Homotopy and homology of digraphs

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BIMSA, March 2024

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1 Chain spaces and path homology on digraphs

1.1 Paths and the boundary operator

Let us fix a finite set V and a field $\mathbb{K}$. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V; it will be also denoted by $e_{i_0...i_p}$.
A *p-path* is any formal linear combinations of elementary p-paths $e_{i_0...i_p}$ with coefficients from $\mathbb{K}$; that is, any p-path u has a form

$$u = \sum_{i_0, i_1, ..., i_p \in V} u^{i_0 i_1 ... i_p} e_{i_0 i_1 ... i_p},$$

where $u^{i_0 i_1 ... i_p} \in \mathbb{K}$. The set of all p-paths is a $\mathbb{K}$-linear space denoted by $\Lambda_p = \Lambda_p(V, \mathbb{K})$.

For example, $\Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \qquad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle.$

Definition. Define for any $p \geq 1$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0...\widehat{i_q}...i_p}, \tag{1}$$

where $\widehat{}$ means omission of the index. For $p = 0$ set $\partial e_i = 0$ (and, hence, $\Lambda_{-1} = \{0\}$).

For example,

$$\partial e_{ij} = e_j - e_i \text{ and } \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}.$$

It is easy to show that $\partial^2 = 0$. Hence, we obtain a chain complex $\Lambda_*(V)$:

$$0 \leftarrow \Lambda_0 \overset{\partial}{\leftarrow} \Lambda_1 \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \Lambda_{p-1} \overset{\partial}{\leftarrow} \Lambda_p \overset{\partial}{\leftarrow} \cdots$$

An elementary p-path $e_{i_0 \ldots i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and *irregular* otherwise. A p-path is called regular (resp. irregular) if it is a linear combination of regular (resp. irregular) elementary paths.

Denote by $\mathcal{R}_p$ the space of all regular p-paths. Then ∂ is well defined on the spaces $\mathcal{R}_p$ if we identify all irregular paths with 0 (which is justified by the fact that if u is irregular then ∂u is also irregular). For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij} \in \mathcal{R}_1,$$

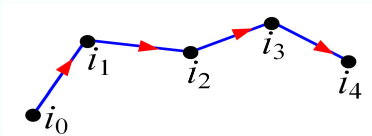
because $e_{ii} = 0$. Hence, we obtain a regular chain complex

$$0 \leftarrow \mathcal{R}_0 \overset{\partial}{\leftarrow} \mathcal{R}_1 \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \cdots$$

1.2 Chain complex on digraphs

A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and $E \subset \{V \times V \setminus \mathrm{diag}\}$ is a set of arrows (directed edges). If $(i, j) \in E$ then we write $i \to j$.

Definition. An elementary p-path $e_{i_0 \ldots i_p}$ in a digraph $G = (V, E)$ is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, \ldots, p - 1$, and *non-allowed* otherwise.



A p-path is called allowed if it is a linear combination of allowed elementary p-paths.

Denote by $\mathcal{A}_p = \mathcal{A}_p(G, \mathbb{K})$ the linear space of all allowed p-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on spaces $\mathcal{A}_p$. However, in general ∂ does not act on the spaces $\mathcal{A}_p$. For example, in the digraph $\overset{a}{\bullet} \to \overset{b}{\bullet} \to \overset{c}{\bullet}$ we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is not allowed.

Consider the following subspace of $\mathcal{A}_p$:

$$\boxed{\Omega_p \equiv \Omega_p(G, \mathbb{K}) := \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}}.$$

Claim. $\partial\Omega_p \subset \Omega_{p-1}$. Indeed, if $u \in \Omega_p$ then $\partial u \in \mathcal{A}_{p-1}$ and $\partial(\partial u) = 0 \in \mathcal{A}_{p-2}$ whence $\partial u \in \Omega_{p-1}$.

For example, we have $\Omega_0 = \mathcal{A}_0 = \langle e_i : i \in V \rangle$ and $\Omega_1 = \mathcal{A}_1 = \{e_{ij} : i \to j\}$.

Definition. The elements of Ω_p are called *∂-invariant* p-paths.

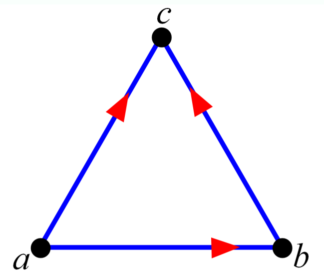
Hence, we obtain a chain complex $\Omega_* = \Omega_* (G, \mathbb{K})$ that reflects a digraph structure:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \cdots \tag{2}$$

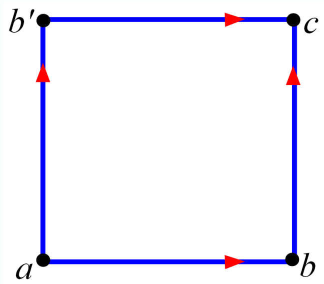
Homology groups of (2) are called *path homologies* of G and are denoted by $H_p(G)$.

1.3 Examples of ∂-invariant paths

A *triangle* is a sequence of three distinct vertices a, b, c such that $a \to b \to c$, $a \to c$.
It determines a ∂-invariant 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$.
The path e_{abc} is also referred to as a triangle.
If $a \to b \to c$ but $a \not\to c$ then $e_{abc} \in \mathcal{A}_2$ but $e_{abc} \notin \Omega_2$.



A *square* is a sequence of four distinct vertices a, b, b', c such that $a \to b \to c$, $a \to b' \to c$ while $a \not\to c$.



It determines a ∂-invariant 2-path

$$u = e_{abc} - e_{ab'c} \in \Omega_2$$

because $u \in \mathcal{A}_2$ and

$$\begin{aligned}\partial u &= (e_{bc} - \underline{e_{ac}} + e_{ab}) - (e_{b'c} - \underline{e_{ac}} + e_{ab'}) \\ &= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1.\end{aligned}$$

The path u is also referred to as a square.

1.4 Digraph maps

We write $a \,\overrightarrow{=}\, b$ if either $a \to b$ or $a = b$.

Definition. A *morphism* from a digraph $G = (V, E)$ to a digraph $G' = (V', E')$ is a map $f : V \to V'$ such that

$$\text{if } a \,\overrightarrow{=}\, b \text{ on } G \quad \text{then} \quad f(a) \,\overrightarrow{=}\, f(b) \ \text{ on } G'. \tag{3}$$

That is, if $a \to b$ in G then either $f(a) \to f(b)$ or $f(a) = f(b)$ in G'. We will refer to such morphisms also as *digraphs maps* and denote them shortly by $f : G \to G'$.

