Heat kernels on manifolds with ends

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Heat kernel in $\mathbb{R}^n$

Heat equation in $\mathbb{R}^n$:

$$\partial_t u = \Delta u$$

where $u = u(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\Delta = \sum_{i=1}^n \partial_{x_i x_i} u$ is the Laplace operator.

The *Gauss-Weierstrass* function

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{1}$$

satisfies the heat equation in $(t, x) \in (0, \infty) \times \mathbb{R}^n$ and tends to δ_y as $t \to 0+$.

The function (1) is called the *heat kernel* or the *fundamental solution* of the heat equation. Other characterizations:

- the integral kernel of the heat semigroup $\left\{e^{t\Delta}\right\}_{t\geq 0}$ in $L^2(\mathbb{R}^n)$;
- the density of the normal distribution with the mean y and variance $2t$;
- the transition density of Brownian motion in $\mathbb{R}^n$.

Observe: if $|x-y| = O\left(\sqrt{t}\right)$ then $p_t(x, y) \simeq t^{-n/2}$.

Elliptic operators in divergence form

Consider in $\mathbb{R}^n$ a *divergence form* elliptic operator

$$L = \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij}(x) \, \partial_{x_j} \right)$$

where the matrix $(a_{ij}(x))_{i.j=1}^{n}$ is symmetric and positive definite. Uniform ellipticity: there is $\lambda \geq 1$ such that, for any x, all the eigenvalues of $(a_{ij}(x))$ lie in $\left[\lambda^{-1}, \lambda\right]$.

Theorem 1 (D.G. Aronson '67) *The fundamental solution $p_t(x,y)$ of $\partial_t u = Lu$ satisfies for all $t > 0$ and $x, y \in \mathbb{R}^n$ the estimates*

$$p_t(x,y) \asymp \frac{C}{t^{n/2}} \exp\left(-\frac{|x-y|^2}{ct} \right)$$

where $C, c > 0$ depend on n, λ only and $\asymp$ means $\leq$ and $\geq$, but with different values of C, c.

The proof is based on the previous works of Jürgen Moser and John Nash.

Laplace-Beltrami operator and Li-Yau estimate

Given a Riemannian manifold (M, g), the *Laplace-Beltrami* operator Δ_g is defined in local coordinates $x_1, ..., x_n$ by

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_{x_i} \left(g^{ij} \sqrt{\deg g} \partial_{x_j} \right) = \operatorname{div}_g \circ \nabla,$$

where $g^{ij} = (g_{ij})^{-1}$ and $g = (g_{ij})$.

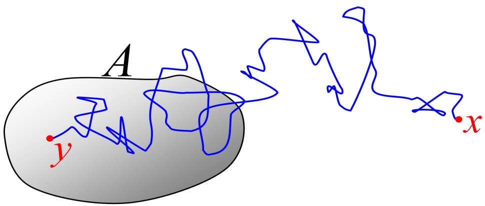
Heat equation $\partial_t u = \Delta_g u$ on $\mathbb{R}_+ \times M$ has the minimal positive fundamental solution $p_t(x, y)$ that is called the *heat kernel* of M. The heat kernel is also:

- the integral kernel of $\left\{ e^{t\Delta_g} \right\}_{t \geq 0}$ in $L^2(M, \mu)$, where μ is Riemannian measure;
- the transition density for Brownian motion $\{X_t\}_{t \geq 0}$ on M:

for any Borel set $A \subset M$,

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y) \, d\mu(y),$$

where $\mathbb{P}_x$ is the probability measure in the space of paths started at x.



Goal: estimates of the heat kernel on a class of Riemannian manifolds.
Notation:
M - a geodesically complete, non-compact Riemannian manifold;
$d(x, y)$ - the geodesic distance on M;
$B(x, r)$ - the geodesic ball of radius r centered at x, and $V(x, r) = \mu(B(x, r))$.

Theorem 2 (E.B. Davies '92) *For arbitrary measurable sets $A, B \subset M$,*

$$\int_A \int_B p_t(x, y)\, d\mu(x)\, d\mu(y) \leq \sqrt{\mu(A)\,\mu(B)} \exp\left(-\frac{d^2(A, B)}{4t}\right).$$

Assumptions about the geometry of M are needed in order to obtain pointwise estimates with a decay as $t \to \infty$.

Theorem 3 (P.Li and S.-T.Yau '86) *If $Ricci_M \geq 0$ then, for some $c, C > 0$ and all $x, y \in M$ and $t > 0$,*

$$p_t(x, y) \asymp \frac{C}{V\left(x, \sqrt{t}\right)} \exp\left(-\frac{d^2(x, y)}{ct}\right). \tag{LY}$$

In $\mathbb{R}^n$: $V\left(x, \sqrt{t}\right) = ct^{n/2}$ so that (LY) matches (1).

