Heat kernels on manifolds and fractals

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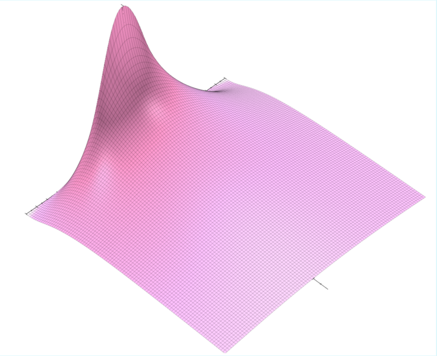
Heat kernel in $\mathbb{R}^n$

The heat equation in $\mathbb{R}^n$: $\partial_t u = \Delta u$ where $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\Delta = \sum_{i=1}^n \partial_{x_i x_i} u$ is the Laplace operator. The *Gauss-Weierstrass* function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \qquad (GW)$$



satisfies the heat equation in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and tends to δ_y as $t \to 0+$.

The function (GW) is called the *heat kernel* or the *fundamental solution* of the heat equation.

Other characterizations of the heat kernel:

=the integral kernel of the heat semigroup $\left\{e^{t\Delta}\right\}_{t\geq 0}$ in $L^2(\mathbb{R}^n)$;

=the density of the normal distribution with the mean y and variance $2t$;

=the transition density of Brownian motion in $\mathbb{R}^n$.

Heat kernel on a manifold

Let M be a geodesically complete Riemannian manifold and Δ - the Laplace-Beltrami operator on M. In the local coordinates it has the form

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^j} \right).$$

Let $p_t(x, y)$ be the heat kernel of M, that is, the smallest positive fundamental solution of $\partial_t u = \Delta u$ on $\mathbb{R}_+ \times M$, where $u = u(t, x)$, $t \in \mathbb{R}$, $x \in M$.

Problem: obtaining estimates of $p_t(x, y)$ *depending on the geometry of* M.

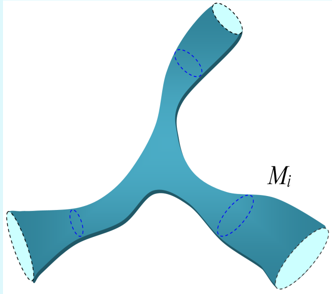
A theorem of Li and Yau 1986 states: if $Ricci_M \geq 0$ then

$$p_t(x, y) \asymp \frac{C}{V(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{ct} \right). \qquad (LY)$$

Here $d(x, y)$ is the geodesic distance, $V(x, r) = \mu(B(x, r))$ is the Riemannian volume of the geodesic ball $B(x, r)$, C, c are positive constants, and $\asymp$ means that both $\leq$ and $\geq$ take place, but with different values of C, c. For example, in $\mathbb{R}^n$ we have $V(x, r) = c_n r^n$, and (LY) matches (GW).

Heat kernels on manifolds with ends

Let M be a connected sum of other manifolds $M_1, ...M_k$. We write $M = M_1 \# ... \# M_k$.
The manifolds M_i are called *ends* of M.
Assuming that the heat kernel on each end M_i satisfies (LY), we ask the following question:

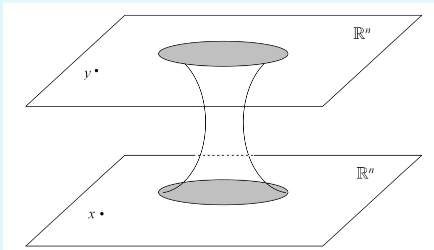


how to estimate the heat kernel $p_t(x, y)$ *on* M?

Some results were obtained in a series of papers of A.Grigor'yan and L.Saloff-Coste from 1999 to 2018.

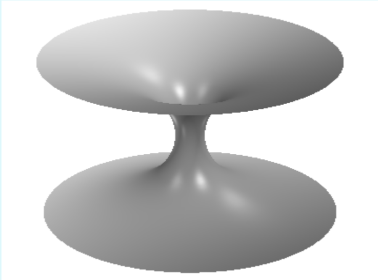
For example, let $M = \mathbb{R}^n \# \mathbb{R}^n$ with $n > 2$.
Let x, y lie on different sheets. Then

$$p_t(x, y) \asymp \frac{C}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}} \right)$$



