Path homology and Hodge Laplacian on digraphs

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Wuppertal, July 2023

Contents

1 Motivation

Let M be a compact Riemannian manifold. The Hodge Laplace operator Δ_p of dimension $p \geq 0$ acts in the space Ω^p of differential p-form on M as follows:

$$\Delta_p \omega = d^* d\omega + dd^* \omega \,,$$

where d is the exterior derivative from the de Rham cochain complex:

$$0 \quad \longrightarrow \quad \Omega^0 \quad \overset{d}{\longrightarrow} \quad \Omega^1 \quad \overset{d}{\longrightarrow} \quad \cdots \quad \overset{d}{\longrightarrow} \quad \Omega^n \quad \overset{d}{\longrightarrow} \quad \Omega^{n+1} \quad \overset{d}{\longrightarrow} \cdots$$

and d^* is the adjoint operator:

$$0 \quad \longleftarrow \quad \Omega^0 \quad \overset{d^*}{\longleftarrow} \quad \Omega^1 \quad \overset{d^*}{\longleftarrow} \quad \cdots \quad \overset{d^*}{\longleftarrow} \quad \Omega^n \quad \overset{d^*}{\longleftarrow} \quad \Omega^{n+1} \quad \overset{d^*}{\longleftarrow} \cdots$$

Our purpose is to define similar notions on digraphs (directed graphs): a chain complex and the corresponding Hodge Laplacian, as well as to investigate the spectral properties of the latter.

2 Chain spaces and path homology on digraphs

2.1 Paths and the boundary operator

Let us fix a finite set V and a field $\mathbb{K}$. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V; it will be denoted by $e_{i_0 \ldots i_p}$.
A *p-path* is any formal linear combinations of elementary p-paths with coefficients from $\mathbb{K}$; that is, any p-path u has a form

$$u = \sum_{i_0, i_1, \ldots, i_p \in V} u^{i_0 i_1 \ldots i_p} e_{i_0 i_1 \ldots i_p},$$

where $u^{i_0 i_1 \ldots i_p} \in \mathbb{K}$. The set of all p-paths is a $\mathbb{K}$-linear space denoted by $\Lambda_p = \Lambda_p(V, \mathbb{K})$.

For example, $\Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \quad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle.$

Definition. Define for any $p \geq 1$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \ldots \widehat{i_q} \ldots i_p}, \tag{1}$$

where $\widehat{\ }$ means omission of the index. For $p = 0$ set $\partial e_i = 0$ (and, hence, $\Lambda_{-1} = \{0\}$).

For example,

$$\partial e_{ij} = e_j - e_i \quad \text{and} \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}.$$

It is easy to show that $\partial^2 = 0$. Hence, we obtain a chain complex $\Lambda_*(V)$:

$$0 \leftarrow \Lambda_0 \overset{\partial}{\leftarrow} \Lambda_1 \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \Lambda_{p-1} \overset{\partial}{\leftarrow} \Lambda_p \overset{\partial}{\leftarrow} \cdots$$

An elementary p-path $e_{i_0 \ldots i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and *irregular* otherwise. A p-path is called regular (resp. irregular) if it is a linear combination of regular (resp. irregular) elementary paths.

Denote by $\mathcal{R}_p$ the space of all regular p-paths. Then ∂ is well defined on the spaces $\mathcal{R}_p$ if we identify all irregular paths with 0 (which is justified by the fact that if u is irregular then ∂u is also irregular). For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij} \in \mathcal{R}_1,$$

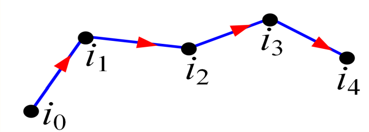
because $e_{ii} = 0$. Hence, we obtain a regular chain complex

$$0 \leftarrow \mathcal{R}_0 \overset{\partial}{\leftarrow} \mathcal{R}_1 \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \cdots$$

2.2 Chain complex on digraphs

A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and $E \subset \{V \times V \setminus \text{diag}\}$ is a set of arrows (directed edges). If $(i, j) \in E$ then we write $i \to j$.

Definition. An elementary p-path $e_{i_0 \dots i_p}$ in a digraph $G = (V, E)$ is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, \dots, p-1$,
and *non-allowed* otherwise.



A p-path is called allowed if it is a linear combination of allowed elementary p-paths.

Denote by $\mathcal{A}_p = \mathcal{A}_p(G, \mathbb{K})$ the linear space of all allowed p-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on spaces $\mathcal{A}_p$. However, in general ∂ does not act on the spaces $\mathcal{A}_p$. For example, in the digraph $\overset{a}{\bullet} \to \overset{b}{\bullet} \to \overset{c}{\bullet}$ we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is not allowed.

Consider the following subspace of $\mathcal{A}_p$:

$$\boxed{\Omega_p \equiv \Omega_p(G, \mathbb{K}) := \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}}.$$

Claim. $\partial\Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial(\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

By construction we have $\Omega_0 = \mathcal{A}_0 = \langle e_i : i \in V\rangle$ and $\Omega_1 = \mathcal{A}_1 = \{e_{ij} : i \to j\}$, while in general $\Omega_p \subset \mathcal{A}_p$.

Definition. The elements of Ω_p are called ∂-*invariant* p-paths.

Hence, we obtain a chain complex $\Omega_* = \Omega_*(G, \mathbb{K})$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \cdots \tag{2}$$

that reflects the digraph structure of G. Homology groups of the chain complex (2) are called *path homologies* of G and are denoted by $H_p(G)$.

There is a dual cochain complex

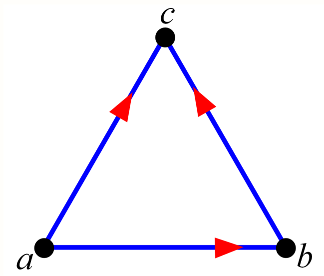
$$0 \rightarrow \Omega^0 \overset{d}{\rightarrow} \Omega^1 \overset{d}{\rightarrow} \cdots \overset{d}{\rightarrow} \Omega^{p-1} \overset{d}{\rightarrow} \Omega^p \overset{d}{\rightarrow} \cdots$$

that is analogous to the de Rham complex but in the setting of digraphs it is more convenient to work with the chain complex (2).

The dimension $\beta_p := \dim H_p(G)$ is called the p-th *Betti number* of G. It is easy to prove that β_0 is equal to the number of connected components of the underlying undirected graph. In particular, for connected graphs, $\beta_0 = 1$.

2.3 Examples of ∂-invariant paths

A *triangle* is a sequence of three distinct vertices a, b, c such that $a \to b \to c,\ a \to c$.
It determines a 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$.
The path e_{abc} is also referred to as a triangle.

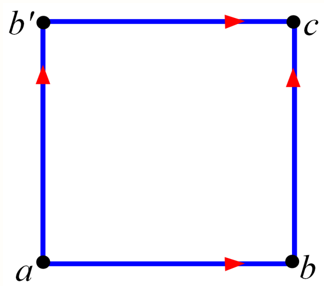


If $a \to b \to c$ but $a \not\to c$ then $e_{abc} \in \mathcal{A}_2$ but $e_{abc} \notin \Omega_2$.