The estimate (LY) holds also on a more general class of manifolds.

Definition. We say that M satisfies *volume doubling condition* if for all $x \in M$ and $r > 0$

$$V(x, 2r) \leq CV(x, r). \tag{VD}$$

Definition. We say that M satisfies the (*weak*) *Poincaré inequality* if there are constants $C > 0$ and $\varepsilon \in (0, 1]$ such that, for any ball $B(x, r)$ and for any function $u \in C^1(B(x, r))$,

$$\inf_{s \in \mathbb{R}} \int_{B(x, \varepsilon r)} (u - s)^2 \, d\mu \leq Cr^2 \int_{B(x,r)} |\nabla u|^2 \, d\mu. \tag{PI}$$

For example, (PI) holds in $\mathbb{R}^n$ with $\varepsilon = 1$.

Theorem 4 (AG '91, L.Saloff-Coste '92)

$$(LY) \Leftrightarrow (VD) + (PI).$$

Theorem 4 implies Theorem 3 as both (VD) and (PI) can be proved on manifolds with *Ricci* ≥ 0.

Theorem 4 allows to obtain further examples of manifolds satisfying (LY).

Let (r, θ) be the polar coordinates on $\mathbb{R}^n$ with $n \geq 2$, where $r > 0$ and $\theta \in \mathbb{S}^{n-1}$. The canonical metric of $\mathbb{R}^n$ is $dr^2 + r^2 d\theta^2$ where $d\theta^2$ is the canonical metric on $\mathbb{S}^{n-1}$.

Fix a real $\alpha > 0$ and define a Riemannian metric g_α on $\mathbb{R}^n$ by

$$g_\alpha = \begin{cases} dr^2 + r^2 d\theta^2 & r \ll 1, \\ dr^2 + r^{2\beta} d\theta^2 & r \gg 1, \end{cases}$$

where $\beta = \frac{\alpha - 1}{n - 1}$. Set

$$\mathcal{R}^\alpha := (\mathbb{R}^n, g_\alpha).$$

It is easy to verify that on $\mathcal{R}^\alpha$,

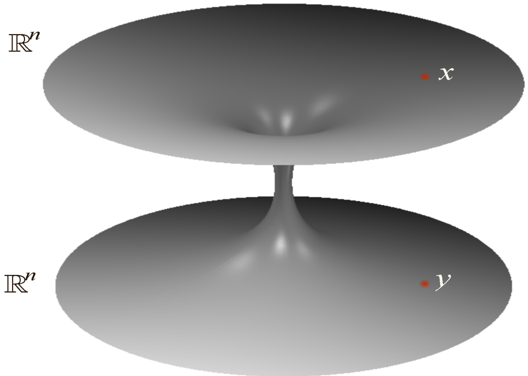
$$V(o, r) \simeq r^\alpha \quad \text{for } r \gg 1.$$

The number α is called "the dimension at ∞" of $\mathcal{R}^\alpha$, while the topological dimension of $\mathcal{R}^\alpha$ is n. Note that $\mathcal{R}^n = \mathbb{R}^n$ while $\mathcal{R}^1$ is a (one-sided) cylinder:

Proposition 5 *The heat kernel on $\mathcal{R}^\alpha$ satisfies (LY) provided $0 < \alpha \leq n$.*

An example where (LY) fails

Let $M = \mathbb{R}^n \# \mathbb{R}^n$ be a connected sum of two copies of $\mathbb{R}^n$ with $n \geq 3$. On this manifold $V(x, r) \simeq r^n$.



The heat kernel on M satisfies the *upper* bound of (LY) but the *lower* bound

$$p_t(x, y) \geq \frac{C}{t^{n/2}} \exp\left(-\frac{d^2(x, y)}{ct}\right)$$

fails if x and y belong to different copies of $\mathbb{R}^n$: the probability of getting from x to y become smaller because any path from x to y has to go through the *bottleneck* of the central part. *This example is a major motivation for what follows.*