$\mathbb{R}^n \# \mathbb{R}^n$

Consider now $M = \mathbb{R}^2 \# \mathbb{R}^2$ or, equivalently, the catenoid



Let x, y again lie on different sheets. If $|x|, |y| \leq \sqrt{t}$ then the heat kernel on M satisfies

$$p_t(x,y) \asymp \frac{C}{t \ln^2 \sqrt{t}} \left(\ln \sqrt{t} + \ln^2 \sqrt{t} - \ln |x| \ln |y| \right).$$

If $|x|, |y| \geq \sqrt{t}$ then

$$p_t(x,y) \asymp \frac{C}{t} \left(\frac{1}{\ln |x|} + \frac{1}{\ln |y|} \right) e^{-\frac{d^2(x,y)}{ct}}.$$

Weighted manifolds

Consider in $\mathbb{R}^n$ the measure $e^{\psi(x)}\,d\mu$ where μ is the Lebesgue measure in $\mathbb{R}^n$ and ψ is a given function. The new measure determines a *weighted Laplacian*

$$L = e^{-\psi}\,\mathrm{div}\left(e^{\psi}\nabla\right) = \Delta + \nabla\psi\cdot\nabla$$

that is symmetric with respect to $e^{\psi}d\mu$. The function ψ may degenerate to $-\infty$ or to $+\infty$ thus creating singularities of L.

Question: how obtain estimates of the heat kernel of L near singularities?

A model example with

$$\psi(x) = -\frac{1}{|x|^{\alpha}}$$

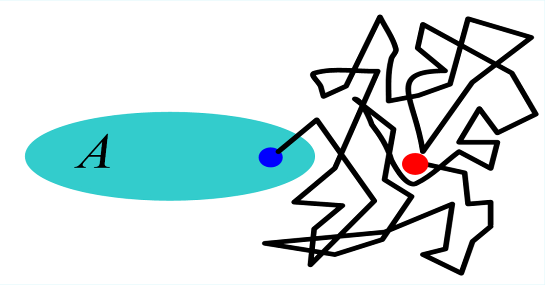
(where $\alpha > 0$) was partially solved in a paper of A.Grigor'yan, S.Ouyang and M.Röckner 2018: it was proved that, for $0 < t < 1$,

$$\sup_x p_t(x,x) \asymp C\exp\left(\frac{c}{t^{\frac{\alpha}{\alpha+2}}}\right).$$

However, optimal off-diagonal estimates of the heat kernel are not yet known, even in this model case.

Jump processes in $\mathbb{R}^n$

The Gauss-Weierstrass function (GW) serves as the transition density of Brownian motion $\{X_t\}$ in $\mathbb{R}^n$. That is,



$$\mathbb{P}_x\left(X_t \in A\right) = \int_A p_t\left(x,y\right) dy.$$

Let $\{Y_t\}$ be the Levy process in $\mathbb{R}^n$ generated by $(-\Delta)^{\beta/2}$ where $\beta \in (0,2)$. This is a jump process that is called *symmetric stable process of index* β. Its transition density $p_t\left(x,y\right)$ satisfies the estimate

$$p_t\left(x,y\right) \asymp \frac{C}{\left(t^{1/\beta} + |x-y|\right)^{n+\beta}} = \frac{C}{t^{n/\beta}}\left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}.$$

In a particular case $\beta = 1$ we obtain the Cauchy distribution:

$$p_t\left(x,y\right) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}} = \frac{c_n}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}} \quad \left(\text{where } c_n = \Gamma\left(\tfrac{n+1}{2}\right)/\pi^{(n+1)/2}\right)$$

Diffusions and jump processes on fractal-like spaces

Let (M, d, μ) be a metric measure space with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$. The heat kernel is defined as the integral kernel of the corresponding heat semigroup. We look for equivalent conditions for the following heat kernel bounds:

- *Sub-Gaussian* bounds in the case of diffusion (=local Dirichlet form):

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right). \tag{1}$$

- *Stable-like* bounds in the case of jump process (=non-local Dirichlet form):

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{2}$$

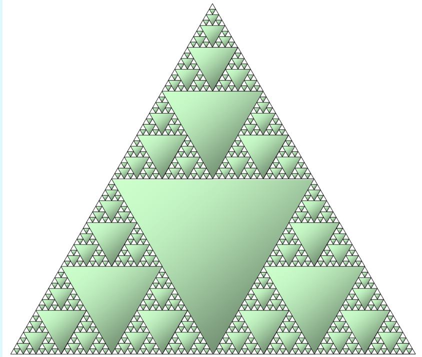
Here α has to be the Hausdorff dimension of (M, d), while β is called the *walk dimension* (in the case (1)) or the *index* (in the case (2)).