A *square* is a sequence of four distinct vertices a, b, b', c such that $a \to b \to c$, $a \to b' \to c$ while $a \not\to c$.
It determines a 2-path $u = e_{abc} - e_{ab'c} \in \Omega_2$ because $u \in \mathcal{A}_2$ and

$$\partial u = (e_{bc} - \underline{e_{ac}} + e_{ab}) - (e_{b'c} - \underline{e_{ac}} + e_{ab'})$$
$$= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1.$$

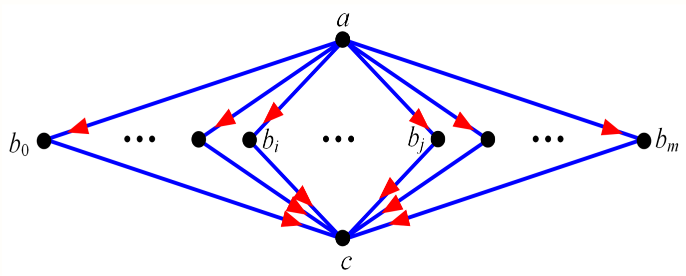
The path u is also referred to as a square.



An *m-square* is a sequence of $m+3$ distinct vertices

$$a,\ b_0, b_1, ..., b_m,\ c$$

such that $a \to b_k \to c\ \ \forall k = 0, \ldots, m$, while $a \not\to c$.

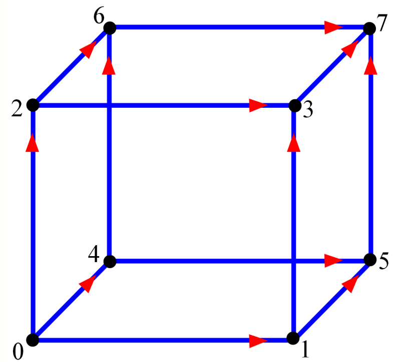


Clearly, a square is an 1-square. Any m-square with $m \geq 2$ is also called a *multisquare*. The m-square determines ∂-invariant 2-paths (squares) as follows:

$$u_{ij} = e_{ab_ic} - e_{ab_jc} \in \Omega_2 \quad \text{for all } i, j = 0, ..., m,$$

and m of these squares are linearly independent: $u_{0j} = e_{ab_0c} - e_{ab_jc},\ \ j = 1, ..., m.$

A 3-*cube* is the following digraph:



It determines a ∂-invariant 3-path

$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \Omega_3$,

that is also called a 3-cube. Indeed, $u \in \mathcal{A}_3$ and

$$\partial u = (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267})$$
$$- (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2.$$

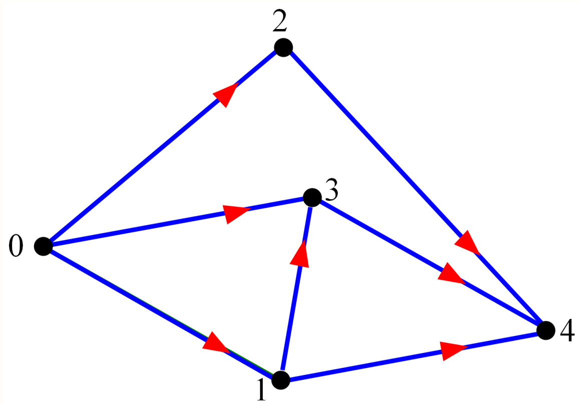
2.4 Structure of Ω_2

As we know, $\Omega_0 = \langle e_i \rangle$ and $\Omega_1 = \langle e_{ij} : i \to j \rangle$. Here we discuss a basis in Ω_2.

Theorem 1. *Space Ω_2 is spanned by all triangles e_{abc}, squares $e_{abc} - e_{ab'c}$ and double arrows e_{aba}. Consequently, the set of all triangles, the set of double arrows and a maximal set of linearly independent squares form a basis in Ω_2.*

Note that all triangles and double edges are linearly independent whereas squares can be dependent.

For example, consider the following digraph:



It contains two triangles e_{013}, e_{134} and three squares: $e_{014} - e_{024}$, $e_{014} - e_{034}$, $e_{024} - e_{034}$. A basis in Ω_2 consists of two triangles and *two* squares:

$$\Omega_2 = \langle e_{013}, e_{134}, e_{014} - e_{024}, e_{014} - e_{034} \rangle .$$

3 Definition of the Hodge operator Δ_p

Set $\mathbb{K} = \mathbb{R}$. Let us fix an arbitrary inner product $\langle \cdot, \cdot \rangle$ in each of the spaces $\mathcal{R}_p$ so that we have an inner product also in all Ω_p. In all examples we use the *natural* inner product where the basis $\{e_{i_0 \ldots i_p}\}$ of the elementary paths in $\mathcal{R}_p$ is orthonormal.

For the operator $\partial : \Omega_p \to \Omega_{p-1}$ consider the adjoint operator $\partial^* : \Omega_{p-1} \to \Omega_p$ given by

$$\langle \partial u, v \rangle = \langle u, \partial^* v \rangle \quad \text{for all } u \in \Omega_p \text{ and } v \in \Omega_{p-1}.$$

Definition. Define the *Hodge-Laplace operator* $\Delta_p : \Omega_p \to \Omega_p$ by

$$\Delta_p u = \partial^* \partial u + \partial \partial^* u. \tag{3}$$

Here we use the following pairs of operators ∂ and ∂^*: $\Omega_{p-1} \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_p$ and $\Omega_p \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_{p+1}$.

It is easy to prove that the operator Δ_p is self-adjoint and non-negative definite. Hence, the spectrum of Δ_p consists of a finite sequence of non-negative real eigenvalues.

Major problem. Develop a technique for determination of $\operatorname{spec} \Delta_p$ (or for computation of the coefficients of the characteristic polynomials of Δ_p) at least for some classes of digraphs.

Example. Let $V=\{1,...,n\}$. The operator Δ_0 acts on functions on V and has in the basis $\{e_i\}$ the following $n\times n$ matrix:

$$\text{matrix of } \Delta_0 = \operatorname{diag}(\deg(i)) - \mathbf{1}_{\{i\to j\}} - \mathbf{1}_{\{j\to i\}}$$

where $\deg(i)$ is the (undirected) degree of the vertex i. If G has no double arrow then

$$\text{the matrix of } \Delta_0 = \operatorname{diag}(\deg(i)) - \mathbf{1}_{\{i\sim j\}} \tag{4}$$

where $i\sim j$ denotes an edge in the underlying undirected graph. Hence, Δ_0 is the usual unnormalized Laplacian on functions on V.

It follows from (4) that

$$\operatorname{trace}\Delta_0 = \sum_{i\in V}\deg(i) = 2|E|. \tag{5}$$

The bottom eigenvalue of Δ_0 is always 0 because $\Delta_0 1 = 0$. It is easy to prove that

$$\lambda_{\max}(\Delta_0) \le 2\max_{i\in V}\deg(i), \tag{6}$$

where $\lambda_{\max}$ denotes the maximal eigenvalue of the operator in question.

Our results below include a formula for $\operatorname{trace}\Delta_1$ *and bounds for* $\operatorname{spec}\Delta_1$.

4 Harmonic paths

Definition. A path $u \in \Omega_p$ is called *harmonic* if $\Delta_p u = 0$.

It is easy to prove that a path $u \in \Omega_p$ is harmonic if and only if $\partial u = 0$ and $\partial^* u = 0$.

Denote by $\mathcal{H}_p$ the set of all harmonic paths in Ω_p, so that $\mathcal{H}_p$ is a subspace of Ω_p.

Theorem 2. (Hodge decomposition) Ω_p *is the following orthogonal sum:*

$$\Omega_p = \partial\Omega_{p+1} \bigoplus \partial^*\Omega_{p-1} \bigoplus \mathcal{H}_p. \tag{7}$$

Corollary 3. *There is a natural linear isomorphism between* $\mathcal{H}_p$ *and the homology group* H_p*:*

$$\mathcal{H}_p \cong H_p. \tag{8}$$

That is, each homology class has a unique harmonic representative.

