t}\right) \le \psi_K(t,x) \le \frac{C\,|x|^2}{V(o,|x|)h(|x|)} \exp\left(-c\frac{|x|^2}{t}\right), \tag{4.45}$$

and if $t \ge 2|x|^2$ then

$$\frac{c}{h(\sqrt{t})}\int_{|x|}^{\sqrt{t}} \frac{sds}{V(o,s)} \le \psi_K(t,x) \le \frac{C}{h(\sqrt{t})}\int_{|x|}^{\sqrt{t}} \frac{sds}{V(o,s)}. \tag{4.46}$$

2. *For all $x \notin K_{2\delta}$ and $t \ge |x|^2$,*

$$c\frac{h(|x|)}{h(\sqrt{t})} \le 1 - \psi_K(t,x) \le C\frac{h(|x|)}{h(\sqrt{t})}. \tag{4.47}$$

3. *For all $x \notin K_{2\delta}$ and $t \ge \delta^2$,*

$$\frac{ch(|x|)\exp\left(-C\frac{|x|^2}{t}\right)}{V(o,\sqrt{t})(h(|x|)+h(\sqrt{t}))h(\sqrt{t})} \le \partial_t\psi_K(t,x) \le \frac{Ch(|x|)\exp\left(-c\frac{|x|^2}{t}\right)}{V(o,\sqrt{t})(h(|x|)+h(\sqrt{t}))h(\sqrt{t})}. \tag{4.48}$$

Here $c, C > 0$ depend on c_1, c_2, C_1, C_2 from (4.27)*, on A from* (RCA) *as well as on K.*

Remark 4.12 For the range $t \ge |x|^2$, (4.48) reads as follows:

$$\frac{ch(|x|)}{V(o,\sqrt{t})h^2(\sqrt{t})} \le \partial_t\psi_K(t,x) \le \frac{Ch(|x|)}{V(o,\sqrt{t})h^2(\sqrt{t})}.$$

Integrating this from t to ∞ gives (4.47) (cf. (4.59)).

Remark 4.13 Theorem 4.11 requires that K contains the ball $B(o,\delta)$ of a large enough radius δ. As we will see from the proof, δ depends on the estimates of the Dirichlet heat kernel p_U in the region $U = M \setminus F$ based on [23, Theorem 4.9]. This makes applications of Theorem 4.11 to concrete situations somewhat difficult. However, combining [23, Theorem 4.9] with [23, Corrolary 3.5], one can drop the assumption that K contains $B(o,\delta)$ replacing it by

$$K \text{ contains } B(o,\varepsilon) \text{ and } M \setminus \overline{B(o,\varepsilon)} \text{ is connected.}$$

Then, statement 1 of Theorem 4.11 holds for all x with $d(x,K) \ge 1$. Statement 2 holds for all x with $|x|$ large enough. In statement 3, the upper bound holds for all x with $d(x,K) \ge 1$ whereas the lower bound holds for all x with $|x|$ large enough.

Proof. Set $U = M \setminus F$. By (3.28) we have, for all $x \notin K_\delta$ and $t > 0$,

$$c\int_0^t \inf_{y\in\partial K_\delta} p_U(s,y,x)ds \le \psi_K(t,x) \le C\int_0^t \sup_{y\in K_\delta\setminus K} p_U(s,y,x)ds. \tag{4.49}$$

For any complete weighted parabolic manifold (M, μ) satisfying (4.27) and (RCA), [23, Theorem 4.9] yields the following estimates for the Dirichlet heat kernel $p_U(t, x, y)$ provided $d(x, F)$ and $d(y, F)$ are large enough:

$$c\frac{D(t,x,y)}{V(y,\sqrt{t})}\exp\left(-C\frac{d^2(x,y)}{t}\right) \le p_U(t,x,y) \le C\frac{D(t,x,y)}{V(y,\sqrt{t})}\exp\left(-c\frac{d^2(x,y)}{t}\right) \tag{4.50}$$

where

$$D(t,x,y) := \frac{h(|x|)h(|y|)}{\left(h(|x|)+h(\sqrt{t})\right)\left(h(|y|)+h(\sqrt{t})\right)}.$$