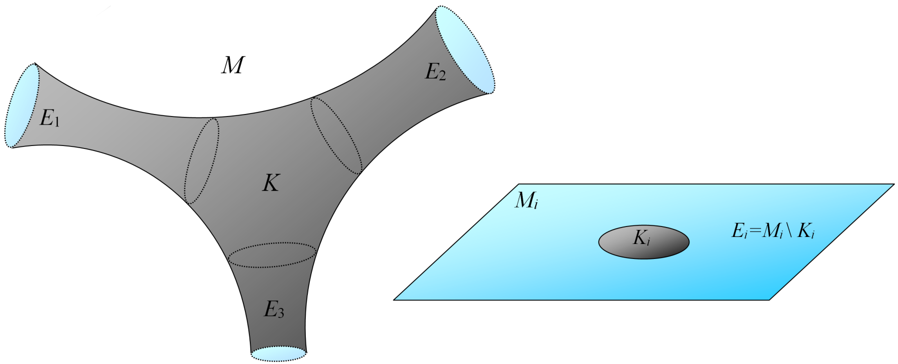
Manifolds with ends

Let $M_1, ..., M_k$ and M be complete, connected, non-compact Riemannian manifolds of the same dimension n. We say that M is a *connected sum* of $M_1, ..., M_k$ and write

$$M = M_1 \# M_2 \# ... \# M_k$$

if $M = K \sqcup E_1 \sqcup ... \sqcup E_k$, where $K \subset M$ is compact and each E_i is isometric to an exterior domain in M_i.

The sets E_i (as well as manifolds M_i) are called the *ends* of M. Manifold M is referred to as a manifold with ends.



The question to be discussed here is:

Assuming that all M_i are complete and satisfy (LY),
how to estimate the heat kernel on $M = M_1 \# M_2 \# ... \# M_k$?

For example, how to estimate the heat kernel on

$$M = \mathcal{R}^{\alpha_1} \# \mathcal{R}^{\alpha_2} \# ... \# \mathcal{R}^{\alpha_k} \ ?$$

Here we assume that all $\mathcal{R}^{\alpha_i}$ have the same topological dimension n and that $0 < \alpha_i \leq n$, so that each $\mathcal{R}^{\alpha_i}$ satisfies (LY).

Even obtaining the heat kernel estimates on $M = \mathbb{R}^n \# \mathbb{R}^n$ is highly non-trivial!

The answer to the above question depends on the property of the ends M_i to be *parabolic* or not.

Parabolic and non-parabolic manifolds

Definition. A Riemannian manifold M is called *parabolic* if any positive superharmonic function on M is constant, and *non-parabolic* otherwise.

Equivalent characterizations of the parabolicity:

- there exists no positive fundamental solution of $-\Delta$;
- $\int^\infty p_t(x,y)\,dt = \infty$ for all/some $x, y \in M$;
- Brownian motion on M is recurrent.

For example, $\mathbb{R}^n$ is parabolic for $n \le 2$ and non-parabolic for $n > 2$.

Proposition 6 *Let M be geodesically complete and satisfy (LY). Then M is parabolic if and only if for all/some $x \in M$*

$$\int^\infty \frac{r\,dr}{V(x,r)} = \infty. \tag{2}$$

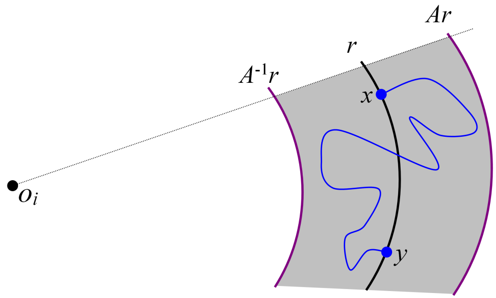
For example, if $V(x,r) \simeq r^\alpha$ for large r, then (2) is satisfies if and only if $\alpha \le 2$. In particular, $\mathcal{R}^\alpha$ is parabolic if and only if $\alpha \le 2$.

Heat kernels on manifolds with ends

Let $M_1, ..., M_k$ be complete non-compact manifolds satisfying (LY). Fix a reference point $o_i \in M_i$ and set $|x| = d_i(x, o_i)$. Assume for simplicity that

$$V_i(o_i, r) \simeq r^{\alpha_i} \text{ for large } r.$$

If M_i is parabolic M_i, then assume in addition that M_i has "*relatively connected annuli*": there is $A > 1$ such that, for all large r and all x, y with $|x| = |y| = r$, the points x, y can be connected by a curve in the annulus $B_i(o_i, Ar) \setminus B_i(o_i, A^{-1}r)$.



Clearly, any $\mathcal{R}^\alpha = (\mathbb{R}^n, g_\alpha)$ with $n \geq 2$ satisfies this property, but $\mathbb{R}^1$ does not.

We present estimates of the heat kernel $p_t(x,y)$ on $M = M_1 \# ... \# M_k$ assuming that $x \in E_i$, $y \in E_j$ with $i \neq j$ and that $|x|, |y|, t$ are large. Estimates for the entire range of t, x, y are available as well.