Given a map $f : V \to V'$, define for any $p \geq 0$ the *induced map*

$$f_* : \Lambda_p(V) \to \Lambda_p(V')$$

by the rule

$$f_*\left(e_{i_0 \ldots i_p}\right) = e_{f(i_0) \ldots f(i_p)}, \tag{4}$$

extended by $\mathbb{K}$-linearity to all elements of $\Lambda_p(V)$. It is obvious that

$$f_*(\mathcal{R}_p(V)) \subset \mathcal{R}_p(V') \ \text{ and } \ f_*(\mathcal{A}_p(G)) \subset \mathcal{A}_p(G').$$

It follows from (1) and (4) that $\partial f_* = f_* \partial$, which implies the following.

Proposition 1 *Let G and G' be two digraphs, and $f: G \to G'$ be a digraph map. Then, for any $p \geq 0$,*

$$f_*\left(\Omega_p\left(G\right)\right) \subset \Omega_p\left(G'\right). \tag{5}$$

Moreover, the map

$$f_*: \Omega_p\left(G\right) \to \Omega_p\left(G'\right)$$

is a morphism of the chain complexes

$$\Omega_*(G) \to \Omega_*(G')$$

and, consequently, a homomorphism of homology groups

$$H_*(G) \to H_*(G')$$

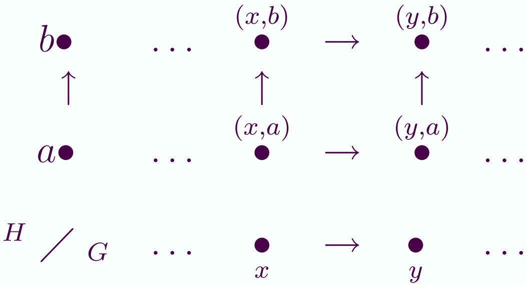
that will also be denoted by f_*.

1.5 Cartesian product

Given two digraphs G and H, define their Cartesian product as a digraph $G\Box H$ as follows:

• the vertices of $G\Box H$ are the couples (x, a) where $x \in V_G$ and $a \in V_H$;

• the arrows of $G\Box H$ are of two types: $(x, a) \to (y, a)$ if $x \to y$ in G (a *horizontal* arrow) and $(x, a) \to (x, b)$ if $a \to b$ in H (a *vertical* arrow):



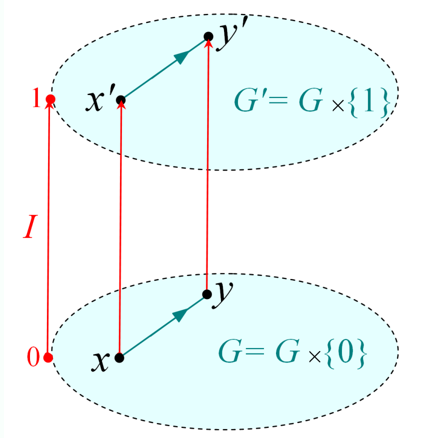
For any digraph G, define the *cylinder* over G by $\widehat{G} = G\Box I$ where $I = ({}^0\bullet \to \bullet^1)$.

We shall put the hat $\widehat{}$ over all notation related to $\widehat{G}$.

Let us identify $G \times 0$ with G and set $G' = G \times 1$.

For any $x \in V$, identify $(x, 0)$ with x and set $x' = (x, 1)$ so that $x \to x'$ in $\widehat{G}$.

For any arrow $x \to y$ in G, we have also $x \to y$ and $x' \to y'$ in $\widehat{G}$.



Digraph $\widehat{G}$

For any path $v \in \Lambda_p$ define the lifted path $\widehat{v} \in \widehat{\Lambda}_{p+1}$ by

$$\widehat{e}_{i_0 \ldots i_p} = \sum_{k=0}^{p} (-1)^k \, e_{i_0 \ldots i_k i'_k \ldots i'_p} \tag{6}$$

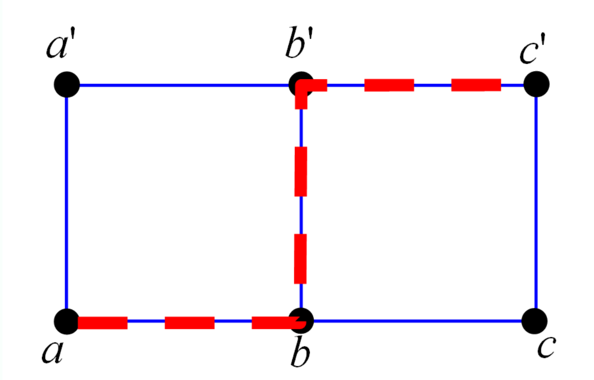
and linearity.

For example, we have

$\widehat{e}_a = e_{aa'}$

$\widehat{e}_{ab} = e_{aa'b'} - e_{abb'}$

$\widehat{e}_{abc} = e_{aa'b'c'} - e_{abb'c'} + e_{abcc'}$



The component $e_{abb'c'}$ of the 3-path $\widehat{e}_{abc}$

If $i_0...i_p$ is allowed in G then, for any k, the path $i_0...i_k i'_k...i'_p$ is allowed in $\widehat{G}$:

$$\begin{array}{ccccccccccccc} & & & & \cdots & \overset{i'_k}{\bullet} & \longrightarrow & \overset{i'_{k+1}}{\bullet} & \longrightarrow & \cdots & \longrightarrow & \overset{i'_p}{\bullet} & \\ & & & & & \uparrow & & \uparrow & & & & & , \\ \overset{i_0}{\bullet} & \longrightarrow & \cdots & \longrightarrow & & \overset{i_k}{\bullet} & \longrightarrow & \overset{i_{k+1}}{\bullet} & & \cdots & & & \end{array}$$

Hence, for any $v \in \mathcal{A}_p$ we have $\widehat{v} \in \widehat{\mathcal{A}}_{p+1}$. Below we will prove that if $v \in \Omega_p$ then $\widehat{v} \in \widehat{\Omega}_{p+1}$.
For any path v in G define its image v' in G' by $\left(e_{i_0...i_p}\right)' = e_{i'_0...i'_p}$.

