On a class of random perturbations of the hierarchical Laplacian*

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Abstract

Let (X,d) be a locally compact separable ultrametric space. Given a measure m on X and a function $C(B)$ defined on the set of all non-singleton balls B of X we consider the hierarchical Laplacian $L = L_C$. The operator L acts in $L^2(X,m)$, is essentially self-adjoint and has a purely point spectrum. Choosing a family $\{\varepsilon(B)\}$ of i.i.d. we define the perturbated function $C(B,\omega)$ and the perturbated hierarchical Laplacian $L^\omega = L_{C(\omega)}$. We study the arithmetic means $\overline{\lambda}(\omega)$ of the L^ω-eigenvalues. Under some mild assumptions the normalized arithmetic means $\left(\overline{\lambda} - \mathbb{E}\overline{\lambda}\right)/\sigma\left[\overline{\lambda}\right]$ converge to $N(0,1)$ in law. We also give examples where the normal convergence fails. We prove existence of the integrated density of states. We introduce the empirical point process N^ω of the L^ω-eigenvalues and, assuming that the density of states exists and is continuous, we prove that the finite dimensional distributions of N^ω converge to the finite dimensional distributions of the Poisson point process. As an example we consider random perturbations of the Vladimirov operator acting in $L^2(X,m)$, where $X = \mathbb{Q}_p$ is the ring of p-adic numbers and m is the Haar measure.

Contents

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1 Introduction

The concept of hierarchical lattice and hierarchical distance was proposed by F.J. Dyson in his famous papers on the phase transition for **1D** ferromagnetic model with long range interaction [12, 13]. The notion of the hierarchical Laplacian L, which is closely related to the Dyson's model was studied in several mathematical papers [7], [18],[19],[20], [24], [2], [25]. These papers contain some basic information about L (the spectrum, the Markov semigroup, resolvent etc) in the case when the state space X is discrete and the hierarchical lattice satisfies some symmetry conditions (homogenuity, self-similarity etc). Under these symmetry conditions the state space (X, m) can be identified with some discrete infinitely generated Abelian group G equipped with a translation invariant ultrametric, the Markov semigroup $P^t = \exp(-tL)$ acting on $L^2(G, m)$ becomes symmetric, translation invariant and isotropic. In particular, $\mathsf{Spec}(L)$ is pure point and all eigenvalues have infinite multiplicity.

The main goal of the papers mentioned above was to study the corresponding Anderson Hamiltonian $H = L + V$; hierarhical Laplacian L plus random potential V. There was a hope to detect for such operators the spectral bifurcation from the pure point spectrum to the continuous one, i.e. to justify the famous Anderson conjecture. Unfortunately, the true result was opposite: under mild technical conditions the hierarchical Anderson Hamiltonian has a pure point spectrum - the phenomenon of localization, see [25] and [19]. Moreover, the local statistics of the spectrum of H is Poissonian, see [20], which is always deemed a manifistation of the spectral localization, see [1] and [23].

We will introduce a new class of operators: the random hierarchical Laplacians L^ω, which demonstrate several new spectral effects. The spectrum of such operators is still pure point (with compactly supported eigenfunctions) but in contrast to the deterministic case there exists the continuous density of states. This density detects the spectral bifurcation from the pure point spectrum to the continuous one. The eigenvalues form locally a Poissonian point process with intensity given by the density of states. We will show that our assumptions on the random variables in the definition of L^ω are almost final. Counterexamples demonstrate that all major results are failed without such assumptions.

A systematic study of a class of isotropic Markov semigroups defined on a ultrametric measure space (X, d, m) has been done in [4] (X is discrete) and in the recent paper [5] (X may contain both isolated and non-isolated points), see also related papers [2] and [16]. This study has been motivated by *Random walks on infinitely generated groups* - the classical subject which can be traced back to the pioneering works of Erdos, Spitzer, Kesten, Molchanov, Lawler and others. It turned out that the two mentioned above studies are closely related to each other. Namely, given an isotropic Markov semigroup (P^t) defined on a ultrametric measure space (X, d, m) with minus Markov generator L, one can show that the operator L coincides with the hierarchical Laplacian L_C on (X, d, m) associated with appropriately defined choice-function $C(B)$ (see the definition below), and vice versa. Then the general theory developed in [4] and [5] applies. In particular, modifying canonically the underlying ultrametric d, we call this new ultrametric d_*, the set $\mathsf{Spec}(L)$ can be described as

$$\mathsf{Spec}(L) = \mathsf{closure}\{1/d_*(x, y) : x \neq y\} \cup \{0\}. \tag{1.1}$$

In this construction the families of d-balls and d_*-balls coincide, whence these two ultrametrics generate the same topology and the same hierarchical structure, and in particular, the same class of hierarchical Laplacians. In turn the equation (1.1) yields the following crucial in our analysis facts, see the paper [6]: Let $S \subseteq [0,\infty)$ be any given closed set which contains 0 as an accumulation point. Assume that X is not compact and that if X contains a non-isolated point then S is unbounded. The following properties hold true:

- There exists a proper ultrametric d' on X and a choice function $C(B)$ defined on the set of balls in (X,d') such that $\mathsf{Spec}(L_C)=S$. The ultrametric d' defines the same topology as d.
- Assume that d is proper and that there exists a partition of X made of d-balls containing infinitely many non-singletons. Then there exists a choice function $C(B)$ defined on the set of balls in (X,d) such that $\mathsf{Spec}(L_C)=S$.

A very simple example shows that the condition "there exists a partition of X made of d-balls containing infinitely many non-singletons" in the second statement can not be dropped: $X=\mathbb{N}$ and $d(m,n)=\max(m,n)$ when $m\neq n$ and 0 otherwise.

In the course of study we assume that (X,d) is a locally compact and separable ultrametric space. Recall that a metric d is called an *ultrametric* if it satisfies the ultrametric inequality

$$d(x,y)\leq \max\{d(x,z),d(z,y)\}, \tag{1.2}$$

that is obviously stronger than the usual triangle inequality. Usually, we also assume that the ultrametric d is *proper*, that is, each closed d-ball is a compact set.

Let m be a Radon measure on (X,d) such that

- $m(B)>0$ for each ball B of positive diameter.
- $m(\{x\})=0$ if and only if x is a non-isolated point.
- $m(X)=\infty$ if X is not compact.

Let $\mathcal{B}$ be the set of all balls having positive measure. Our assumptions imply that the set $\mathcal{B}$ is at most countable. Let $C:\mathcal{B}\to(0,\infty)$ be a function which satisfies the following assumptions (in short, a choice-function):

- For any ball $B\in\mathcal{B}$,
$$\lambda(B):=\sum_{T\in\mathcal{B}:\ B\subseteq T} C(T)<\infty.$$
- For any non-isolated $x\in X$,
$$\sup\{\lambda(B): B\in\mathcal{B} \text{ and } x\in B\}=\infty.$$

Let $\mathcal{D}$ be the set of all locally constant functions having compact support. Given the data (X,d,m,C) we define (pointwise) the hierarchical Laplacian

$$L_C f(x):=-\sum_{B\in\mathcal{B}:\ x\in B} C(B)\left(P_B f-f(x)\right),\quad f\in\mathcal{D}, \tag{1.3}$$

where

$$P_B f:=\frac{1}{m(B)}\int_B f\,dm.$$

The operator $(L_C, \mathcal{D})$ acts in $L^2 = L^2(X, m)$, is symmetric and admits a complete system of eigenfunctions $\{f_{B,B'}\}_{B \in \mathcal{B}}$, namely,

$$f_{B,B'} = \frac{1}{m(B)} \mathbf{1}_B - \frac{1}{m(B')} \mathbf{1}_{B'}, \tag{1.4}$$

where $B \subset B'$ are nearest neighboring balls; when $m(X) < \infty$, we also set $f_{X,X'} = 1/m(X)$. The eigenvalue $\lambda(B')$ corresponding to $f_{B,B'}$ is

$$\lambda(B') = \sum_{T \in \mathcal{B}:\ B' \subseteq T} C(T); \tag{1.5}$$

when $m(X) < \infty$, we also set $X' = X \cup \{\varpi\}$ and $\lambda(X') = 0$. In particular, we conclude that $(L_C, \mathcal{D})$ is an essentially self-adjoint operator in L^2. By abuse of notation, we shall write $(L_C, \mathsf{Dom}_{L_C})$ for its unique self-adjoint extension. For all that we refer to [6].

Observe that to define the functions $C(B)$, $\lambda(B)$ and in particular the operator $(L_C, \mathsf{Dom}_{L_C})$ we do not need to specify the ultrametric d. What is needed is the family of balls $\mathcal{B}$ which evidently can be the same for two different ultrametrics d and d'.

On the other hand, given the data (X, d, m) and choosing the function

$$C(B) = \frac{1}{\mathsf{diam}(B)} - \frac{1}{\mathsf{diam}(B')},$$

where $B \subset B'$ are any two nearest neighboring balls, we obtain the hierarchical Laplacian $(L_C, \mathsf{Dom}_{L_C})$ satisfying

$$\lambda(B) = \frac{1}{\mathsf{diam}(B)}.$$

We will refere to $(L_C, \mathsf{Dom}_{L_C})$ as to the standard hierarchical Laplacian associated with the data (X, d, m).

Let us describe the main body of the paper. In Section 2 we specify some spectral properties of the hierarchical Laplacian L_C assuming that the ultrametric measure space (X, d, m) where it acts and the Laplacian by itself satisfy certain symmetry conditions (homogenuity, self-semilarity). As an example we consider the operator $\mathfrak{D}^\alpha$ of the p-adic fractional derivative of order $\alpha > 0$. This operator related to the concept of p-adic Quantum Mechanics was introduced by V.S. Vladimirov, see [31], [32] and [33]. $\mathfrak{D}^\alpha$ is the hierarchical Laplacian which acts in $L^2(\mathbb{Q}_p, m)$, where $\mathbb{Q}_p$ is the field of p-adic numbers and m is the Haar measure. The set $\mathsf{Spec}(\mathfrak{D}^\alpha)$ consists of eigenvalues $p^{k\alpha}, k \in \mathbb{Z}$, each of which has infinite multiplicity and contains 0 as an accumulation point.

In Section 3, given a homogeneous Laplacian L_C and a family $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ of symmetric i.i.d., we define a perturbated choice-function $C(B, \omega)$ and a perturbated Laplacian $L^\omega = L_{C(\omega)}$. The new operator L^ω is the main object in the study of the spectral statistics.

For each $\omega \in \Omega$, the operator L^ω is the hierarchical Laplacian whence it has a pure point spectrum. On the other hand, for some ω it may fail to be homogeneous. In particular, the set of its eigenvalues may form dense subsets in certain intervals. We study the arithmetic means $\overline{\lambda}_O(\omega)$, $O \in \mathcal{B}$, of the eigenvalues of the operator L^ω. Under some mild assumptions the normalized arithmetic means $\left(\overline{\lambda}_O - \mathbb{E}\overline{\lambda}_O\right)/\sigma\left[\overline{\lambda}_O\right]$ converge to $N(0,1)$ in law, as $O \to X$. We also give examples where the normal convergence fails.

In Section 4 we study the problem of existence of the integrated density of states, i.d.s. for short. It turns out that the i.d.s., whenever it exists, has a remarkable structure, it can be represented via infinite convolution of probability measures. More precisely, the i.d.s.

coincides with the distribution of the random variable $X = \sum A_k X_k$, where X_k are i.i.d. and the coefficients $A_k > 0$ satisfy $\sum A_k = 1$. Various properties of probability measures of the form $\mu = \mathbb{P}_X$ (the infinite convolutions) have been studied by many authors since 1930's, see e.g. [22], [14], [27], [28], [29] and references therein. This classical study can be traced back to the pioneering works of Erdös, Kerschner, Lévy, Jessen and Wintner. We apply the known results about infinite convolutions to study random perturbations by Bernoulli random variables of the Vladimirov operator $\mathfrak{D}^{\alpha}$. It turns out that the i.d.s. has an L^2-density for almost all $0 < \alpha \leq \log 2/\log p$ and is purely singular for $\alpha > \log 2/\log p$.

In the concluding Section 5 we apply the results of the previous sections to study the the empirical process

$$N_O^{\omega} = \sum_{B \subseteq O} \delta_{\lambda(B,\omega)}$$

associated to the eigenvalues $\lambda(B,\omega)$ of the perturbated operator L^{ω}, where δ_a is the probability measure taking value 1 at $\{a\}$. Let I be a finite interval, we set $N_O(I) : \omega \to N_O^{\omega}(I)$. We show that, when $\mathbb{E}N_O(I)$, converges to some value $\lambda = \lambda(I) > 0$ as $O \to \varpi$, the random variable $N_O(I)$ converges to the Poisson random variable $\mathcal{P}_{\lambda}$ in law as $O \to \varpi$. We provide various examples to illustrate our results.

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2 Homogeneous Laplacian

In this section we specify some spectral properties of the hierarchical Laplacian L_C assuming that the ultrametric measure space (X, d, m) where it acts and the Laplacian by itself satisfy the following symmetry conditions:

- The group of isometries of (X, d) acts transitively on X.
- Both the reference measure m and the choice-function $C(B)$ are invariant under the action of isometries.