Non-parabolic case (all M_i are non-parabolic)

Theorem 7 (AG and L.Saloff-Coste '09) *Assume that all $\alpha_i > 2$ and set*

$$\alpha = \min_{1 \leq i \leq k} \alpha_i \ .$$

For $x \in E_i$ and $y \in E_j$ with $i \neq j$ we have

$$p_t(x,y) \asymp C \left(\frac{1}{t^{\alpha/2} |x|^{\alpha_i - 2} |y|^{\alpha_j - 2}} + \frac{1}{t^{\alpha_j/2} |x|^{\alpha_i - 2}} + \frac{1}{t^{\alpha_i/2} |y|^{\alpha_j - 2}} \right) e^{-\frac{d^2(x,y)}{ct}}. \qquad (3)$$

In particular, (3) holds for $M = \mathcal{R}^{\alpha_1} \# ... \# \mathcal{R}^{\alpha_k}$ provided all $\alpha_i > 2$.
For $M = \mathbb{R}^n \# \mathbb{R}^n$ with $n > 2$, the estimate (3) becomes

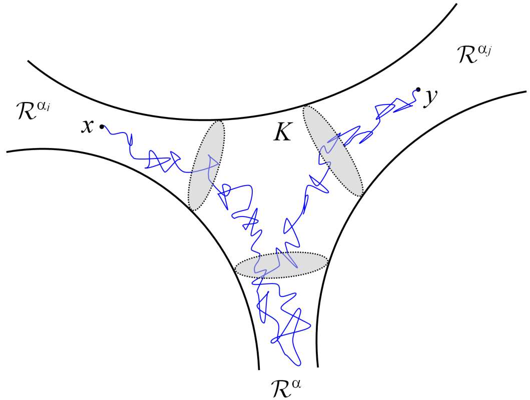
$$p_t(x,y) \asymp \frac{C}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}} \right) e^{-\frac{d^2(x,y)}{ct}}.$$

Long time regime: x, y are fixed and $t \to \infty$. Then (3) amounts to

$$p_t(x,y) \simeq t^{-\alpha/2}. \tag{4}$$

Hence, the long time decay of p_t is determined by the *minimal* volume growth exponent $\alpha = \min \alpha_i$. Note that $V(x,r) \simeq r^{\max \alpha_i}$.

The estimate (4) has the following probabilistic meaning: in order to get from x to y in time t, Brownian motion on M spends most time on the *smallest* end $\mathcal{R}^\alpha$. The reason for that is that the return probability in that end is the *largest*.



Medium time regime: $|x| \simeq |y| \simeq \sqrt{t} \to \infty$. Then (3) implies

$$p_t(x,y) \simeq t^{-\left(\frac{\alpha_i+\alpha_j}{2}-1\right)}.$$

Since $\frac{\alpha_i+\alpha_j}{2} - 1 > \frac{\alpha}{2}$, we obtain $p_t(x,y) \ll t^{-\alpha/2}$, which is due to a *bottleneck effect*.

Mixed case (there are parabolic and non-parabolic M_i)

Theorem 8 *Assume that all $\alpha_i \neq 2$ and there are values $\alpha_i > 2$ and $\alpha_i < 2$. Set*

$$\widetilde{\alpha}_i := \begin{cases} 4-\alpha_i, & \alpha_i < 2 \\ \alpha_i, & \alpha_i > 2 \end{cases}$$

and

$$\alpha := \min_{1\le i\le k} \widetilde{\alpha}_i.$$

For $x \in E_i$ and $y \in E_j$ with $i \neq j$ we have

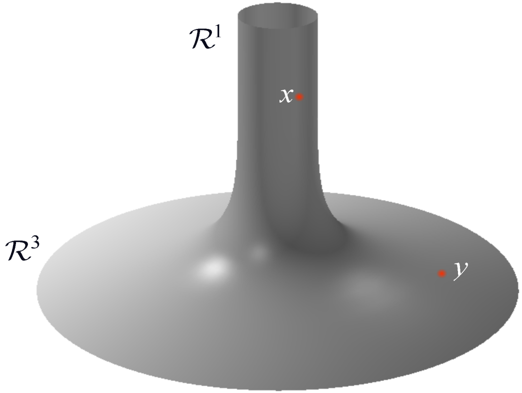
$$p_t(x,y) \simeq C\left(\frac{1}{t^{\alpha/2}\,|x|^{\widetilde{\alpha}_i-2}\,|y|^{\widetilde{\alpha}_j-2}} + \frac{1}{t^{\widetilde{\alpha}_i/2}\,|y|^{\widetilde{\alpha}_j-2}} + \frac{1}{t^{\widetilde{\alpha}_j/2}\,|x|^{\widetilde{\alpha}_i-2}}\right) \times |x|^{(2-\alpha_i)_+}\ |y|^{(2-\alpha_j)_+}\ e^{-\frac{d^2(x,y)}{ct}} \tag{5}$$