Consequently, $\dim \mathcal{H}_p = \beta_p$. In other words, the multiplicity of 0 as an eigenvalue of Δ_p is equal to the Betti number β_p.

That is, $\lambda_{\min}(\Delta_p) = 0$ if $\beta_p > 0$ and $\lambda_{\min}(\Delta_p) > 0$ if $\beta_p = 0$.

5 Matrix of Δ_p

Let $\{\alpha_i\}$ be an orthonormal basis in Ω_p, $\{\beta_m\}$ be an orthonormal basis in Ω_{p-1} and $\{\gamma_n\}$ be an orthonormal basis in Ω_{p+1} :

$$\begin{array}{ccccc} \Omega_{p-1} & \underset{\partial}{\overset{\partial^*}{\rightleftarrows}} & \Omega_p & \underset{\partial}{\overset{\partial^*}{\rightleftarrows}} & \Omega_{p+1} \\ \{\beta_m\} & & \{\alpha_i\} & & \{\gamma_n\} \end{array}$$

Lemma 4. *The matrix of Δ_p in the basis $\{\alpha_i\}$ has the following entries:*

$$\langle \Delta_p \alpha_i, \alpha_j \rangle = \sum_m \langle \partial \alpha_i, \beta_m \rangle \langle \partial \alpha_j, \beta_m \rangle + \sum_n \langle \alpha_i, \partial \gamma_n \rangle \langle \alpha_j, \partial \gamma_n \rangle . \tag{9}$$

Example. For the 1-torus $T = \{0 \to 1 \to 2 \to 0\}$ we have

$$\text{the matrix of } \Delta_1 = \begin{pmatrix} & e_{01} & e_{12} & e_{20} \\ e_{01} & 2 & -1 & -1 \\ e_{12} & -1 & 2 & -1 \\ e_{20} & -1 & -1 & 2 \end{pmatrix}.$$

The eigenvalues of Δ_1 are $\{0, 3_2\}$, where "3_2" means that 3 is the eigenvalues with multiplicity 2.

Example. For a triangle $G = \{0 \to 1 \to 2,\ 0 \to 2\}$ we have

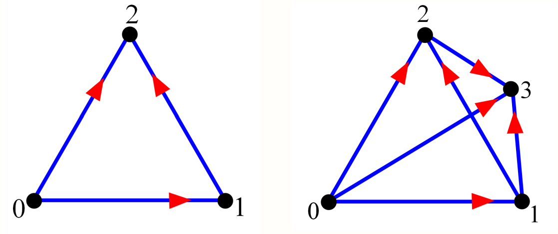
$$\text{the matrix of } \Delta_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Example. For any integer $n \geq 1$, denote by K_n a complete digraph with the set of n vertices $V = \{0, ..., n-1\}$ and arrows $i \to j \Leftrightarrow i < j$.

That is, K_n is a directed $(n-1)$-simplex.

The space $\Omega_p = \mathcal{A}_p$ is generated by all elementary allowed p-paths: $\{e_{i_0 \ldots i_p} : i_0 < i_1 < \ldots < i_p\}$ so that

$$|\Omega_p| = \binom{n}{p+1}.$$



K_3 K_4

A computation shows that, for any $1 \leq p < n$,

$$\text{the matrix of } \Delta_p(K_n) = \operatorname{diag}(n).$$

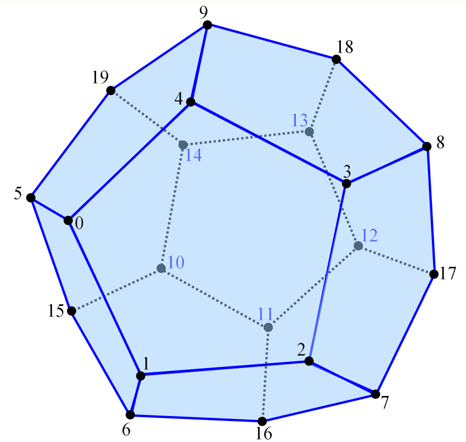
Consequently, $\operatorname{spec} \Delta_p(K_n)$ consists of one eigenvalue n with the multiplicity $\binom{n}{p+1}$.

Example. Let G be a square $\{0 \to 1 \to 3,\, 0 \to 2 \to 3\}$. Then

$$\text{the matrix of } \Delta_1 = \frac{1}{2}\begin{pmatrix} 5 & 1 & -1 & -1 \\ 1 & 5 & -1 & -1 \\ -1 & -1 & 5 & 1 \\ -1 & -1 & 1 & 5 \end{pmatrix},$$

and the eigenvalues of Δ_1 are $\{2_3,\, 4\}$.

Example. Let G be a dodecahedron ($V = 20$, $E = 30$), where the arrows go in the direction of increasing numbers.



The eigenvalues of Δ_1 are

$$\{0_{11},\ 2_5,\ 3_4,\ 5_4,\ (3 \pm \sqrt{5})_3\}.$$

The matrix of Δ_1 is a follows:

the matrix of $\Delta_1 =$

2	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	2	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	2	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	-1	0	2	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	-1	0	0	0	0	0	-1	0	2	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0	2	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
0	0	0	0	-1	0	0	0	0	0	1	0	2	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	-1	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	1	0	0	0	0	1	0	-1
0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	1	2	0	0	0	0	0	0	0
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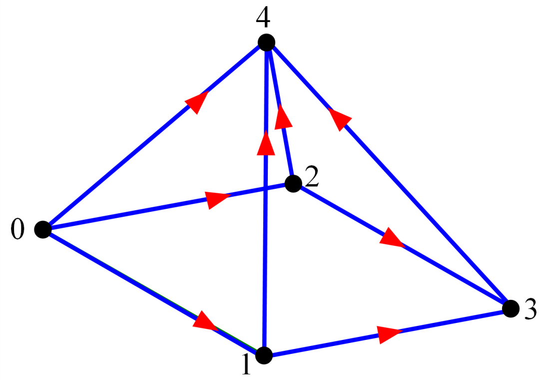
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	2	1	-1	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	0	1	2	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	-1	0	2	1	-1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	0	1	2	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	-1	0	0	0	0	0	-1	0	2

Example. Consider the following pyramid:



The eigenvalues of Δ_1 are $\{3_5,\ 5_3\}$.

The matrix of Δ_1 is as follows:

$$\text{the matrix of } \Delta_1 = \frac{1}{2}\begin{pmatrix} 7 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 7 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 7 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 8 \end{pmatrix}.$$

6 A formula for $\operatorname{trace}\Delta_1$

From now on $\langle \cdot, \cdot \rangle$ is the natural inner product in all Ω_p. Recall that by (5) $\operatorname{trace}\Delta_0 = 2E$, where E is now the number of arrows in the digraph. Here is a similar result for $\operatorname{trace}\Delta_1$.

Theorem 5. *Let T be the number of all triangles in a digraph, S be the maximal number of linearly independent squares, and D be the number of all double arrows $a \rightleftarrows b$. Then*

$$\operatorname{trace}\Delta_1 = 2E + 3T + 2S + 4D. \tag{10}$$

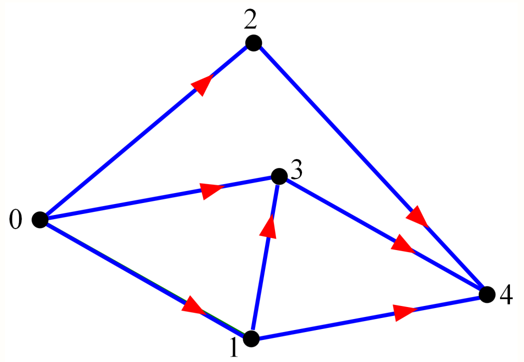
For example, consider the "broken" pyramid:

(as on p.10). It has 2 triangles and 3 squares

of which 2 are linearly independent.