Taking δ large enough, we can assume that (4.50) holds for all $x, y \notin K$ and $t > 0$. If in addition $x \notin K_{2\delta}$ and $y \in K_\delta \setminus K$ then (4.50) and (4.34) imply

$$c\frac{\widetilde{D}(t,x)}{V(o,\sqrt{t})}\exp\left(-C\frac{|x|^2}{t}\right) \le p_U(t,x,y) \le C\frac{\widetilde{D}(t,x)}{V(o,\sqrt{t})}\exp\left(-c\frac{|x|^2}{t}\right) \tag{4.51}$$

where

$$\widetilde{D}(t,x) := \frac{h(|x|)}{(h(|x|)+h(\sqrt{t}))h(\sqrt{t})}. \tag{4.52}$$

Set

$$I_a(r,t) := \int_0^t \frac{h(r)}{(h(r)+h(\sqrt{s}))h(\sqrt{s})}\frac{\exp\left(-a\frac{r^2}{s}\right)}{V(o,\sqrt{s})}ds.$$

Since the functions V and h are doubling, one easily checks that, for $0 < t < 2r^2$,

$$\frac{cr^2}{V(o,r)h(r)}\exp\left(-4a\frac{r^2}{t}\right) \le I_a(r,t) \le \frac{Cr^2}{V(o,r)h(r)}\exp\left(-a\frac{r^2}{2t}\right). \tag{4.53}$$

For $t \ge 2r^2$, we have instead,

$$ce^{-a}h(r)\int_{r^2}^t \frac{ds}{V(o,\sqrt{s})h^2(\sqrt{s})} \le I_a(r,t) \le Ch(r)\int_{r^2}^t \frac{ds}{V(o,\sqrt{s})h^2(\sqrt{s})}.$$

Moreover, for $r \ge 1$, we obtain

$$\begin{aligned}
\frac{1}{2}\int_{r^2}^t \frac{ds}{V(o,\sqrt{s})h^2(\sqrt{s})} &= \int_r^{\sqrt{t}} \frac{\rho d\rho}{V(o,\rho)\left(1+\int_1^\rho \frac{\sigma d\sigma}{V(o,\sigma)}\right)^2} \\
&= \frac{1}{1+\int_1^r \frac{\sigma d\sigma}{V(o,\sigma)}} - \frac{1}{1+\int_1^{\sqrt{t}} \frac{\sigma d\sigma}{V(o,\sigma)}} \\
&= \frac{1}{h(r)h(\sqrt{t})}\int_r^{\sqrt{t}} \frac{\sigma d\sigma}{V(o,\sigma)}.
\end{aligned} \tag{4.54}$$

Thus, for $t \ge 2r^2$ and $r \ge 1$,

$$\frac{c_a}{h(\sqrt{t})}\int_r^{\sqrt{t}} \frac{sds}{V(o,s)} \le I_a(r,t) \le \frac{C_a}{h(\sqrt{t})}\int_r^{\sqrt{t}} \frac{sds}{V(o,s)}. \tag{4.55}$$

Collecting together (4.49), (4.51), (4.52), (4.53), and (4.55), we finish the proof of (4.45) and (4.46).

To prove (4.47) let us apply the estimate (3.29) which yields, for all $x \notin K_\delta$ and $t > 0$,

$$c \int_t^\infty \inf_{y \in K_\delta \setminus K_{\delta/2}} p_\Omega(s,x,y)ds \le 1 - \psi_K(t,x) \le C \int_t^\infty \sup_{y \in \partial K} p_U(s,x,y)ds. \tag{4.56}$$

If $x \notin K_{2\delta}$ and $t \ge |x|^2$ then (4.51) and (4.52) imply

$$\sup_{y \in \partial K} p_U(t,x,y) \le \frac{C\, h(|x|)}{V(o,\sqrt{t})h^2(\sqrt{t})}. \tag{4.57}$$