This theorem contains the estimate (3) of non-parabolic case because if $\alpha_i > 2$ then $\widetilde{\alpha}_i = \alpha_i$ and $|x|^{(2-\alpha_i)_+} = 1$.

Observe that always $\widetilde{\alpha}_i > 2$, and the minimal $\widetilde{\alpha}_i$ is determined by the value of α_i that is *nearest* to 2! **Hence, the long time decay of the heat kernel $p_t(x,y) \simeq t^{-\alpha/2}$ is determined by the nearest to 2 value of α_i.**

This rules applies also to Theorem 7 where the nearest to 2 exponent α_i is the minimal one. As we will see below, this rule is valid also in the parabolic case.

As an example, consider $M = \mathcal{R}^1 \# \mathcal{R}^3$, where $x \in \mathcal{R}^1$ and $y \in \mathcal{R}^3$.



In this case $\alpha_1 = 1$, $\alpha_2 = 3$ whence $\widetilde{\alpha}_1 = \widetilde{\alpha}_2 = 3$. It follows from (5) that

$$p_t(x,y) \asymp \frac{C}{t^{3/2}} \left(1 + \frac{|x|}{|y|}\right) e^{-\frac{d^2(x,y)}{ct}}.$$

For $t \to \infty$ we obtain $p_t(x,y) \simeq t^{-3/2}$. In the case $|y| \simeq 1$, $|x| \simeq \sqrt{t} \to \infty$ we obtain $p_t(x,y) \simeq t^{-1} \gg t^{-3/2}$ – a kind of anti-bottleneck effect!

Parabolic case (all M_i are parabolic)

The next two theorems were obtained by AG, S.Ishiwata and L.Saloff-Coste in 2015.

Theorem 9 (Subcritical case) *Assume that* $0 < \alpha_i < 2$ *for all* $i = 1, ..., k$ *and set*

$$\alpha = \max_{1 \leq i \leq k} \alpha_i \ .$$

For $x \in E_i$ *and* $y \in E_j$ *with* $i \neq j$ *we have*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/2}} e^{-\frac{d^2(x,y)}{ct}}.$$

In this case the long time behavior of the heat kernel $p_t(x, y) \simeq t^{-\alpha/2}$ is determined by the *maximal* volume growth exponent α_i, which is again nearest to 2. There is no bottleneck effect in this case.

In the next statement we use the following notation:

$$Q(x, t) = \frac{1}{\ln |x|} + \frac{1}{\ln t}\left(\ln \frac{\sqrt{t}}{|x|}\right)_+ \simeq \begin{cases} \frac{1}{\ln |x|}, & \text{if } |x| \geq \sqrt{t} \\ \frac{1}{\ln t} \ln \frac{e\sqrt{t}}{|x|}, & \text{if } |x| \leq \sqrt{t}, \end{cases}$$

Theorem 10 (Critical case) *Assume that* $0 < \alpha_i \leq 2$ *for all* $i = 1, ..., k$ *and that* $\alpha_l = 2$ *for some* l*. For* $x \in E_i$ *and* $y \in E_j$ *with* $i \neq j$ *the following is true:*

(a) *If* $\alpha_i < 2$ *and* $\alpha_j < 2$ *then in the case* $|x| + |y| \geq \sqrt{t}$

$$p_t(x, y) \asymp \frac{C \ln t}{t} e^{-\frac{d^2(x,y)}{ct}},$$

and in the case $|x| + |y| < \sqrt{t}$

$$p_t(x, y) \asymp \frac{C}{t} \left(1 + \ln t \, \left[\left(\frac{|x|}{\sqrt{t}} \right)^{2-\alpha_i} + \left(\frac{|y|}{\sqrt{t}} \right)^{2-\alpha_j} \right] \right).$$

