HEAT KERNEL ESTIMATES FOR AN OPERATOR WITH A SINGULAR DRIFT AND ISOPERIMETRIC INEQUALITIES

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ABSTRACT. We prove upper and lower bounds of the heat kernel for the operator $\Delta - \nabla(\frac{1}{|x|^\alpha}) \cdot \nabla$ in $\mathbb{R}^n \setminus \{0\}$ where $\alpha > 0$. We obtain these bounds from an isoperimetric inequality for a measure $\mathrm{e}^{-\frac{1}{|x|^\alpha}} dx$ on $\mathbb{R}^n \setminus \{0\}$. The latter amounts to a certain functional isoperimetric inequality for the radial part of this measure.

CONTENTS

1. INTRODUCTION

Consider the following differential operator $\mathcal{L} = \Delta + \nabla\psi \cdot \nabla$ defined on $M := \mathbb{R}^n \setminus \{0\}$, with a singular potential

$$\psi(x) = -\frac{1}{|x|^\alpha}, \quad \alpha > 0.$$

The purpose of this paper is to obtain uniform bounds for the heat kernel $p_t(x, y)$ of $\mathcal{L}$ that would take into account the singularity of ψ at the origin. In order to define what is the heat kernel of $\mathcal{L}$ let us observe that $\mathcal{L}$ can be written in the form

$$\mathcal{L} = \mathrm{e}^{-\psi}\mathrm{div}(\mathrm{e}^{\psi}\nabla)$$

which implies that $\mathcal{L}$ is symmetric with respect to the following measure:

$$d\mu(x) = \mathrm{e}^{\psi(x)}\, dx = \mathrm{e}^{-\frac{1}{|x|^\alpha}}\, dx. \tag{1.1}$$

That is, the operator $\mathcal{L}$ is formally self-adjoint on $L^2 = L^2(M, \mu)$. Following the terminology of [10], $\mathcal{L}$ is the Laplace operator of the weighted manifold[1] (M, μ). Using the Friedrichs extension of this operator, one defines the associated heat semigroup $P_t = \mathrm{e}^{t\mathcal{L}}$,

Date: March 1, 2015.

2010 *Mathematics Subject Classification.* 58J35, 26D10, 28A75.

Key words and phrases. Isoperimetric inequalities, functional inequalities, weighted measure, singular drift.

Supported by SFB 701 of the German Research Council.

[1]A weighted manifold is a couple (M, μ) where M is a Riemannian manifold and μ is a measure on M with a smooth positive density with respect to the Riemannian measure.

$t \geq 0$, acting in L^2. The heat kernel of $\mathcal{L}$ is then the integral kernel of P_t, that is, a function $p_t(x, y)$ defined on $\mathbb{R}_+ \times M \times M$ such that, for all $f \in L^2$, $t \geq 0$, $x \in M$,

$$P_t f(x) = \int_M p_t(x, y) f(y)\, d\mu(y).$$

By general regularity theory, the heat kernel always exists and is a smooth positive function of (t, x, y) (cf. [10, 11]).

The motivation for considering heat kernels of operators as $\mathcal{L}$ with singular drift comes from [13], where global existence and uniqueness of strong solutions for stochastic differential equations (SDE) with singular drifts was proved. The most important applications are the analysis of particle systems with physically realistic, hence singular interactions (cf. [13, Section 9]). One example is a diffusion in a frozen random environment given by a countable set γ of particles in $\mathbb{R}^n$, distributed according to a Ruelle Gibbs measure, i.e. the diffusion solves the SDE

$$dX(t) = b(X(t))dt + dW(t),$$

with

$$b(x) := -\sum_{y\in\gamma} \nabla V(x - y), \quad x \in \mathbb{R}^n,$$

and $V\colon \mathbb{R}^n \to \mathbb{R}$ is a pair potential describing the interaction of the moving particle $X(t)$, $t \geq 0$, with those in γ. V is typically very singular at $x = 0$ (e.g. of Lenard-Jones type) modelling the strong repulsion between two particles. One of the main and most interesting open questions about the solution $X(t)$, $t \geq 0$, is whether (depending on the location of the points in γ and the strength of the singularity of V) it exhibits sub- or super-diffusive behavior. So, a good way to start is to examine the heat kernel of the corresponding generator $\mathcal{L}_b = \Delta + \langle b, \nabla\rangle$, which is symmetric on $L^2\left(\mathbb{R}^n, \exp\left(-\sum_{y\in\gamma} V(x-y)dx\right)\right)$. Therefore, in this paper, as a first step, we study the model case described above, where $b = \nabla\psi$ and we have only one particle, i.e. $\gamma = \{0\}$.

Our main results — Theorems 6.2 and 7.3 below, provide the following bounds for the heat kernel of $\mathcal{L}$ for all $0 < t < 1$:

$$\sup_{x,y} p_t(x, y) \leq C \exp\left(\frac{C}{t^{\frac{\alpha}{\alpha+2}}}\right) \tag{1.2}$$

and

$$\sup_{x} p_t(x, x) \geq c \exp\left(\frac{c}{t^{\frac{\alpha}{\alpha+2}}}\right) \tag{1.3}$$

where C, c are some positive constants. It is important that these estimates correctly capture the term $\exp\left(\frac{\text{const}}{t^{\frac{\alpha}{\alpha+2}}}\right)$, describing the short time on-diagonal behavior of the heat kernel, that is determined by the singularity of the drift.

Presently a variety of methods are available for obtaining heat kernel estimates. A challenging feature of the above problem is that the methods based on the curvature bounds fail here (cf. [15]). We use instead the approach developed by the first-named author [11, 8, 10] that is based on isoperimetric and Faber-Krahn inequalities. Given a weighted manifold (M, μ) and a function $\Lambda : (0, +\infty) \to [0, +\infty)$, we say that (M, μ) satisfies the Faber-Krahn inequality with function Λ if, for any precompact open set $U \subset M$, the following inequality holds

$$\lambda_1(U) \geq \Lambda(\mu(U)), \tag{1.4}$$

where $\lambda_1(U)$ denotes the bottom of the spectrum of $\mathcal{L}$ in $L^2(U, \mu)$ with the Dirichlet boundary condition on ∂U. By a result of [8], the Faber-Krahn inequality implies a certain upper bound of the heat kernel.

For any Borel set $A \subset M$, define its perimeter $\mu^+(A)$ by

$$\mu^+(A) = \liminf_{r \to 0^+} \frac{\mu(A^r) - \mu(A)}{r},$$

where A^r is the r-neighborhood of A with respect to the Riemannian metric of M.

By [8, Proposition 2.4], if for any precompact open set $U \subset M$ with smooth boundary,

$$\mu^+(U) \geq J(\mu(U)), \tag{1.5}$$

where J is a function on $[0, +\infty)$, such that $\frac{J(v)}{v}$ is monotone decreasing, then the Faber-Krahn inequality holds with the function

$$\Lambda(v) = \frac{1}{4} \left(\frac{J(v)}{v} \right)^2.$$

We say, that J is a lower isoperimetric function of μ if (1.5) is satisfied for all Borel sets $U \subset M$; and that J is a lower isoperimetric function of μ of restricted type, if (1.5) is satisfied for all precompact open sets $U \subset M$ with smooth boundaries. The latter is sufficient for obtaining Faber-Krahn inequality.

Our main technical result, Theorem 5.3, yields the following lower isoperimetric function of the measure (1.1) of restricted type:

$$J(v) = C\, v \left(\log \frac{1}{v} \right)^{1+\frac{1}{\alpha}}$$

for small enough values of v, which then leads to the upper bound (1.2) of the heat kernel.

The lower bound (1.3) is obtained in Theorem 7.3 using the fact that the Faber-Krahn inequality (1.4) is sharp (up to constant multiple) on the balls centered at the origin.

Let us recall some previous results on isoperimetric inequalities (for more information on this active field, we refer the reader to [1, 3, 12, 17] and the references therein). For any weighted manifold (M, μ) let I_μ denote the isoperimetric function of μ, that is, the largest possible lower isoperimetric function. For some specific measures on Euclidean space, the respective isoperimetric functions are known exactly. For example, the isoperimetric function for the Lebesgue measure λ in $\mathbb{R}^n$ is given by

$$I_\lambda(v) = n\omega_n^{1/n} v^{(n-1)/n},$$

where ω_n is the $(n-1)$-volume of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

Due to the celebrated result of Borell [4] and Sudakov-Tsirel'son [18], the isoperimetric function I_{γ^n} of the Gaussian measure

$$\gamma^n(dx) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{2} \right) dx$$

satisfies

$$I_{\gamma^n}(v) \geq c\,(v \wedge (1-v)) \sqrt{\log \frac{1}{v \wedge (1-v)}},$$

where $c > 0$ is some constant independent of n.

Various generalizations of this result have been studied. In particular, in [12] a lower bound is given for the isoperimetric function of the probability measure

$$\nu^{n,\alpha}(dx) := \frac{1}{Z_{n,\alpha}} \mathrm{e}^{-|x|^\alpha}\, dx \tag{1.6}$$

on $\mathbb{R}^n$ with $\alpha \geq 1$ (where $Z_{n,\alpha}$ is the normalization constant such that $\nu^{n,\alpha}(\mathbb{R}^n) = 1$):

$$I_{\nu^{n,\alpha}}(v) \geq Cn^{\frac{1}{2}-\frac{1}{\alpha}}(v \wedge (1-v)) \left(\log \frac{1}{v \wedge (1-v)}\right)^{1-\frac{1}{\alpha \wedge 2}}$$

for some constant $C > 0$ independent of n.