The heat kernel p_Ω admits the estimates similar to (4.50). Hence, if δ is large enough, $x \notin K_{2\delta}$, $y \in K_\delta \setminus K_{\delta/2}$ and $t \ge |x|^2$, we obtain

$$\inf_{y \in K_\delta \setminus K_{\delta/2}} p_\Omega(t,x,y) \ge \frac{c\, h(|x|)}{V(o,\sqrt{t})h^2(\sqrt{t})}. \tag{4.58}$$

Substituting (4.57) and (4.58) into (4.56) and using the identity

$$\int_t^\infty \frac{ds}{V(o,\sqrt{s})h^2(\sqrt{s})} = \frac{2}{h(\sqrt{t})}, \tag{4.59}$$

which is proved in the same way as (4.54), we obtain (4.47).

To prove (4.48), observe that $p_\Omega \le p_U$ whence by (4.50), for all $x, y \notin K$ and $t > 0$,

$$p_\Omega(t,x,y) \le C \frac{D(t,x,y)}{V(y,\sqrt{t})} \exp\left(-c\frac{d^2(x,y)}{t}\right). \tag{4.60}$$

By [9, Theorem 4], (4.60) implies

$$|\partial_t p_\Omega(t,x,y)| \le C \frac{D(t,x,y)}{tV(y,\sqrt{t})} \exp\left(-c\frac{d^2(x,y)}{t}\right).$$

In particular, for $x \notin K_{2\delta}$, $y \in K_\delta \setminus K$ we obtain

$$p_\Omega(t,x,y) \le \frac{C\, h(|x|) \exp\left(-c\frac{|x|^2}{t}\right)}{V(o,\sqrt{t})(h(|x|)+h(\sqrt{t}))h(\sqrt{t})}$$

and

$$|\partial_t p_\Omega(t,x,y)| \le \frac{C\, h(|x|) \exp\left(-c\frac{|x|^2}{t}\right)}{tV(o,\sqrt{t})(h(|x|)+h(\sqrt{t}))h(\sqrt{t})}. \tag{4.61}$$

Substituting the above estimates into (3.6), we obtain the upper bound in (4.48). If $t \ge |x|^2$, $x \notin K_{2\delta}$, $y \in K_\delta \setminus K_{\delta/2}$, and δ is large enough then (4.61) and (4.58) imply

$$|\partial_t p_\Omega(t,x,y)| \le \frac{C\, h(|x|)}{tV(o,\sqrt{t})h^2(\sqrt{t})} \le \frac{C\, h(|x|)}{\delta^2 V(o,\sqrt{t})h^2(\sqrt{t})} << p_\Omega(t,x,y).$$

Therefore, for $t \ge |x|^2$ we obtain from (3.7)

$$\partial_t \psi_K(t,x) \ge \frac{c\, h(|x|)}{V(o,\sqrt{t})h^2(\sqrt{t})}, \tag{4.62}$$

which is equivalent to the lower bound in (4.48).

Assume now $|x|^2 \geq t \geq \delta^2$. Let z be the point on a geodesic line connecting o and x, such that $|z| = \sqrt{t/2}$. Since $\partial_t \psi_k$ is a nonnegative solution to the heat equation in a $c\sqrt{t}$-neighborhood of the geodesic line connecting z and x, the parabolic Harnack inequality implies (see [23, (2.18)])

$$\partial_t \psi_K(t,x) \geq c\partial_t \psi_K(\frac{t}{2}, z) \exp\left(-C\frac{d(x,z)^2}{t}\right).$$

Applying (4.62) to estimate $\partial_t \psi_K(\frac{t}{2}, z)$, we obtain

$$\partial_t \psi_K(\frac{t}{2}, z) \geq \frac{c}{V(o,\sqrt{t})h(\sqrt{t})}$$

whence

$$\partial_t \psi_K(t,x) \geq \frac{c}{V(o,\sqrt{t})h(\sqrt{t})} \exp\left(-C\frac{|x|^2}{t}\right).$$

We are left to observe that in the range $|x|^2 \geq t$ this estimate is equivalent to the lower bound in (4.48). ∎

5 Examples

For two positive functions f, g, the relation $f \approx g$ means that there are positive constants c, C such that $c \leq f/g \leq C$, for a specified range of the arguments.