Hence, $E = 7$, $T = S = 2$, $D = 0$, whence

$$\operatorname{trace}\Delta_1 = 2 \cdot 7 + 3 \cdot 2 + 2 \cdot 2 = 24.$$



In fact, the full spectrum of Δ_1 on this digraph is $\left\{2,\ 3,\ 4_2,\ 5,\ 3 \pm \sqrt{3}\right\}$.

Problem. Find a formula for $\operatorname{trace}\Delta_p$ similar to (10).

7 An upper bound of $\lambda_{\max}(\Delta_1)$

Denote by $\lambda_{\max}(A)$ the maximal eigenvalue of a symmetric operator A. By (6) we have

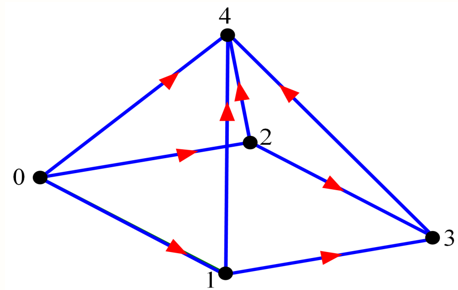
$$\lambda_{\max}(\Delta_0) \leq 2 \max_{i \in V} \deg(i) .$$

For any arrow $i \to j$ in G denote by $\deg_\Delta(ij)$ the number of triangles containing the arrow $i \to j$, and by $\deg_\Box(ij)$ the number of squares containing $i \to j$.

Theorem 6. *Assume that the digraph G contains no multisquares* (see p. 9). *Then*

$$\lambda_{\max}(\Delta_1) \leq 2 \max_{i} \deg(i) + 3 \max_{i \to j} \deg_\Delta(ij) + 2 \max_{i \to j} \deg_\Box(ij) . \tag{11}$$

For example, consider the pyramid:



We have in this case $\max \deg(i) = 4$,

$\max \deg_\triangle(ij) = 2$, $\max \deg_\Box(ij) = 1$.

Hence, by (11)

$$\lambda_{\max} \leq 2 \cdot 4 + 3 \cdot 2 + 2 \cdot 1 = 16.$$

In fact, we have $\lambda_{\max} = 5$ (see p. 18) so that the estimate (11) is rather rough in this case.

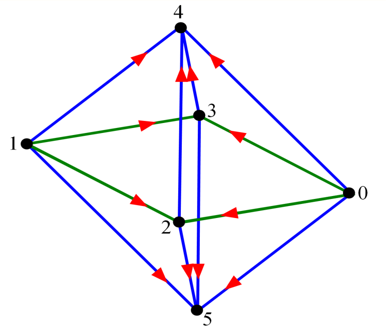
Problem. How sharp is the upper bound of $\lambda_{\max}(\Delta_1)$ in (11)? Is it attained on some digraphs?

Problem. Extend (11) to a general case when G may contain multisquares.

Problem. Obtain a reasonable upper bound for $\lambda_{\max}(\Delta_p)$.

8 Examples of computations of $\operatorname{trace}\Delta_1$ and $\operatorname{spec}\Delta_1$

Example. Consider an octahedron based on a diamond:

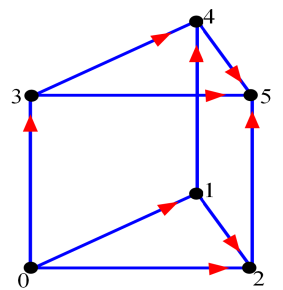


For this digraph $E = 12, \quad T = 8, \quad S = 0.$

Hence, $\operatorname{trace}\Delta_1 = 2E + 3T = 48.$

The eigenvalues of Δ_1 are $\{2_3,\ 4_6,\ 6_3\}$.

Example. Consider a prism:



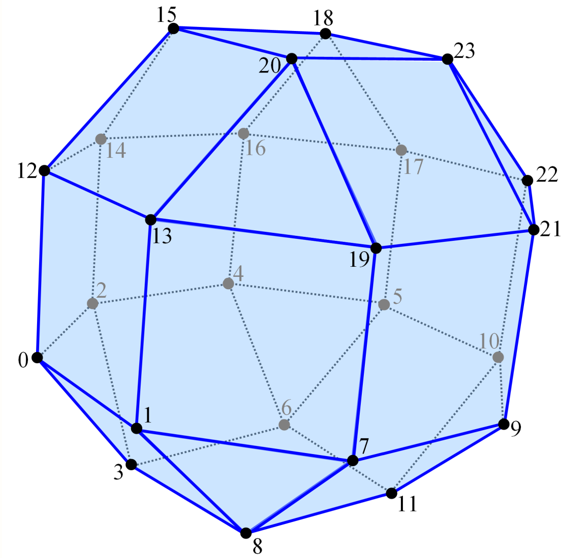
Since $E = 9,\ T = 2,\ S = 3,$ we have

$$\operatorname{trace}\Delta_1 = 2E + 3T + 2S = 30$$

The eigenvalues of Δ_1 are $\{2,\ (\frac{5}{2})_2,\ 3_3,\ 4,\ 5_2\}$.

Example. Consider a rhombicuboctahedron where the arrows go in the direction of increasing numbers:

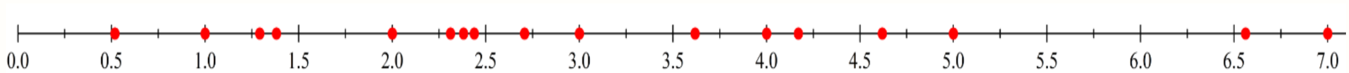
We have here $V = 24$ and $E = 48$. There is 8 triangles and 18 squares corresponding exactly to the faces of the polyhedron.



Therefore, $T = 8$, $S = 18$ and

$\operatorname{trace} \Delta_1 = 2E + 3T + 2S = 156.$

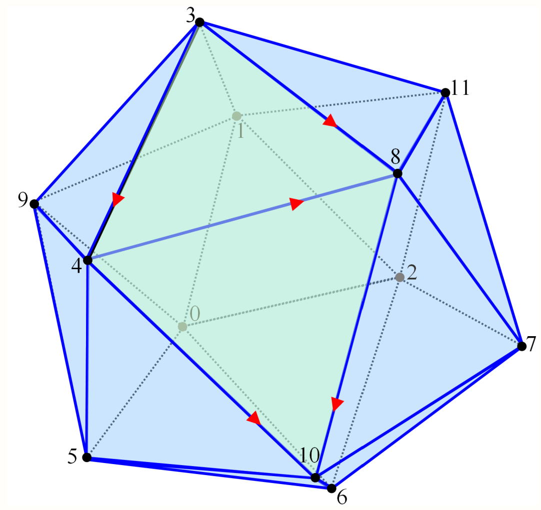
A computation of the eigenvalues of Δ_1 gives $\lambda_{\min} = 0.518...$ and $\lambda_{\max} = 7_2$. There are multiple eigenvalues: $1_3, 2_3, 3_3, 4_4, 5_6$, etc. The full spectrum of Δ_1 is shown here:



For rhombicuboctahedron the matrix of $\Delta_1 = \dfrac{1}{2}\times$

6	2	1	-2	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	7	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	-1	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	7	0	-1	0	0	-2	-1	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-2	0	0	7	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	1	0	0	0	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	-1	0	7	0	0	0	1	0	0	0	0	0	0	0	-2	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	7	1	-1	-2	0	-1	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-2	0	0	0	1	6	-1	0	-2	-1	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-2	0	0	-1	-1	6	1	0	1	0	2	-1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	-1	0	1	-2	0	1	6	0	0	0	0	-1	2	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-2	0	0	7	0	-2	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-1	-1	1	0	0	7	0	0	-2	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-2	0	6	1	-1	0	0	0	2	0	-1	0	0	0	0	1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	2	0	0	0	1	7	-1	0	0	0	0	0	-1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1	-1	0	-2	-1	-1	6	0	0	1	0	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	0	0	0	0	0	7	1	-1	0	-1	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-2	0	0	0	0	0	0	0	0	0	0	1	6	-1	-2	-1	0	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0
0	0	0	0	-2	0	0	-1	-1	0	0	0	0	1	-1	-1	6	0	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	-2	0	7	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	2	-1	-1	1	0	7	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	-1	-1	1	0	0	2	0	0	7	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	-1	-1	0	0	0	0	0	-1	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	6	0	0	0	-1	0	0	1	0	0	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	0	0	0	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	6	-1	0	0	0	0	0	1	0	0	0	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	1	2	0	0	0	0	0	0	0	0	0	0	0	-1	6	-1	0	0	0	0	0	0	0	0	0	1	0	0	-1	-2	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-1	0	1	2	0	0	0	0	0	0	0	0	0	0	-1	6	0	0	-1	0	0	0	0	0	0	0	0	0	1	0	-1	-2	0	0	0	0	0	0	0
0	0	0	-1	0	0	0	0	0	0	0	0	0	0	2	1	0	0	0	0	0	-1	0	0	0	6	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-2	-1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	2	0	0	0	0	0	0	-1	6	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	-1	-2	0
0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	0	2	0	0	0	0	-1	0	-1	6	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	-2
-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	6	2	1	-2	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	2	7	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	1	0	7	0	-1	0	0	-2	-1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	0	-2	0	0	7	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	-1	0	-1	0	7	0	0	0	1	0	0	0	0	0	0	0	-2	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	7	1	-1	-2	0	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	-2	0	0	0	1	6	-1	0	-2	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	-1	-1	6	1	0	1	0	2	-1	0	0	-1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-1	0	1	-2	0	1	6	0	0	0	0	-1	2	0	-1	0	0	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	-2	0	0	7	0	-2	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	-1	-1	1	0	0	7	0	0	-2	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	1	0	0	0	0	0	0	0	0	0	-2	0	6	1	-1	0	0	0	2	0	-1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	2	0	0	0	1	7	-1	0	0	0	0	0	-1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	-2	-1	-1	6	0	0	1	0	2	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	7	1	-1	0	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	1	6	-1	-2	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	-1	-1	0	0	0	0	1	-1	-1	6	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	-2	0	7	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	-1	-1	1	0	7	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	-1	-1	1	0	0	2	0	0	7

Example. Consider the icosahedron, where the arrows go in the direction of increasing numbers:



We have here $V = 12$, $E = 30$.

There are 20 triangles corresponding to the faces of the polyhedron and 5 squares:

$$e_{0\,1\,11} - e_{0\,2\,11},\ e_{0\,2\,7} - e_{0\,6\,7},\ e_{0\,5\,10} - e_{0\,6\,10}$$
$$e_{2\,7\,10} - e_{2\,6\,10},\ e_{3\,4\,10} - e_{3\,8\,10}.$$

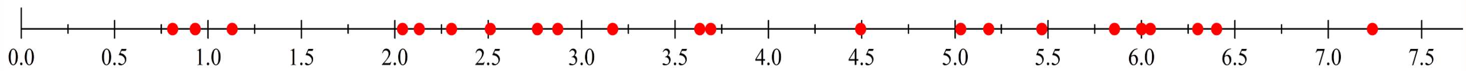
Therefore, $T = 20$, $S = 5$ and

$$\operatorname{trace}\Delta_1 = 2E + 3T + 2S = 130.$$

A computation yields that

$$\lambda_{\min} = 0.810\ldots \quad \text{and} \quad \lambda_{\max} = (5 + \sqrt{5})_3.$$

Other multiple eigenvalues are 6_5 and $(5 - \sqrt{5})_3$. The full spectrum of Δ_1 is shown here:



For icosahedron

the matrix of $\Delta_1 = \frac{1}{2}\times$

9	-1	2	0	2	0	-2	0	-1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	10	2	2	-1	0	0	0	-1	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
2	2	9	0	-1	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	-1	0	-1	0	0	0	0	0
0	2	0	8	2	0	0	0	0	0	0	0	0	0	2	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0
2	-1	-1	2	10	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	0	0	0	0	0
0	0	0	0	0	8	2	2	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-2	0	0	0	0	2	8	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	8	2	0	0	0	0	0	0	0	0	0	2	0	0	2	0	0	0	0	0	0	0	0
-1	-1	0	0	0	0	0	2	9	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	2
0	-1	0	0	-1	-2	0	0	0	10	-1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-2	-1	0	0	0
0	0	0	0	0	-2	0	0	0	-1	9	2	0	0	0	0	0	0	0	0	2	0	0	0	-1	0	-1	0	0	0
-1	-1	0	0	0	0	0	0	-1	0	2	9	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	2
0	0	0	0	0	0	-2	0	0	0	0	0	9	-1	0	2	-2	0	0	-1	0	0	0	0	0	0	0	0	-1	0
0	0	0	0	0	0	-2	0	0	0	0	0	-1	9	2	0	0	0	0	-1	0	0	0	0	0	2	0	0	-1	0
0	0	0	2	0	0	0	0	0	0	0	0	0	2	8	2	0	0	0	0	0	2	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	8	2	0	0	0	0	0	0	2	0	0	0	-2
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0	-1	0	0	-1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	-2	0	0	9	0	-2	0	-2	0	0
0	0	-1	0	-1	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	2	0	0	-1	0	10	0	-1	0	2	0
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0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	-1	0	0	2	0	2	0	0	0	9	2
0	0	0	0	0	0	0	0	2	0	0	2	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	2	8

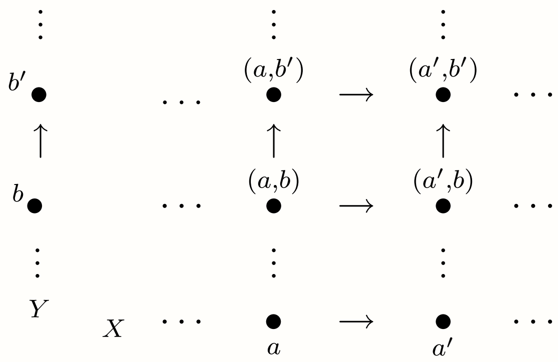
9 Cartesian product and Künneth formula

Denote a digraph and its set of vertices by the same letters to simplify notation.

Given two digraphs X and Y, define their Cartesian product as a digraph $Z = X\Box Y$ as follows:

- the vertices of Z are couples (a, b) where $a \in X$ and $b \in Y$;

- the edges in Z are of two types: $(a, b) \to (a', b)$ where $a \to a'$ in X (a *horizontal* edge) and $(a, b) \to (a, b')$ where $b \to b'$ in Y (a *vertical* edge):



Theorem 7. (Künneth formula for product) *Let X, Y be two digraphs, set $Z = X \Box Y$. Then, for any $r \geq 0$,*

$$\Omega_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} \Omega_p(X) \otimes \Omega_q(Y), \tag{12}$$

where the isomorphism is given by $u \otimes v \mapsto u \times v$ for $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$.

Here $u \times v$ denotes a certain *cross product* of paths.