(b) *If* $\alpha_i = 2$ *and* $\alpha_j < 2$ *then*

$$p_t(x, y) \asymp \frac{C}{t} \left(1 + Q(x, t) \ln t \, \left(\frac{|y|}{|y| + \sqrt{t}} \right)^{2-\alpha_j} \right) e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, if $|x|, |y| \geq \sqrt{t}$ *then*

$$p_t(x, y) \asymp \frac{C}{t} \left(1 + \frac{\ln t}{\ln |x|} \right) e^{-\frac{d^2(x,y)}{ct}}$$

and if $|x|, |y| \leq \sqrt{t}$ *then*

$$p_t(x,y) \asymp \frac{C}{t}\left(1 + \ln \frac{e\sqrt{t}}{|x|}\left(\frac{|y|}{\sqrt{t}}\right)^{2-\alpha_j}\right).$$

(c) *If* $\alpha_i = \alpha_j = 2$ *then*

$$p_t(x,y) \asymp \frac{C}{t}\left(Q(x,t)\,Q(y,t) + Q(x,t)\frac{\ln|y|}{\ln|y| + \ln t} + Q(y,t)\frac{\ln|x|}{\ln|x| + \ln t}\right) e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, if $|x|, |y| \geq \sqrt{t}$ *then*

$$p_t(x,y) \asymp \frac{C}{t}\left(\frac{1}{\ln|x|} + \frac{1}{\ln|y|}\right) e^{-\frac{d^2(x,y)}{ct}},$$

and if $|x|, |y| \leq \sqrt{t}$ *then*

$$p_t(x,y) \asymp \frac{C}{t\ln^2 t}\left(\ln\frac{e\sqrt{t}}{|x|}\ln\frac{e\sqrt{t}}{|y|} + \ln|y|\ln\frac{e\sqrt{t}}{|x|} + \ln|x|\ln\frac{e\sqrt{t}}{|y|}\right).$$

Note that in the setting of Theorem 10 the long time behavior of the heat kernel is simple:

$$p_t(x,y) \simeq \frac{1}{t} \simeq \frac{1}{V\left(o,\sqrt{t}\right)} \quad \text{as } t \to \infty,$$

and is determined by the value $\alpha_l = 2$, which is again the nearest to 2 volume growth exponent.

In the medium time regime $|x| \simeq |y| \simeq \sqrt{t} \to \infty$, we have the following.

In the case (a), that is, $\alpha_i, \alpha_j < 2$:

$$p_t(x,y) \simeq \frac{\ln t}{t}.$$

In the case (b), that is, $\alpha_i = 2$, $\alpha_j < 2$:

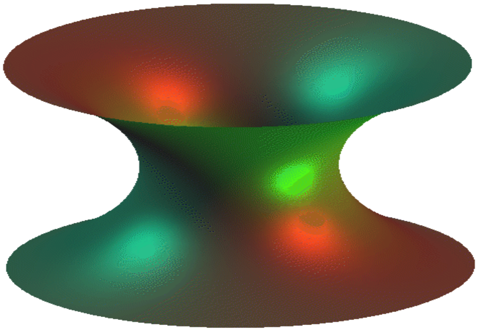
$$p_t(x,y) \simeq \frac{1}{t}.$$

In the case (c), that is, $\alpha_i = \alpha_j = 2$:

$$p_t(x,y) \simeq \frac{1}{t\ln t}.$$

Some examples

Let $M = \mathbb{R}^2 \# \mathbb{R}^2$.
This manifold is equivalent to the catenoid. Let x, y belong to the different sheets.



Then by Theorem 10(c) we have

$$p_t(x,y) \simeq \frac{C}{t}\left(Q(x,t)Q(y,t) + Q(x,t)\frac{\ln|y|}{\ln|y| + \ln t} + Q(y,t)\frac{\ln|x|}{\ln|x| + \ln t}\right)e^{-\frac{d^2(x,y)}{ct}}.$$

If $t \to +\infty$ then $p_t(x,y) \simeq t^{-1}$.
If $|x| \geq \sqrt{t}$ and $|y| \geq \sqrt{t}$ then

$$p_t(x,y) \asymp \frac{C}{t}\left(\frac{1}{\ln|x|} + \frac{1}{\ln|y|}\right)e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, if $|x| \simeq |y| \simeq \sqrt{t}$ then $p_t(x,y) \simeq \frac{1}{t \ln t}$.