5.1 Surfaces of revolution

Consider the polar coordinates $x = (r, \theta)$ around the origin in $\mathbb{R}^2$ and the following Riemannian metric

$$dr^2 + f^2(r)d\theta^2$$

where $f(r)$ is a smooth positive function on $(0, +\infty)$. Let $M = \{(r,\theta) : r \geq 1\}$ be the manifold with boundary equipped with this metric, and let μ be the Riemannian measure on M.

Obviously, (M, o) satisfies (RCA), for any point $o \in M$. It is proved in [22] that the two sided Gaussian bound (4.27) holds on M, in particular, for the following two classes of f:

- (a) $f(r) = r^\alpha$ with $\alpha \in (-1, 1]$;
- (b) $f(r) = r(1 + \log r)^{-\beta}$ with $\beta > 0$.

We assume in the sequel that f is one of the functions in (a), (b). Observe that if $\alpha = 1$ then M is the exterior of a ball in $\mathbb{R}^2$. Let $K = \delta M = \{(r,\theta) : r = 1\}$. For any point $o \in K$ and $s \geq 1$, we have $V(o,s) \approx sf(s) \leq s^2$ so that (M,μ) is parabolic. Computing function h by (4.44), we obtain, for large values of the argument τ,

$$h(\tau) \approx \begin{cases} \tau^{1-\alpha}, & \text{case } (a),\ \alpha < 1 \\ \log \tau, & \text{case } (a),\ \alpha = 1 \\ (\log \tau)^{1+\beta}, & \text{case } (b). \end{cases}$$

Applying Theorem 4.11, we obtain the following estimates for $x = (r, \theta)$, assuming r is large enough.

Case (a), $\alpha < 1$. We have for all $t > 0$

$$c \exp\left(-C\frac{r^2}{t}\right) \leq \psi_K(t,x) \leq C \exp\left(-c\frac{r^2}{t}\right),$$

and, for all $t \geq r^2$,

$$1 - \psi_K(t,x) \asymp \left(\frac{r}{\sqrt{t}}\right)^{1-\alpha} \quad \text{and} \quad \partial_t \psi_K(t,x) \simeq \frac{1}{t}\left(\frac{r}{\sqrt{t}}\right)^{1-\alpha}.$$

Case (a), $\alpha = 1$. We have:

(i) For all $t < 2r^2$,

$$\frac{c}{\log r}\exp\left(-C\frac{r^2}{t}\right) \leq \psi_K(t,x) \leq \frac{C}{\log r}\exp\left(-c\frac{r^2}{t}\right).$$

(ii) For $t \geq 2r^2$

$$\psi_K(t,x) \simeq \frac{\log\sqrt{t} - \log r}{\log\sqrt{t}} \tag{5.1}$$

and

$$1 - \psi_K(t,x) \simeq \frac{\log r}{\log t} \quad \text{and} \quad \partial_t \psi_K \simeq \frac{\log r}{t(\log t)^2}.$$

(iii) If $t \geq 2r^2$ and in addition $a := \sqrt{t}/r = \text{const}$ then (5.1) implies

$$\psi_K(t,x) \simeq \frac{\log a}{\log r}.$$

If $t \geq r^{2+\varepsilon}$, $\varepsilon > 0$, then $\psi_K(t,x) \simeq 1$.