Note that all measures in $\mathbb{R}^n$ mentioned above are spherically symmetric, so that they can be split into a product of an one dimensional measure in the radial direction and the canonical measure on $\mathbb{S}^{n-1}$ in the angular direction. The isoperimetric function of the measure on $\mathbb{S}^{n-1}$ is classical. The isoperimetric inequality for the radial part of the measure μ is also straightforward. Gluing the radial and angular isoperimetric inequalities presents certain challenges[2]. For that purpose, we use a so called *functional* isoperimetric inequality. Such inequality was proved for the Gaussian measure by Bobkov [2] and for the measure (1.6) by Huet [12]. This inequality enjoys the following distinctive feature: if it is known in the radial and angular directions, it implies easily an isoperimetric inequality in the whole $\mathbb{R}^n$.

Hence, the main problem that we face on this road to the goal is obtaining the functional isoperimetric inequalities separately for radial and angular parts of the measure $d\mu(x) = \mathrm{e}^{-\frac{1}{|x|^\alpha}}dx$ (i.e. (1.1)). For the angular part, we do it in Theorem 4.3 using [1, Theorem 2] (quoted in Theorem 4.1).

The methods previously used for the measures γ^n and $\nu^{n,\alpha}$ do not work for the measure $d\mu(x) = \mathrm{e}^{-\frac{1}{|x|^\alpha}}dx$, as they require the measure μ to be finite. We have developed a new method that constitutes the most interesting part of this paper and is presented in Theorem 2.1 (and its application to the radial part ν of the measure μ is given in Theorem 3.4). The main difficulty lies in obtaining the functional isoperimetric inequality for the radial part of μ.

Theorem 2.1 may be used in the future work to obtain isoperimetric inequality for a more general radial measure on $\mathbb{R}^n$, which would lead to the estimates of the heat kernel of $\mathrm{e}^{t\mathcal{L}_b}$ with a more general potential.

The organization of this paper follows the above scheme of the proof. In Section 2 we deduce a functional isoperimetric inequality for measures on $\mathbb{R}_+$ from the normal isoperimetric inequality. In Section 3 we obtain the functional isoperimetric inequality for the radial part of the measure μ. In Section 4 we verify the functional isoperimetric inequality for the canonical measure on the unit sphere. In Section 5 we combine these two inequalities to obtain a full functional isoperimetric inequality for the measure μ and, hence, the isoperimetric inequality for μ. Finally, in Section 6 we apply our isoperimetric inequality to obtain the heat kernel upper estimate, and in Section 7 we prove the lower estimate.

Notation. 1. For any two nonnegative functions f, g, the relation $f \approx g$ means that f and g are comparable, that is, there exists a constant $C > 0$ such that

$$\frac{1}{C}g \leq f \leq Cg$$

for a specified range of the arguments of f, g.

2. Letter C, C_1, C_2, C' etc. are used to denote various positive constants whose values can change at each occurrence, unless otherwise specified.

3. We frequently use the function $I(v) = v\left(\log \frac{1}{v}\right)^\beta$ defined for $0 < v \leq 1$. Since $\lim_{v\to 0} I(v) = 0$, we always assume without further explanation that this function is extended to all $0 \leq v \leq 1$ by setting $I(0) = 0$.

[2]Isoperimetric inequality for the Riemannian product of Riemannian manifold was proved in [7, 14], but these results do not apply in our cases.

2. One-dimensional functional isoperimetric inequalities

In this section we prove the following theorem that is the key to our main result.

Theorem 2.1. *Let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative continuous function on $\mathbb{R}_+$ and consider the Borel measure $d\nu(r) = \phi(r)\,dr$ on $\mathbb{R}_+$. Let I, J, K, L be four nonnegative continuous functions on $\mathbb{R}_+$ with the following properties:*

(*i*) *For all $a, b \geq 0$,*

$$I(ab) \leq bJ(a) + K(aL(b)); \tag{2.1}$$

(*ii*) *J is a lower isoperimetric function for the measure ν;*
(*iii*) *K is non-decreasing and concave;*
(*iv*) *L is concave.*

Then, for all nonnegative continuously differentiable functions f on $\mathbb{R}_+$ with bounded support, we have

$$I\left(\int_{\mathbb{R}_+} f\,d\nu\right) \leq K\left(\int_{\mathbb{R}_+} L(f)\,d\nu\right) + \int_{\mathbb{R}_+} |f'|\,d\nu. \tag{2.2}$$

Remark. The conditions and statement of Theorem 2.1 are similar to that of [1, Theorem 2] (cf. Theorem 4.1 below). The difference is that [1, Theorem 2] works with probability measures on arbitrary spaces, while Theorem 2.1 applies for general measures on the half-line. The main difference is that Theorem 2.1 is suited to measures of infinite mass. Thus we develop a different method. The approach of the proof for [1, Theorem 2] was an extension of Bobkov's technique and combined the co-area formula and convexity type argument in the set of probability measures. The method in the present paper does not use co-area formula (i.e. horizontal slicing of graphs of functions) but rather a sort of equipartitions of the base space (i.e. vertical slicing).

In this paper we shall only use the special case of Theorem 2.1 when $J = L = \mathrm{const}\, I$ and $K = \mathrm{id}$. For convenience of the reader, let us state Theorem 2.1 in this case.

Theorem 2.2. *Let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative continuous function on $\mathbb{R}_+$ and consider the Borel measure $d\nu(r) = \phi(r)\,dr$ on $\mathbb{R}_+$. Let I be a nonnegative function on $\mathbb{R}_+$ with the following properties:*

(*i*) *For some constant $C > 0$ and for all $a, b \geq 0$,*

$$CI(ab) \leq bI(a) + aI(b). \tag{2.3}$$

(*ii*) *I is a concave lower isoperimetric function for ν.*

Then, for all nonnegative continuously differentiable functions f on $\mathbb{R}_+$with bounded support, we have

$$CI\left(\int_{\mathbb{R}_+} f\,d\nu\right) \leq \int_{\mathbb{R}_+} I(f)\,d\nu + \int_{\mathbb{R}_+} |f'|\,d\nu. \tag{2.4}$$

The proof of Theorem 2.1 will consist of a series of lemmas. In fact, we shall prove an extension of (2.2) for a class of step functions f. Let f be a real-valued function on $\mathbb{R}_+$ with bounded support. Define the *weighted total variation* of f with respect to the measure ν by

$$V_\nu(f) = \sup_{\{\xi_0, \xi_1, \cdots, \xi_n\}} \sum_{k=1}^{n} |f(\xi_k) - f(\xi_{k-1})| \phi(\xi_{k-1}),$$

where sup is taken over all finite increasing sequences $\{\xi_0, \xi_1, \cdots, \xi_n\}$ of nonnegative reals with arbitrary $n \in \mathbb{N}$ such that $\mathrm{supp} f \subset [\xi_0, \xi_n]$. For example, if f is continuously differentiable then

$$V_\nu(f) = \int_{\mathbb{R}_+} |f'|\, d\nu.$$

A function f on $\mathbb{R}_+$ is called an *elementary step function* if it has the form

$$f = b\mathbf{1}_{[r,s)}$$

for some real constant b and $0 \le r < s$. A function f on $\mathbb{R}_+$ is called a *step function* if it is a finite sum of elementary step functions. Clearly, any step function can be represented in the following form

$$f = \sum_{k=1}^{n} b_k \mathbf{1}_{[x_{k-1},x_k)}, \tag{2.5}$$

where $0 = x_0 < x_1 < x_2 < \cdots < x_n$, and b_k are real constants. For the step function (2.5) we obviously have

$$V_\nu(f) = \sum_{k=1}^{n} |b_{k+1} - b_k| \phi(x_k),$$

where we set $b_{n+1} = 0$.

For the proof of Theorem 2.1, we shall first prove that any nonnegative step function f satisfies the following inequality

$$I\left(\int_{\mathbb{R}_+} f\, d\nu\right) \le K\left(\int_{\mathbb{R}_+} L(f)\, d\nu\right) + V_\nu(f). \tag{2.6}$$

We start with elementary step functions.

Lemma 2.3. *Under the hypotheses of Theorem 2.1, inequality (2.6) holds for any elementary step function of the form $f = b\mathbf{1}_{[r,s)}$, where $b \ge 0$ and $0 \le r < s$.*

Proof. Let $a = \nu([r,s))$. It is clear that

$$I\left(\int_{\mathbb{R}_+} f\, d\nu\right) = I(b\nu([r,s))) = I(ab)$$

and

$$K\left(\int_{\mathbb{R}_+} L(f)\, d\nu\right) \ge K(L(b)\nu([r,s))) = K(aL(b)).$$

Using that J is a lower isoperimetric function for ν, we obtain, for the case $r > 0$

$$V_\nu(f) = b\left(\phi(r) + \phi(s)\right) = b\nu^+([r,s)) \ge bJ\left(\nu([r,s))\right) = bJ(a),$$

and for the case $r = 0$

$$V_\nu(f) = b\phi(s) = b\nu^+([0,s)) \ge bJ\left(\nu([0,s))\right) = bJ(a).$$

Hence, (2.6) follows from (2.1). ∎

Before we can treat an arbitrary step function, let us prove the following lemma.

Lemma 2.4. *Let $f_1, f_2, \cdots, f_n$ be nonnegative functions on $\mathbb{R}_+$ with bounded supports such that (2.6) holds for all f_k, $k = 1, 2, \cdots, n$. Assume also that*

$$\int_{\mathbb{R}_+} f_1\, d\nu = \int_{\mathbb{R}_+} f_2\, d\nu = \cdots = \int_{\mathbb{R}_+} f_n\, d\nu.$$

Choose a sequence $\{p_k\}_{k=1}^n$ of nonnegative reals such that $\sum_{k=1}^n p_k = 1$, and set

$$f = \sum_{k=1}^{n} p_k f_k. \tag{2.7}$$

If

$$V_\nu(f) = \sum_{k=1}^{n} p_k V_\nu(f_k), \tag{2.8}$$

then (2.6) *holds also for* f*.*

Note that (2.7) implies the inequality

$$V_\nu(f) \le \sum_{k=1}^{n} p_k V_\nu(f_k),$$

whereas the equality (2.8) holds only in specific situations, one of which will be described below.