Lemma 2 *For any p-path v on G with $p \geq 0$*

$$\partial \widehat{v} + \widehat{\partial v} = v' - v. \tag{7}$$

Proof. It suffices to prove (7) for $v = e_{i_0 \ldots i_p}$. For $p = 0$ set $v = e_i$ so that $\partial v = 0$ and $\widehat{v} = e_{ii'}$ whence

$$\partial \widehat{v} + \widehat{\partial v} = e_{i'} - e_i + 0 = v' - v.$$

For $p \geq 1$ we have

$$\begin{aligned}
\partial \widehat{v} &= \sum_{k=0}^{p} (-1)^k \, \partial e_{i_0 \ldots i_k i'_k \ldots i'_p} \\
&= \sum_{k=0}^{p} (-1)^k \left[\sum_{l=0}^{l} (-1)^l \, e_{i_0 \ldots \widehat{i_l} \ldots i_k i'_k \ldots i'_p} + \sum_{l=k}^{p} (-1)^{l+1} \, e_{i_0 \ldots i_k i'_k \ldots \widehat{i'_l} \ldots i'_p} \right] \\
&= \sum_{0 \leq l \leq k \leq p} (-1)^{k+l} \, e_{i_0 \ldots \widehat{i_l} \ldots i_k i'_k \ldots i'_p} + \sum_{0 \leq k \leq l \leq p} (-1)^{k+l+1} \, e_{i_0 \ldots i_k i'_k \ldots \widehat{i'_l} \ldots i'_p}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\partial v} &= \left(\sum_{l=0}^{p} (-1)^l \, e_{i_0 \dots \widehat{i_l} \dots i_p} \right)^{\widehat{}} \\
&= \sum_{l=0}^{p} (-1)^l \left[\sum_{k=l+1}^{p} (-1)^{k-1} e_{i_0 \dots \widehat{i_l} \dots i_k i'_k \dots i'_p} + \sum_{k=0}^{l-1} (-1)^k e_{i_0 \dots i_k i'_k \dots \widehat{i'_l} \dots i'_p} \right] \\
&= \sum_{0 \le l < k \le p} (-1)^{k+l-1} e_{i_0 \dots \widehat{i_l} \dots i_k i'_k \dots i'_p} + \sum_{0 \le k < l \le p} (-1)^{k+l} e_{i_0 \dots i_k i'_k \dots \widehat{i'_l} \dots i'_p}.
\end{aligned}$$

We see that in the sum $\partial \widehat{v} + \widehat{\partial v}$ all the terms with $k \neq l$ cancel out and we obtain

$$\partial \widehat{v} + \widehat{\partial v} = \sum_{k=0}^{p} e_{i_0 \dots i_{k-1} i'_k \dots i'_p} - \sum_{k=0}^{p} e_{i_0 \dots i_k i'_{k+1} \dots i'_p} = e_{i'_0 \dots . i'_p} - e_{i_0 \dots i_p} = v' - v.$$

■

Corollary 3 *If* $v \in \Omega_p$ *then* $\widehat{v} \in \widehat{\Omega}_{p+1}$.

Proof. We already know that $\widehat{v} \in \mathcal{A}_{p+1}$, and we need to prove that $\partial \widehat{v} \in \widehat{\mathcal{A}}_p$. Since $v \in \mathcal{A}_p$ and $\partial v \in \mathcal{A}_{p-1}$, we have $v' \in \widehat{\mathcal{A}}_p$ and $\widehat{\partial v} \in \widehat{\mathcal{A}}_p$ whence it follows from (7) that also $\partial \widehat{v} \in \widehat{\mathcal{A}}_p$. ■

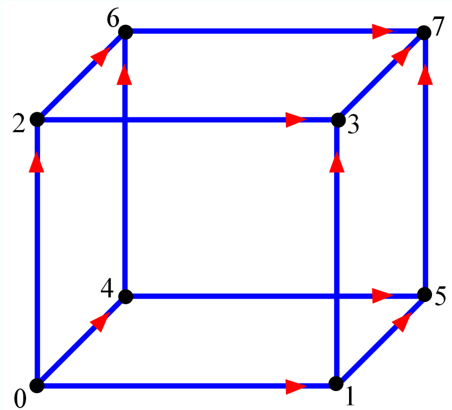
Example. The cylinder over the digraph $I = ({}^0\bullet \to \bullet^1)$ is a square

$$\begin{array}{ccc} {}^{2=0'}\bullet & \longrightarrow & \bullet^{1'=3} \\ \uparrow & & \uparrow \\ {}^{0}\bullet & \longrightarrow & \bullet^{1} \end{array} \tag{8}$$

Lifting a ∂-invariant 1-path $e_{01} \in \Omega_1$ we obtain a ∂-invariant 2-path on the square:

$$\widehat{e}_{01} = e_{00'1'} - e_{011'} = e_{023} - e_{013}.$$

The cylinder over the square (8) is a 3-cube: where we take $i' = i + 4$.



Lifting the ∂-invariant 2-path $v = e_{023} - e_{013}$ we obtain a ∂-invariant 3-path on the 3-cube:

$$\begin{aligned} \widehat{v} &= e_{00'2'3'} - e_{022'3'} + e_{0233'} - (e_{00'1'2'} - e_{011'2'} + e_{0133'}) \\ &= e_{0467} - e_{0267} + e_{0237} - e_{0457} + e_{0157} - e_{0137}. \end{aligned}$$

2 Homotopy theory of digraphs

2.1 The notion of homotopy

For any $n \geq 1$ define a *linear digraph* I_n as any digraph with vertices $\{0, 1, \ldots, n\}$ such that if $|i - j| = 1$ then either $i \to j$ or $j \to i$, and if $|i - j| \neq 1$ then there is no arrow between i and j.

For example, here is a linear digraph I_3: $\underset{0}{\bullet} \to \underset{1}{\bullet} \leftarrow \underset{2}{\bullet} \to \underset{3}{\bullet}$

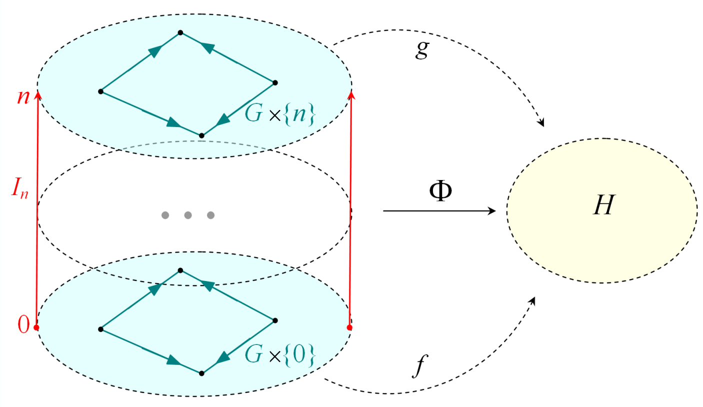
Definition. Let G and H be digraphs. Two digraph maps $f, g\colon G \to H$ are called *homotopic* if there exists a linear digraph I_n with some $n \geq 1$ and a digraph map

$$\Phi\colon G \Box I_n \to H$$

such that

$$\Phi|_{G \times 0} = f \quad \text{and} \quad \Phi|_{G \times n} = g. \tag{9}$$

In this case we write $f \simeq g$. Clearly, this is an equivalence relation.