The ultrametric measure space (X, d, m) and the hierarchical Laplacian L_C on it satisfying these two conditions we call *homogeneous.*

The first assumption implies that (X, d) is either discrete or perfect. Basic examples which we have in mind are

- $X = \mathbb{Z}_p$ - the ring of $p-$ adic integers.
- $X = \mathbb{Q}_p$ - the ring of $p-$ adic numbers.
- $X = \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^{\infty})$ - the multiplicative group of p^nth roots of unity, where n runs through the set of all nonnegative integers, considered in the discrete topology.

As it was noticed in [10],[11], our assumptions imply that the measure space (X, m) can be identified with a locally compact totally disconnected Abelian group G equipped with its Haar measure. Notice that the group G is not unique. As a possible choice of the group G when for instance, X is *perfect and non-compact*, one can take the following Abelian group

$$G = \operatorname*{limind}_{l \to -\infty} \left(\prod_{k \geq l} \mathbb{Z}(n_k) \right), \tag{2.1}$$

where $\mathbb{Z}(n_k)$ are cyclic groups and $\{ n_k\}_{k\in\mathbb{Z}}$ is an appropriately chosen double sided sequence of integers. The canonical ultrametric structure on G is defined by the descending sequence of its compact subgroups

$$G_l = \prod_{k \geq l} \mathbb{Z}(n_k).$$

Namely, the groups G_l, $l \in \mathbb{Z}$, and their cosets $\{a + G_l\}$ form the collection $\mathcal{B}$ of all clopen balls.

There is a natural ultrametric structure associated to the double sided chain of subgroups G_l of G. One defines the *absolute value* $|a|$ for the elements a of G,

$$|a| = \begin{cases} 0 & \text{if} \quad a = 0 \\ m(G_l) & \text{if} \quad a \in G_l \backslash G_{l+1} \end{cases}.$$

The absolute value $|a|$ satisfies the ultrametric inequality

$$|a + b| \leq \max\{|a|, |b|\}.$$

It is also clear that $|a| = |-a|$ and that $d(a, b) := |a - b|$ is an ultrametric that gives (G, m) the structure of a homogeneous ultrametric measure space as defined above. In particular, for any ball B we have

$$m(B) = \mathsf{diam}(B).$$

Choosing the Haar measure m such that $m(G_0) = 1$ we compute $m(G_l)$ for any $l \neq 0$ as follows

$$m(G_l) = \begin{cases} n_l ... n_{-1} & \text{if} \quad l < 0 \\ (n_{l-1} ... n_0)^{-1} & \text{if} \quad l > 0 \end{cases}.$$

Recall that in the classical setting $X = \mathbb{Q}_p$ we have $G_0 = \mathbb{Z}_p$, $G_l = p^l \mathbb{Z}_p$ and

$$|a| = p^{-n(a)}, \text{ where } n(a) = \max\{l : a \in G_l\}.$$

The quantity $|a|$ becomes a pseudonorm, that is,

$$|ab| \leq |a| \, |b| \, ;$$

it is a norm if p is a prime number - the basic property in the p - adic analysis and its applications.

We recall that to an ultrametric space (X, d) one can associate in a standard fashion a tree $\mathcal{T}(X)$ (see Fig. 1). The vertices of the tree are metric balls, and hence in our case the cosets $\{a + G_l : a \in G, l \in \mathbb{Z}\}$. The ascending sequence of subgroups $\{G_l : l = 0, -1, -2, ...\}$ identifies a special boundary point, which we denote ϖ. With respect to this special point we consider the horocycles of the tree. A *horocycle* in this case is the set of vertices consisting of the balls of a given diameter; in other words the cosets relative to the same subgroup G_l. Thus, for fixed l, the horocycle $H_l = \{a + G_l : a \in G\}$. The boundary $\partial\mathcal{T}(G)$ can be identified with the one-point compactification $G \cup \{\varpi\}$ of G. We refere to [10],[11] and [6] for a complete treatment of the association between an ultrametric space and the tree of its metric balls. The most complete sourse for the basic definitions related to the geometry of trees is [8], see also [34].

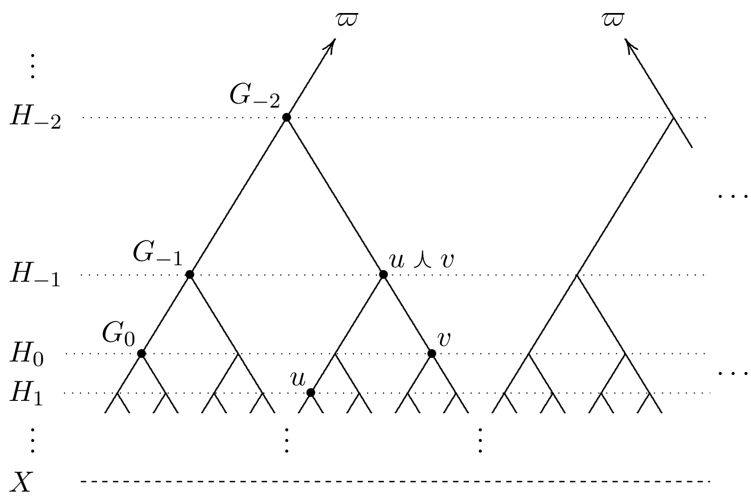

Figure 1. Tree of balls $\mathcal{T}(X)$ with forward degree $n_l = 2$.

Let L_C be a homogeneous hierarchical Laplacian on the ultrametric measure space (G, d, m) defined by the choice-function $C(B)$, that is

$$L_C f(x) = -\sum_{B\in\mathcal{B}:\ x\in B} C(B)\left(P_B f - f(x)\right).$$

Thanks to the homogenuity property, $C(A) = C(B)$ for any two balls which belong to the same horocycle H. The same of course true for the eigenvalues $\lambda(A)$ and $\lambda(B)$. We set $c_H = C(B)$ and $\lambda_H = \lambda(B)$, for any ball $B \in H$. When $H = H_k$ we will also write $c_k = c_{H_k}$ and $\lambda_k = \lambda_{H_k}$. In this notation

$$\lambda_k = \sum_{l\leq k} c_l.$$

Each ball $B \in H_k$ has n_k nearest neighbors $B_i \subset B$. The eigenfunctions $f_{B_i,B}$ corresponding to $\lambda(B)$ have the following form

$$f_{B_i,B} = \frac{1_{B_i}}{m(B_i)} - \frac{1_B}{m(B)}.$$

The system of functions $\{f_{B_i,B} : i = 1, ..., n_k\}$ is not orthogonal, its linear span $\mathcal{H}(B) \subset L^2$ has dimension $n_k - 1$. Setting

$$f_i := f_{B_i,B},\ \text{and}\ m := m(B_i),\ i = 1, ..., n_k - 1,$$

and applying the Gramm-Schmidt procedure we obtain an orthogonal basis $\{u_i : i = 1, ..., n_k - 1\} \subset \mathcal{H}(B) : u_1 = f_1$ and for $i \geq 2$,

$$u_i = f_1 + ... + f_{i-1} + (n_k - i + 1)\, f_i,\ \ \|u_i\|^2 = \frac{(n_k - i)(n_k - i + 1)}{m}.$$

For two different balls S and T the eigenspaces $\mathcal{H}(S)$ and $\mathcal{H}(T)$ are orthogonal. It follows that the eigenspace $\mathcal{H}_k$ corresponding to the horocycle H_k (equivalently, to the eigenvalue λ_k) is of the form

$$\mathcal{H}_k = \bigoplus_{B \in H_k} \mathcal{H}(B).$$

The system of eigenfunctions $\{f_{B_i,B}\}_{B\in\mathcal{B}}$ is complete, whence

$$\bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k = L^2(G,m).$$

Among the variety of homogeneous hierarchical Laplacians L_C on (G,d,m) we would like to mention a one-parametric family $\{\mathfrak{B}^\alpha\}_{\alpha>0}$. The hierarchical Laplacian $\mathfrak{B}^\alpha$ is defined by the choice-function

$$C_\alpha(B) = (\mathsf{diam}(B))^{-\alpha} - \left(\mathsf{diam}(B')\right)^{-\alpha}, \tag{2.2}$$

where $B \subset B'$ are nearest neighbouring balls. Hence for any ball $B \in \mathcal{B}$, the eigenvalue $\lambda_\alpha(B)$ of $\mathfrak{B}^\alpha$ corresponding to B is

$$\lambda_\alpha(B) = (\mathsf{diam}(B))^{-\alpha}.$$

The eigenvalue $\lambda_\alpha(k)$ of $\mathfrak{B}^\alpha$ corresponding to the horocycle H_k is computed then as follows: $\lambda_\alpha(0) = 1$ and

$$\lambda_\alpha(k) = \begin{cases} (n_k ... n_{-1})^{-\alpha} & \text{if} \quad k<0 \\ (n_{k-1} ... n_0)^{\alpha} & \text{if} \quad k>0 \end{cases}.$$

We recall from [5] that the set $\mathcal{D}$ of compactly supported locally constant functions is in the domain of the operator $\mathfrak{B}^\alpha$. It is remarkable however, although not difficult to prove, that the following properties hold:

$$\mathfrak{B}^\beta : \mathcal{D} \to \mathsf{Dom}(\mathfrak{B}^\alpha)$$

and on $\mathcal{D}$,

$$\mathfrak{B}^\alpha \circ \mathfrak{B}^\beta = \mathfrak{B}^{\alpha+\beta} \text{ and } (\mathfrak{B}^\alpha)^\beta = \mathfrak{B}^{\alpha\beta}.$$

Moreover, when $G = \mathbb{Q}_p$, the operator $\mathfrak{D}^\alpha = p^\alpha \mathfrak{B}^\alpha$ is the operator of fractional derivative of order α as defined and studied via Fourier transform in [30], [31], [33] and [17]. In this case, for $f \in \mathcal{D}$,

$$\mathfrak{D}^\alpha f(x) = \frac{p^\alpha - 1}{1-p^{-\alpha-1}} \int_G \frac{f(x)-f(y)}{|x-y|^{1+\alpha}} dm(y) \tag{2.3}$$

and therefore

$$\mathfrak{B}^\alpha f(x) = \frac{1-p^{-\alpha}}{1-p^{-\alpha-1}} \int_G \frac{f(x)-f(y)}{|x-y|^{1+\alpha}} dm(y).$$

At last notice that similar identifications (based on cyclic groups $\mathbb{Z}(n)$ as building blocks) can be carried over in the remaining two cases: (X,d) is *infinite and discrete*, and (X,d) is *compact and perfect*. For instance, the infinite (non-Abelian) symmetric group S_∞ equipped with its canonocal ultrametric structure defined by the family $\{S_n\}$ of its finite symmetric subgroups S_n can be identified (as ultrametric measure space) with discrete Abelian group $G = \bigoplus_{l>1} \mathbb{Z}(l)$. The group G is equipped with its canonocal ultrametric structure defined by the family $\{G_n\}$ of its finite subgroups $G_n = \prod_{1<l\leq n} \mathbb{Z}(l)$. As a second example, we consider a compact and perfect ultrametric space $X = \mathbb{Z}_p \subset \mathbb{Q}_p$ - the ring of p-adic integers, which we can identify (as ultrametric measure spaces) with compact Abelian group $G = \prod_{k\geq 1} \mathbb{Z}(l_k)$, with all $l_k = p$.The ultrametric structure in this case is defined by the descending family of small subgroups $G_l = \prod_{k\geq l} \mathbb{Z}(l_k) \subset G$.

3 Random perturbations

Let (X, d, m) be a non-compact homogeneous ultrametric space. Let L_C be the homogeneous hierarchical Laplacian acting on X and defined by the choice-function $C(B)$. Let $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ be a family of symmetric i.i.d. random variables defined on the probability space $(\Omega, \mathbb{P})$ and taking values in some small interval $[-\epsilon, \epsilon] \subset (-1, 1)$. We define the perturbated choice-function $C(B, \omega)$ and the perturbated hierarchical Laplacian as follows:

$$C(B, \omega) = C(B)(1 + \varepsilon(B, \omega))$$

and

$$L^\omega f(x) := L_{C(\omega)} f(x) = -\sum_{B\in\mathcal{B}:\ x\in B} C(B, \omega)\left(P_B f - f(x)\right).$$

Evidently L^ω may well be non-homogeneous for some $\omega \in \Omega$. Still it has a pure point spectrum for all ω but the structure of the closed set $\mathsf{Spec}(L^\omega)$ can be quite complicated, see [6] for various examples.

Two stationary families. Let us fix a horocycle $H = H_l$, for some $l \in \mathbb{Z}$. Let $\lambda_H = \lambda_l$ be the eigenvalue of the homogeneous Laplacian L_C corresponding to the horocycle H. Let $B \in H$ and $\{B_k\}_{k\leq l}$ be the unique infinite geodesic path in $\mathcal{T}(X)$ from ϖ to B. We compute the eigenvalue $\lambda(B, \omega)$ of the operator $L^\omega = L_{C(\omega)}$,

$$\begin{aligned}\lambda(B, \omega) &= \sum_{k\leq l} C(B_k, \omega) = \sum_{k\leq l} c_k\left(1 + \varepsilon(B_k, \omega)\right) \\ &= \lambda_l\left(1 + \sum_{k\leq l} a_k \varepsilon(B_k, \omega)\right) := \lambda_l\left(1 + U(B, \omega)\right),\end{aligned}$$

where $a_k = c_k/\lambda_l$, and

$$U(B, \omega) = \sum_{k\leq l} a_k \varepsilon(B_k, \omega). \tag{3.1}$$

Notice that $\sum_{k\leq l} a_k = 1$ and that $\{U(B)\}_{B\in H}$ are (dependent) identically distributed symmetric random variables taking values in some symmetric interval $I \subsetneqq (-1, 1)$.