Proof. It is clear that, for all $k = 1, 2, \cdots, n$, we have

$$\int_{\mathbb{R}_+} f_k\, d\nu = \int_{\mathbb{R}_+} f\, d\nu.$$

By hypotheses, we have, for all $k = 1, 2, \cdots, n$,

$$I\left(\int_{\mathbb{R}_+} f_k\, d\nu\right) \le K\left(\int_{\mathbb{R}_+} L(f_k)\, d\nu\right) + V_\nu(f_k).$$

Using the monotonicity of the function K, the concavity of K and L, and (2.8) we obtain

$$\begin{aligned}
I\left(\int_{\mathbb{R}_+} f\, d\nu\right) &= \sum_{k=1}^{n} p_k I\left(\int_{\mathbb{R}_+} f_k\, d\nu\right) \\
&\le \sum_{k=1}^{n} p_k \left(K\left(\int_{\mathbb{R}_+} L(f_k)\, d\nu\right) + V_\nu(f_k)\right) \\
&\le K\left(\int_{\mathbb{R}_+} L\left(\sum_{k=1}^{n} p_k f_k\right)\, d\nu\right) + \sum_{k=1}^{n} p_k V_\nu(f_k) \\
&= K\left(\int_{\mathbb{R}_+} L(f)\, d\nu\right) + V_\nu(f),
\end{aligned}$$

which was to be proved. ■

Lemma 2.5. *Let* f *be a step function of the following form*

$$f = \sum_{k=1}^{n} b_k \mathbf{1}_{[x_{k-1}, x_k)}, \tag{2.9}$$

where $0 = x_0 < x_1 < x_2 < \cdots < x_n$ *and* $b_k \ge 0$ *for all* $k = 1, 2, \cdots, n$*. Then* f *can be represented in the form*

$$f = \sum_{k=1}^{n} p_k f_k, \tag{2.10}$$

where each f_k *is a nonnegative elementary function and the following relations are satisfied:*

$$\sum_{k=1}^{n} p_k = 1, \quad p_k \ge 0, \tag{2.11}$$

$$\int_{\mathbb{R}_+} f_k\, d\nu = \int_{\mathbb{R}_+} f\, d\nu, \tag{2.12}$$

and

$$V_\nu(f) = \sum_{k=1}^{n} p_k V_\nu(f_k). \tag{2.13}$$

Proof. In the case $n = 1$ we just need to take $f_1 = f$ and $p_1 = 1$. Assume that $n > 1$ and make the induction step from $n - 1$ to n. We can assume that f as in (2.9) is not elementary. For convenience, we set $b_0 = b_{n+1} = 0$. Let b_{k_0} be the maximal value of $\{b_k \colon k = 1, 2, \dots, n\}$. Without loss of generality we assume that

$$b_{k_0-1} \le b_{k_0+1}$$

because the case when $b_{k_0-1} \ge b_{k_0+1}$ can be treated similarly. If $b_{k_0+1} = b_{k_0}$, then we can reduce the number of intervals and use the inductive hypothesis. Hence, we can assume that

$$b_{k_0+1} < b_{k_0}.$$

Let us define a function h as follows

$$h = f\mathbf{1}_{\mathbb{R}_+\setminus[x_{k_0-1},x_{k_0})} + b_{k_0+1}\mathbf{1}_{[x_{k_0-1},x_{k_0})}, \tag{2.14}$$

that is, h is equal to f outside $[x_{k_0-1}, x_{k_0})$ and is equal to b_{k_0+1} on $[x_{k_0-1}, x_{k_0})$.

Define also a function g by

$$g = c\mathbf{1}_{[x_{k_0-1},x_{k_0})},$$

where the constant c is chosen to satisfy the following condition

$$\int_{\mathbb{R}_+} g \, d\nu = \int_{\mathbb{R}_+} f \, d\nu, \tag{2.15}$$

that is,

$$c = \frac{1}{\nu([x_{k_0-1}, x_{k_0}))} \int_{\mathbb{R}_+} f \, d\nu = b_{k_0} + \frac{1}{\nu([x_{k_0-1}, x_{k_0}))} \int_{\mathbb{R}_+\setminus(x_{k_0-1},x_{k_0}]} f \, d\nu.$$

It is clear that $c > b_{k_0}$ since outside $[x_{k_0-1}, x_{k_0})$ the function $f \ge 0$ is not identically zero.

It follows that

$$g > f \text{ on } [x_{k_0-1}, x_{k_0}).$$

On the other hand, we have

$$f = b_{k_0} > b_{k_0+1} = h \text{ on } [x_{k_0-1}, x_{k_0}).$$

Hence, we obtain

$$g > f > h \text{ on } [x_{k_0-1}, x_{k_0}).$$

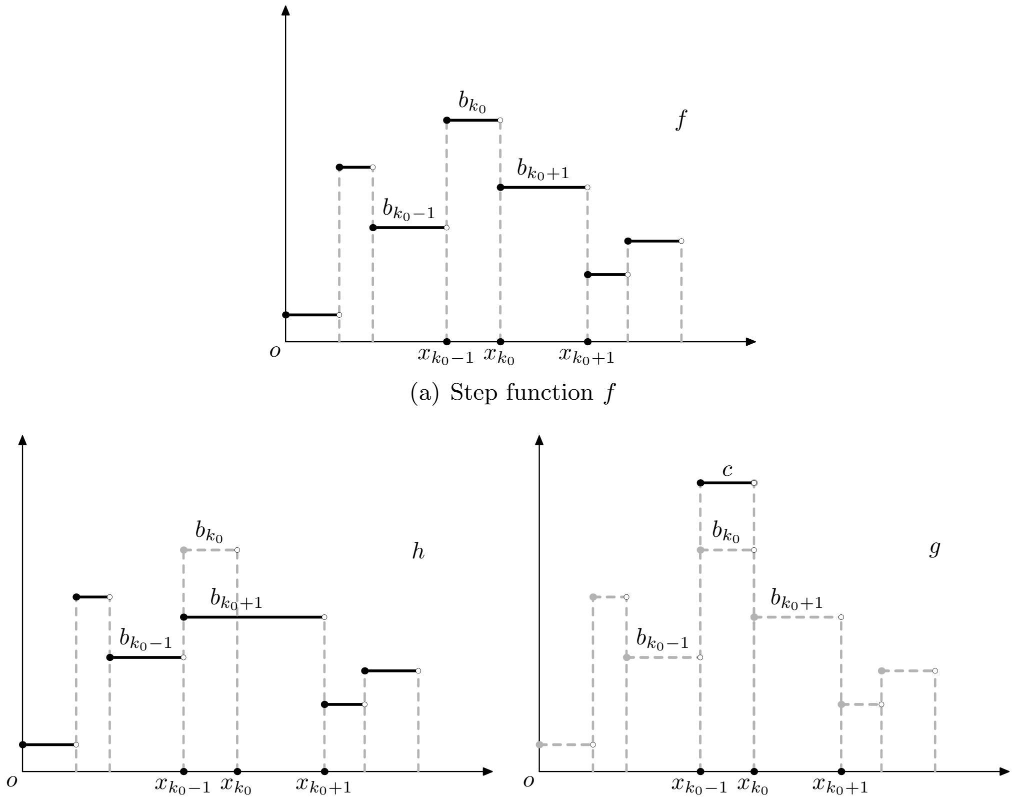
Therefore, there is a constant $p \in (0, 1)$ such that

$$f = pg + h \tag{2.16}$$

on $[x_{k_0-1}, x_{k_0})$. Noting that $h = f$ and $g = 0$ outside $[x_{k_0-1}, x_{k_0})$, we see that (2.16) holds on $\mathbb{R}_+$.

The function h is constant on each interval $[x_{k-1}, x_k)$. On $[x_{k_0-1}, x_{k_0})$ and $[x_{k_0}, x_{k_0+1})$, h is equal to b_{k_0+1}. Therefore, by merging these two intervals, h can be represented as a step function, based on $n - 1$ intervals, that is,

$$h = \sum_{k=1}^{k_0-1} b_k\mathbf{1}_{[x_{k-1},x_k)} + b_{k_0+1}\mathbf{1}_{[x_{k_0-1},x_{k_0+1})} + \sum_{k=k_0+2}^{n} b_k\mathbf{1}_{[x_{k-1},x_k)}.$$



(a) Step function f

(b) Step function h

(c) Step function g

FIGURE 1. Functions $f = pg + h$, h and g

By the induction hypothesis, there exist $n-1$ nonnegative elementary step functions h_i and constants q_i, $i = 1, 2, \cdots, n-1$, such that

$$h = \sum_{i=1}^{n-1} q_i h_i, \tag{2.17}$$

and

$$\sum_{i=1}^{n-1} q_i = 1, \quad q_i \geq 0, \tag{2.18}$$

$$\int_{\mathbb{R}_+} h_i \, d\nu = \int_{\mathbb{R}_+} h \, d\nu, \tag{2.19}$$

$$V_\nu(h) = \sum_{i=1}^{n-1} q_i V_\nu(h_i). \tag{2.20}$$

It follows from (2.17) that

$$f = pg + \sum_{i=1}^{n-1} q_i h_i = pg + \sum_{i=1}^{n-1} q_i (1-p) \frac{h_i}{1-p}.$$