Let $M = \mathcal{R}^1 \# \mathcal{R}^2$. By Theorem 10$(b)$ we obtain, for $x \in \mathcal{R}^1$ and $y \in \mathcal{R}^2$,

$$p_t(x,y) \asymp \frac{C}{t} \left(1 + \ln t \frac{|x|}{|x| + \sqrt{t}} Q(y,t) \right) e^{-\frac{d^2(x,y)}{ct}}$$

If $|x|, |y| > \sqrt{t}$ then

$$p_t(x,y) \asymp \frac{C}{t} e^{-\frac{d^2(x,y)}{ct}},$$

If $|x|, |y| \leq \sqrt{t}$ then

$$p_t(x,y) \simeq \frac{1}{t} \left(1 + \frac{|x|}{\sqrt{t}} \ln \frac{e\sqrt{t}}{|y|} \right).$$

For $t \to \infty$ we obtain

$$p_t(x,y) \simeq t^{-1}.$$

If $y \simeq 1$ and $|x| \simeq \sqrt{t} \to \infty$ then

$$p_t(x,y) \simeq \frac{\ln t}{t}.$$

Let $M = \mathcal{R}^2 \# \mathcal{R}^3$. This is a mixed case that is covered by an extension of Theorem 8. It yields the following estimate for $x \in \mathcal{R}^2$ and $y \in \mathcal{R}^3$:

$$p_t(x,y) \asymp C\left(\frac{\ln|x|}{t\ln^2 t\ \ |y|} + \frac{1}{t^{3/2}} Q\left(x,t\right)\right) e^{-\frac{d^2(x,y)}{ct}}.$$

For $t \to \infty$ we have

$$p_t\left(x,y\right) \simeq \frac{1}{t\ln^2 t}.$$

For $|x| \simeq |y| \simeq \sqrt{t} \to \infty$ we obtain

$$p_t\left(x,y\right) \simeq \frac{1}{t^{3/2}\ln t},$$

so that there is a bottleneck effect. For $|y| \simeq 1$ and $|x| \simeq \sqrt{t} \to \infty$ we obtain

$$p_t\left(x,y\right) \simeq \frac{1}{t\ln t},$$

that is, an anti-bottleneck effect.

Let $M = \mathcal{R}^1 \# \mathcal{R}^2 \# \mathcal{R}^3$. For $x \in \mathcal{R}^1$ and $y \in \mathcal{R}^2$ we have

$$p_t(x,y) \simeq C\left(\frac{\ln|y|}{t\ln^2 t} + \left(\frac{|x|}{t^{3/2}} + \frac{1}{t\ln^2 t}\right) Q(y,t)\right) e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, for $t \to \infty$

$$p_t(x,y) \simeq \frac{1}{t\ln^2 t},$$

For $|x| \simeq |y| \simeq \sqrt{t}$ we have an anti-bottleneck effect:

$$p_t(x,y) \simeq \frac{1}{t\ln t}.$$

For $x \in \mathcal{R}^1$ and $y \in \mathcal{R}^3$ we have

$$p_t(x,y) \asymp C\left(\frac{1}{t^{3/2}}\left(1 + \frac{|x|}{|y|}\right) + \frac{1}{|y|\, t\ln^2 t}\right) e^{-\frac{d^2(x,y)}{ct}}.$$

For $|x| \simeq |y| \simeq \sqrt{t}$ we have a bottleneck effect:

$$p_t(x,y) \simeq \frac{1}{t^{3/2}}.$$

Approach to the proof

The following approach works in non-parabolic case (Theorem 7) and in parabolic case (Theorems 9, 10).

We start with estimates for $p_t(o,o)$ where $o \in K$ is a fixed reference point. In the non-parabolic case we use *Faber-Krahn type* inequalities to obtain upper bound of $p_t(o,o)$. The Li-Yau upper bound for the heat kernel $p_t^{(i)}$ on M_i implies certain FK inequality on M_i. The "weakest" of FK inequalities across all ends M_i gives a FK inequality on M, which implies the upper bound of $p_t(o,o)$, matching the weakest upper bound among all $p_t^{(i)}(o_i,o_i)$.

For the lower bounds of $p_t(x,y)$ we use $p_t(x,y) \geq p_t^{E_i}(x,y)$, where $p_t^{E_i}$ is the Dirichlet heat kernel in E_i. By non-parabolicity of M_i, $p_t^{E_i}(x,y)$ satisfies (LY) away from ∂E_i, which implies the lower bound of $p_t(o,o)$ matching the strongest lower bound among all $p_t^{(i)}(o_i,o_i)$.

To estimate $p_t(x,y)$ for arbitrary x,y, we use the *hitting probability.* For any closed set $A \subset M$, define the function

$$\psi_A(t,x) = \mathbb{P}_x(X_s \in A \text{ for some } s \leq t)$$

In fact, $\psi_A(t,x)$ solves in $\mathbb{R}_+ \times A^c$ the heat equation with the initial condition $\psi_A(0,\cdot) = 0$ and the boundary condition $\psi_A(t,\cdot) = 1$ on ∂A.