Case (b). We have:

(i) If $t < 2r^2$, then

$$\frac{c}{\log r}\exp\left(-C\frac{r^2}{t}\right) \leq \psi_K(t,x)\frac{C}{\log r}\exp\left(-c\frac{r^2}{t}\right).$$

(ii) If $t \geq 2r^2$ then

$$\psi_K(t,x) \simeq \frac{(\log\sqrt{t})^{1+\beta} - (\log r)^{1+\beta}}{(\log\sqrt{t})^{1+\beta}}, \tag{5.2}$$

as well as

$$1 - \psi_K(t,x) \simeq \left(\frac{\log r}{\log t}\right)^{1+\beta} \quad \text{and} \quad \partial_t \psi_K(t,x) \simeq \frac{(\log r)^{1+\beta}}{t(\log t)^{2+\beta}}.$$

(iii) If $t \geq 2r^2$ and in addition $a := \sqrt{t}/r = \text{const}$ then (5.2) implies

$$\psi_K(t,x) \simeq \frac{\log a}{\log r}.$$

If $t \geq r^{2+\varepsilon}$, $\varepsilon > 0$, then (5.2) implies $\psi_K(t,x) \asymp 1$.

5.2 Bodies of revolution

Let (r, u, v) be the Cartesian coordinates in $\mathbb{R}^3$. Given a smooth positive function $f(r)$ on $(0, +\infty)$, consider the following domain of revolution in $\mathbb{R}^3$ (see Fig. 5):

$$M = \{(r, u, v) \in \mathbb{R}^3 : r \geq 0, \ \sqrt{u^2 + v^2} \leq f(r)\}.$$

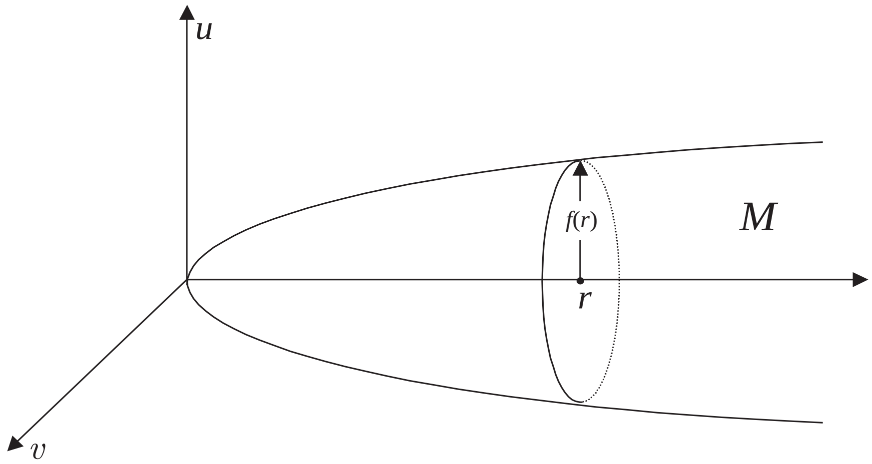

Figure 5 The domain of revolution.

If f possesses a certain regularity at $r = 0$ (in particular, $f(0) = 0$) then M can be regarded as a manifold with boundary. Let us endow M with the Euclidean metric and the Lebesgue measure μ. Assume in the sequel that f is concave, that is $f'' \leq 0$. Then M is convex as a subset of $\mathbb{R}^3$, and the result of [25] and [14] implies that M satisfies (4.27).

Let $o = (0, 0, 0)$ and

$$K = \{x = (r, u, v) \in M : 0 \leq r \leq 1\}.$$

Clearly, (M, o) satisfies (RCA) and we have for any $\tau > 0$, $V(o, \tau) \approx \tau f^2(\tau)$. Set

$$f(r) = \sqrt{r \log^{\alpha}(2 + r)}.$$

Then, for all $s \geq 1$,

$$V(o, s) \approx s^2 (1 + \log s)^{\alpha}.$$

In particular, M is parabolic if and only if $\alpha \leq 1$. We will use Theorems 4.5 and 4.11 to obtains estimates for $\psi_K(t, x)$ where $x = (r, u, v)$ and r is large enough.

Case $\alpha > 1$. In this case, (M, μ) is non-parabolic, and Theorem 4.5 gives the following estimates.