Setting

$$f_n = g, \; p_n = p, \; f_i = \frac{h_i}{1-p}, \quad p_i = q_i(1-p) \;\; \text{for } i = 1, 2, \cdots, n-1,$$

we obtain

$$f = \sum_{k=1}^{n} p_k f_k.$$

Moreover, we have

$$\sum_{k=1}^{n} p_k = p + \sum_{i=1}^{n-1} q_i(1-p) = p + (1-p) = 1,$$

and

$$\int_{\mathbb{R}_+} f_n \, d\nu = \int_{\mathbb{R}_+} g \, d\nu = \int_{\mathbb{R}_+} f \, d\nu.$$

Since by (2.15)

$$\int_{\mathbb{R}_+} h \, d\nu = \int_{\mathbb{R}_+} (f - pg) \, d\nu = (1-p) \int_{\mathbb{R}_+} f \, d\nu,$$

we obtain, for any $k = 1, 2, \cdots, n-1$,

$$\int_{\mathbb{R}_+} f_k \, d\nu = \int_{\mathbb{R}_+} \frac{h_k}{1-p} \, d\nu = \frac{1}{1-p} \left(\int_{\mathbb{R}_+} h \, d\nu \right) = \int_{\mathbb{R}_+} f \, d\nu.$$

By the construction of h and g, at each point x_k the jumps of h and g have the same sign as that of f, so that V_μ acts linearly on the sum $f = h + pg$, consequently

$$V_\nu(f) = V_\nu(h) + pV_\nu(g).$$

By (2.20) we obtain

$$\begin{aligned}
V_\nu(f) &= \sum_{i=1}^{n-1} q_i V_\nu(h_i) + pV_\nu(g) \\
&= \sum_{i=1}^{n-1} \frac{p_i}{1-p} V_\nu(h_i) + p_n V_\nu(f_n) \\
&= \sum_{i=1}^{n-1} p_i V_\nu \left(\frac{1}{1-p} h_i \right) + p_n V_\nu(f_n) \\
&= \sum_{i=1}^{n-1} p_i V_\nu(f_i) + p_n V_\nu(f_n) \\
&= \sum_{i=1}^{n} p_i V_\nu(f_i),
\end{aligned}$$

which finishes the proof. ■

Corollary 2.6. *Under the hypotheses of Theorem 2.1, inequality (2.6) holds for all non-negative step functions on $\mathbb{R}_+$.*

Proof. By Lemma 2.5, we can represent any nonnegative step function f as the sum of nonnegative elementary step functions such that the conditions of Lemma 2.4 are satisfied. Since for any nonnegative elementary function inequality (2.6) holds by Lemma 2.3, we conclude by Lemma 2.4, that f satisfies (2.6). ■

Lemma 2.7. *Let f be a nonnegative continuously differentiable function on $\mathbb{R}_+$ with support in an interval $[0, l]$. Consider the step function*

$$f_n = \sum_{k=1}^{n} f(x_k) \mathbf{1}_{[x_{k-1}, x_k)}, \tag{2.21}$$

where $x_k = \frac{k}{n} l$. *Then the sequence* $\{f_n\}$ *converges to* f *as* $n \to \infty$ *uniformly on* $\mathbb{R}_+$ *and*

$$\lim_{n\to\infty} V_\nu(f_n) = \int_{\mathbb{R}_+} |f'| \, d\nu. \tag{2.22}$$

Proof. The uniform convergence of $\{f_n\}$ to f is obvious. We only need to show (2.22). By the mean value theorem, for every $k = 1, 2, \cdots, n$, there exists some $\xi_k \in [x_k, x_{k+1}]$ such that

$$f(x_{k+1}) - f(x_k) = f'(\xi_k)(x_{k+1} - x_k) = f'(\xi_k)\frac{l}{n}.$$

It follows that

$$V_\nu(f_n) = \sum_{k=1}^{n} |f(x_{k+1}) - f(x_k)|\phi(x_k) = \sum_{k=1}^{n} |f'(\xi_k)|\phi(x_k)\frac{l}{n}. \tag{2.23}$$

Since the function $|f'|\phi$ is Riemann integrable, we have as $n \to \infty$

$$\sum_{k=1}^{n} |f'(x_k)|\phi(x_k)\frac{l}{n} \to \int_{\mathbb{R}_+} |f'|\phi \, dx = \int_{\mathbb{R}_+} |f'| d\nu.$$

On the other hand, we have

$$\left| \sum_{k=1}^{n} |f'(\xi_k)|\phi(x_k)\frac{l}{n} - \sum_{k=1}^{n} |f'(x_k)|\phi(x_k)\frac{l}{n} \right| \le \sup_k |f'(\xi_k) - f'(x_k)| \sum_{k=1}^{n} \phi(x_k)\frac{l}{n}.$$

By the continuity of f', the sup-term on the right hand side tends to 0 as $n \to \infty$. Since the sum-term tends to $\int_0^l \phi(x)dx < \infty$, the whole expression tends to 0, which finishes the proof. ■

Proof of Theorem 2.1. Let f be a nonnegative continuously differentiable function on $\mathbb{R}_+$ with bounded support. Define f_n by (2.21). By Corollary 2.6, inequality (2.6) holds for each function f_n. Letting $n \to \infty$, by Lemma 2.7 we obtain that f satisfies (2.2), which finishes the proof. ■

3. Functional isoperimetric inequality for the radial measure

We here apply Theorem 2.2 to obtain a functional isoperimetric inequality for the measure

$$d\nu(r) = r^{n-1} \mathrm{e}^{-\frac{1}{r^\alpha}} dr \tag{3.1}$$

on $(0, \infty)$, where $\alpha > 0$ and $n \ge 1$. Note that ν is the radial part of the measure

$$d\mu(x) = \mathrm{e}^{-\frac{1}{|x|^\alpha}} dx \tag{3.2}$$

on $\mathbb{R}^n \setminus \{0\}$.

The isoperimetric function for the measure ν can be obtained from the following result (see e.g. [5, Proposition 3.1]).

Proposition 3.1. *Let* ϕ *be a positive continuous non-decreasing function defined on* $(0, +\infty)$. *Consider the Borel measure* $d\nu(r) = \phi(r)\, dr$ *on* $(0, +\infty)$. *Then for any Borel set* $A \subset (0, +\infty)$ *we have*

$$\nu^+(A) \ge \nu^+((0, r)), \tag{3.3}$$

where $r \ge 0$ *is chosen such that*

$$\nu((0, r)) = \nu(A).$$

Furthermore, if $\lim_{r\to 0} \phi(r) = 0$, *then the isoperimetric function* I_ν *is given by the identity*

$$I_\nu(v) = \phi(r),$$

where $v = \nu((0, r))$.

Now we can determine a lower isoperimetric function for the measure defined in (3.1).

Proposition 3.2. *There exist some constants $c, c' > 0$, and $0 < v_0 < 1$ such that the function J, defined by*

$$J(v) = \begin{cases} cv\left(\log \dfrac{1}{v}\right)^{1+\frac{1}{\alpha}}, & 0 \le v \le v_0, \\ c' v^{\frac{n-1}{n}}, & v > v_0, \end{cases} \tag{3.4}$$

satisfies the following properties:

(i) J is a lower isoperimetric function for the measure ν given by (3.1).
(ii) J is concave, increasing and continuous on $(0, +\infty)$.

Remark. As we will see from the proof,

$$v_0 = \mathrm{e}^{-n\left(1+\frac{1}{\alpha}\right)}.$$

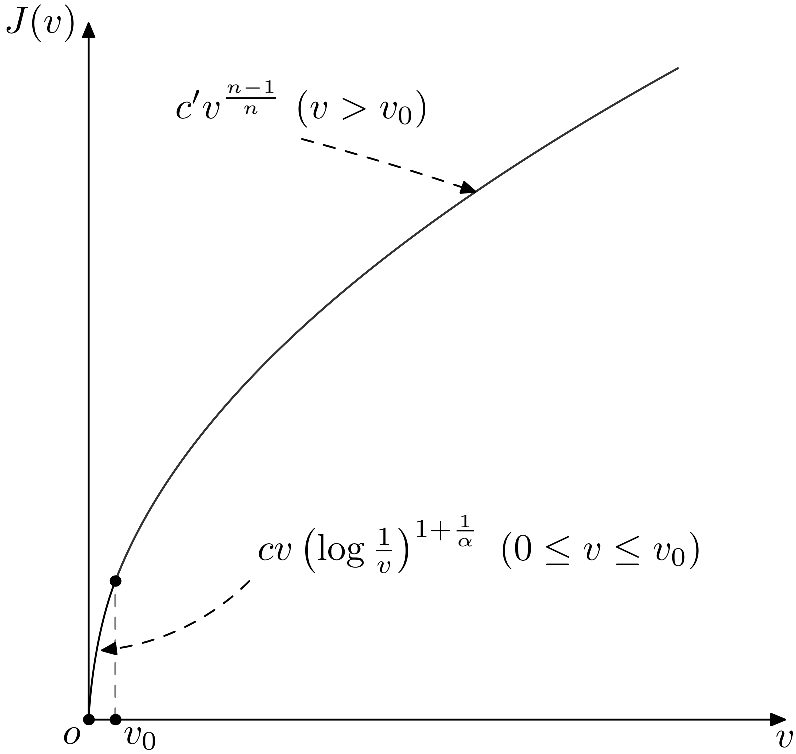


FIGURE 2. Function J defined by (3.4)

Proof. Since the function

$$\phi(r) := r^{n-1} \mathrm{e}^{-\frac{1}{r^\alpha}}$$

is increasing in r, and $\lim_{r\to 0} \phi(r) = 0$, by Proposition 3.1 we obtain

$$I_\nu(v) = \phi(R),$$

where $R > 0$ is such that

$$v = \nu((0, R)) = \int_0^R \phi(r)\, dr = \int_0^R r^{n-1} \mathrm{e}^{-\frac{1}{r^\alpha}}\, dr.$$

It is clear that, for large enough R, we have $\phi(R) \approx R^{n-1}$, consequently $v \approx R^n$. Hence, for large enough v we obtain

$$I_\nu(v) \approx v^{\frac{n-1}{n}}. \tag{3.5}$$

In order to estimate v for small R, we shall use the following claim.