We want to study the family of random variables $\{\lambda(B, \omega)\}_{B\in H}$, resp. $\{U(B)\}_{B\in H}$. As the horocycle $H = H_l$ is fixed it is useful to identify the balls $B \in H$ with elements $g \in G$ of the (discrete!) Abelian group $G = \bigoplus_{k<l} \mathbb{Z}(n_k)$. Having such identification in mind it is now straightforvard to show that the family $\{U(B)\}_{B\in H} = \{U(g)\}_{g\in G}$ is stationary, that is, for any $g, g_1, ..., g_s$ in G,

$$\{U(g + g_1), ..., U(g + g_s)\} \stackrel{d}{=} \{U(g_1), ..., U(g_s)\}.$$

For the general theory of stationary processes we refere to [15] and [14].

One easely compute the corelation function $\mathcal{K}_U(g,g') = \mathbb{E}U(g)U(g')$. We have

$$\mathcal{K}_U(g,g') = \mathcal{K}_U(0,g-g') := \mathcal{K}_U(g-g')$$

and

$$\mathcal{K}_U(g) = \sum_{k\geq|g|} a^2_{l-k}, \tag{3.2}$$

where

$$|g| = \min\left\{ n : g \in \{0\} \times \bigoplus_{-n\leq k<l} \mathbb{Z}(n_k) \subset G \right\}.$$

The function $\mathcal{K}_U(g)$ is positive definite, whence by Bochner's theorem there is a finite measure $\mathcal{F}_U$ (the spectral measure) defined on the compact group $\widehat{G} = \prod_{k<l} \mathbb{Z}(n_k)$, such that

$$\mathcal{K}_U(g) = \int_{\widehat{G}} \langle g, \gamma \rangle \, d\mathcal{F}_U(\gamma).$$

We have

$$\int_G \mathcal{K}_U(g)dg = a^2_l + n_{l-1}a^2_{l-1} + n_{l-1}n_{l-2}a^2_{l-2} + \ldots.$$

In particular, if the function $\mathcal{K}_U$ is integrable, the spectral measure $\mathcal{F}_U$ is absolutely continuous w.r.t. Haar measure on $\widehat{G}$ and admits a continuous density $F_U(\gamma)$ which can be computed as the inverse Fourier transform of $\mathcal{K}_U(g)$,

$$F_U(\gamma) = a^2_l + n_{l-1}a^2_{l-1}1_{A(G_{l-1})}(\gamma) + n_{l-1}n_{l-2}a^2_{l-2}1_{A(G_{l-2})}(\gamma) + \ldots.$$

In the above equation $A(G_{l-i}) \subset \widehat{G}$ is the anigillator of the subgroup G_{l-i}, that is,

$$A(G_{l-i}) = \{0\} \times \prod_{k<l-i} \mathbb{Z}(n_k).$$

Since for $B \in H$,

$$\lambda(B,\omega) = \lambda_H \left(1 + U(B,\omega)\right),$$

the family of random variables

$$\lambda(g,\omega) = \lambda_H \left(1 + U(g,\omega)\right)$$

is stationary as well. In particular, its correlation function $\mathcal{K}_\lambda(g)$ satisfies

$$\mathcal{K}_\lambda(g) = \lambda^2_H \mathcal{K}_U(g), \quad g \in G,$$

we will use this property in our further computations.

The Law of Large Numbers. Let us choose a reference point $o \in X$, say the neutral element 0 in our group-identification $X \equiv G$, and consider the family $\mathcal{O}$ of all balls O centred at o. We fix a horocycle H and study limit behavior as $O \to \varpi$ along $\mathcal{O}$ of the arithmetic means

$$\overline{\lambda}_H(O,\omega) = \frac{1}{|\mathcal{B}_H(O)|} \sum_{B \in \mathcal{B}_H(O)} \lambda(B,\omega),$$

where $\mathcal{B}_H(O)$ is the set of all balls B in O which belong to H, and $|\mathcal{B}_H(O)|$ stands for the cardinality of the finite set $\mathcal{B}_H(O)$.

Recall that for any two balls A and B which belong to the same horocycle H the eigenvalues $\lambda(A)$ and $\lambda(B)$ of the homogeneous Laplacian L_C coincide; we denote their common value λ_H.

Theorem 3.1 *For any given horocycle H, as $O \to \varpi$,*

$$\overline{\lambda}_H(O,\omega) \longrightarrow \lambda_H \ \ a.s.$$

Proof. Let $B \in H_l$, for some $l \in \mathbb{Z}$. Let $\{B_k\}_{k \le l}$ be the unique infinite geodesic path in $\mathcal{T}(X)$ from ϖ to B. We have already computed the eigenvalue $\lambda(B,\omega)$ of the operator $L_{C(\omega)}$ corresponding to the ball B,

$$\lambda(B,\omega) = \lambda_l \left(1 + U(B,\omega)\right),$$

where $a_k = c_k/\lambda_l$, and

$$U(B,\omega) = \sum_{k \le l} a_k \varepsilon(B_k,\omega). \tag{3.3}$$

Let us compute the arithmetic mean $\overline{\lambda}_{H_l}(O,\omega)$ assuming that $O \in H_L$ and $L << l$. To simplify our notation we set $\overline{\lambda}_L(\omega) := \overline{\lambda}_{H_l}(O,\omega)$.

$$\begin{aligned} \overline{\lambda}_L(\omega) &= \frac{1}{|\mathcal{B}_H(O)|} \sum_{B \in \mathcal{B}_H(O)} \lambda(B,\omega) \\ &= \frac{\lambda_l}{n_{l-1}...n_L} \sum_{B \in \mathcal{B}_{H_l}(O)} (1 + U(B,\omega)) \\ &= \lambda_l \left(1 + \overline{U}_L(\omega)\right), \end{aligned} \tag{3.4}$$

where

$$\overline{U}_L(\omega) = \frac{1}{n_{l-1}...n_L} \sum_{B \in \mathcal{B}_{H_l}(O)} U(B,\omega). \tag{3.5}$$

Thus to prove the theorem we are left to show that $\overline{U}_L(\omega) \to 0$ as $L \to -\infty$ almost surely ω. Let $\{O_k\}_{k \le l}$ be the infinite geodesic path from ϖ to O. By substitution (3.3) to the equation (3.5) we obtain

$$\begin{aligned}
\overline{U}_L &= \frac{1}{n_{l-1}...n_L}\sum_{B\in H_l:B\subset O}\sum_{k\le l} a_k\varepsilon(B_k) = \frac{1}{n_{l-1}...n_L}\sum_{k\le l} a_k \sum_{B\in H_l:B\subset O}\varepsilon(B_k)\\
&= \frac{1}{n_{l-1}...n_L}\Bigg(a_l\sum_{B\in H_l:B\subseteq O}\varepsilon(B) + a_{l-1}n_{l-1}\sum_{B\in H_{l-1}:B\subseteq O}\varepsilon(B)\\
&\quad + a_{l-2}n_{l-1}n_{l-2}\sum_{B\in H_{l-2}:B\subseteq O}\varepsilon(B) + ... + a_L n_{l-1}n_{l-2}...n_L\varepsilon(O_L)\Bigg)\\
&\quad + a_{L-1}\varepsilon(O_{L-1}) + a_{L-2}\varepsilon(O_{L-2}) + ... \, .
\end{aligned}$$

Let us introduce two random variables

$$\mathcal{U}_L = \frac{a_l}{n_{l-1}...n_L}\sum_{B\in H_l:B\subseteq O}\varepsilon(B) + \frac{a_{l-1}}{n_{l-2}...n_L}\sum_{B\in H_{l-1}:B\subseteq O}\varepsilon(B) + ... + a_L\varepsilon(O_L) \tag{3.6}$$

and

$$\mathcal{V}_L = a_{L-1}\varepsilon(O_{L-1}) + a_{L-2}\varepsilon(O_{L-2}) + ... \, . \tag{3.7}$$

Random variables $\mathcal{U}_L$ and $\mathcal{V}_L$ are independent, have zero mean and

$$\overline{U}_L = \mathcal{U}_L + \mathcal{V}_L. \tag{3.8}$$

Moreover, $\mathcal{V}_L \to 0$ uniformly as $L \to -\infty$. Hence we are left to show that

$$\mathcal{U}_L \to 0 \text{ a.s. as } L \to -\infty.$$

Let $\sigma = \sqrt{\mathsf{Var}[\varepsilon]}$ and $\sigma[\mathcal{U}_L] = \sqrt{\mathsf{Var}[\mathcal{U}_L]}$. Using (3.6) we compute $\sigma[\mathcal{U}_L]$,

$$\sigma[\mathcal{U}_L]^2 = \sigma^2\left(\frac{a_l^2}{n_{l-1}...n_L} + \frac{a_{l-1}^2}{n_{l-2}...n_L} + ... + a_L^2\right). \tag{3.9}$$

It follows that, since all $n_k \ge 2$,

$$\begin{aligned}
\sum_{L\le l}\sigma[\mathcal{U}_L]^2 &= \sigma^2 a_l^2\left(1 + \frac{1}{n_{l-1}} + \frac{1}{n_{l-1}n_{l-2}} + ...\right)\\
&\quad + \sigma^2 a_{l-1}^2\left(1 + \frac{1}{n_{l-2}} + \frac{1}{n_{l-2}n_{l-3}} + ...\right) + ...\\
&\le 2\sigma^2\left(a_l^2 + a_{l-1}^2 + ...\right) < 2\sigma^2\left(a_l + a_{l-1} + ...\right)^2 = 2\sigma^2.
\end{aligned}$$

At last, Chebyshev inequality and Borell-Cantelli lemma yield the desired result. ■

Central Limit Theorem. We study limit behaviour as $O \to \varpi$ of the normalized arithmetic means

$$\Lambda_H(O) = \frac{\overline{\lambda}_H(O) - \lambda_H}{\sigma\left[\overline{\lambda}_H(O)\right]}.$$

Recall that for random variable Y we denote $\sigma[Y]$ its mean-square displacement $\sqrt{\mathsf{Var}[Y]}$. In the course of study we will assume that the following condition holds:

$$1/\kappa \le C(B)\left(\mathsf{diam}(B)\right)^{\delta/2} \le \kappa, \tag{3.10}$$

for any ball $B \in \mathcal{B}$ and some $\delta, \kappa > 0$.

It is easy to see that (3.10) is equivalent to the following condition

$$1/2\kappa \leq \lambda(B)\,(\mathsf{diam}(B))^{\delta/2} \leq 2\kappa. \tag{3.11}$$

Evidently (3.10) and (3.11) with $\delta = 2\alpha$ hold true for the operator $\mathfrak{B}^\alpha$ introduced in the previous section at (2.2). Actually in this case we have

$$\lambda(B)\,(\mathsf{diam}(B))^{\delta/2} = 1.$$

Let $N(0,1)$ be the standard normal random variable. The main result of this subsection is the following theorem.

Theorem 3.2 *Assume that* $\delta \geq 1$, *then as* $O \to \varpi$,

$$\Lambda_H\,(O) \to N(0,1) \text{ in law.}$$

Proof. Let $H = H_l$ and $O = O_L \in H_L$ for some $L << l$. As in the proof of Theorem 3.1 we fix l and let $L \to -\infty$. To simplify our notation we set $\Lambda_H(O) := \Lambda_L$. By the equation (3.4), we have

$$\Lambda_L = \frac{\overline{U}_L}{\sigma\left[\overline{U}_L\right]},$$

whence we are left to show that as $L \to -\infty$,

$$\frac{\overline{U}_L}{\sigma\left[\overline{U}_L\right]} \to N(0,1) \text{ in law.} \tag{3.12}$$

As in the equation (3.8) we write $\overline{U}_L = \mathcal{U}_L + \mathcal{V}_L$. Since $\mathcal{U}_L$ and $\mathcal{V}_L$ are independent,

$$\sigma[\overline{U}_L]^2 = \sigma[\mathcal{U}_L]^2 + \sigma[\mathcal{V}_L]^2.$$

Claim 1. For l fixed and $L \to -\infty$,

$$\sigma[\mathcal{V}_L]^2 \asymp \sigma^2\,(n_{L-1}n_L...n_0)^{-\delta} \tag{3.13}$$

and

$$\sigma[\mathcal{U}_L]^2 \asymp \begin{cases} \sigma^2\,(n_L...n_0)^{-1} & \text{if} \quad \delta > 1 \\ -\sigma^2 L\,(n_L...n_0)^{-1} & \text{if} \quad \delta = 1 \\ \sigma^2\,(n_L...n_0)^{-\delta} & \text{if} \quad \delta < 1 \end{cases}, \tag{3.14}$$

where $x \asymp y$ means that the ratio x/y is uniformly bounded from above and from below.