(i) If $t < r^2$ then

$$\frac{c}{(\log r)^{\alpha}} \exp\left(-C\frac{r^2}{t}\right) \leq \psi_K(t, x) \leq \frac{C}{(\log r)^{\alpha}} \exp\left(-c\frac{r^2}{t}\right).$$

(ii) If $t \geq r^2$ then

$$\psi_K(t, x) \simeq \frac{1}{(\log r)^{\alpha}} + \left[\frac{1}{(\log r)^{\alpha-1}} - \frac{1}{(\log \sqrt{t})^{\alpha-1}}\right],$$

as well as

$$\psi_K(x) - \psi_K(t, x) \simeq \frac{1}{(\log t)^{\alpha-1}} \quad \text{and} \quad \partial_t \psi_K \simeq \frac{1}{t\,(\log t)^{\alpha}}.$$

(*iii*) If $t \geq r^2$ and $a := \sqrt{t}/r = \text{const}$ then

$$\psi_K(t,x) \simeq \frac{\log a}{(\log r)^\alpha}.$$

If $t \geq r^{2+\varepsilon}$, $\varepsilon > 0$, then

$$\psi_K(t,x) \simeq \frac{1}{(\log r)^{\alpha-1}}$$

Case $\alpha < 1$. In this case (M, μ) is parabolic. Computing the function $h(r)$ by (4.44) we obtain for large τ

$$h(\tau) \simeq 1 + \int_1^\tau \frac{ds}{s(1+\log s)^\alpha} \simeq (\log \tau)^{1-\alpha}.$$

Hence, we obtain by Theorem 4.11:

(*i*) If $t < 2r^2$, then

$$\frac{c}{\log r} \exp\left(-C\frac{r^2}{t}\right) \leq \psi_K(t,x) \leq \frac{C}{\log r} \exp\left(-c\frac{r^2}{t}\right).$$

(*ii*) If $t \geq 2r^2$ then

$$\psi_K(t,x) \simeq \frac{(\log \sqrt{t})^{1-\alpha} - (\log r)^{1-\alpha}}{(\log \sqrt{t})^{1-\alpha}}$$

as well as

$$1 - \psi_K(t,x) \simeq \left(\frac{\log r}{\log t}\right)^{1-\alpha} \quad \text{and} \quad \partial_t \psi_K \simeq \frac{(\log r)^{1-\alpha}}{t(\log t)^{2-\alpha}}.$$

(*iii*) If $t \geq 2r^2$ and $a := \sqrt{t}/r = \text{const}$ then

$$\psi_K(t,x) \simeq \frac{\log a}{\log r}.$$

If $t \geq r^{2+\varepsilon}$, $\varepsilon > 0$, then $\psi_K(t,x) \simeq 1$.

Case $\alpha = 1$. Computing the function $h(r)$ by (4.44) we obtain for large τ

$$h(\tau) \simeq 1 + \int_1^\tau \frac{ds}{s(1+\log s)} \simeq \log\log\tau.$$

Theorem 4.11 then yields:

(*i*) If $t < 2r^2$, then

$$\frac{c\exp\left(-C\frac{r^2}{t}\right)}{\log r \log\log r} \leq \psi_K(t,x) \leq \frac{C\exp\left(-c\frac{r^2}{t}\right)}{\log r \log\log r}.$$

(*ii*) If $t \geq 2r^2$ then

$$\psi_K(t,x) \simeq \frac{\log\log\sqrt{t} - \log\log r}{\log\log\sqrt{t}}$$

as well as

$$1 - \psi_K(t,x) \simeq \frac{\log\log r}{\log\log t} \quad \text{and} \quad \partial_t \psi_K(t,x) \simeq \frac{\log\log r}{t\log t(\log\log t)^2}.$$

(ii) Let $t \geq 2r^2$. If $a := \sqrt{t}/r = \text{const}$ then

$$\psi_K(t,x) \simeq \frac{\log a}{\log r \log\log r}.$$

If $a := \log \sqrt{t} / \log r = \text{const}$ then

$$\psi_K(t,x) \simeq \frac{\log a}{\log\log r}.$$

If $\log \sqrt{t} \geq (\log r)^{1+\varepsilon}$, $\varepsilon > 0$, then $\psi_K(t,x) \simeq 1$.

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