Claim. Let F be a smooth enough positive function on $(0, +\infty)$ such that

$$a := \lim_{x\to+\infty} \frac{F(x)F''(x)}{F'^2(x)} > 0 \quad \text{and} \quad \int_x^\infty \frac{dr}{F(r)} < \infty. \tag{3.6}$$

Then

$$\int_x^\infty \frac{dr}{F(r)} \sim \frac{a^{-1}}{F'(x)} \quad \text{as } x \to \infty. \tag{3.7}$$

Indeed, the estimate (3.7) follows from l'Hospital's rule since

$$\lim_{x\to+\infty} \frac{\int_x^\infty \frac{dr}{F(r)}}{\frac{1}{F'(x)}} = \lim_{x\to+\infty} \frac{\frac{1}{F(x)}}{\frac{F''(x)}{F'^2}} = \lim_{x\to+\infty} \frac{F'^2(x)}{F(x)F''(x)} = a^{-1}.$$

The function $F(x) = x^{n+1} e^{x^\alpha}$ clearly satisfies (3.6), and we obtain for small enough R

$$\begin{aligned} v &= \int_0^R r^{n-1} e^{-r^{-\alpha}} \, dr = \int_{1/R}^\infty \frac{1}{x^{n-1}} e^{-x^\alpha} \frac{1}{x^2} \, dx \\ &= \int_{1/R}^\infty \frac{dx}{x^{n+1} e^{x^\alpha}} \approx \left. \frac{1}{(x^{n+1} e^{x^\alpha})'} \right|_{x=\frac{1}{R}} \\ &\approx R^{n+\alpha} e^{-\frac{1}{R^\alpha}} \end{aligned} \tag{3.8}$$

It follows from (3.8) that

$$\log v \approx (n+\alpha)\log R - \frac{1}{R^\alpha} \approx -\frac{1}{R^\alpha}, \tag{3.9}$$

consequently

$$\phi(R) = R^{n+\alpha} e^{-\frac{1}{R^\alpha}} R^{-(1+\alpha)} \approx v R^{-(1+\alpha)} \approx v \left(\log \frac{1}{v}\right)^{1+\frac{1}{\alpha}}.$$

Hence, for small enough v, we obtain

$$I_\nu(v) \approx v \left(\log \frac{1}{v}\right)^{1+\frac{1}{\alpha}}. \tag{3.10}$$

Combining (3.5) and (3.10), we obtain that the function J from (3.4) is a lower isoperimetric function for the measure ν, for sufficiently small constants $v_0 \in (0,1)$ and $c, c' > 0$.

Consider the functions

$$I(v) = v \left(\log \frac{1}{v}\right)^{1+\frac{1}{\alpha}} \quad \text{and} \quad I_1(v) = c_1 v^{\frac{n-1}{n}}.$$

Let us show that the constants $c_1 > 0$ and $v_0 \in (0,1)$ can be chosen so that the following function

$$\tilde{J}(v) := \begin{cases} I(v), & 0 \le v \le v_0, \\ I_1(v), & v \ge v_0 \end{cases} \tag{3.11}$$

is concave, increasing and continuous on $\mathbb{R}_+$. Then the function $J = \text{const}\,\tilde{J}$ with small enough const > 0 will satisfy both the conditions (i) and (ii).

The function $I_1(v)$ is clearly increasing and concave on $(0,+\infty)$. For the function $I(v)$ we have

$$I'(v) = \left(\log\frac{1}{v}\right)^{\frac{1}{\alpha}}\left(\log\frac{1}{v} - \left(1+\frac{1}{\alpha}\right)\right),$$

$$I''(v) = -\left(1+\frac{1}{\alpha}\right)\frac{1}{v}\left(\log\frac{1}{v}\right)^{\frac{1}{\alpha}-1}\left(\log\frac{1}{v} - \frac{1}{\alpha}\right),$$

so that $I(v)$ is increasing and concave on the interval $(0, \mathrm{e}^{-(1+\frac{1}{\alpha})})$. Now we choose $c_1 > 0$ and $v_0 \le \mathrm{e}^{-(1+\frac{1}{\alpha})}$ so that the function $\tilde{J}$ is of the class $C^1(0,+\infty)$ and, hence, increasing and concave on $(0,+\infty)$. To that end, the following two identities must be satisfied

$$I(v_0) = I_1(v_0),$$

$$I'(v_0) = I_1'(v_0),$$

which yields the following equations for c_1 and v_0:

$$v_0\left(\log\frac{1}{v_0}\right)^{1+\frac{1}{\alpha}} = c_1 v_0^{\frac{n-1}{n}},$$

$$\left(\log\frac{1}{v_0}\right)^{\frac{1}{\alpha}}\left(\log\frac{1}{v_0} - \left(1+\frac{1}{\alpha}\right)\right) = \frac{n-1}{n}c_1 v_0^{-\frac{1}{n}}.$$

Multiplying the second equation by $v_0 \log\frac{1}{v_0}$ and combining this with the first, we obtain

$$\log\frac{1}{v_0} - \left(1+\frac{1}{\alpha}\right) = \frac{n-1}{n}\log\frac{1}{v_0}, \tag{3.12}$$

whence

$$v_0 = \mathrm{e}^{-n\left(1+\frac{1}{\alpha}\right)}. \tag{3.13}$$

The value of c_1 is then trivially determined from the one of the above equations. The proof is finished by the observation that $v_0 \le \mathrm{e}^{-\left(1+\frac{1}{\alpha}\right)}$. ■

Proposition 3.3. *The function J defined by (3.4) satisfies the following property: there exists some constant $C_J > 0$ such that*

$$C_J J(ab) \le bJ(a) + aJ(b) \tag{3.14}$$

for all $a, b \ge 0$.

Proof. If $a = 0$ or $b = 0$ then (3.14) is trivial. Assume in the sequel that $a, b > 0$. Consider the function

$$F(v) = \frac{J(v)}{v} = \begin{cases} c\left(\log\frac{1}{v}\right)^{1+\frac{1}{\alpha}}, & 0 < v \le v_0, \\ c' v^{-\frac{1}{n}}, & v \ge v_0. \end{cases}$$

Obviously (3.14) is equivalent to

$$F(ab) \le C_J^{-1}\left(F(a) + F(b)\right), \tag{3.15}$$

for all $a, b > 0$. Without loss of generality, let us verify (3.14) for $a \le b$. We shall consider the following four cases.

Case 1. Assume that $b \ge 1$. Since F is monotone decreasing and $ab \ge a$, we obtain

$$F(ab) \le F(a) \le F(a) + F(b). \tag{3.16}$$

In all the next cases we assume $b < 1$.

Case 2. Assume that $a \le v_0 \le b$. In this case we have $a^2 \le ab$ and, hence,

$$F(ab) \le F(a^2). \tag{3.17}$$

Since $a^2 < a < v_0$, we have

$$F(a^2) = c\left(\log \frac{1}{a^2}\right)^{1+\frac{1}{\alpha}} = 2^{1+\frac{1}{\alpha}} F(a). \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$F(ab) \le 2^{1+\frac{1}{\alpha}} F(a) \le 2^{1+\frac{1}{\alpha}} \left(F(a) + F(b)\right). \tag{3.19}$$

Case 3. Assume that $v_0 \le a \le b$. In this case we have $ab \ge v_0^2$ and, hence,

$$F(ab) \le F(v_0^2). \tag{3.20}$$

On the other hand, since $a, b < 1$, we have

$$F(a) + F(b) \ge F(1) + F(1) = 2F(1).$$

Combining this with (3.20) we obtain

$$F(ab) \le \frac{F(v_0^2)}{2F(1)} \left(F(a) + F(b)\right). \tag{3.21}$$

Case 4 (main). Assume that $a \le b \le v_0$. Since $ab < v_0$, we obtain

$$\begin{aligned} F(ab) &= c\left(\log \frac{1}{ab}\right)^{1+\frac{1}{\alpha}} = c\left(\log \frac{1}{a} + \log \frac{1}{b}\right)^{1+\frac{1}{\alpha}} \\ &\le 2^{\frac{1}{\alpha}} c\left(\left(\log \frac{1}{a}\right)^{1+\frac{1}{\alpha}} + \left(\log \frac{1}{b}\right)^{1+\frac{1}{\alpha}}\right) \\ &= 2^{\frac{1}{\alpha}} \left(F(a) + F(b)\right) \end{aligned} \tag{3.22}$$

Combining (3.16), (3.19), (3.21) and (3.22) we obtain (3.15) and hence (3.14) with

$$C_J = \min\left(2^{-\left(1+\frac{1}{\alpha}\right)}, \frac{2F(1)}{F(v_0^2)}\right), \tag{3.23}$$

which finishes the proof. ■

By Theorem 2.1 and Propositions 3.2, 3.3 we obtain the following result.