Consequently, we have

$$H_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} H_p(X) \otimes H_q(Y) \tag{13}$$

and

$$\beta_r(Z) = \sum_{\{p,q\geq 0: p+q=r\}} \beta_p(X)\, \beta_q(Y).$$

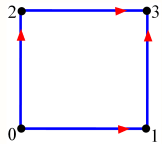
Hence, the multiplicity of 0 as an eigenvalue of $\Delta_r(Z)$ can be expressed via the multiplicities of 0 for $\Delta_p(X)$ and $\Delta_q(Y)$.

Example. Consider a digraph $I = {}^{0}\bullet \to \bullet^{1}$. For any $n \geq 1$, define the digraph

$$n\text{- cube} = I^{\square n} = \underbrace{I \square I \square ... \square I}_{n \text{ times}} .$$

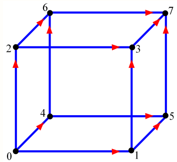
We have for the n-cube: $V = 2^n$, $E = n2^{n-1}$, $S = 2^{n-3} n (n-1)$ and $T = D = 0$. Hence,

$$\operatorname{trace} \Delta_1 = 2E + 2S = 2^{n-2} n (n+3) .$$



For 2-cube $\operatorname{trace} \Delta_1 = 2 \cdot 5 = 10$.

The eigenvalues of Δ_1 are $\{2_3, 4\}$.



For 3-cube $\operatorname{trace} \Delta_1 = 2 \cdot 3 \cdot 6 = 36$.

The eigenvalues of Δ_1 are $\{2_6, 3_2, 4_3, 6\}$.

For 4-cube $\operatorname{trace} \Delta_1 = 2^2 \cdot 4 \cdot 7 = 112$, and the eigenvalues of Δ_1 are

$$\{2_{10}, 3_8, 4_9, 6_4, 8\} .$$

For 5-cube $\operatorname{trace} \Delta_1 = 2^3 \cdot 5 \cdot 8 = 320$, and the eigenvalues of Δ_1 are

$$\{2_{15}, 3_{20}, 4_{25}, 5_4, 6_{10}, 8_5, 10\} .$$

Problem. Determine spec Δ_1 on n-cube. In particular, prove that

$$\lambda_{\min} = 2_{\frac{n(n+1)}{2}} \quad \text{and} \quad \lambda_{\max} = 2n.$$

Prove also that the eigenvalues of Δ_1 are all even integers from 2 to $2n$ and all odd integers from 3 to n. Determine their multiplicities.

The difficulty here is that the method of separation of variables does not work.

Problem. Determine spec Δ_p on n-cube.

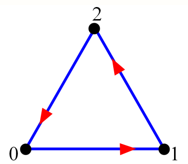
Example. Consider the n-torus $T^{\square n}$, where $T = \{0 \to 1 \to 2 \to 0\}$.

For $G = T^{\square n}$ we have

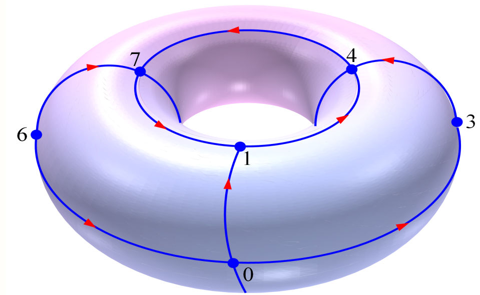
$V = 3^n, \quad E = n3^n, \quad S = \frac{n(n-1)}{2}3^n,$

$T = D = 0,$

whence trace $\Delta_1 = 2E + 2S = n\,(n+1)\,3^n$.



T



$T^{\square 2}$

For 1-torus T: $\operatorname{trace}\Delta_1 = 1 \cdot 2 \cdot 3^1 = 6$, and the eigenvalues of Δ_1 are

$$\{0, 3_2\}\,.$$

For 2-torus $T^{\Box 2}$: $\operatorname{trace}\Delta_1 = 2 \cdot 3 \cdot 3^2 = 54$, and the eigenvalues of Δ_1 are

$$\{0_2,\ (\tfrac{3}{2})_4,\ 3_8,\ 6_4\}.$$

For 3-torus $T^{\Box 3}$: $\operatorname{trace}\Delta_1 = 2 \cdot 81 + 2 \cdot 81 = 324$, and the eigenvalues of Δ_1 are

$$\left\{0_3,\ (\tfrac{3}{2})_{12},\ 3_{30},\ (\tfrac{9}{2})_{16},\ 6_{12},\ 9_8\right\}.$$

Problem. Compute $\operatorname{spec}\Delta_1$ on n-torus. In particular, prove that

$$\lambda_{\max}(\Delta_1) = (3n)_{2^n}\,.$$

It is known that $\lambda_{\min}(\Delta_1) = 0_n$, which is a consequence of $\beta_1(T^{\Box n}) = n$. In fact, we have

$$\beta_p(T^{\Box n}) = \binom{n}{p},$$

so that $\lambda_{\min}(\Delta_p) = 0_{\binom{n}{p}}$ for all $0 \le p \le n$.

Problem. Compute $\operatorname{spec}\Delta_p$ on n-torus. In particular, what is $\lambda_{\max}(\Delta_p)$?

10 Augmented chain complex

In the next sections we use the *augmented* chain complex:

$$\mathbb{K} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots \tag{14}$$

Here we redefine $\partial : \Omega_0 \to \mathbb{K} =: \widetilde{\Omega}_{-1}$ by $\partial e_i = e$, where e denotes the unity of $\mathbb{K}$ (previously we defined $\partial : \Omega_0 \to \{0\}$ as $\partial = 0$). For consistency of notation, set $\widetilde{\Omega}_p = \Omega_p$ for all $p \geq 0$.

The homology groups of the augmented chain complex (14) are denoted by $\widetilde{H}_p$ and are called the *reduced* homology groups.

Consider the Hodge Laplacian $\widetilde{\Delta}_p : \widetilde{\Omega}_p \to \widetilde{\Omega}_p$ associated with this complex:

$$\widetilde{\Delta}_p u = \partial^* \partial u + \partial \partial^* u.$$

Of course, we have $\widetilde{\Delta}_p = \Delta_p$ for $p \geq 1$ but $\widetilde{\Delta}_p \neq \Delta_p$ for $p = -1$ and $p = 0$.

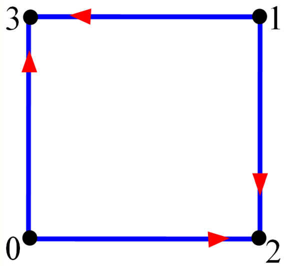
For example, for $p = -1$ we have $\widetilde{\Delta}_{-1} e = |V| \, e$. For $p = 0$ we have $\widetilde{\Delta}_0 e_i = \Delta_0 e_i + \sum_j e_j$. Therefore, the matrix of $\widetilde{\Delta}_0$ in the basis $\{e_i\}$ is obtained from the matrix of Δ_0 (see (4)) by adding 1 to *each* entry.

11 Join of digraphs

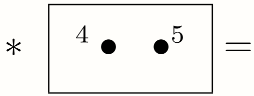
Let X, Y be two digraphs.

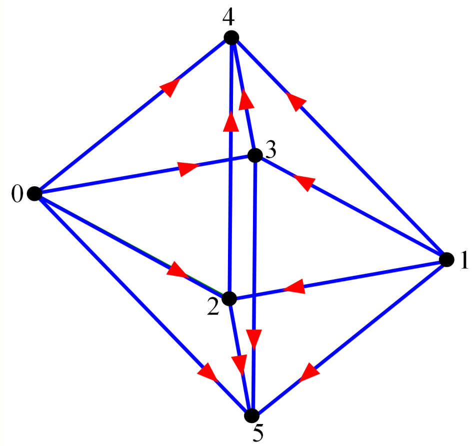
Definition. The *join* $X * Y$ of X and Y is a digraph whose set of vertices is a disjoint union of the sets of vertices of X and Y, and the set of arrows consists of all arrows of X and Y as well as from all arrows $x \to y$ where $x \in X$ and $y \in Y$.