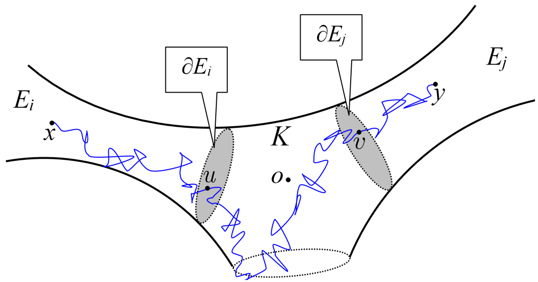
For all $x \in E_i$ and $y \in E_j$ with $i \neq j$, the following holds:

$$\begin{aligned} p_t(x,y) &\leq 2\psi_{\partial E_i}(t,x)\psi_{\partial E_j}(t,y) \sup_{s\in[t/4,t]} \sup_{u\in\partial E_i, v\in\partial E_j} p_s(u,v) \\ &+ \left(\psi_{\partial E_i}(t,x) \sup_{s\in[t/4,t]} \partial_s\psi_{\partial E_j}(s,y) + \psi_{\partial E_j}(t,y) \sup_{s\in[t/4,t]} \partial_s\psi_{\partial E_i}(s,x)\right) \\ &\times \int_0^t \sup_{u\in\partial E_i, v\in\partial E_j} p_s(u,v)ds, \end{aligned}$$

and there is a similar lower bound.

Note that $\psi_{\partial E_i}$ depends only on the intrinsic geometry of M_i and can be estimated using (LY) on M_i.

By local Harnack inequality, $p_s(u,v)$ can be estimated via $p_s(o,o)$, which gives desired estimates for $p_t(x,y)$



In the parabolic case this scheme works except for the crucial upper bound for $p_t(o,o)$. Indeed, the FK method gives the upper bound of $p_t(o,o)$ using the smallest volume growth exponent α_i whereas in the parabolic case we expect to use the largest exponent α_i, that is, we need a stronger upper bound.

In fact, in the parabolic case we prove the following upper bound:

$$p_t(o,o) \leq \frac{C}{V\left(o,\sqrt{t}\right)}, \tag{6}$$

using a new method involving the *resolvents* on each end:

$$R_\lambda^{(i)}(x,y) = \int_0^\infty e^{-t\lambda} p_t^{(i)}(x,y)\,dt,$$

where $\lambda > 0$. The parabolicity of M_i implies that $R_\lambda^{(i)}(x,y) \to \infty$ as $\lambda \to 0$, and the rate of increase of $R_\lambda^{(i)}(x,y)$ as $\lambda \to 0$ is related to the rate of decay of $p_t^{(i)}(x,y)$ as $t \to \infty$.

We show that the resolvent $R_\lambda(x,y)$ on M satisfies a certain integral equation containing $R_\lambda^{(i)}(x,y)$. This allows to estimate the rate of growth of $R_\lambda(x,y)$ as $\lambda \to 0$ and then to recover the upper bound (6). In the critical case we use also the estimates of $\partial_\lambda R_\lambda(x,y)$.

Once the upper bound (6) is known, it implies automatically the matching lower bound

$$p_t(o,o) \geq \frac{c}{V\left(o,\sqrt{t}\right)},$$

by a theorem of AG and T.Coulhon '97.

Finally, the mixed case of Theorem 8 can be reduced to the non-parabolic case by a *Doob transform.* We construct a positive harmonic function h on $M = M_1\#...\#M_k$ such that $h \to \infty$ on each parabolic end and $h \simeq 1$ on each non-parabolic end. Consider a new measure $\widetilde{\mu}$ on M given by $d\widetilde{\mu} = h^2 d\mu$, where μ is the Riemannian measure, and the associated *weighted Laplacian*

$$\widetilde{\Delta} = \frac{1}{h^2}\operatorname{div}\left(h^2\nabla\right) = \frac{1}{h}\circ\Delta\circ h.$$

The heat kernel $\widetilde{p}_t(x,y)$ of $\widetilde{\Delta}$ is related to $p_t(x,y)$ by

$$p_t(x,y) = \widetilde{p}_t(x,y)\,h(x)\,h(y).$$

It turns out that each weighted manifold $(M_i,\widetilde{\mu})$ satisfies (LY) and has the volume growth exponent $\widetilde{\alpha}_i > 2$. In particular, $(M_i,\widetilde{\mu})$ is non-parabolic! By extension of Theorem 7 to weighted manifolds, we obtain the estimates of $\widetilde{p}_t(x,y)$, whence the estimates of $p_t(x,y)$ follow.