To prove (3.13) we apply (3.7). Since $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ are i.i.d.,

$$\sigma[\mathcal{V}_L]^2 = \sigma^2\left(a_{L-1}^2 + a_{L-2}^2 + ...\right) = \frac{\sigma^2}{\lambda_l^2}\left(c_{L-1}^2 + c_{L-2}^2 + ...\right),$$

and (3.10) yields

$$\begin{aligned}\sigma[\mathcal{V}_L]^2 &\asymp \sigma^2\left((n_{L-1}n_L...n_0)^{-\delta} + (n_{L-2}n_{L-1}n_L...n_0)^{-\delta} + ...\right) \\ &= \sigma^2\,(n_{L-1}n_L...n_0)^{-\delta}\left(1 + (n_{L-2})^{-\delta} + (n_{L-2}n_{L-3})^{-\delta} + ...\right),\end{aligned}$$

for any fixed l. Since all $n_k \geq 2$ the result follows.

For (3.14) we apply (3.9) and (3.10). Since l is fixed, we have

$$\begin{aligned}\sigma[\mathcal{U}_L]^2 &= \sigma^2 \left(\frac{a_l^2}{n_{l-1}...n_L} + \frac{a_{l-1}^2}{n_{l-2}...n_L} + ... + a_L^2 \right) \\ &= \frac{\sigma^2}{n_{l-1}...n_L} \left(a_l^2 + a_{l-1}^2 n_{l-1} + ... + a_L^2 n_{l-1}...n_L \right) \\ &\asymp \frac{\sigma^2}{n_0...n_L} \left(1 + \left(\frac{a_{l-1}}{a_l} \right)^2 n_{l-1} + ... + \left(\frac{a_L}{a_l} \right)^2 n_{l-1}...n_L \right) \\ &\asymp \frac{\sigma^2}{n_0...n_L} \left(1 + (n_{l-1})^{1-\delta} + ... + (n_{l-1}...n_L)^{1-\delta} \right).\end{aligned}$$

The desired result follows.

Let now $\delta \geq 1$. By the Claim 1,

$$\sigma[\overline{U}_L] \sim \sigma[\mathcal{U}_L] \text{ as } L \to -\infty,$$

whence to prove (3.12) we are left to show that

$$\frac{\mathcal{U}_L}{\sigma[\mathcal{U}_L]} \to N(0,1) \text{ in law.}$$

Claim 2. Assume that $\delta \geq 1$ and that l is fixed. Then,

$$\lim_{L \to -\infty} \max_{L \leq k \leq l} \frac{\sigma a_k}{\sigma[\mathcal{U}_L] \prod_{L \leq i \leq k-1} n_i} = 0 \tag{3.15}$$

(with the agreement that $\prod_{i \in \varnothing} b_i := 1$).

Indeed, define $\epsilon = \min(1, \delta - 1)$ and consider k such that $L \leq k \leq l$. Since l is fixed we can assume that $k \leq 0$. Denote by $A(\delta)$ the fraction in the left-hand-side of the equation (3.15). By the Claim 1, when $\delta > 1$,

$$\begin{aligned}A(\delta)^2 &\asymp \frac{\sigma^2 (n_k...n_0)^{-\delta}}{\sigma^2 (n_L...n_0)^{-1} (n_L...n_{k-1})^2} \\ &= \frac{1}{(n_k...n_0)^{\delta-1} (n_L...n_{k-1})} \leq \frac{1}{(n_L...n_0)^{\epsilon}}\end{aligned}$$

and, when $\delta = 1$,

$$\begin{aligned}A(\delta)^2 &\asymp -\frac{\sigma^2 (n_k...n_0)^{-1}}{\sigma^2 L (n_L...n_0)^{-1} (n_L...n_{k-1})^2} \\ &= -\frac{1}{L (n_L...n_{k-1})} \leq -\frac{1}{L}.\end{aligned}$$

The result follows.

Observe that when $\delta < 1$, we obtain

$$\max_{L \leq k \leq l} \frac{\sigma a_k}{\sigma[\mathcal{U}_L] \prod_{L \leq i \leq k-1} n_i} \geq \frac{\sigma a_L}{\sigma[\mathcal{U}_L]} \geq \frac{c\sigma (n_L...n_0)^{-\delta/2}}{\sigma (n_L...n_0)^{-\delta/2}} = c,$$

for some $c > 0$. In particular, in this case the equation (3.15) does not hold.

Let ϕ and Φ be characteristic functions of the random variables $\varepsilon = \varepsilon(B)$ and $\mathcal{U}_L/\sigma[\mathcal{U}_L]$ respectively. By the equation (3.6),

$$\Phi(x) = \prod_{L \leq k \leq l} \phi\left(\frac{a_k x}{\sigma[\mathcal{U}_L] n_L ... n_{k-1}}\right)^{n_L ... n_{k-1}}$$

(with the agreement that $n_L ... n_{k-1} = 1$ when $k = L$).

Since the random variable ε has two moments,

$$\phi(z) = 1 - \frac{1}{2}(\sigma z)^2 (1 + \beta(\sigma z)), \tag{3.16}$$

where $\beta(z) \to 0$ as $z \to 0$. Let us set

$$\Delta_k = \frac{\sigma a_k}{\sigma[\mathcal{U}_L] n_L ... n_{k-1}}.$$

Observe that

$$\sum_{L \leq k \leq l} n_L ... n_{k-1} \Delta_k^2 = 1.$$

Applying now (3.16) we obtain

$$\begin{aligned} \log \Phi(x) &= \sum_{L \leq k \leq l} n_L ... n_{k-1} \log \left[1 - \frac{x^2}{2} \Delta_k^2 (1 + \beta(\Delta_k x))\right] \\ &\sim -\frac{x^2}{2} \left[\sum_{L \leq k \leq l} n_L ... n_{k-1} \Delta_k^2 (1 + \beta(\Delta_k x))\right] \\ &= -\frac{x^2}{2} \left[1 + \sum_{L \leq k \leq l} n_L ... n_{k-1} \Delta_k^2 \beta(\Delta_k x)\right]. \end{aligned}$$

Finally, the Claim 2 and the inequality

$$\sum_{L \leq k \leq l} n_L ... n_{k-1} \Delta_k^2 \beta(\Delta_k x) \leq \max_{L \leq k \leq l} \beta(\Delta_k x)$$

evidently yield the desired result. The proof of the theorem is finished. ■

The operator $\mathfrak{B}^\alpha$. As an example we consider the space $X = \mathbb{Q}_p$ equipped with its standard ultrametric structure defined by the descending sequence of compact subgroups $G_l = p^l \mathbb{Z}_p$. Let $\mathfrak{B}^\alpha$ be the homogeneous Laplacian introduced at (2.2) and $\mathfrak{B}^\alpha(\omega)$ its random perturbation by symmetric i.i.d. $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ as defined and studied above. As we noticed $\mathfrak{B}^\alpha$ satisfies the condition (3.11) with $\alpha = \delta/2$. In particular, for any $\alpha \geq 1/2$ the normalized arithmetic means $\Lambda_H^\alpha(O)$ of $\mathfrak{B}^\alpha(\omega)$-eigenvalues converge in law as $O \to \varpi$ to the standard normal random variable $N(0, 1)$. In this subsection we study convergence of the normalized arithmetic means $\Lambda_H^\alpha(O)$ assuming that $0 < \alpha < 1/2$.

Theorem 3.3 *For any $0 < \alpha < 1/2$ there exists a random variable Λ_H^α such that as $O \to \varpi$,*

$$\Lambda_H^\alpha(O) \to \Lambda_H^\alpha \text{ in law.}$$

The random variable Λ_H^α is not Gaussian. It has C^∞- distribution function $\mathcal{F}_{\Lambda_H^\alpha} \in D(2)$ - the domain of attraction of the normal law. $\mathcal{F}_{\Lambda_H^\alpha}$ is unimodal whenever the common distribution function $\mathcal{F}_\varepsilon$ of i.i.d. $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ is unimodal.

Proof. We follow line-by-line the proof of the Theorem 3.2. Without loss of generality we may assume that the horocycle $H = H_0$ and $O = O_L \in H_L$ for some $L < -1$. To simplify our notation we set $\Lambda_L := \Lambda_H^\alpha(O)$. As $\Lambda_L = \overline{U}_L/\sigma\left[\overline{U}_L\right]$, where $\overline{U}_L$ is defined at (3.5), we write $\overline{U}_L = \mathcal{U}_L + \mathcal{V}_L$ and

$$\sigma[\overline{U}_L]^2 = \sigma[\mathcal{U}_L]^2 + \sigma[\mathcal{V}_L]^2.$$

Since $\lambda_k = p^{\alpha k}$ and $c_k = (p^\alpha - 1)\, p^{\alpha(k-1)}$ we obtain

$$a_k = c_k/\lambda_0 = (p^\alpha - 1)\, p^{\alpha(k-1)},\ k \le 0.$$

Using the above data we estimate $\sigma[\mathcal{U}_L]$ and $\sigma[\mathcal{V}_L]$ at $-\infty$. We have

$$\begin{aligned}\sigma[\mathcal{U}_L]^2 &= \sigma^2\left(\frac{a_0^2}{p^{-L}} + \frac{a_{-1}^2}{p^{-L-1}} + ... + a_L^2\right) \qquad (3.17)\\ &= \sigma^2\,(p^\alpha-1)^2 \sum_{0\le l\le -L} p^{L+l-2\alpha(l+1)}\\ &\sim \frac{\sigma^2\,(p^\alpha-1)^2}{1-p^{2\alpha-1}}p^{2\alpha(L-1)} = \frac{\sigma^2}{1-p^{2\alpha-1}}a_L^2\end{aligned}$$

and

$$\begin{aligned}\sigma[\mathcal{V}_L]^2 &= \sigma^2\left(a_{L-1}^2 + a_{L-2}^2 + ...\right) \qquad (3.18)\\ &= \sigma^2\,(p^\alpha-1)^2 \sum_{l\ge -L+1} p^{-2\alpha(l+1)}\\ &\sim \frac{\sigma^2\,(p^\alpha-1)^2}{1-p^{-2\alpha}}p^{2\alpha(L-2)} = \frac{\sigma^2}{1-p^{-2\alpha}}a_{L-1}^2.\end{aligned}$$

Let $\{\varepsilon_i\}_{i\ge 0}$ be i.i.d. random variables independent of $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ and having the same common distribution as $\{\varepsilon(B)\}_{B\in\mathcal{B}}$. By (3.7) and (3.18) the random variable $\mathcal{V}_L/\sigma[\mathcal{V}_L]$ converges in law to the random variable

$$V = \sqrt{1-p^{-2\alpha}}\left(\frac{\varepsilon_0}{\sigma\left[\varepsilon_0\right]} + p^{-\alpha}\frac{\varepsilon_1}{\sigma\left[\varepsilon_2\right]} + ... + p^{-k\alpha}\frac{\varepsilon_k}{\sigma\left[\varepsilon_k\right]} + ...\right).$$

By Cramér's theorem V is not Gaussian.

Let $\{\varepsilon_{ij}\}_{i,j\ge 0}$ be i.i.d. random variables independent of both $\{\varepsilon_i\}_{i\ge 0}$ and $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ and having the same common distribution as $\{\varepsilon(B)\}_{B\in\mathcal{B}}$. Define the random variables

$$S_k = \sum_{0\le j\le p^k-1} \varepsilon_{kj},\ k = 0, 1, 2, ... \,.$$

By (3.6) and (3.17) the random variable $\mathcal{U}_L/\sigma[\mathcal{U}_L]$ converges in law to the random variable

$$U = \sqrt{1-p^{2\alpha-1}}\sum_{k\ge 0} p^{(2\alpha-1)k/2}\frac{S_k}{\sigma\left[S_k\right]}.$$

By Cramér's theorem U is not Gaussian.

Finally, the random variable

$$\Lambda_L = \frac{\overline{U}_L}{\sigma\left[\overline{U}_L\right]} = \frac{\sigma[\mathcal{U}_L]}{\sigma\left[\overline{U}_L\right]}\frac{\mathcal{U}_L}{\sigma[\mathcal{U}_L]} + \frac{\sigma[\mathcal{V}_L]}{\sigma\left[\overline{U}_L\right]}\frac{\mathcal{V}_L}{\sigma[\mathcal{V}_L]}$$

converges in law to the random variable

$$\Lambda = \sqrt{\frac{1-p^{-2\alpha}}{1-p^{-1}}}U + \sqrt{\frac{p^{-2\alpha}-p^{-1}}{1-p^{-1}}}V.$$

Since U and V are independent and non-Gaussian, Λ is not Gaussian as well.