In the case $n = 1$ we refer to the map Φ as an *one-step homotopy* between f and g and write $f \overset{\text{1-step}}{\simeq} g$.

It is easy to see that $f, g : G \to H$ are homotopic if and only if there is a finite sequence of digraph maps $f = f_0, f_1, ..., f_n = g$ from G to H such that

$$f_k \overset{\text{1-step}}{\simeq} f_{k+1}.$$

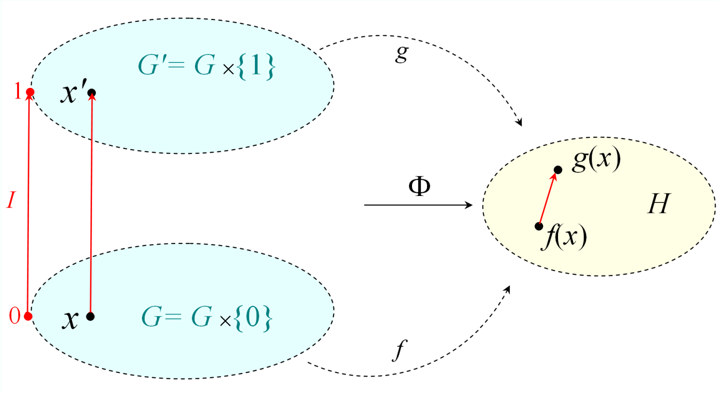
Let $\Phi : G\Box I_1 \to H$ be an one-step homotopy between f and g and let $I_1 = ({}^0\bullet \to \bullet^1) = I$.

Then $G\Box I$ is the cylinder $\widehat{G}$, and the map $\Phi : G\Box I \to H$ is determined by its restrictions $\Phi|_G = f$ and $\Phi|_{G'} = g$.

For a vertical arrow $x \to x'$ we have

$$\Phi(x) = f(x) \quad \text{and} \quad \Phi(x') = g(x)$$

so that the requirement $\Phi(x) \overrightarrow{=} \Phi(x')$ becomes $f(x) \overrightarrow{=} g(x)$ in H.

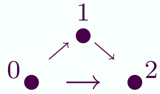


The map Φ:$\widehat{G} \to H$

Considering similarly the case $I_1 = \{{}^0\bullet \leftarrow \bullet^1\}$, we obtain that $f \overset{\text{1-step}}{\simeq} g$ if and only if

$$\text{either } f(x) \overrightarrow{=} g(x) \text{ for all } x \in V_G$$
$$\text{or } g(x) \overrightarrow{=} f(x) \text{ for all } x \in V_G.$$

Example. Consider the digraphs

$G =$  and $H = {}^{a}\bullet \rightarrow \bullet^{b}$

and the mappings $f, g : V_G \rightarrow V_H$ given by the table:

$x \in V_G$	$f(x)$	$g(x)$
0	a	a
1	a	b
2	b	b

It is easy to see that both f and g are digraph maps from G to H. Moreover, f and g are one-step homotopic, because $f(x) \overrightarrow{=} g(x)$ for all $x \in V_G$.

Definition. Two digraphs G and H are called *homotopy equivalent* if there exist digraph maps

$$f : G \rightarrow H, \quad g : H \rightarrow G \tag{10}$$

such that

$$f \circ g \simeq \mathrm{id}_H, \qquad g \circ f \simeq \mathrm{id}_G\,. \tag{11}$$

In this case we shall write $G \simeq H$. The maps f and g as in (12) are called *homotopy inverses* of each other.

2.2 Homotopy preserves homologies

Now we can prove the main result about connections between homotopy and homology on digraphs.

Theorem 4 *Let G, H be two digraphs.*

(i) *Let $f, g : G \to H$ be two digraph maps. If $f \simeq g$ then the induced maps*

$$f_* : H_p(G) \to H_p(H) \quad \textit{and} \quad g_* : H_p(G) \to H_p(H)$$

of the homology groups are identical, that is, $f_* = g_*$ in homologies.

(ii) *If the digraphs G and H are homotopy equivalent, then all their homology groups are isomorphic.*

Proof. (i) Let $\Phi : G\Box I_n \to H$ be a homotopy between f and g. It suffices to treat the case $n = 1$ as the general case then follows by induction. Let $I_1 = I = (0 \to 1)$ so that $G\Box I_1 = G\Box I = \widehat{G}$ (the case $I_1 = I^-$ can be treated similarly). The maps f and g induce morphisms of chain complexes

$$f_*, g_* : \Omega_*(G) \to \Omega_*(H),$$

and Φ induces a morphism

$$\Phi_*\colon \Omega_*(\widehat{G}) \to \Omega_*(H).$$

As before, we identify G with $G \times 0$ and set $G' = G \times 1$. For any path $v \in \Omega_*(G)$ considering as a path in $\widehat{G}$ we have $\Phi_*(v) = f_*(v)$ and $\Phi_*(v') = g_*(v')$.

In order to prove that f_* and g_* induce the identical homomorphisms $H_*(G) \to H_*(H)$, it suffices to construct a chain homotopy between the chain complexes $\Omega_*(G)$ and $\Omega_*(H)$, that is, the $\mathbb{K}$-linear mappings

$$L_p : \Omega_p(G) \to \Omega_{p+1}(H)$$

such that

$$\partial L_p + L_{p-1}\partial = g_* - f_*$$

(note that all the terms here are mapping from $\Omega_p(G)$ to $\Omega_p(H)$) as on the following diagram:

$$\begin{array}{ccccc}
\Omega_{p-1}(G) & \overset{\partial}{\longleftarrow} & \Omega_p(G) & \longleftarrow & \Omega_{p+1}(G) \\
 & \overset{L_{p-1}}{\searrow} & \downarrow f_* \downarrow g_* & \overset{L_p}{\searrow} & \\
\Omega_{p-1}(H) & \longleftarrow & \Omega_p(H) & \underset{\partial}{\longleftarrow} & \Omega_{p+1}(H)
\end{array}$$

Let us define the mapping L_p as follows

$$L_p(v) = \Phi_*(\widehat{v}) \quad \text{for any } v \in \Omega_p(G),$$

where $\widehat{v} \in \Omega_{p+1}(\widehat{G})$ is the lifting of v to the graph $\widehat{G}$ defined in Section 1.5. Using $\partial\Phi_* = \Phi_*\partial$ (see Proposition 1) and the product rule (7) of Lemma 2, we obtain

$$\begin{aligned}(\partial L_p + L_{p-1}\partial)(v) &= \partial(\Phi_*(\widehat{v})) + \Phi_*(\widehat{\partial v}) \\ &= \Phi_*(\partial\widehat{v}) + \Phi_*(\widehat{\partial v}) \\ &= \Phi_*(\partial\widehat{v} + \widehat{\partial v}) \\ &= \Phi_*(v' - v) \\ &= g_*(v) - f_*(v).\end{aligned}$$