Note that the walk dimension is an invariant of the metric space (M, d) alone!

There are many reasons to consider such estimates. Firstly, they are known to hold on various fractals, like the Sierpinski gasket or carpet. Secondly, if the heat kernel satisfies a self-similar estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

with some function Φ then this estimate has to be either (1) or (2) (A.Grigor'yan & T.Kumagai, 2008)



Sierpinski gasket

Sub-Gaussian estimate. The major question to be addressed here is to find some practical conditions on (M, d, μ) and $(\mathcal{E}, \mathcal{F})$ that should be equivalent to (1). On a complete Riemannian manifold the Gaussian heat kernel bounds (that corresponds to $\beta = 2$ in (1)) are equivalent to the conjunction of the following two properties:

- the volume regularity $\mu(B(x,r)) \asymp Cr^\alpha$, where $B(x,r)$ are d-balls;
- the Poincaré inequality

$$\int_{B(x,2r)} |\nabla f|^2 \, d\mu \geq \frac{c}{r^2} \inf_{\xi \in \mathbb{R}} \int_{B(x,r)} (f-\xi)^2 \, dx.$$

In the general setting (including the case $\beta > 2$ that typically occurs in fractals), we have a conjecture.

Conjecture. The sub-Gaussian heat kernel estimate (1) is equivalent to the conjunction of the three properties:

- the volume regularity $\mu\left(B\left(x,r\right)\right) \asymp Cr^{\alpha}$;
- the β-Poincaré inequality

$$\int_{B(x,2r)} d\Gamma\left(f,f\right) \geq \frac{c}{r^{\beta}} \inf_{\xi\in\mathbb{R}} \int_{B(x,r)} \left(f-\xi\right)^2 dx \ ;$$

- the capacity estimate

$$\operatorname{cap}(B\left(x,r\right), B\left(x,2r\right)) \leq Cr^{\alpha-\beta},$$

where cap is the variational capacity associated with $(\mathcal{E},\mathcal{F})$.

A similar equivalence has been proved by A.Grigor'yan, J.Hu and K.-S.Lau 2015 but with a more complicated third condition that involves a *generalized capacity* containing an additional weight function. However, the generalized capacity condition is difficult to verify in applications.

Stable-like estimates. The question about equivalent conditions arises also for the stable-like estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{3}$$

Conjecture. Estimate (3) is equivalent to the conjunction of three conditions:

- the volume regularity $\mu(B(x,r)) \asymp Cr^{\alpha}$;

- the estimate of the jump kernel J of the Dirichlet form $(\mathcal{E},\mathcal{F})$

$$J(x,y) \asymp \frac{C}{d(x,y)^{\alpha+\beta}},$$

(similarly to the jump kernel of the symmetric β-stable process in $\mathbb{R}^n$);

- the capacity condition

$$\operatorname{cap}(B(x,r), B(x,2r)) \leq Cr^{\alpha-\beta}.$$

A similar equivalence has been proved by A.Grigor'yan, E.Hu and J.Hu 2018 assuming the third condition with a generalized capacity in place of capacity (like in the local case).

Construction of jump processes on fractals. Let (M, d, μ) be a metric measure space, and assume that measure μ is α-regular. For any $\beta > 0$ define the following bilinear form in $L^2(M, \mu)$:

$$\mathcal{E}_\beta(f, g) = \int_M \int_M \frac{(f(x) - f(y))(g(x) - g(y))}{d(x, y)^{\alpha+\beta}} d\mu(x) d\mu(y). \tag{4}$$

The following question arises immediately: for which β the bilinear form $\mathcal{E}_\beta$ with a proper domain is a regular Dirichlet form?

If so then $\mathcal{E}_\beta$ defines on M a jump process of the index β.

In $\mathbb{R}^n$ this is the case if and only if $\beta < 2$, while on the Sierpinski gasket $\mathcal{E}_\beta$ is a regular Dirichlet form if and only if $\beta < \beta_*$, where $\beta_* = \frac{\ln 5}{\ln 2}$ is the walk dimension of SG. This result is obtained by using subordination techniques, which requires a priori construction of a diffusion on SG and the sub-Gaussian heat kernel estimates.

Problem. To develop tools for construction of jump type Dirichlet forms on metric measure spaces, *without* using diffusion. It is particularly important to obtain possible values of the index β and to understand the nature of the supremum value β_* of the index, because β_* is an invariant of (M, d).