For a random variable X we denote $\Phi_X(\xi) = \mathbb{E}(\exp(i\xi X))$ its characteristic function. As $\{\varepsilon_{kj}\}$ are i.i.d. $\Phi_{\varepsilon_{kj}}$ does not depend on i, j; we set $\Phi = \Phi_{\varepsilon_{kj}}$. By (3.16), for any $0 < \epsilon < 1$ we find $\delta > 0$ such that

$$\begin{aligned}
|\Phi_U(\xi)| &= \prod_{k\geq 0}\left|\Phi_{S_k}\left(\frac{p^{(2\alpha-1)k/2}}{\sigma\left[S_k\right]}\xi\right)\right| = \prod_{k\geq 0}\left|\Phi\left(\frac{p^{(2\alpha-1)k/2}}{\sigma\left[S_k\right]}\xi\right)\right|^{p^k} \\
&\leq \prod_{k:\ p^{(2\alpha-1)k/2}\xi/\sigma[S_k]<\delta}\left|\Phi\left(\frac{p^{(2\alpha-1)k/2}}{\sigma\left[S_k\right]}\xi\right)\right|^{p^k} \\
&\leq \prod_{k:\ p^{(2\alpha-1)k/2}\xi/\sigma[S_k]<\delta}\left(1-\frac{\xi^2}{2}\frac{\sigma^2 p^{(2\alpha-1)k}(1-\epsilon)}{\sigma\left[S_k\right]^2}\right)^{p^k}.
\end{aligned}$$

Since $\sigma\left[S_k\right]^2 = \sigma^2 p^k$, we obtain

$$\begin{aligned}
|\Phi_U(\xi)| &\leq \exp\left(-\frac{\xi^2}{2}(1-\epsilon)\sum_{k:\ p^{(\alpha-1)k}\xi/\sigma<\delta} p^{(2\alpha-1)k}\right) \\
&\leq \exp\left(-A\xi^{\beta}\right),
\end{aligned}$$

for some $A > 0$, $\beta = (1-\alpha)^{-1} \in (1, 2)$ and for all $\xi > \delta\sigma$. For $\xi \leq \delta\sigma$, we will get

$$|\Phi_U(\xi)| \leq \exp\left(-B\xi^2\right),$$

for some $B > 0$. Thus, for all ξ we obtain,

$$|\Phi_U(\xi)| \leq \exp\left(-C\min(\xi^2, \xi^{\beta})\right),$$

for some $C > 0$.

The random variables U and V are independent, $\Lambda = \lambda_1 U + \lambda_2 V$, whence we have

$$\Phi_\Lambda(\xi) = \Phi_U(\lambda_1\xi)\Phi_V(\lambda_2\xi).$$

In particular, Φ_Λ satisfies the inequality similar to that of Φ_U. This proves that the distribution function $\mathcal{F}_\Lambda$ of Λ is in the class C^∞. Since Λ has second moment, $\mathcal{F}_\Lambda \in D(2)$ - the domain of attraction of the normal law.

At last, assume that the common distribution function $\mathcal{F}_\varepsilon$ of the i.i.d. $\{\varepsilon_{ij}\}$ is unimodal. As a convolution of symmetric unimodal distribution functions $\mathcal{F}_U$ (resp. $\mathcal{F}_V$) is symmetric and unimodal, whence $\mathcal{F}_\Lambda$ does. The proof is finished. ■

4 The integrated density of states

Let L_C be the homogeneous hierarchical Laplacian and $L_{C(\omega)}$ its random perturbation as defined and studied in the previous section. Let $O \in \mathcal{B}$ be an ultrametric ball and $H \in \mathcal{T}$ a horocycle. Let $\mathcal{B}_H(O)$ be the set of all balls $B \subseteq O$ which belong to the horocycle H.

Let δ_a be the probability distribution concentrated at $a \in \mathbb{R}$. We fix a horocycle H and study limit behaviour of the normalized empirical process

$$\mathcal{M}_O^\omega = \frac{1}{|\mathcal{B}_H(O)|} \sum_{B \in \mathcal{B}_H(O)} \delta_{\lambda(B,\omega)} \tag{4.1}$$

as O tend to ϖ.

Theorem 4.1 *There exists a probability measure $\mathcal{M}$ such that as O tend to ϖ, for almost all $\omega \in \Omega$,*

$$\mathcal{M}_O^\omega \to \mathcal{M} \text{ in the Bernoulli topology.} \tag{4.2}$$

Proof. As in the proof of Theorem 3.1, for any $B \in H$, we write

$$\lambda(B, \omega) = \lambda_H (1 + U(B, \omega)),$$

where $U(B, \omega)$ is defined at (3.3). The random variables $\{U(B)\}_{B \in H}$ are identically distributed (dependent) random variables; denote $\mathcal{N}$ their common distribution. Since the horocycle H is fixed we are left to study the normalized empirical process

$$\mathcal{N}_O^\omega = \frac{1}{|\mathcal{B}_H(O)|} \sum_{B \in \mathcal{B}_H(O)} \delta_{U(B,\omega)}.$$

Claim. As O tend to ϖ, for almost all $\omega \in \Omega$,

$$\mathcal{N}_O^\omega \to \mathcal{N} \text{ in the Bernoulli topology.} \tag{4.3}$$

Without loss of generality we may assume that $H = H_0$ and $O \in H_L$; in that case we will write $\mathcal{N}_O = \mathcal{N}_L^\omega$. Let Φ_L^ω (resp. Φ) be the characteristic function of the probability measure $\mathcal{N}_L^\omega$ (resp. $\mathcal{N}$),

$$\Phi_L^\omega(\theta) = \frac{1}{n_L n_{L+1} \ldots n_{-1}} \sum_{B \in \mathcal{B}_H(O)} \exp\{i\theta U(B,\omega)\}. \tag{4.4}$$

We have

$$\mathbb{E}(\Phi_L^\omega(\theta)) = \Phi(\theta).$$

Let us show that for every θ,

$$\Phi_L^\omega(\theta) \to \Phi(\theta) \text{ a.s. } \omega. \tag{4.5}$$

Since all probability measures $\mathcal{N}_L^\omega$, $\omega \in \Omega$, are supported by some finite interval $[-a, a]$, the family of functions $\{\Phi_L^\omega\}$ is equicontinuous. Hence the exceptional null-set in (4.5) can be chosen the same for all θ. Thus, given (4.5) holds true, the Claim follows by the the Lévy continuity theorem.

For $B \in \mathcal{B}_H(O)$, let $\{B_k\}_{k\le 0}$ be the unique infinite geodesic path in the tree of balls $\mathcal{T}$ from ϖ to B. We write

$$U(B,\omega) = \left(\sum_{L\le k\le 0} + \sum_{k<L}\right) a_k \varepsilon(B_k,\omega) := U_L(B,\omega) + V_L(B,\omega).$$

Since for any two balls B and B' in $\mathcal{B}_H(O)$,

$$V_L(B,\omega) = V_L(B',\omega) := V_L(\omega),$$

we have

$$\Phi_L^\omega(\theta) = \frac{\exp\{i\theta V_L(\omega)\}}{n_L n_{L+1} \ldots n_{-1}} \sum_{B\in\mathcal{B}_H(O)} \exp\{i\theta U_L(B,\omega)\}.$$

As $L \to -\infty$, uniformly in $\omega \in \Omega$,

$$\exp\{i\theta V_L(\omega)\} \to 1, \text{ for every } \theta,$$

whence we are left study limit behaviour of the random variables

$$\omega \to \Psi_L^\omega(\theta) = \frac{1}{n_L n_{L+1} \ldots n_{-1}} \sum_{B\in\mathcal{B}_H(O)} \exp\{i\theta U_L(B,\omega)\}.$$

The random variables $\{U_L(B,\omega)\}_{B\in\mathcal{B}_H(O)}$ are still dependent, identically distributed random variables; let $\Psi_L(\theta)$ be their common characteristic function. We have

$$\mathbb{E}(\Psi_L^\omega(\theta)) = \Psi_L(\theta).$$

Let us show that for any fixed θ,

$$\sum_{L<0} \sigma\left[\Psi_L^\omega(\theta)\right]^2 < \infty. \tag{4.6}$$

Given (4.6) holds true, Chebychev inequality and Borel-Cantelli lemma yield

$$\Psi_L^\omega(\theta) - \Psi_L(\theta) \to 0 \text{ a.s. } \omega.$$

Since as $L \to -\infty$ the random variables $\Phi_L^\omega(\theta) - \Psi_L^\omega(\theta)$ and the functions $\Phi(\theta) - \Psi_L(\theta)$ tend to zero pointwise, we will finally get (4.5).

Let X, Y be independent random variables. Assume that $|X| = 1$ and $|Y| \le 1$, then

$$\sigma\left[XY\right]^2 \le \sigma\left[Y\right]^2 + 2\left(1 - |\mathbb{E}(X)|\right). \tag{4.7}$$

Let ϕ be the common characteristic function of i.i.d. $\{\varepsilon(B)\}_{B\in\mathcal{B}}$. By substitution the data

$$X = \exp\{i\theta a_L \varepsilon(B_L)\}$$

and

$$Y = \frac{1}{n_L n_{L+1} \ldots n_{-1}} \sum_{B\in\mathcal{B}_H(O)} \exp\{i\theta U_{L+1}(B)\}$$

in (4.7) we obtain

$$\sigma\left[\Psi_L(\theta)\right]^2 \le \sigma\left[Y\right]^2 + 2\left(1 - |\phi(a_L\theta)|\right) \le \sigma\left[Y\right]^2 + a_L^2\theta^2.$$

Let $\{O_l : 0 \geq l \geq -\infty\}$ be an infinite geodesic path in $\mathcal{T}$ with $O_0 \in H$, $O_L = O$ and $O_{-\infty} = \varpi$. Let $\{O^i_{L+1} : 1 \leq i \leq n_L,\}$, with $O^1_{L+1} = O_{L+1}$, be n_L ultrametric balls which belong to the horocycle H_{L+1} and are subballs of the ball $O = O_L$. We write

$$Y = \frac{1}{n_L}\sum_{i=1}^{n_L} Y_i,$$

where

$$Y_i = \frac{1}{n_{L+1}...n_{-1}} \sum_{B\in\mathcal{B}_H(O^i_{L+1})} \exp\{i\theta U_{L+1}(B)\}.$$

The random variables $\{Y_i : 1 \leq i \leq n_L\}$ are independent and identically distributed, and $Y_1 = \Psi_{L+1}(\theta)$. As a result we obtain the following inequality

$$\sigma[\Psi_L(\theta)]^2 \leq \frac{1}{n_L}\sigma[\Psi_{L+1}(\theta)]^2 + a_L^2\theta^2,$$

which evidently proves (4.6). ■

Definition 4.2 *The measure $\mathcal{M}$ defined at (4.2) is called the integrated density of states (i.d.s. for short) corresponding to the horocycle H. If the measure $\mathcal{M}$ is absolutelly continuous w.r.t. Lebesgue measure, i.e.*

$$\mathcal{M}(I) = \int_I \mathfrak{m}(\tau)d\tau,$$

the function $\mathfrak{m}(\tau)$ is called the density of states (d.s. for short) corresponding to the horocycle H.

The question whether the d.s. $\mathfrak{m}(\tau)$ related to the data (C, ϵ) exists, is continuous, belongs to the class C^∞ etc. is basic in various applications, see Theorem 5.1 below.

Remark 4.3 *Recall that the measures $\mathcal{M}$ and $\mathcal{N}$ defined at (4.2) and (4.3) respectively are related by the equation*

$$\mathcal{M} = \mathcal{N} \circ \vartheta^{-1},$$

where $\vartheta : x \to ax + b$ with $a = b = \lambda_H$. In particular, $\mathcal{M}$ is absolutely continuous w.r.t. Lebesgue measure if and only if $\mathcal{N}$ does.

The measure $\mathcal{N}$ has a remarkable feature - it belongs to the class $\mathfrak{J}$ of probability measures each of which can be represented as the distribution of some random variable U of the form

$$U = \sum_{k\geq 0} b_k\varepsilon_k, \tag{4.8}$$

where $\{\varepsilon_k\}_{k\geq 0}$ are symmetric i.i.d. with values in some finite interval $I \subset \mathbb{R}^1$ and $b_k > 0$ satisfy $\sum b_k = 1$.

Various properties of $\mathfrak{J}$-distributions (infinite convolutions) have been studied by many authors since 1930's, see e.g. [22], [14], [27], [28], [29], [26] and references therein. We would like to mention two remarkable properties of $\mathfrak{J}$-distributions. The first one is due to Lévy (1937) and the second one is due to Jessen and Wintner (1935), see e.g. [22], Thm. 3.7.6 and 3.7.7 respectively.

- Each $\mathfrak{I}$-distribution $\mathcal{N}$ in its Lebesgue decomposition contains no discrete component.
- Assume that $\{\varepsilon_k\}$ at (4.8) are discrete, then the measure $\mathcal{N}$ is either singular or it is absolutely continuous (w.r.t. Lebesgue measure).

Examples of singular $\mathfrak{I}$-distributions will be given later (infinite Bernoulli convolutions). We consider first a simple class of absolutely continuous $\mathfrak{I}$-distributions. Let $\mathcal{N}$ be a $\mathfrak{I}$-distribution as defined at (4.8). Assume that the common characteristic function ϕ of i.i.d. $\{\varepsilon_k\}$ satisfies

$$|\phi(x)| \leq x^{-D} \text{ at } \infty,$$

for some $D > 0$. Then evidently the characteristic function $\Phi(x)$ of the measure $\mathcal{N}$, as an infinite product of characteristic functions, satisfies

$$|\Phi(x)| \leq x^{-B} \text{ at } \infty,$$

for any $B > 0$, whence $\mathcal{N}$ admits a C^∞-density w.r.t. Lebesgue measure. The proposition below gives a little refinement of the above observation.