(ii) Let $f : G \to H$ and $g : H \to G$ be digraph maps such that

$$f \circ g \simeq \mathrm{id}_H, \qquad g \circ f \simeq \mathrm{id}_G. \tag{12}$$

Then they induce the following mappings

$$H_p(G) \xrightarrow{f_*} H_p(H) \xrightarrow{g_*} H_p(G) \xrightarrow{f_*} H_p(H).$$

By (i) and (13) we have $f_* \circ g_* = \mathrm{id}$ and $g_* \circ f_* = \mathrm{id}$, which implies that f_* and g_* are mutually inverse isomorphisms of $H_p(G)$ and $H_p(H)$. ■

2.3 Retraction

A (induced) sub-digraph H of a digraph G is a digraph such that $V_H \subset V_G$, and $x \to y$ in H if and only if $x \to y$ in G.

Definition. Let G be a digraph and H be its sub-digraph. A *retraction* of G onto H is a digraph map $r : G \to H$ such that $r|_H = \mathrm{id}_H$.

Let $r : G \to H$ be a retraction and let $i : H \to G$ be the natural inclusion map. By definition of retraction we have $r \circ i = \mathrm{id}_H$. Therefore, if

$$i \circ r \simeq \mathrm{id}_G, \tag{13}$$

then i and r are homotopy inverses and we obtain that $G \simeq H$. A retraction $r : G \to H$ with the property (14) is called a *deformation retraction.*

Proposition 5 *Let $r : G \to H$ be a retraction of a digraph G onto a sub-digraph H such that*

$$\text{either } x \overrightarrow{=} r(x) \text{ for all } x \in V_G \text{ or } r(x) \overrightarrow{=} x \text{ for all } x \in V_G. \tag{14}$$

Then r is a deformation retraction and, consequently, the digraphs G and H are homotopy equivalent.

Proof. Set $f = \mathrm{id}_G$ and $g = i \circ r$. For any $x \in V_G$ we have $f(x) = x$ and $g(x) = r(x)$. The condition (15) means that f and g satisfy (**??**), whence $f \overset{1\text{-step}}{\simeq} g$. Hence, we obtain (14) and, consequently, $G \simeq H$. ■

Example. Let us show that the square

$$G = \begin{array}{ccc} {}^2\bullet & \longrightarrow & \bullet^3 \\ \uparrow & & \uparrow \\ {}^0\bullet & \longrightarrow & \bullet^1 \end{array}$$

is also contractible. It suffices to show that $G \simeq H$ where H is the following subgraph

$$H = {}^0\bullet \longrightarrow \bullet^1 .$$

Consider a retraction $r : G \to H$ given by

$$r(0) = r(2) = 0 \text{ and } r(1) = r(3) = 1.$$

Clearly, it satisfies $r(x) \overrightarrow{=} x$ for all $x \in V_G$ and we conclude by Proposition 5 that $G \simeq H$. Since H is contractible, we obtain that G is also contractible.

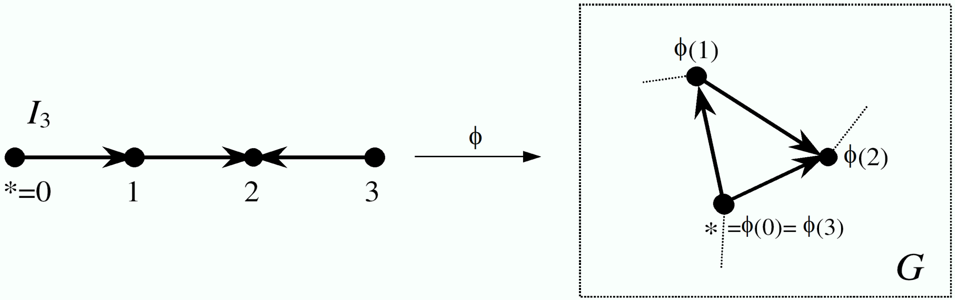
Example. For any $n \geq 1$, consider the n-dimensional cube $I^n = \underbrace{I \Box I \Box \ldots \Box I}_{n \text{ times}}$. As in the previous example, one constructs an obvious deformation retraction of I^n onto I^{n-1} thus proving that $I^n \simeq I^{n-1}$. By induction we obtain that all cubes I^n are contractible.

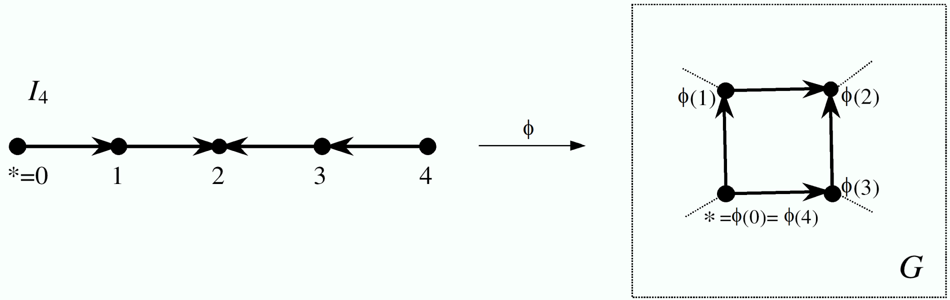
3 Fundamental group of a digraph

A *based* digraph G^* is a digraph G with a fixed base vertex $* \in V_G$. A based digraph map $f : G^* \to H^*$ is a digraph map $f : G \to H$ such that $f(*) = *$. Any linear digraph I_n will always be considered as a based digraph with the base point 0.

3.1 C-homotopy

A *loop* in a digraph G is any digraph map $\phi : I_n \to G$ with $\phi(0) = \phi(n)$. A *based loop* on a based digraph G^* is a loop $\phi : I_n \to G^*$, such that $\phi(0) = \phi(n) = *$.





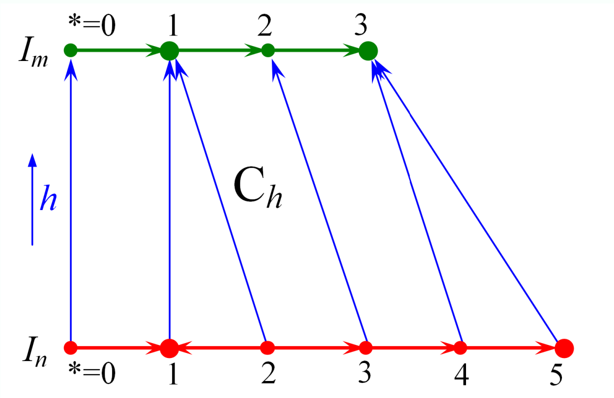
A digraph map $h : I_n \to I_m$ is called *shrinking* if $h(0) = 0$, $h(n) = m$, and $h(i) \leq h(j)$ whenever $i \leq j$ (which is only possible when $m \leq n$).