Theorem 3.4. *The function J given by (3.4) is a lower isoperimetric function for the measure $d\nu(r) = r^{n-1} e^{-\frac{1}{r^\alpha}} dr$ on $\mathbb{R}_+$. Moreover, for any nonnegative continuously differentiable function f on $\mathbb{R}_+$ with bounded support we have*

$$C_J J\left(\int_{\mathbb{R}_+} f \, d\nu\right) \le \int_{\mathbb{R}_+} J(f) \, d\nu + \int_{\mathbb{R}_+} |f'| d\nu, \tag{3.24}$$

where C_J is the constant from (3.14).

4. Functional isoperimetric inequality on a sphere

We shall use the following result of [1] about isoperimetric inequalities for probability measures that we state here in a specific setting adapted to our needs.

Theorem 4.1. ([1, Theorem 2]) *Let L be a nonnegative function on $[0,1]$ with the following properties:*

(i) L is continuous, concave and symmetric with respect to $1/2$, and $L(0) = L(1) = 0$.
(ii) For some constant $C_L > 0$ and for all $a, b \in [0,1]$,

$$C_L L(ab) \leq bL(a) + aL(b). \tag{4.1}$$

Let (N, σ) be a weighted manifold and $\sigma(N) = 1$. If L is a lower isoperimetric function for the measure σ, then, for any locally Lipschitz function $f\colon N \to [0,1]$, we have

$$C_L L\left(\int_N f\, d\sigma\right) \leq \int_N L(f)\, d\sigma + \int_N |\nabla f|\, d\sigma. \tag{4.2}$$

Let σ_{n-1} denote the canonical spherical measure on $\mathbb{S}^{n-1}$. Set $\omega_n = \sigma_{n-1}\left(\mathbb{S}^{n-1}\right)$ and consider the normalized spherical measure

$$\tilde{\sigma}_{n-1} = \frac{1}{\omega_n}\sigma_{n-1}.$$

Before we apply Theorem 4.1 to $(\mathbb{S}^{n-1}, \tilde{\sigma}_{n-1})$, we need to construct a function L satisfying appropriate conditions.

Proposition 4.2. *Choose some $\beta > 1$ and $n \geq 1$, set $v_0 = \mathrm{e}^{-n\beta}$ and consider the functions I and L on $[0,1]$ defined by*

$$I(v) = v\left(\log\frac{1}{v}\right)^{\beta} \tag{4.3}$$

and

$$L(v) = c\begin{cases} I(v), & 0 \leq v \leq v_0, \\ I(v_0), & v_0 < v < 1 - v_0, \\ I(1-v), & 1 - v_0 \leq v \leq 1, \end{cases} \tag{4.4}$$

where c is a positive constant. Then L satisfies the following properties:

(i) L is continuous, concave and symmetric with respect to $1/2$.
(ii) If c is sufficiently small, then L is a lower isoperimetric function for the measure σ_{n-1} on $\mathbb{S}^{n-1}$.
(iii) There exists a constant $C_L > 0$ such that

$$C_L L(ab) \leq bL(a) + aL(b) \tag{4.5}$$

for all $0 < a, b < 1$.

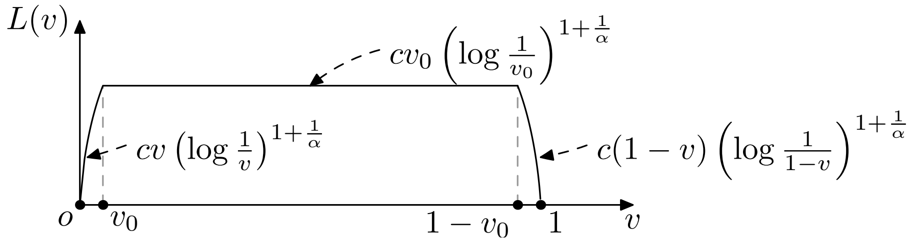


Figure 3. Function L defined by (4.4)

Remark. We shall use Proposition 4.2 with $\beta = 1 + \frac{1}{\alpha}$, where α is the constant in the definitions (3.1) and (1.1) of the measures ν and μ, respectively. By Theorem 3.4 we have a lower isoperimetric function J for the measure μ that is given by (3.4). In the next section we shall combine the isoperimetric functions J and L in order to obtain a lower isoperimetric function of the measure μ. Note that the parameter v_0 in (3.4) and (4.4) has the same value given by (3.13). It will be convenient to assume that the constants c in (3.4) and (4.4) also have the same value, which can always be achieved. Hence, we have

$$J(v) = L(v) = cI\left(v\right), \quad \text{for all } 0 \le v \le v_0. \tag{4.6}$$

Proof. (i) From (4.4) it is clear that L is continuous and symmetric. The concavity follows from Proposition 3.2.

(ii) Set

$$I_{\mathbb{S}^{n-1}}(v) = \begin{cases} c_n v^{\frac{n-2}{n-1}}, & 0 \le v \le 1/2, \\ c_n (1-v)^{\frac{n-2}{n-1}}, & 1/2 < v \le 1, \end{cases}$$

where $c_n > 0$ is a constant. It is well known that $I_{\mathbb{S}^{n-1}}$ is a lower isoperimetric function on $\mathbb{S}^{n-1}$ with respect to σ_{n-1}, provided c_n is sufficiently small.

If $c > 0$ is sufficiently small then we have for all $v \in (0, \frac{1}{2})$

$$cv \left(\log \frac{1}{v}\right)^{\beta} \le c_n v^{\frac{n-2}{n-1}}$$

and, hence, $L\left(v\right) \le I_{\mathbb{S}^{n-1}}\left(v\right)$ for all $v \in (0,1)$. Consequently, L is a lower isoperimetric function.

(iii) If a or b are equal to 0 or 1, then (4.5) is trivially satisfied, so we can assume in the sequel $a, b \in (0,1)$. Define $F\colon (0,1) \to \mathbb{R}$ by

$$F\left(v\right) = \begin{cases} \left(\log \dfrac{1}{v}\right)^{\beta}, & 0 < v \le v_0, \\ \left(\log \dfrac{1}{v_0}\right)^{\beta}, & v_0 \le v \le 1 - v_0, \\ \dfrac{1-v}{v_0}\left(\log \dfrac{1}{1-v}\right)^{\beta}, & 1 - v_0 \le v < 1. \end{cases} \tag{4.7}$$

Then F is positive, continuous and decreasing on $(0,1)$, and

$$\frac{L\left(v\right)}{v} \approx F\left(v\right) \quad \text{for } v \in (0,1)\,.$$

Hence, (4.5) is equivalent to

$$F(ab) \le \operatorname{const}\left(F(a) + F(b)\right), \tag{4.8}$$

for all $a, b \in (0,1)$. Since F is decreasing, it is sufficient to show that

$$F(a^2) \le \operatorname{const} F(a). \tag{4.9}$$

for all $a \in (0,1)$. Indeed, if (4.9) holds, then for all $0 < a \le b < 1$

$$F(ab) \le F(a^2) \le \operatorname{const} F(a) \le \operatorname{const}\left(F(a) + F(b)\right).$$

It is easy to show that (4.9) holds since the ratio $F(a)/F(a^2)$ converges to $\left(1/2\right)^{\beta}$ as $a \to 0$ and to $1/2$ as $a \to 1$. ■

Applying Theorem 4.1 we obtain the following result.

Theorem 4.3. *Let L be defined as in (4.4). Then any C^1 function $f\colon \mathbb{S}^{n-1} \to [0,1]$ satisfies the following inequality*

$$\omega_n C_L L\left(\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} f\,d\sigma_{n-1}\right) \leq \int_{\mathbb{S}^{n-1}} L(f)\,d\sigma_{n-1} + \int_{\mathbb{S}^{n-1}} |\nabla f|\,d\sigma_{n-1}, \tag{4.10}$$

where $\omega_n = \sigma_{n-1}\left(\mathbb{S}^{n-1}\right)$ and C_L is the constant from (4.5).

Proof. (4.10) is a direct consequence of the following inequality

$$C_L L\left(\int_{\mathbb{S}^{n-1}} f\,d\tilde{\sigma}_{n-1}\right) \leq \int_{\mathbb{S}^{n-1}} L(f)\,d\tilde{\sigma}_{n-1} + \int_{\mathbb{S}^{n-1}} |\nabla f|\,d\tilde{\sigma}_{n-1},$$

that in turn follows from Theorem 4.1 and the properties of L stated in Proposition 4.2. ■

5. Isoperimetric inequality for a weighted measure on $\mathbb{R}^n \setminus \{0\}$

In this section we again consider the measure

$$d\mu(x) = \mathrm{e}^{-\frac{1}{|x|^\alpha}}\,dx$$

on $M := \mathbb{R}^n \setminus \{0\}$, where $\alpha > 0$. Consider also the radial part of μ, that is, the measure ν on $\mathbb{R}_+$ given by

$$d\nu(r) = r^{n-1}\mathrm{e}^{-\frac{1}{r^\alpha}}\,dr.$$

For any $R > 0$, set

$$B_R := \{x \in \mathbb{R}^n \setminus \{0\}\colon\ |x| < R\}.$$

Let $\bar{B}_R$ denote the closure of B_R in $\mathbb{R}^n$, i.e.

$$\bar{B}_R := \{x \in \mathbb{R}^n\colon\ |x| \leq R\}.$$

Theorem 5.1. *Let f be a C^1 function on M with support in $\bar{B}_R$ for some $R > 0$. Assume that*

$$0 \leq f \leq \frac{v_0}{\nu((0,R))} \wedge v_0, \tag{5.1}$$

where $v_0 = \mathrm{e}^{-n\left(1+\frac{1}{\alpha}\right)}$ (cf. (3.13)). Then

$$\omega_n C_J C_L I\left(\frac{1}{\omega_n}\int_M f\,d\mu\right) \leq \int_M I(f)\,d\mu + \frac{1}{c}\left(1 + C_J R\right)\int_M |\nabla f|\,d\mu, \tag{5.2}$$

where

$$I(v) = v\left(\log\frac{1}{v}\right)^{1+\frac{1}{\alpha}},$$

C_J, C_L are the constants from Theorems 3.4 and 4.3 respectively, and c is the constant from (4.6).