Here is an example of join:



a diamond





an octahedron

Theorem 8. (Künneth formula for the join) *Let X, Y be two digraphs, set $Z = X * Y$. We have the following isomorphism: for any $r \geq -1$,*

$$\widetilde{\Omega}_r(Z) \cong \bigoplus_{\{p,q\geq -1: p+q=r-1\}} \left(\widetilde{\Omega}_p(X) \otimes \widetilde{\Omega}_q(Y)\right) \tag{15}$$

that is given by the map $u \otimes v \mapsto u * v$ with $u \in \widetilde{\Omega}_p(X)$ and $v \in \widetilde{\Omega}_q(Y)$

Here $u * v$ denotes the *join* of two paths that is defined by $e_{i_0\ldots i_p} * e_{j_0\ldots j_q} = e_{i_0\ldots i_p j_0\ldots j_q}$.

It follows that, for any $r \geq 0$,

$$\widetilde{H}_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r-1\}} \widetilde{H}_p(X) \otimes \widetilde{H}_q(Y)$$

and

$$\widetilde{\beta}_r(Z) = \sum_{\{p,q\geq 0: p+q=r-1\}} \widetilde{\beta}_p(X)\, \widetilde{\beta}_q(Y).$$

12 Spectrum of $\widetilde{\Delta}_p$ on the join

The advantage of the augmented chain complex (14) lies in the following statements.

Lemma 9. *Let X, Y be two digraphs. Then, for $u \in \Omega_p(X)$, $v \in \Omega_q(Y)$ with $p, q \geq -1$, we have*

$$\widetilde{\Delta}_r(u * v) = (\widetilde{\Delta}_p u) * v + u * \widetilde{\Delta}_q v, \tag{16}$$

where $r = p + q + 1$.

The Künneth formula (15) and the product rule (16) allow to prove the following result.

Theorem 10. *Let X, Y be two digraphs. We have, for any $r \geq 0$,*

$$\operatorname{spec} \widetilde{\Delta}_r(X * Y) = \bigsqcup_{\{p,q\geq -1 : p+q=r-1\}} \left(\operatorname{spec} \widetilde{\Delta}_p(X) + \operatorname{spec} \widetilde{\Delta}_q(Y) \right). \tag{17}$$

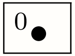
Here we denote by $\operatorname{spec} A$ a sequence of all the eigenvalues of the operator A counted with multiplicities. The sum of two such sequences consists of all pairwise sums of the elements of the first and the second sequences. In particular, if one of the sequences is empty then its sum with another sequence is also empty.

The disjoint union of sequences means the union of all the elements of the sequences, summing up the multiplicities of their common elements.

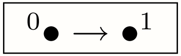
13 Spectrum of Δ_p on digraphs D_m^{*n}

For any $m \in \mathbb{N}$ denote by D_m a digraph with m vertices and no arrows: $\{\bullet \ \bullet \ \cdots \ \bullet\}$. We compute here $\operatorname{spec} \Delta_p \left(D_m^{*n}\right)$ where D_m^{*n} is the n-th join power of D_m.

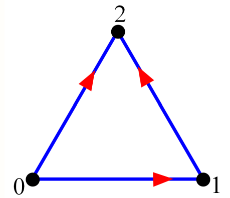
For $m = 1$, $D_1^{*n} = K_n$ where K_n is a complete digraph with n vertices (=($n-1$)-simplex)



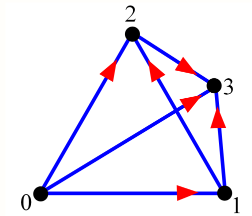
$D_1 = K_1$



$D_1^{*2} = K_2$



$D_1^{*3} = K_3$



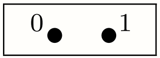
$D_1^{*4} = K_4$

For $m = 2$, $D_2^{*n} =: S^{n-1}$ that can be regarded as a digraph sphere of dimension $n - 1$.

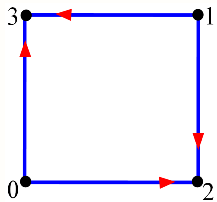
In particular, for digraph D_2^{*n}

$$\beta_0 = \beta_{n-1} = 1$$

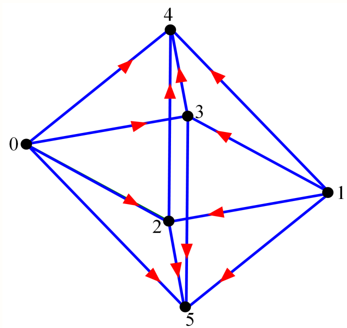
while all other β_p vanish like for a topological sphere $\mathbb{S}^{n-1}$.



$D_2 = S^0$



$D_2^{*2} = S^1$
a diamond



$D_2^{*3} = S^2$
an octahrdron

As it follows from the Künneth formula for join,

$$\dim \Omega_{r-1}\left(D_m^{*n}\right) = \binom{n}{r} m^r. \tag{18}$$

In particular, the Hodge Laplacian Δ_{r-1} on D_m^{*n} is non-trivial only if $n \geq r$.

Theorem 11. *We have, for all $n, m \geq 1$ and $r \geq 2$,*

$$\operatorname{spec} \Delta_{r-1}(D_m^{*n}) = \left\{ ((n-k)m)_{\binom{r}{k}\binom{n}{r}(m-1)^k} \right\}_{k=0}^{r}. \tag{19}$$

More explicitly, (19) can be stated as follows: if $n < r$ then

$$\operatorname{spec} \Delta_{r-1}(D_m^{*n}) = \emptyset,$$

while for $n \geq r$ the spectrum of $\Delta_{r-1}(D_m^{*n})$ consists of the following $r+1$ eigenvalues

$$(n-r)m,\ (n-r+1)m,\ (n-r+2)m, \ldots, (n-1)m,\ nm, \tag{20}$$

having the following multiplicities:

$$\binom{n}{r}(m-1)^r,\quad r\binom{n}{r}(m-1)^{r-1},\ \binom{r}{2}\binom{n}{r}(m-1)^{r-2}, \ldots, r\binom{n}{r}(m-1),\ \binom{n}{r}. \tag{21}$$

Example. Let $m = 1$. Then $D_1^{*n} = K_n$. In this case all the multiplicities in (21) are 0 except for the last one $\binom{n}{r}$. Hence, $\operatorname{spec}\Delta_{r-1}(K_n)$ consists of a single eigenvalue n with multiplicity $\binom{n}{r}$, as we have seen above (p. 15).