The *cylinder* C_h of the map h is the digraph with the set of vertices $V_{C_h} = V_{I_n} \sqcup V_{I_m}$ and with the set of arrows E_{C_h} that consists of all the arrows of I_n and I_m and of the arrows

$$i \to h(i) \text{ for all } i \in I_n.$$

Similarly define the inverse cylinder C_h^- using

$$h(i) \to i \text{ for all } i \in I_n.$$



Definition. Consider two based loops

$$\phi\colon I_n \to G^* \quad \text{and} \quad \psi\colon I_m \to G^*.$$

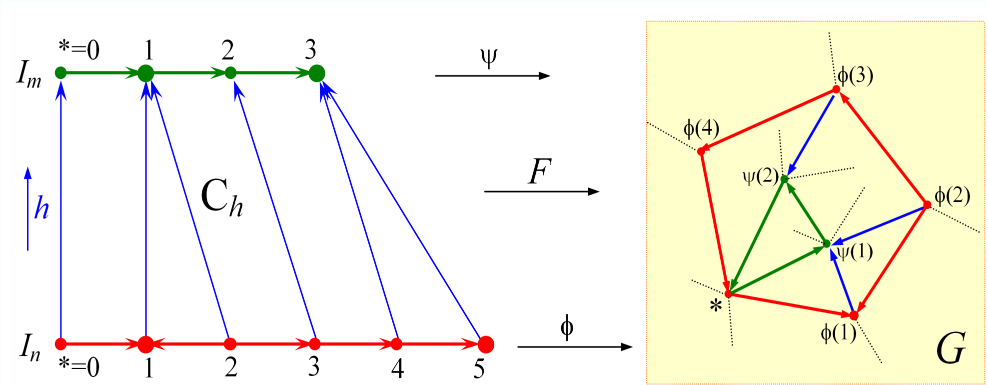
We say that there is *one-step direct C-homotopy* from ϕ to ψ and write with $\phi \overset{C}{\to} \psi$ if there exists a shrinking map $h : I_n \to I_m$ such that the map $F : \mathrm{C}_h \to G$ given by

$$F|_{I_n} = \phi \quad \text{and} \quad F|_{I_m} = \psi, \tag{15}$$

is a digraph map, that is, $\phi(i) \overrightarrow{=} \psi(h(i))$ for all $i \in I_n$.

If F is a digraph map from C_h^- to G then we call it an one-step *inverse* C-homotopy and write $\phi \overset{C}{\leftarrow} \psi$.

Example. An example of one-step direct C-homotopy is shown here:

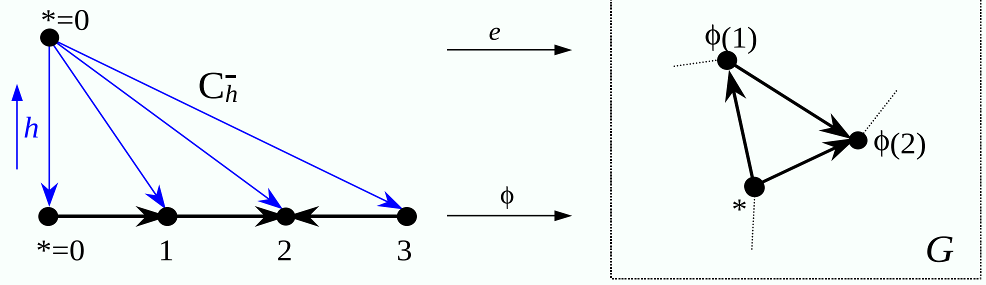


If $n=m$ then $h=\mathrm{id}_{I_n}$ and an one-step C-homotopy is a homotopy.

Definition. We call two loops ϕ,ψ *C-homotopic* and write $\phi \overset{C}{\simeq} \psi$ if there exists a finite sequence $\{\phi_k\}_{k=0}^{m}$ of loops in G^* such that $\phi_0=\phi$, $\phi_m=\psi$ and, for any $k=0,...,m-1$, holds $\phi_k \overset{C}{\rightarrow} \phi_{k+1}$ or $\phi_k \overset{C}{\leftarrow} \phi_{k+1}$.

Clearly, $\phi \overset{C}{\simeq} \psi$ is an equivalence relation. The C-homotopy class of a based loop ϕ will be denoted by $[\phi]$. We say that a loop ϕ is *C-contractible* if $\phi \overset{C}{\simeq} e$, that is, $[\phi]=[e]$.

Example. A *triangular loop* is a loop $\phi : I_3 \rightarrow G^*$ with $I_3 = (0 \rightarrow 1 \rightarrow 2 \leftarrow 3)$.

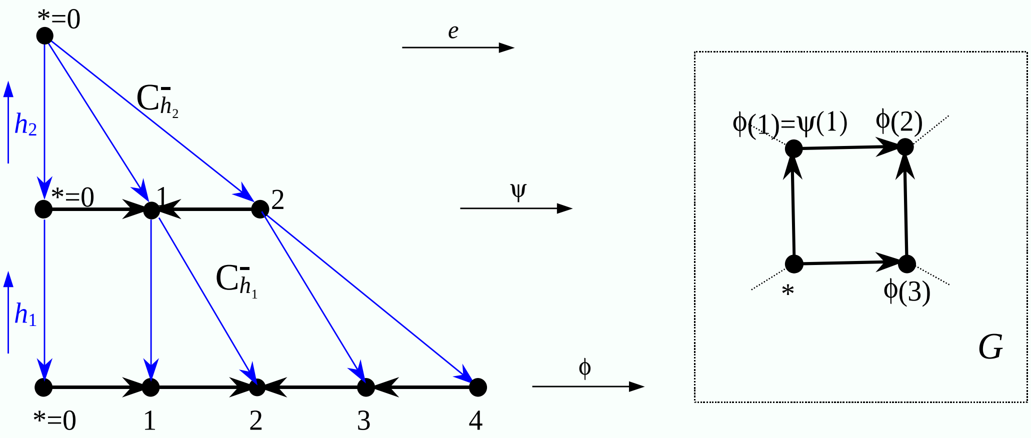


The triangular loop is C-contractible because the following shrinking map

$$h : I_3 \to I_0, \quad h(k) = 0 \text{ for all } k = 0, ..., 3,$$

provides an inverse one-step C-homotopy between ϕ and e.