Proof. Let us use polar coordinates (r,θ) in $M = \mathbb{R}^n \setminus \{0\}$, where $r > 0$ is the polar radius and $\theta \in \mathbb{S}^{n-1}$ is the polar angle (that is, for any $x \in M$ we have $r = |x|$ and $\theta = x/|x|$). Let f be a C^1 function on M with support in $\bar{B}_R$ that satisfies (5.1). Consider the following function F on $\mathbb{S}^{n-1}$:

$$F(\theta) = \int_{\mathbb{R}_+} f(r,\theta)\,d\nu(r).$$

By (5.1) we have

$$0 \leq F \leq v_0 \tag{5.3}$$

and, consequently,

$$0 \leq \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} F(\theta)\,d\sigma_{n-1}(\theta) \leq v_0. \tag{5.4}$$

Applying the estimate (4.10) of Theorem 4.3 to F and noting that the function L on the range of F can be replaced by J or cI (cf. (4.6)), we obtain

$$\omega_n C_L cI\left(\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} F\,d\sigma_{n-1}\right) \le \int_{\mathbb{S}^{n-1}} J(F)\,d\sigma_{n-1} + \int_{\mathbb{S}^{n-1}} |\nabla_\theta F|\,d\sigma_{n-1}. \tag{5.5}$$

For the term in the left hand side we have

$$\int_{\mathbb{S}^{n-1}} F d\sigma_{n-1} = \int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}_+} f\,d\nu d\sigma_{n-1} = \int_M f\,d\mu. \tag{5.6}$$

For the right hand side of (5.5), we apply Theorem 3.4 to the function $f(\theta,\cdot)$ and obtain

$$\begin{aligned} C_J J(F(\theta)) &= C_J J\left(\int_{\mathbb{R}_+} f(r,\theta)\,d\nu(r)\right) \\ &\le \int_{\mathbb{R}_+} J(f)\,d\nu + \int_{\mathbb{R}_+} |f_r|\,d\nu \\ &= \int_{\mathbb{R}_+} cI(f)\,d\nu + \int_{\mathbb{R}_+} |f_r|\,d\nu, \end{aligned} \tag{5.7}$$

where we have used that $J(f) = cI(f)$, which in turn is true by (4.6), because $0 \le f \le v_0$. Combining (5.5), (5.7), and using that

$$|\nabla_\theta F| \le \int_{\mathbb{R}_+} |\nabla_\theta f|\,d\nu,$$

we obtain

$$\begin{aligned} &\omega_n C_L C_J cI\left(\frac{1}{\omega_n}\int_M f\,d\mu\right) \\ &\le \int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}_+} cI(f)\,d\nu\,d\sigma_{n-1} + \int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}_+} |f_r|\,d\nu\,d\sigma_{n-1} + C_J\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}_+} |\nabla_\theta f| d\nu\,d\sigma_{n-1} \end{aligned} \tag{5.8}$$

Note that

$$|\nabla f|^2 = f_r^2 + \frac{1}{r^2}|\nabla_\theta f|^2,$$

whence

$$|f_r| + C_J|\nabla_\theta f| \le |\nabla f| + C_J r|\nabla f|.$$

Since f is supported in $\bar{B}_R$, the value of the polar radius r in the integrals of (5.8) is bounded by R. Hence,

$$|f_r| + C_J|\nabla_\theta f| \le (1 + C_J R)|\nabla f|,$$

whence we obtain

$$\omega_n C_L C_J cI\left(\frac{1}{\omega_n}\int_M f\,d\mu\right) \le c\int_M I(f)\,d\mu + (1 + C_J R)\int_M |\nabla f|\,d\mu.$$

Dividing both sides by c, we obtain (5.2). ∎

Now we shall apply the functional isoperimetric inequality (5.2) in order to prove an isoperimetric inequality for the measure μ. We use the same notation as above and start with the following lemma.

Lemma 5.2. *There are constants $R > 0$ and $C > 0$ such that, for any open set $A \subset B_R$ with Lipschitz boundary,*

$$\mu^+(A) \ge CI(\mu(A)). \tag{5.9}$$

Proof. Let $\{f_\varepsilon\}_{\varepsilon>0}$ be a family of functions in $C^1(M)$ such that

(a) $0 \le f_\varepsilon \le 2$ and $f_\varepsilon = 0$ outside A^ε;
(b) f_ε converges pointwise to $\mathbf{1}_A$ as $\varepsilon \to 0$;
(c) $\int_M |\nabla f_\varepsilon| d\mu$ converges to $\mu^+(A)$ as $\varepsilon \to 0$.

Set $\tilde{f}_\varepsilon = \frac{v_0}{2} f_\varepsilon$. Then $\tilde{f}_\varepsilon$ satisfies (5.1). Hence by (5.2) we have

$$\omega_n C_J C_L I\left(\frac{1}{\omega_n}\int_M \tilde{f}_\varepsilon \, d\mu\right) \le \int_M I(\tilde{f}_\varepsilon)\, d\mu + \frac{1}{c}\left(C_J R + 1\right)\int_M |\nabla \tilde{f}_\varepsilon|\, d\mu. \tag{5.10}$$

Passing to the limit as $\varepsilon \to 0$, using the dominated convergence theorem and (a)–(c), we obtain

$$\omega_n C_J C_L I\left(\frac{v_0}{2\omega_n}\mu(A)\right) \le I\left(\frac{v_0}{2}\right)\mu(A) + \frac{v_0}{2c}\left(C_J R + 1\right)\mu^+(A). \tag{5.11}$$

Let us show that if R is small enough, then

$$I\left(\frac{v_0}{2}\right)\mu(A) \le \frac{1}{2}\omega_n C_J C_L I\left(\frac{v_0}{2\omega_n}\mu(A)\right). \tag{5.12}$$

Indeed, using $I(v) = v\left(\log\frac{1}{v}\right)^\beta$ where $\beta = 1 + \frac{1}{\alpha}$, we obtain that (5.12) is equivalent to

$$\left(\log\frac{2}{v_0}\right)^\beta \le \frac{1}{2} C_J C_L \left(\log\frac{2\omega_n}{v_0 \mu(A)}\right)^\beta,$$

which in turn is equivalent to

$$\mu(A) \le \omega_n \left(\frac{v_0}{2}\right)^N,$$

where $N = \left(\frac{1}{2} C_J C_L\right)^{-1/\beta} - 1$. Since $\mu(A) \le \mu(B_R)$, this inequality will be satisfied provided

$$\mu(B_R) \le \omega_n \left(\frac{v_0}{2}\right)^N. \tag{5.13}$$

Hence, for the value of R that satisfies (5.13), we obtain

$$\frac{1}{2}\omega_n C_J C_L I\left(\frac{v_0}{2\omega_n}\mu(A)\right) \le \frac{v_0}{2c}\left(C_J R + 1\right)\mu^+(A),$$

whence (5.9) follows. ■

Now we are ready to prove a full isoperimetric inequality for μ. This is the main technical result of this paper.

Theorem 5.3. *For the manifold $M = \mathbb{R}^n \setminus \{0\}$ with measure $d\mu(x) = e^{-\frac{1}{|x|^\alpha}} dx$, there exist constants $C > 0$ and $\tau \in (0,1)$ depending on n, α and such that the following function*

$$\tilde{I}(v) = C \begin{cases} v\left(\log\frac{1}{v}\right)^{1+\frac{1}{\alpha}}, & 0 \le v \le \tau, \\ v^{\frac{n-1}{n}}, & v > \tau \end{cases} \tag{5.14}$$

is a lower isoperimetric function for the measure μ on M of restricted type.

Proof. We shall use the function $I(v) = v\left(\log\frac{1}{v}\right)^{1+\frac{1}{\alpha}}$ as before. By Lemma 5.2, there exist some $R > 0$ and a constant $C_0 > 0$ such that for all open sets $A \subset B_R$ with Lipschitz boundary

$$\mu^+(A) \ge C_0 I(\mu(A)). \tag{5.15}$$

Since for all $|x| > R$ we have $e^{-\frac{1}{|x|^\alpha}} \approx 1$, the measure ν outside B_R is in finite ratio with Lebesgue measure, which implies that for all Borel sets $A \subset B_R^c := M \setminus B_R$,

$$\mu^+(A) \ge C_1 (\mu(A))^{\frac{n-1}{n}}, \tag{5.16}$$

for some constant $C_1 > 0$. We need to prove that for any precompact open set $\Omega \subset M$ with smooth boundary

$$\mu^+(\Omega) \geq C\tilde{I}(\mu(\Omega)). \tag{5.17}$$

For any such $\Omega \subset M$, set

$$\Omega_0 = B_R \cap \Omega, \quad \Omega_1 = B_R^c \cap \Omega.$$

Let us first prove that

$$3\mu^+(\Omega) \geq C_0 I(\mu(\Omega_0)) + C_1\mu(\Omega_1)^{\frac{n-1}{n}}. \tag{5.18}$$

Set

$$\Gamma_0 = \partial\Omega \cap B_R, \quad \Gamma_1 = \partial\Omega \cap B_R^c, \quad \Sigma = \Omega \cap \partial B_R$$

and let σ denote the $(n-1)$-dimensional measure induced by μ, that is, σ has density $\mathrm{e}^{-\frac{1}{|x|^\alpha}}$ with respect to the $(n-1)$-Hausdorff measure $\mathcal{H}_{n-1}$. First observe that

$$\sigma(\Gamma_1) \geq \sigma(\Sigma). \tag{5.19}$$

Indeed, consider the projection $\Pi : x \mapsto \frac{Rx}{|x|}$ of Γ_1 onto ∂B_R. Clearly, the image $\Pi(\Gamma_1)$ covers Σ. Since Γ_1 lies outside B_R, the mapping Π reduces the measure $\mathcal{H}_{n-1}$, and since the weight function $\mathrm{e}^{-\frac{1}{|x|^\alpha}}$ is increasing in $|x|$, the same reduction holds a fortiori for the measure σ, which proves (5.19).