Proposition 4.4 *Let $\mathcal{N}$ be a $\mathfrak{I}$-distribution. Assume that the common characteristic function ϕ of i.i.d. $\{\varepsilon_k\}$ tend to zero at infinity and that*

$$b_k \geq C \exp(-Dk),$$

for some $C, D > 0$. Then $\mathcal{N}$ admits a C^∞-density w.r.t. Lebesgue measure.

Proof. Let Φ be the characteristic function of $\mathcal{N}$. For a given $\epsilon > 0$ choose $N = N(\epsilon) > 1$ such that $|\phi(z)| \leq \epsilon$ for all $z \geq N$, and write

$$|\Phi(x)| \leq \prod_{k:\ b_k x \geq N} \phi(b_k x) \leq \exp\left(-\#\{k : b_k x \geq N\} \log \frac{1}{\epsilon}\right).$$

By the assumption,

$$\#\{k : b_k x \geq N\} \geq \log \frac{1}{e}\left(\frac{cx}{N}\right)^{1/D},$$

whence

$$|\Phi(x)| \leq A x^{-B} \text{ at } \infty$$

with

$$A = \left[e\,(N/C)^{1/D}\right]^{\log \frac{1}{\epsilon}} \text{ and } B = \frac{1}{D} \log \frac{1}{\epsilon}.$$

Since $\epsilon > 0$ can be chosen arbitrary small the result follows. ■

Various examples of characteristic functions ϕ of singular distributions $\{\varepsilon_k\}$ which satisfy the condition of Proposition 4.4 are given in [22], Sec.3, and also in [14], Sec. 6 and 7. Here is an example of Kerschner(1936): a is a rational number such that $0 < a < 1/2$ and $a \neq 1/n$, where $n \geq 3$ is an integer. Then

$$\phi(x) = \prod_{k=1}^{\infty} \cos(xa^k)$$

is the characteristic function of a singular symmetric $\mathfrak{J}$-distribution satisfying

$$|\phi(x)| \leq \frac{1}{(\log x)^{\gamma}} \text{ at } +\infty, \tag{4.9}$$

for some $\gamma > 0$.

Certain applications of Proposition 4.4 which we have in mind are homogeneous ultrametric spaces X such that the sequence $\{n_H\}$ of forward degrees, defined by the tree of balls $\mathcal{T}(X)$, is bounded and the homogeneous hierarchical Laplacians L_C on X satisfy the condition (3.11). For instance, one can consider $X = \mathbb{Q}_p$ and L_C is the operator of fractional derivative introduced at (2.2).

Modifying the proof of Proposition 4.4 one can obtain results which can be applied when the sequence $\{n_H\}$ is unbounded, e.g. when X is the infinite symmetric group $S_\infty = \cup_{n\in\mathbb{N}} S_n$. In this case, for H consisting of the symmetric group S_l and its cosets aS_l, the n_H equals l.

Proposition 4.5 *Assume that the common characteristic function ϕ of i.i.d. $\{\varepsilon_k\}$ satisfies (4.9) and that $b_k \geq C/k!$ for some $C > 0$. Then the characteristic function $\Phi(x)$ of the corresponding $\mathfrak{J}$-distribution $\mathcal{N}$ satisfies*

$$|\Phi(x)| \leq x^{-(\gamma-\epsilon)} \textit{ at } +\infty,$$

for any $0 < \epsilon < \gamma$. In particular, if $\gamma > 1$, the distribution $\mathcal{N}$ admits a C^k-density with $k \leq \gamma - 1$.

Proof. By the assumption,

$$n(t) := \#\{k : b_k \geq t\} \geq \#\{k : C/k! \geq t\} \sim \frac{\log 1/t}{\log\log 1/t} \text{ at } 0.$$

For $x > 1$ set $\overline{\Phi}(x) = (\log x)^{\gamma}$ and choose $r(x)$ such that

$$\log r(x) \sim \frac{\log x}{\log\log x} \text{ at } +\infty.$$

For x big enough define

$$A(x) := n\left(\frac{r(x)}{x}\right) \log \overline{\Phi}(r(x)).$$

By the assumption,

$$\liminf_{x\to+\infty} \frac{A(x)}{\gamma \log x} \geq 1. \tag{4.10}$$

At last we estimate the characteristic function $\Phi(x)$ of the $\mathfrak{J}$-distribution $\mathcal{N}$,

$$\begin{aligned} |\Phi(x)| &= \prod_k |\phi(b_k x)| \leq \prod_{k:\, b_k x \geq r(x)} |\phi(b_k x)| \leq \prod_{k:\, b_k x \geq r(x)} \left(\overline{\Phi}(b_k x)\right)^{-1} \\ &\leq \left(\overline{\Phi}(r(x))\right)^{-n\left(\frac{r(x)}{x}\right)} = \exp\{-A(x)\}. \end{aligned}$$

Applying (4.10) we obtain the desired result. ■

Infinite Bernoulli convolutions. Let $\{\varepsilon_k\}_{k\geq 0}$ be i.i.d. random variables taking values ± 1 with equal probability $1/2$. Define a one-parametric family of random variables

$$U_\lambda = \sum_{k\geq 0} \lambda^k \varepsilon_k,\ \ 0<\lambda<1. \tag{4.11}$$

Let $\mathcal{N}_\lambda$ be the distribution of U_λ. The measure $\mathcal{N}_\lambda$ is the infinite convolution product of discrete measures $\left(\delta_{-\lambda^k}+\delta_{\lambda^k}\right)/2$. It is called the *infinite Bernoulli convolution.* The characteristic function Φ_λ of $\mathcal{N}_\lambda$ can be represented as a convergent infinite product

$$\Phi_\lambda(\theta)=\prod_{k=0}^{\infty}\cos\left(\lambda^k\theta\right).$$

We describe some of the previous work on the infinite Bernoulli convolutions. We refere to the survey of B. Solomyak [27] which contains a comprehensive list of references to the relevant literature.

- Jessen and Wintner (1935) showed that $\mathcal{N}_\lambda$ is either absolutely continuous or purely singular, depending on λ.
- Kerschner and Wintner (1935) observed that $\mathcal{N}_\lambda$ is singular for $\lambda\in(0,2^{-1})$, since it is supported on a Cantor set of zero Lebesgue measure.
- Wintner (1935) noted that $\mathcal{N}_\lambda$ is uniform on $[-2,2]$ for $\lambda=2^{-1}$, and for $\lambda=2^{-1/k}$ with integer $k\geq 1$ it is absolutely continuous, with a C^{k-1}-density.
- Erdös (1939) has shown that $\mathcal{N}_\lambda$, $\lambda\in(2^{-1},1)$, is singular if $1/\lambda$ is a PV-number (an algebraic integer whose Galois conjugates are strictly less than one in modulus; the golden ratio $\left(1+\sqrt{5}\right)/2$ is an example). No other $\lambda\in(2^{-1},1)$ with singular $\mathcal{N}_\lambda$ are known.
- Solomyak (1995) proved that $\mathcal{N}_\lambda$ is absolutely continuous for almost every $\lambda\in(2^{-1},1)$ - Garsia conjecture (1962). A stronger conjecture that this is true for all but countably many $\lambda\in(2^{-1},1)$ is still very much open.

Bernoulli perturbations of the operator $\mathfrak{B}^\alpha$. As an example we consider the homogeneous Laplacian $\mathfrak{B}^\alpha$ introduced at (2.2). The operator $\mathfrak{B}^\alpha$ acts in $L^2(\mathbb{Q}_p,m)$, where $\mathbb{Q}_p$ is the ring of p-adic numbers (p is not necessary a prime number), and m is the Haar measure,

$$\mathfrak{B}^\alpha f(x)=\frac{1-p^{-\alpha}}{1-p^{-\alpha-1}}\int_{\mathbb{Q}_p}\frac{f(x)-f(y)}{|x-y|^{1+\alpha}}dm(y).$$

Let $\mathfrak{B}^\alpha(\omega)$ be the perturbation of the operator $\mathfrak{B}^\alpha$ by symmetric i.i.d. Bernoulli random variables $\{\theta\varepsilon(B)\}_{B\in\mathcal{B}}$, $0<\theta<1$. Let $\mathcal{N}_\lambda$ be the infinite Bernoulli convolution with $\lambda=p^{-\alpha}$. Set

$$a=\theta\left(1-p^{-\alpha}\right)\lambda_H,\ \ b=\lambda_H \text{ and } \vartheta(x)=ax+b.$$

Applying Theorem 4.1, properties of infinite Bernoulli convolutions listed above and the fact that the eigenvalue λ_H of the operator $\mathfrak{B}^\alpha$ corresponding to the horocycle $H=H_l$ is equal to $p^{\alpha l}$ we obtain the following result.

Theorem 4.6 *Let $\mathcal{M}_\alpha$ be the integrated density of states associated with the operator $\mathfrak{B}^\alpha(\omega)$ via Theorem 4.1 and Definition 4.2 . Let $\mathcal{N}_\lambda$ be the infinite Bernoulli convolution with $\lambda = p^{-\alpha}$. We have,*

$$\mathcal{M}_\alpha = \mathcal{N}_\lambda \circ \vartheta^{-1}.$$

In particular, with respect to Lebesgue measure, $\mathcal{M}_\alpha$ is:

- singular, for all $\alpha > (\log 2)/(\log p)$,
- uniform, for $\alpha = (\log 2)/(\log p)$, and
- absolutely continuous, having L^2-density, for a.e. $0 < \alpha \leq (\log 2)/(\log p)$. Moreover, for $\alpha = (\log 2)/(k \log p)$, $k \in \mathbb{N}$, it admits a C^{k-1}-density.

5 The Poisson Convergence

Let us fix a horocycle H. The eigenvalues $\lambda(B,\omega)$, $B \in \mathcal{B}_H(O)$, themselves can be represented by the following *empirical process*

$$N_O^\omega(I) = \sum_{B \in \mathcal{B}_H(O)} \delta_{\lambda(B,\omega)}(I),\ I \in \mathcal{B}(\mathbb{R}).$$

The *intensity measure* $\mu_O(I)$ - the expected number of $\lambda(B,\omega)$, $B \in \mathcal{B}_H(O)$, which fall in the set I - is computed as

$$\mu_O(I) = \mathbb{E}N_O(I) = |\mathcal{B}_H(O)|\,\mathbb{P}(\omega \in \Omega : \lambda(B,\omega) \in I);$$

recall that the right-hand-side of the above equality does not depend of $B \in \mathcal{B}_H(O)$. We fix numbers $c, \tau_0 > 0$ and consider a small interval

$$I = \left\{ \tau : |\tau - \tau_0| \leq \frac{c}{2\,|\mathcal{B}_H(O)|} \right\}. \tag{5.1}$$

Assuming that the density of states $\mathfrak{m}(\tau)$ as defined in (4.2) exists, is continuous at $\tau = \tau_0$ and that $\mathfrak{m}(\tau_0) > 0$ we obtain

$$\lim_{O \to \varpi} \mu_O(I) = c\mathfrak{m}(\tau_0) := \lambda_c > 0. \tag{5.2}$$

In particular, if the $\lambda(B,\omega)$, $B \in \mathcal{B}_H(O)$, were i.i.d., then the equation (5.2) would yield the classical convergence of $N_O = N_O^\omega(I)$ to the Poisson random variable $\mathcal{P}_\lambda$ with intensity $\lambda = \lambda_c$. More precisely, in the i.i.d. case we would then have, see [21], [9], [3],

$$\|\mathcal{L}(N_O) - \mathcal{L}(\mathcal{P}_\lambda)\|_{TV} \leq \frac{\min(\lambda, \lambda^2)}{2\,|\mathcal{B}_H(O)|}.$$

Remember that $\mathcal{L}(X)$ stands for the law of the random variable X and $\|\mu - \nu\|_{TV}$ is the distance between μ and ν in total variation.

However, in our case the $\lambda(B,\omega)$, $B \in \mathcal{B}_H(O)$, are *dependent* random variables, whence the classical theory does not apply directly and needs some justifications and complements. Basic ingredients in our study are the stationarity property of the family $\{\lambda(B,\omega)\}_{B \in H}$ and certain estimates of its correlation function, see Section 3. We will prove the following statement.

Theorem 5.1 *Assume that the condition (3.11) holds with $\delta > 2$, and that the common law of i.i.d. $\{\varepsilon(B)\}$ admits a bounded density. Then, in the notation introduced above, as $O \to \varpi$,*

$$\mathcal{L}(N_O) \to \mathcal{L}(\mathcal{P}_\lambda) \text{ in the Bernoulli topology.}$$

Before embarking on the proof notice that, thanks to our assumptions, the density of states $\mathfrak{m}(\tau)$ exists and belongs to C^∞, see Proposition 4.4. In particular, $\lambda = \lambda_c$ at (5.2) is well defined for any $c > 0$. Next, for any $B \in H$, the eigenvalue $\lambda(B, \omega)$ can be written in the form

$$\lambda(B, \omega) = \lambda_H \left(1 + U(B, \omega)\right), \tag{5.3}$$

where

$$U(B, \omega) = \sum_{B \subseteq B_k} a_k \varepsilon(B_k, \omega), \text{ and } a_k = C(B_k)/\lambda_H. \tag{5.4}$$

Since $\sum a_k = 1$ and $|\varepsilon(B, \omega)| \leq \epsilon$, for all B,ω and some $0 < \epsilon < 1$,

$$|U(B, \omega)| \leq \sup_{k,\omega} |\varepsilon(B_k, \omega)| = \epsilon.$$

The common distribution function $\mathcal{N}(t)$ of the family $\{U(B, \omega)\}_{B \in H}$ is absolutely continuous and its density $\mathfrak{n}(t)$ relates to the integrated density of states $\mathfrak{m}(\tau)$ by

$$\mathfrak{n}(t) = \lambda_H \mathfrak{m}(\lambda_H t + \lambda_H).$$

In particular, $\mathfrak{n}(t)$ is supported by the interval $[-\epsilon, \epsilon]$, is continuous and strictly positive at $t_0 = \tau_0/\lambda_H - 1$.