Example. Let $m = 2$. Then $D_2^{*n} = S^{n-1}$. In this case (19) becomes

$$\operatorname{spec}\Delta_{r-1}(S^{n-1}) = \left\{(2(n-k))_{\binom{r}{k}\binom{n}{r}}\right\}_{k=0}^{r}.$$

For example, we have

$$\operatorname{spec}\Delta_1(S^{n-1}) = \left\{(2(n-2))_{\binom{n}{2}}, (2(n-1))_{2\binom{n}{2}}, (2n)_{\binom{n}{2}}\right\},$$

$$\operatorname{spec}\Delta_2(S^{n-1}) = \left\{(2(n-3))_{\binom{n}{3}}, (2(n-2))_{3\binom{n}{3}}, (2\,(n-1))_{3\binom{n}{3}}, (2n)_{\binom{n}{3}}\right\},$$

$$\operatorname{spec}\Delta_3(S^{n-1}) = \left\{(2(n-4))_{\binom{n}{4}}, (2(n-3))_{4\binom{n}{4}}, (2(n-2))_{6\binom{n}{4}}, (2(n-1))_{4\binom{n}{4}}, (2n)_{\binom{n}{4}}\right\}.$$

In particular,

$$\operatorname{spec}\Delta_1(S^1) = \{0,\ 2_2,\ 4\},$$

$$\operatorname{spec}\Delta_1(S^2) = \{2_3, 4_6,\ 6_3\}, \quad \operatorname{spec}\Delta_2(S^2) = \{0,\ 2_3,\ 4_3,\ 6\},$$

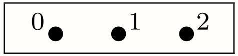
$$\operatorname{spec}\Delta_1(S^3) = \{4_6,\ 6_{12},\ 8_6\}, \operatorname{spec}\Delta_2(S^3) = \{2_4,\ 4_{12},\ 6_{12},\ 8_4\}, \operatorname{spec}\Delta_3(S^3) = \{0,\ 2_4,\ 4_6,\ 6_4,\ 8\}.$$

Example. Let $m = 3$.

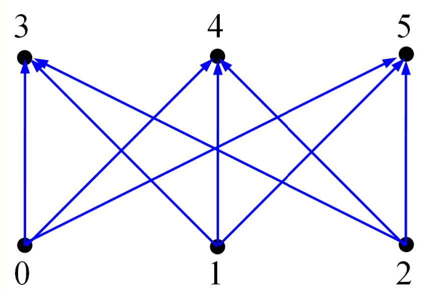
In this case $D_3^{*2} = K_{3,3}$:

The identity (19) yields

$\operatorname{spec} \Delta_1(K_{3,3}) = \{0_4, 3_4, 6\}$.



D_3



$D_3^{*2} = K_{3,3}$

Remark. The digraphs from the family $\{D_m^{*n}\}$ enjoy another remarkable feature: they all have a constant *combinatorial curvature*.

14 Trapezohedra

For any integer $m \geq 2$, define a *trapezohedron* T_m of order m as follows:

T_m is a digraph of $2m+2$ vertices

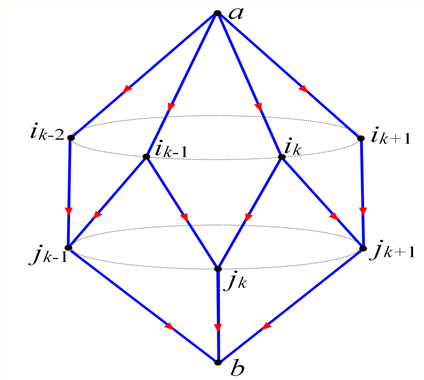
$$a,\ b,\ i_0, ..., i_{m-1},\ j_0, j_1, ..., j_{m-1}$$

and $4m$ arrows

$$a \to i_k \to j_k \to b, \quad i_k \to j_{k+1}$$

for all $k = 0, \ldots, m-1 \bmod m$.

A fragment of T_m is shown here:



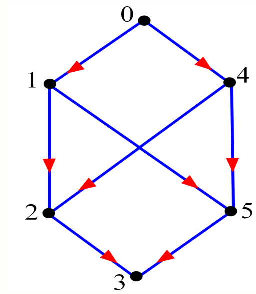
The trapezohedron gives rise to a ∂-invariant 3-path: $\tau_m = \sum_{k=0}^{m-1} \left(e_{a i_k j_k b} - e_{a i_k j_{k+1} b}\right)$.

Theorem 12. *If a digraph G has neither double arrows nor multisquares then there is a basis in $\Omega_3(G)$ that consists of trapezohedral paths τ_m with $m \geq 2$ and their images under digraph morphisms.*

For $G = T_m$ we have $V = 2m+2$, $E = 4m$, $S = 2m$ while $T = D = 0$. Hence, we obtain

$$\operatorname{trace} \Delta_1(T_m) = 2E + 2S = 12m.$$

Here is the trapezohedron T_2:

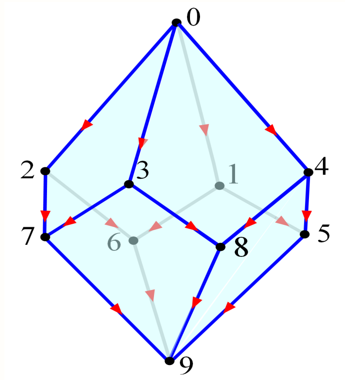


We have trace $\Delta_1 = 12 \cdot 2 = 24$, and the eigenvalues of Δ_1 are

$$\{2,\ 3_5,\ \tfrac{7}{2} \pm \tfrac{1}{2}\sqrt{17}\}.$$

The trapezohedron T_3 coincides with a 3-cube.
In this case trace $\Delta_1 = 36$ and the eigenvalues of Δ_1 are $\{2_6,\ 3_2,\ 4_3,\ 6\}$.

Here is the trapezohedron T_4:



In this case trace $\Delta_1 = 12 \cdot 4 = 48$.

The characteristic polynomial of Δ_1 is

$$(z-2)\,(z-3)^4\,(z-5)\,(z^2-9z+16)(z^2-4z+\tfrac{7}{2})^2(z^2-6z+7)^2,$$

the eigenvalues are $\{2,\ 3_4,\ 5,\ \tfrac{9}{2} \pm \tfrac{1}{2}\sqrt{17},\ (2 \pm \tfrac{1}{2}\sqrt{2})_2,\ (3 \pm \sqrt{2})_2\}$.

For T_5 the characteristic polynomial of Δ_1 is

$$(z-2)\,(z-\tfrac{5}{2})^4\,(z-6)\,(z^2-10z+20)(z^2-7z+11)^2(z^2-5z+5)^2(z^2-4z+\tfrac{11}{4})^2,$$

and the eigenvalues of Δ_1 are

$$\{2,\ (\tfrac{5}{2})_4,\ 6,\ 5\pm\sqrt{5},\ (\tfrac{7}{2}\pm\tfrac{1}{2}\sqrt{5})_2,\ (\tfrac{5}{2}\pm\tfrac{1}{2}\sqrt{5})_2,\ \ (2\pm\tfrac{1}{2}\sqrt{5})_2\},$$

For T_6 the characteristic polynomial of Δ_1 is

$$(z-2)^5\,(z-3)^7\,(z-4)^2\,(z-7)\,(z-8)\,(z^2-3z+\tfrac{3}{2})^2(z^2-6z+6)^2,$$

and the eigenvalues of Δ_1 are

$$\{2_5,\ 3_7,\ 4_2,\ 7,\ 8,\ (\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{3})_2,\ (3\pm\sqrt{3})_2\},$$

For T_7 the characteristic polynomial of Δ_1 is

$$(z-2)\,(z-8)\,(z^2-12z+28)(z^3-6z^2+\tfrac{41}{4}z-\tfrac{29}{8})^2(z^3-10z^2+31z-29)^2$$
$$\times\,(z^3-7z^2+\tfrac{63}{4}z-\tfrac{91}{8})^2(z^3-8z^2+19z-13)^2.$$

It has eigenvalues 2, 8, $6\pm 2\sqrt{2}$ while all other eigenvalues are zeros of cubic equations.

Proposition 13. *For any $m\geq 2$, the operator Δ_1 on the trapezohedron T_m has eigenvalues $\lambda=2$ and $\lambda=m+1$.*

Problem. Determine $\operatorname{spec}\Delta_1$ on the trapezohedron T_m for any m.

Problem. Determine $\operatorname{spec}\Delta_2$ on the trapezohedron T_m. It is known that $|\Omega_2|=2m$, $|\Omega_3|=1$ and that $\operatorname{spec}\Delta_3=\{2\}$.

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