Example. A *square loop* is a loop $\phi : I_4 \to G$ with $I_4 = (0 \to 1 \to 2 \leftarrow 3 \leftarrow 4)$.The square loop can be C-contracted to e in two steps:

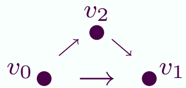


3.2 Local description of C-homotopy

Any loop $\phi\colon I_n \to G$ determines a sequence $\theta_\phi = \{\phi(i)\}_{i=0}^n$ of vertices of G. We consider the sequence θ_ϕ as a *word* over the alphabet V_G.

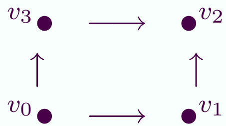
Theorem 6 *Two based loops $\phi : I_n \to G^*$ and $\psi : I_m \to G^*$ are C-homotopic if and only if the word θ_ψ can be obtained from θ_ϕ by a finite sequence of the following transformations (or their inverses):*

(i) ...abc... $\mapsto$ ...ac... where (a, b, c) is any permutation of a triple (v_0, v_1, v_2) of vertices forming a triangle in G:



(and the dots "..." denote the unchanged parts of the words).

(ii) ...abc... $\mapsto$ ...adc... where (a, b, c, d) is any cyclic permutation (or an inverse cyclic permutation) of a quadruple (v_0, v_1, v_2, v_3) of vertices forming a square in G:



(iii) $...abcd... \mapsto ...ad...$ *where* (a, b, c, d) *is as in* (ii).

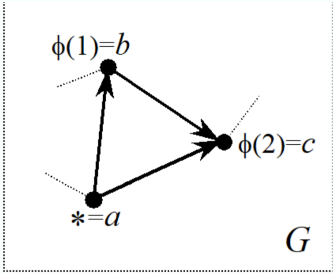
(iv) $...aba... \mapsto ...a...$ *if* $a \to b$ *or* $b \to a$.

(v) $...aa... \mapsto ...a...$

Examples.

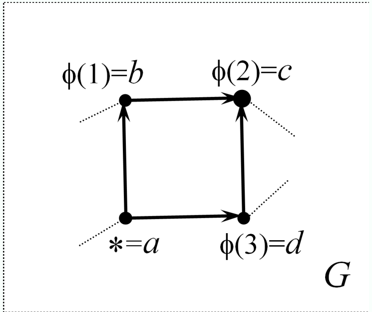
1. A triangular loop $\phi : I_3 \to G$ is contractible because

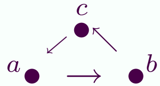
$$\theta_\phi = abca \overset{(i)}{\sim} aca \overset{(iv)}{\sim} a$$



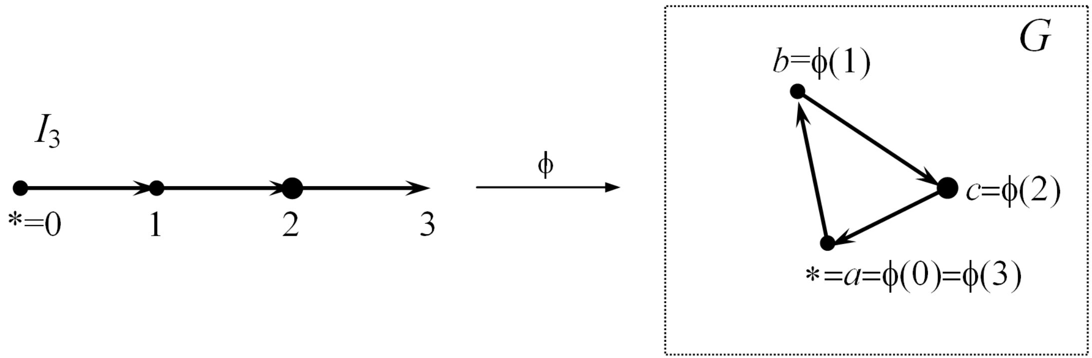
2. A square loop $\phi : I_4 \to G$ is contractible because

$$\theta_\phi = abcda \overset{(iii)}{\sim} ada \overset{(iv)}{\sim} a.$$



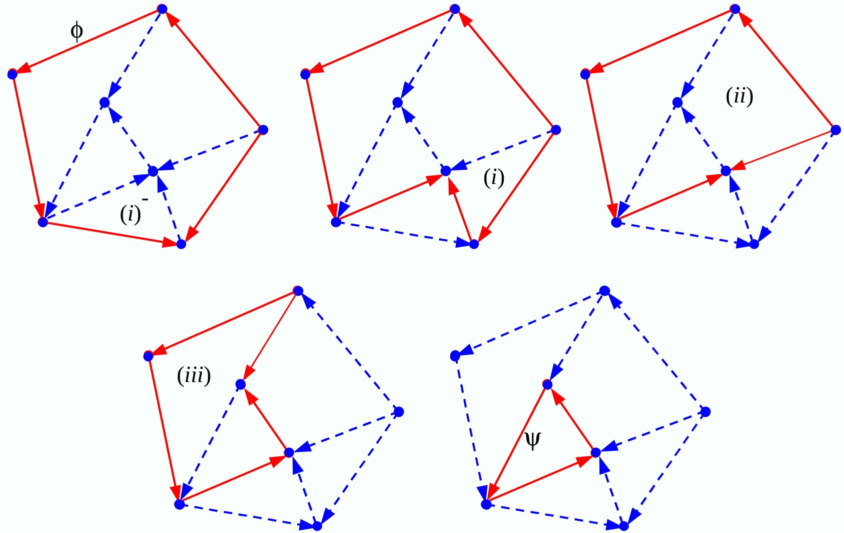
3. Consider a cyclic digraph $G =$ 

and the following loop $\phi : I_3 \to G$:



We have $\theta_\phi = abca$. It is clear that this word does not allow any of the transformations of Theorem 6, which implies that ϕ is not C-contractible.

4. Consider the loops ϕ and ψ as p.27. It is shown here how to transform θ_ϕ to θ_ψ using the word transformations of Theorem 6.



Transforming a 5-cycle θ_ϕ to a 3-cycle θ_ψ using successively $(i)^-$ (the inverse of (i)), $(i)\,,(ii)$ and $(iii)\,.$

3.3 Group structure in π_1

For any two linear digraphs I_n and I_m, define the linear digraph $I_n \vee I_m$ that is obtained from I_n and I_m by identification of the vertex $n \in I_n$ with the vertex $0 \in I_m$.