Let $\widetilde{N}_O^\omega$ be the empirical process defined by the family $\{U(B, \omega)\}$, $B \in \mathcal{B}_H(O)$. Let us choose an interval $\widetilde{I}$ as

$$\widetilde{I} = \left\{ t : |t - t_0| \leq \frac{\widetilde{c}}{2\,|\mathcal{B}_H(O)|} \right\}, \ \widetilde{c} = c/\lambda_H,$$

and set $\widetilde{N}_O := \widetilde{N}_O^\omega(\widetilde{I})$. The equation (5.3) yields

$$\mathbb{P}\{N_O = k\} = \mathbb{P}\left\{\widetilde{N}_O = k\right\},$$

and therefore

$$\lim_{O \to \varpi} \mathbb{P}\{N_O = k\} = \lim_{O \to \varpi} \mathbb{P}\left\{\widetilde{N}_O = k\right\}.$$

Having all these observations in mind we will prove the evident U-version of Theorem 5.1.

We fix a horocycle H and let O tend to ϖ. Evidently when studying the family $U(B, \omega)$, $B \in H$, we can replace the original ultrametric space X by certain discrete ultrametric space. For instance, we can replace X by the discrete Abelian group

$$G = \bigoplus_{k \geq 1} \mathbb{Z}(n_k)$$

equipped with its canonical ultrametric structure defined by the family $\{G_l\}_{l \geq 0}$ of its finite subgroups

$$G_0 = \{0\}, \ G_l = \prod_{1 \leq k \leq l} \mathbb{Z}(n_k).$$

With such agreement in mind we have: $H = H_0$ is the set of all singletones, H_1 is the set of all ultrametric balls of the form $g+G_1$ etc. When $B = \{g\}$ we shell write $U(B,\omega) = U_g(\omega)$. For any $B_i \supseteq B$, we set $\varepsilon(B_i,\omega) = \varepsilon_{ig}(\omega)$. Thus we define a stationary family $\{U_g\}_{g\in G}$,

$$U_g(\omega) = \sum_{i=0}^{\infty} a_i \varepsilon_{ig}(\omega).$$

Let $Z_l^c(\omega)$ be the number of $U_g(\omega)$, $g \in G_l$, which fall in the interval

$$I_l^c = \left\{ t : |t - t_0| \leq \frac{c}{2\pi_l} \right\}, \ \pi_l = n_1 ... n_l.$$

We set $\lambda_c = c\mathfrak{n}\,(t_0)$ and prove that

$$\lim_{l\to\infty} \mathbb{P}\left\{Z_l^c = k\right\} = \frac{(\lambda_c)^k}{k!} \exp\left(-\lambda_c\right). \tag{5.5}$$

Writing Z_l^c in the form

$$Z_l^c(\omega) = \sum_{g\in G_l} \delta_{U_g(\omega)}(I_l^c)$$

we compute

$$\lim_{l\to\infty} \mathbb{E}\left(Z_l^c\right) = \lim_{l\to\infty} \pi_l \mathbb{P}(U_g \in I_l^c) = \lambda_c.$$

It follows that the family $\{\mathcal{L}(Z_l^c)\}_{l\in\mathbb{N}}$ is *tight*, whence it is relatively compact in the week topology. Let $\mathcal{L}(Z)$ be an accumulation point of $\{\mathcal{L}(Z_l^c)\}_{l\in\mathbb{N}}$. *We claim that Z is infinitely divisible.* Indeed, using the ultrametric structure of G we can write

$$Z_l^c = \sum_{g\in\mathbb{Z}(n_l)} \tau_g\left(Z_{l-1}^{c/n_l}\right),$$

where

$$\tau_g\left(Z_{l-1}^{c/n_l}\right) := \sum_{a\in G_{l-1}} \delta_{U_{g+a}}(I_{l-1}^{c/n_l}).$$

The $\tau_g\left(Z_{l-1}^{c/n_l}\right)$, $g \in \mathbb{Z}(n_l)$, are identically distributed (dependent) random variables. However the dependence between them becomes weaker as l tend to infinity, that is, for each $g \in G_l$ we can write

$$U_g = \sum_{i=0}^{l-1} a_i\varepsilon_{ig} + \sum_{i=l}^{\infty} a_i\varepsilon_{ig} := \widetilde{U_g} + K_l. \tag{5.6}$$

The common part K_l of the random variables U_g is independent of the family $\{\widetilde{U_g}\}$ and can be estimated as

$$|K_l(\omega)| \leq \epsilon \sum_{i=l}^{\infty} a_i = O(a_l) = O(\pi_l^{-\delta/2}).$$

It particular, assuming that $\delta > 2$, we obtain

$$k_l := \sup_\omega |K_l(\omega)| = o(\pi_l^{-1}). \tag{5.7}$$

Let us compute the characteristic function

$$\Phi_Z(\gamma) = \mathbb{E}\left(\exp\left(-\gamma Z\right)\right), \gamma \geq 0,$$

of the random variable $Z = Z_l^c$. We have

$$Z_l^c = \sum_{a\in\mathbb{Z}(n_l)} \sum_{g\in G_{l-1}} \delta_{U_{a+g}}(I_l^c) = \sum_{a\in\mathbb{Z}(n_l)} \tau_a\left(Z_{l-1}^{c/n_l}\right),$$

whence

$$\begin{aligned}\Phi_{Z_l^c}(\gamma) &= \mathbb{E}\left[\mathbb{E}\left(\exp\left(-\gamma Z_l^c\right)|K_l\right)\right]\\ &= \mathbb{E}\left[\left(\mathbb{E}\left(\exp\left(-\gamma Z_{l-1}^{c/n_l}\right)|K_l\right)\right)^{n_l}\right]\\ &= \int d\mu_l(k)\left(\mathbb{E}\left(\exp\left(-\gamma Z_{l-1}^{c/n_l}\right)|K_l=k\right)\right)^{n_l},\end{aligned}$$

where μ_l is the law of K_l. Using (5.6), we compute

$$\mathbb{E}\left(\exp\left(-\gamma Z_{l-1}^{c/n_l}\right)|K_l=k\right) = \mathbb{E}\{\exp[-\gamma\sum_{g\in G_{l-1}}\delta_{\widetilde{U_g}}(I_{l-1}^{c/n_l}-k)]\}.$$

Remark that by (5.7), $\epsilon_l := \pi_l k_l = o(1)$, whence for any $g \in G_{l-1}$ and $k \in [-k_l, k_l]$, we will have

$$\left\{U_g \in I_{l-1}^{(c-2\epsilon_l)/n_l}\right\} \subseteq \left\{\widetilde{U}_g \in I_{l-1}^{c/n_l} - k\right\} \subseteq \left\{U_g \in I_{l-1}^{(c+2\epsilon_l)/n_l}\right\},$$

and

$$\left(\Phi_{Z_{l-1}^{(c+2\epsilon_l)/n_l}}(\gamma)\right)^{n_l} \leq \Phi_{Z_l^c}(\gamma) \leq \left(\Phi_{Z_{l-1}^{(c-2\epsilon_l)/n_l}}(\gamma)\right)^{n_l}. \tag{5.8}$$

1st Case. *Assume that $n_l = n$ along some infinite sequence $\{l_k\}$.* Then, along this sequence,

$$\mathbb{E}\left(Z_{l-1}^{(c\pm2\epsilon_l)/n}\right) = \pi_{l-1}\mathbb{P}\left(U_g(\omega) \in I_{l-1}^{(c\pm2\epsilon_l)/n}\right) \to \lambda_c/n,$$

whence the family $\{\mathcal{L}(Z_{l-1}^{(c\pm2\epsilon_l)/n})\}$ is tight. Recall that the family $\{\mathcal{L}(Z_l^c)\}$ is tight as well. Choose a sequence $\{l_k'\} \subset \{l_k\}$ such that along this new sequence

$$Z_l^c \to Z \text{ and } Z_{l-1}^{(c\pm2\epsilon_l)/n} \to Z^{\pm} \text{ in law.}$$

Since for all $\omega \in \Omega$,

$$Z_{l-1}^{(c-2\epsilon_l)/n}(\omega) \leq Z_{l-1}^{(c+2\epsilon_l)/n}(\omega)$$

and the strong inequality occurs if and only if ω belongs to the event

$$\Omega_l = \{U_g(\omega) \in I_{l-1}^{(c+2\epsilon_l)/n}\backslash I_{l-1}^{(c-2\epsilon_l)/n} \text{ for some } g \in G_{l-1}\}$$

whose probability is estimated as

$$\begin{aligned}\mathbb{P}(\Omega_l) &\leq 2\,|G_{l-1}|\,(4\epsilon_l/\pi_l)(\mathfrak{n}\,(t_0)+o(1))\\ &= O(\epsilon_l) \to 0 \text{ as } l \to \infty,\end{aligned}$$

we must have $Z^+ = Z^-$ a.s. Let $Z' = Z^{\pm}$, passing to the limit in the equation (5.8) we obtain

$$\Phi_Z(\gamma) = \left(\Phi_{Z'}(\gamma)\right)^n. \tag{5.9}$$

Applying the same procedure as before we will have

$$\Phi_{Z'}(\gamma) = \left(\Phi_{Z''}(\gamma)\right)^n$$

etc, whence Z is infinite divisible as claimed.

2nd Case. *Assume that* $n_l \to \infty$. Let G_l be partitioned into disjoint subsets A_i, $i = 1, 2, 3$, that are made of G_{l-1}-cosets each and $|A_1| = |A_2| = [\pi_l/2]$. We let

$$Z^c_{l,i} = \sum_{g \in A_i} \delta_{U_g(\omega)}(I^c_l), \; i = 1, 2, 3,$$

so that

$$Z^c_l = Z^c_{l,1} + Z^c_{l,2} + Z^c_{l,3}.$$

Note that as $l \to \infty$,

$$\mathbb{E}\left(Z^c_{l,3}\right) \leq \frac{1}{n_l} \mathbb{E}\left(Z^c_l\right) \to 0$$

and

$$\mathbb{E}\left(Z^c_{l,i}\right) \to \lambda_c/2.$$

In particular, $Z^c_{l,3} \to 0$ in probability. Clearly, $Z^c_{l,1}$ and $Z^c_{l,2}$ have the same distribution. The families of laws $\{\mathcal{L}(Z^c_{l,i})\}$ are tight. Reasoning as in the 1st case, we will show that along some subsequence $Z^c_l \to Z$, $Z^c_{l,1} \to Z'$ and $Z^c_{l,2} \to Z''$in law, and that $\mathcal{L}(Z') = \mathcal{L}(Z'')$. The equation (5.9) holds with $n = 2$, whence Z is infinite divisible as desired.

Since Z is non-negative and integer valued its characteristic function has the following representation

$$\Phi_Z(\gamma) = \exp\left\{-a\gamma - \int \left(1 - e^{-\gamma x}\right) m(dx)\right\}, \; \gamma \geq 0, \tag{5.10}$$

where m is a finite measure on $\mathbb{N}$ and $a \in \mathbb{N} \cup \{0\}$. Since the range of Z is the whole of $\mathbb{N} \cup \{0\}$, it must be the case that $a = 0$. Note that

$$\mathbb{E}(Z) = \int x m(dx) \tag{5.11}$$

and

$$\mathbb{E}(Z^2) = \int x^2 m(dx) + \left(\int x m(dx)\right)^2 \tag{5.12}$$

We claim that the measure m *is concentrated at* $\{1\}$. Suppose we can show that

$$\limsup_{l \to \infty} \mathbb{E}\left(Z^c_l\right)^2 \leq \lambda_c + \left(\lambda_c\right)^2. \tag{5.13}$$

It would then follow that the family $\{Z^c_l\}$ is uniformly integrable, so that along some subsequence (l_k)

$$\mathbb{E}\left(Z\right) = \lim \mathbb{E}\left(Z^c_l\right) = \lambda_c. \tag{5.14}$$

Furthermore, by Fatou's lemma and by (5.13),

$$\mathbb{E}\left(Z\right)^2 \leq \limsup_{l \to \infty} \mathbb{E}\left(Z^c_l\right)^2 \leq \lambda_c + \left(\lambda_c\right)^2,$$

so that, by (5.12) and (5.14) we would obtain

$$\int x^2 m(dx) \leq \lambda_c = \int x m(dx).$$

Since m is concentrated on $\mathbb{N}$ it would follow that $m = \lambda_c \delta_{\{1\}}$ and thus Z is Poissonian with parameter λ_c. This evidently would prove the claim (5.5). It remains therefore to prove (5.13).

Without loss of generality we may assume that $n_l \equiv n$. Let $g \wedge g'$ be the confluent of g and g', that is, the minimal ball in G which contains both g and g'. We have

$$\begin{aligned}
\mathbb{E}\left(Z_l^c\right)^2 &= \sum_{g,g' \in G_l} \mathbb{P}\left(U_g \in I_l^c, U_{g'} \in I_l^c\right) \\
&= \sum_{0 \le j \le l} \sum_{g \wedge g' \in \mathcal{B}_{H_j}(G_l)} \mathbb{P}\left(U_g \in I_l^c, U_{g'} \in I_l^c\right) \\
&= \sum_{1 \le j \le l} n^{l-j} \sum_{g \wedge g' = G_j} \mathbb{P}\left(U_g \in I_l^c, U_{g'} \in I_l^c\right) + n^l \mathbb{P}\left(U_0 \in I_l^c\right).
\end{aligned}$$

Since for any two couples (g, g') and (f, f') such that $g \wedge g' = f \wedge f'$,

$$\mathbb{P}\left(U_g \in I_l^c, U_{g'} \in I_l^c\right) = \mathbb{P}\left(U_f \in I_l^c, U_{f'} \in I_l^c\right),$$

we obtain

$$\sum_{g \wedge g' = G_j} \mathbb{P}\left(U_g \in I_l^c, U_{g'} \in I_l^c\right) = n^j (n^j - n^{j-1}) \mathbb{P}\left(U_{g_j} \in I_l^c, U_{g'_j} \in I_l^c\right),$$

where g_j, g'_j are chosen such that $g_j \wedge g'_j = G_j$. Whence we have

$$\mathbb{E}\left(Z_l^c\right)^2 = J + J',$$

where

$$J = (n-1) \sum_{1 \le j \le l} n^{l+j-1} \mathbb{P}\left(U_{g_j} \in I_l^c, U_{g'_j} \in I_l^c\right)$$

and

$$J' = n^l \mathbb{P}\left(U_0 \in I_l^c\right).$$

Choosing $\{\varepsilon_i\}$ and $\{\varepsilon'_i\}$ to be two independent families of i.i.d. having the same common distribution as $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ we write

$$U_{g_j} = \sum_{i=0}^{j-1} a_i \varepsilon_i + \sum_{i=j}^{\infty} a_i \varepsilon_i := \widetilde{U}_j + K_j$$

and similarly

$$U_{g'_j} = \sum_{i=0}^{j-1} a_i \varepsilon'_i + \sum_{i=j}^{\infty} a_i \varepsilon_i = \widetilde{U}'_j + K_j,$$

We already know that

$$\lim_{l \to \infty} n^l \mathbb{P}\left(U_0 \in I_l^c\right) = \lambda_c, \tag{5.15}$$

whence we are left to show that

$$\limsup_{l \to \infty} J \le (\lambda_c)^2. \tag{5.16}$$

To simplify our notation we set

$$\mathcal{P}_{l-i} = \mathbb{P}\left(U_{g_{l-i}} \in I_l^c, U_{g'_{l-i}} \in I_l^c\right).$$

We choose $0 < m < l$ and split the sum in J in two terms

$$\begin{aligned} J &= (n-1)\sum_{0\le i<l} n^{2l-i-1}\mathcal{P}_{l-i} \\ &= (n-1)n^{2l}\left(\sum_{0\le i\le m} + \sum_{m<i<l}\right) n^{-(i+1)}\mathcal{P}_{l-i} := J_m + J^m. \end{aligned}$$

We write

$$U_{g_{l-i}} = a_0\varepsilon_0 + A_{l-i} + K_{l-i}$$

and similarly

$$U_{g'_{l-i}} = a_0\varepsilon'_0 + A'_{l-i} + K_{l-i}.$$

Since $\{\varepsilon_0, \varepsilon'_0, A_{l-i}, A'_{l-i}, K_{l-i}\}$ are independent, we write $\mathcal{P}_{l-i}$ as

$$\int \mathbb{P}\left(a_0\varepsilon_0 \in I_l^c - a - k\right)\mathbb{P}\left(a_0\varepsilon'_0 \in I_l^c - a' - k\right) d\mu(a)d\mu(a')d\nu(k), \tag{5.17}$$

where μ is the common distribution of i.i.d. $\{A_{l-1}, A'_{l-1}\}$ and ν is the distribution of K_{l-i}.

Assume now that the common distribution function of $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ *admits a bounded density* $\varepsilon(x)$. Then the equation (5.17) yields

$$\mathcal{P}_{l-i} \le n^{-2l}\,\|\varepsilon\|_\infty^2\, c^2/a_0^2$$

and therefore, as $m \to \infty$,

$$\begin{aligned} J^m &\le (n-1)n^{2l}\sum_{m<i<l} n^{-(i+1)}n^{-2l}\,\|\varepsilon\|_\infty^2\, c^2/a_0^2 \\ &< \|\varepsilon\|_\infty^2\, c^2/a_0^2(n-1)\sum_{i>m} n^{-(i+1)} = n^{-(m+1)}\,\|\varepsilon\|_\infty^2\, c^2/a_0^2 \to 0. \end{aligned}$$

To estimate J_m we choose $0 < \theta < 1$ and applying the same procedure of decoupling as before we obtain

$$\begin{aligned} \mathcal{P}_{l-i} &= \int \left(\mathbb{P}\left(\widetilde{U}_{l-i} \in I_l^c - k\right)\right)^2 d\nu(k) \\ &\le \sup_{|k|\le\theta/n^l}\left(\mathbb{P}\left(\widetilde{U}_{l-i} \in I_l^c - k\right)\right)^2 + \mathbb{P}\left(|K_{l-i}| > \theta/n^l\right) \\ &\le \left(\mathbb{P}\left(U_g \in I_l^{c+4\theta}\right)\right)^2 + \mathbb{P}\left(|K_{l-i}| > \theta/n^l\right). \end{aligned}$$

Hence we have

$$\begin{aligned} J_m \le\; & \left(\mathbb{P}\left(U_g \in I_l^{c+4\theta}\right)\right)^2 (n-1)n^{2l}\sum_{0\le i\le m} n^{-(i+1)} \\ & +(n-1)n^{2l}\sum_{0\le i\le m} n^{-(i+1)}\mathbb{P}\left(|K_{l-i}| > \theta/n^l\right). \end{aligned}$$

Further, as $l \to \infty$ and m, θ are fixed,

$$\left(\mathbb{P}\left(U_g \in I_l^{c+4\theta}\right)\right)^2 (n-1)n^{2l} \sum_{0\le i\le m} n^{-(i+1)} \to (\lambda_{c+4\theta})^2 (1 - n^{-m}).$$

Next we fix $p > 1$, $m < l$ and apply Chebychev inequality for $i \le m$,

$$\mathbb{P}\left(|K_{l-i}| > \theta/n^l\right) \le \theta^{-p} n^{pl} \|\varepsilon\|_\infty^p \left(\sum_{k\ge l-m} a_k\right)^p,$$

Assume now that the condition (3.11) *holds with* $\delta/2 = 1+\gamma$, $\gamma > 0$. By (3.11), as $l \to \infty$ and m is fixed,

$$\frac{1}{a_{l-m}} \sum_{k\ge l-m} a_k = O(1).$$

Choosing p big enough (such that $p\gamma > 2$) we get

$$\begin{aligned}(n-1)n^{2l} \sum_{0\le i\le m} n^{-(i+1)} \mathbb{P}\left(|K_{l-i}| > \theta/n^l\right) &\le n^{2l} \max_{0\le i\le m} \mathbb{P}\left(|K_{l-i}| > \theta/n^l\right)\\ &\le Cn^{2l+pl} a_{l-m}^p \le C' n^{-l\beta},\end{aligned}$$

where $C, C', \beta > 0$ do not depend on l. Finally, all the above yields

$$\limsup_{l\to\infty} J \le (\lambda_{c+4\theta})^2 (1 - n^{-m}).$$

Letting $\theta \to 0$ and $m \to \infty$ in the above inequality we get the desired result, that is, the inequality (5.16). The proof of Theorem 5.1 is finished.

Binomial perturbations of the operator $\mathfrak{B}^\alpha$. Let ε be the common distribution of the i.i.d. $\{\varepsilon(B)\}_{B\in\mathcal{B}}$. The assumption of Theorem 5.1 that ε admits a bounded density can not be entirelly droped. More precisely, *we will show that if ε contains a discrete component while the other assumptions of Theorem 5.1 hold true the Poisson convergence may fail.*

As an example we choose the operator of fractional derivative $\mathfrak{B}^\alpha$ and consider its perturbation $\mathfrak{B}^\alpha(\omega)$ defined by the i.i.d. symmetrized Binomial random variables, that is, we take $\varepsilon = B_1 + ... + B_n$, where $\{B_i\}$ are i.i.d. symmetric Bernoulli random variables, i.e. B_i take values ± 1 with probability $1/2$ each.

The condition (3.11) holds with $\delta > 2$ if $\alpha = \delta/2 > 1$. By our choice, the common distribution function $\mathcal{N}(t)$ of the $U_g(\omega)$, $g \in G$, is the n-fold convolution of the distribution function $\mathcal{N}_\lambda(t)$, $\lambda = p^{-\alpha}$, of the infinite Bernoulli convolution defined at (4.11). Since $\alpha > 1$, we have $0 < \lambda < 1/2$. Therefore $\mathcal{N}_\lambda(t)$ is purely singular. On the other hand, by [29], Proposition 6.1, the Fourier transform $\Phi_\lambda(x)$ of the function $\mathcal{N}_\lambda(t)$ satisfies

$$|\Phi_\lambda(x)| \le \frac{C}{1 + |x|^\gamma},$$

for some $C = C(\lambda) > 0$ and $\gamma = \gamma(\lambda) > 0$, and for almost all $0 < \lambda < 1$. Thus we can choose $\alpha > 1$ such that $\lambda = p^{-\alpha}$ does not belong to the exceptional set and then $n = n(\alpha)$ big enough such that the Fourier transform $\Phi(x)$ of the function $\mathcal{N}(t)$ satisfies

$$|\Phi(x)| = |\Phi_\lambda(x)|^n \le \frac{C'}{1 + |x|^2},$$

for some $C' > 0$. In particular, according to our choice, $\mathcal{N}(t)$ is absolutely continuous and has a continuous density. This shows that the density of states *exists* and is a *continuous* function, whereas ε is *discrete.* Thus $\mathfrak{B}^\alpha(\omega)$ with appropriately chosen $\alpha > 1$ and $\varepsilon(\omega)$ is as desired.

Now let us return to our general setting and prove that the Poisson convergence fails. Without loss of generality we can assume that $\varepsilon(\{0\}) := p_0 > 0$. We also assume that all forward degrees n_j are the same and equal n. Keeping all the notation from the proof of Theorem 5.1 we write

$$U_g = \sum_{i=0}^{\infty} a_i \varepsilon_{ig} = a_0 \varepsilon_{0g} + \widetilde{U_g}$$

and

$$\begin{aligned} Z_l^c &= \sum_{g\in G_l} 1_{\{U_g \in I_l^c\}} \geq \sum_{g \in G_l} 1_{\{\widetilde{U_g} \in I_l^c\}} 1_{\{\varepsilon_{0g}=0\}} \\ &= \sum_{g \in G_l/G_1} 1_{\{U_g \in I_{l-1}^{c/n}\}} \sum_{a \in g} 1_{\{\varepsilon_{0a}=0\}}. \end{aligned} \tag{5.18}$$

For each $g \in G_l/G_1$ we define a random variable

$$\mathcal{B}_g = \sum_{a\in g} 1_{\{\varepsilon_{0a}=0\}}.$$

The $\{\mathcal{B}_g\}_{g \in G_l/G_1}$ are i.i.d. Binomial random variables with parameters (p_0, n). Further, setting

$$\widetilde{Z}_{l-1}^{c/n} = \sum_{g \in G_l/G_1} 1_{\{U_g \in I_{l-1}^{c/n}\}}$$

and

$$\mathcal{Z}_l^c = \sum_{g \in G_l/G_1} 1_{\{U_g \in I_{l-1}^{c/n}\}} \sum_{a\in g} 1_{\{\varepsilon_{0a}=0\}},$$

we will obtain

$$\mathcal{Z}_l^c \stackrel{d}{=} \sum_{j\geq 0}^{\tau} \mathcal{B}_j,$$

where $\{\mathcal{B}_i\}_{i=0}^\infty$ are i.i.d. Binomial random variables with parameters (p_0, n) which are independent of $\tau = \widetilde{Z}_{l-1}^{c/n}$.

If we assume that the Poisson convergense holds then, as in the proof of Theorem 5.1, we can choose a subsequence $\{l_k\}$ such that along this subsequence $Z_l^c \to Z^c$ and $\widetilde{Z}_{l-1}^{c/n} \to \widetilde{Z}^{c/n}$, where both Z^c and $\widetilde{Z}^{c/n}$ are Poisson random variables with intensities λ_c and $\lambda_{c/n}$ respectively. In particular, by (5.18), we will have

$$\mathbb{P}(Z^c \geq 2) \geq \mathbb{P}(\widetilde{Z}^{c/n} \geq 1)\mathbb{P}(\mathcal{B}_0 \geq 2).$$

Contradiction, because as $c \to 0$ the left-hand-side of the above inequality is of order c^2 whereas the right-hand-side is of order c.

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