For any linear digraph I_n define a linear digraph $\hat{I}_n$ as follows:

$$i \to j \text{ in } \hat{I}_n \;\Leftrightarrow\; (n-i) \to (n-j) \text{ in } I_n.$$

Definition. (i) For two based loops $\phi : I_n \to G$ and $\psi : I_m \to G$ define their *concatenation* $\phi \vee \psi : I_n \vee I_m \to G$ by

$$\phi \vee \psi(i) = \begin{cases} \phi(i), & 0 \le i \le n \\ \psi(i-n), & n \le i \le n+m. \end{cases}$$

(ii) For any based loop $\phi : I_n \to G$ define its *inversion* $\hat{\phi} : \hat{I}_n \to G$ by $\hat{\phi}(i) = \phi(n-i)$.

Denote by $\pi_1(G^*)$ the set of all equivalence classes $[\phi]$ for all based loops ϕ in G^*. Now we can define a product in $\pi_1(G^*)$ as follows.

Definition. For any two based loops ϕ, ψ in G^* define the product of the equivalence classes $[\phi]$ and $[\psi]$ by $[\phi] \cdot [\psi] = [\phi \vee \psi]$.

Theorem 7 *Let G, H be digraphs.*

(i) The product in $\pi_1(G^*)$ is well defined. The set $\pi_1(G^*)$ with the product $[\phi] \cdot [\psi]$, the neutral element $[e]$ and inversion $[\hat{\phi}]$ is a group.

(ii) Any based digraph map $f : G^* \to H^*$ induces a group homomorphism

$$\begin{aligned} f &: \pi_1(G^*) \to \pi_1(H^*) \\ f([\phi]) &= [f \circ \phi], \end{aligned}$$

which depends only on homotopy class of f.

(iii) Let G, H be connected. If $G \simeq H$ then the fundamental groups $\pi_1(G^*)$ and $\pi_1(H^*)$ are isomorphic (for any choice of the base vertices).

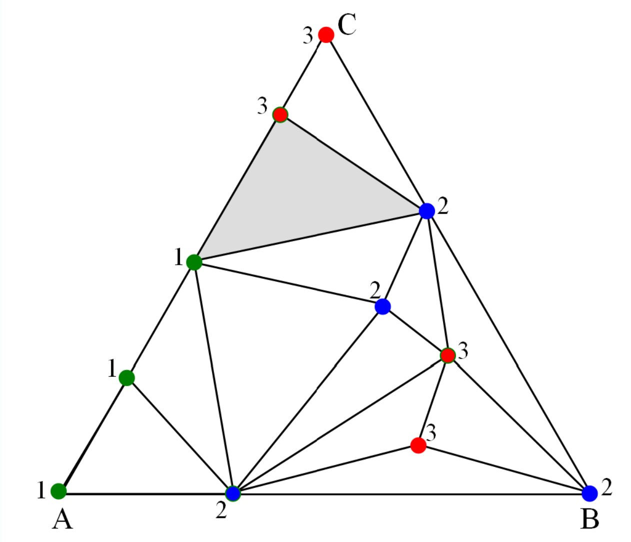
3.4 Application to graph coloring

An an illustration of the theory of digraph homotopy, we give here a new proof of the classical lemma of Sperner, using the notion the fundamental group and C-homotopy.

Consider a triangle ABC on the plane $\mathbb{R}^2$ and its triangulation T. Assume that the set of vertices of T is colored with three colors $1, 2, 3$ in such a way that

- A, B, C are colored with $1, 2, 3$ respectively;
- each vertex on any side of ABC is colored with one of the two colors of the endpoints of the side.

The classical lemma of Sperner says:
there exists in T a 3-color triangle, that is, a triangle whose vertices are colored with three different colors.



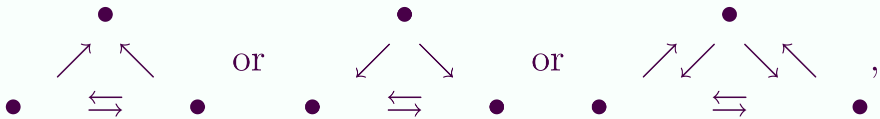
A Sperner coloring

To prove this, let us first modify the triangulation T so that there are no vertices on the sides AB, AC, BC except for A, B, C. Indeed, if X is a vertex on AB then we move X a bit inside the triangle ABC. This gives rise to a new triangle in the triangulation T that is formed by X and its former neighbors, say Y and Z, on the arrow AB (while keeping all other triangles). However, since all X, Y, Z are colored with two colors, no 3-color triangle emerges after that move. By induction, we remove all the vertices from the sides of ABC.

The triangulation T can be regarded as a graph. Let us make it into a digraph G by choosing the direction on the arrows as follows. If the vertices a, b are connected by an arrow in T then choose direction between a, b using the colors of a, b and the following rule:

$$\begin{array}{ccc} 1 \to 2, & 2 \to 3, & 3 \to 1 \\ 1 \leftrightarrows 1, & 2 \leftrightarrows 2, & 3 \leftrightarrows 3 \end{array} \tag{16}$$

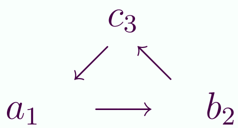
Assume now that there is no 3-color triangle in T. Then each triangle from T looks in G like



in particular, each of them contains a triangle in the sense of Theorem 6.

Consider a 3-loop $\phi : I_3 \to G^*$ with the word $\theta_\phi = ABCA$. Using the transformation (ii) of Theorem 6 and the partition of G into the triangle digraphs, we can contract the word $ABCA$ to an empty word. Hence, $\phi \overset{C}{\sim} e$.

Consider the cycle digraph H with the vertices a, b, c as follows



(17)

where the vertex a is colored by 1, b by 2 and c by 3. Define a map $f : G \to H$ by the rule that $f(x)$ has the same color in H as x in G.

By the choice of directions on the arrows of G, f is a digraph map. The loop $f \circ \phi$ on H has the word

$$\theta_{f\circ\phi} = abca,$$

which is not contractible on H as we have seen above. However, by Theorem 8, f induces homomorphism of $\pi_1(G)$ to $\pi_1(H)$. Therefore, $\phi \overset{C}{\simeq} e$ implies that also $f \circ \phi \overset{C}{\simeq} e$, which contradicts the previous observation.

3.5 Hurewicz theorem

One of our main results is the following discrete version of Hurewicz theorem.

Theorem 8 *For any based connected digraph G^* we have an isomorphism*

$$\pi_1(G^*) \,/[\pi_1(G^*), \pi_1(G^*)] \cong H_1(G, \mathbb{Z})$$

where $[\pi_1(G^*), \pi_1(G^*)]$ is a commutator subgroup.