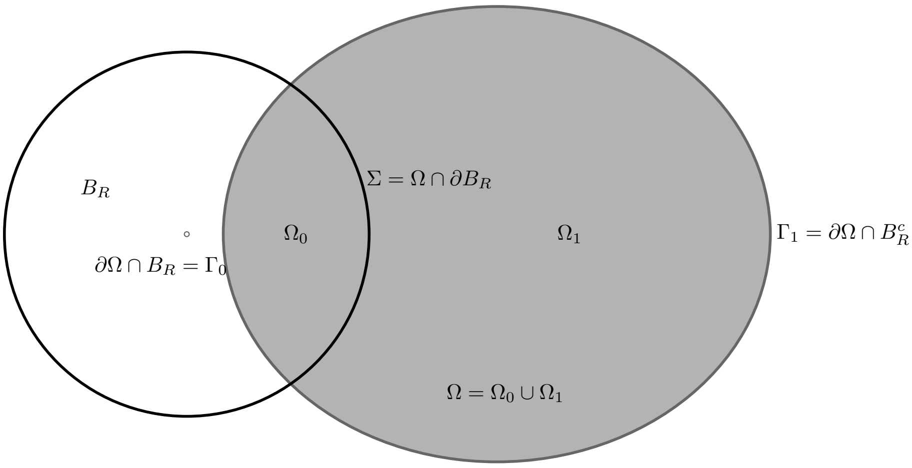


FIGURE 4. Decompostion of Ω and $\partial\Omega$

By (5.15) we have

$$\sigma(\Gamma_0) + \sigma(\Sigma) = \mu^+(\Omega_0) \geq C_0 I(\mu(\Omega_0)). \tag{5.20}$$

By (5.16) we have

$$\sigma(\Gamma_1) + \sigma(\Sigma) = \mu^+(\Omega_1) \geq C_1(\mu(\Omega_1))^{\frac{n-1}{n}}. \tag{5.21}$$

Adding up (5.20) and (5.21) and replacing $\sigma(\Sigma)$ by $\sigma(\Gamma_1)$ according to (5.19), we obtain

$$\sigma(\Gamma_0) + 3\sigma(\Gamma_1) \geq C_0 I(\mu(\Omega_0)) + C_1(\mu(\Omega_1))^{\frac{n-1}{n}},$$

whence (5.18) follows, as $\mu^+(\Omega) = \sigma(\Gamma_0) + \sigma(\Gamma_1)$.

Now from (5.18) we deduce the required isoperimetric inequality (5.17). Set $\tau = \mu(B_R)$ and consider three cases.

(a) Assume that $0 \leq \mu(\Omega) \leq \tau$. Clearly, there is a constant $C_2 > 0$ such that

$$v^{\frac{n-1}{n}} \geq C_2 I(v) \text{ for all } 0 \leq v \leq \tau. \tag{5.22}$$

From (5.18) and (5.22) we obtain

$$\begin{aligned} 3\mu^+(\Omega) &\geq C_0 I\left(\mu(\Omega_0)\right) + C_2 C_1 I\left(\mu(\Omega_1)\right) \\ &\geq C I\left(\mu\left(\Omega_0\right) \vee \mu\left(\Omega_1\right)\right) \\ &\geq C I\left(\frac{1}{2}\mu\left(\Omega\right)\right) \\ &\geq \frac{1}{2} C I\left(\mu\left(\Omega\right)\right), \end{aligned}$$

where $C = (C_0 \wedge (C_1 C_2))$ and we have used that $I\left(\frac{1}{2}v\right) \geq \frac{1}{2} I\left(v\right)$. Renaming $\frac{1}{2}C$ by C, we obtain (5.17).

(*b*) Assume that $\mu(\Omega) \geq 2\tau$. Since $\mu\left(\Omega_0\right) \leq \tau$, we have in this case

$$\mu(\Omega_1) \geq \frac{1}{2}\mu(\Omega). \tag{5.23}$$

Therefore, we obtain from (5.18)

$$\mu^+(\Omega) \geq \frac{1}{3} C_1 \mu(\Omega_1)^{\frac{n-1}{n}} \geq \frac{1}{3} C_1 \left(\frac{1}{2}\mu(\Omega)\right)^{\frac{n-1}{n}} = C\mu(\Omega)^{\frac{n-1}{n}}, \tag{5.24}$$

with $C = C_1 \frac{1}{3}\left(\frac{1}{2}\right)^{\frac{n-1}{n}}$, which proves (5.17) in this case.

(*c*) Assume that $\tau \leq \mu(\Omega) \leq 2\tau$. In this case we have either $\mu\left(\Omega_0\right) \geq \frac{\tau}{2}$ or $\mu\left(\Omega_1\right) \geq \frac{\tau}{2}$. In both cases, from (5.18) we obtain that

$$\mu^+\left(\Omega\right) \geq C_0 I\left(\frac{\tau}{2}\right) \wedge C_1 \left(\frac{\tau}{2}\right)^{\frac{n-1}{n}} = C\left(2\tau\right)^{\frac{n-1}{n}} \geq C\mu\left(\Omega\right)^{\frac{n-1}{n}},$$

where the constant C is defined by the middle identity.

Hence, (5.17) is satisfied in all cases, which was to be proved. ■

6. An upper bound of the heat kernel

The following result was proved in [8, Theorem 2.1] for the case of Riemannian manifolds and extended in [10, Theorem 5.1] to arbitrary weighted manifolds. In fact, it also holds in the framework of Dirichlet form [16].

Theorem 6.1. *Let (M, μ) be a weighted manifold and assume that (M, μ) satisfies the Faber-Krahn inequality (1.4) with a function Λ, where $\Lambda\colon (0, +\infty) \to (0, +\infty)$ is a decreasing function such that*

$$\int_0^1 \frac{dv}{v\Lambda(v)} < \infty. \tag{6.1}$$

Then the heat kernel $p_t\left(x, y\right)$ of (M, μ) satisfies the following upper bound

$$\sup_{x,y\in M} p_t(x, y) \leq \frac{4}{\zeta\left(\frac{1}{2}t\right)} \tag{6.2}$$

for all $t > 0$, where the function ζ is defined by

$$t = \int_0^{\zeta(t)} \frac{dv}{v\Lambda(v)}. \tag{6.3}$$

Since $\mathbb{R} \setminus \{0\}$ with measure $d\mu(x) = e^{-\frac{1}{|x|^\alpha}} dx$ is a weighted manifold, we can combine this theorem with the isoperimetric inequality (5.14), to obtain the following result.

Theorem 6.2. *Set $M = \mathbb{R}^n \setminus \{0\}$ and consider the measure $d\mu(x) = e^{-\frac{1}{|x|^\alpha}}dx$ on M for some $\alpha > 0$. Then there are positive constants C, C_0, depending only on n and α, such that the heat kernel of (M, μ) satisfies the following inequality*

$$\sup_{x,y\in M} p_t(x,y) \le C \begin{cases} \exp\left(\frac{C_0}{t^{\frac{\alpha}{\alpha+2}}}\right), & 0 < t < 1, \\ t^{-\frac{n}{2}}, & t > 1. \end{cases} \tag{6.4}$$

Proof. The isoperimetric inequality (5.14) of Theorem 5.3 implies the Faber-Krahn inequality with the function

$$\Lambda(v) = \frac{1}{4}\left(\frac{\tilde{I}(v)}{v}\right)^2 = C \begin{cases} \left(\log \frac{1}{v}\right)^{2+\frac{2}{\alpha}}, & 0 < v \le \tau, \\ v^{-\frac{2}{n}}, & v > \tau \end{cases} \tag{6.5}$$

(cf. [8, Proposition 2.4]). Observe that the function in (6.5) satisfies condition (6.1) so that Theorem 6.1 applies and yields the upper bound (6.2). Let us estimate the function $\zeta(t)$ that enters the right hand side of (6.2).

For small enough $t > 0$ by (6.3) we have

$$t = \int_0^{\zeta(t)} \frac{dv}{v\Lambda(v)} = -\frac{1}{C}\int_0^{\zeta(t)} \frac{d\log 1/v}{\left(\log\frac{1}{v}\right)^{2+\frac{2}{\alpha}}} = \frac{\left(1+\frac{2}{\alpha}\right)}{C}\left(\log\frac{1}{\zeta(t)}\right)^{-\left(1+\frac{2}{\alpha}\right)},$$

whence

$$\zeta(t) = \exp\left(-\frac{C_0}{t^{\frac{\alpha}{\alpha+2}}}\right),$$

where $C_0 = C_0(C, \alpha) > 0$. For a large enough t we have

$$t = \int_0^{\zeta(t)} \frac{dv}{v\Lambda(v)} \approx \int_0^{V(t)} \frac{dv}{v^{1-\frac{2}{n}}} \approx \zeta(t)^{-\frac{2}{n}},$$

whence

$$\zeta(t) \approx t^{\frac{n}{2}}.$$

Substituting these estimates of ζ into (6.2) we obtain (6.4) for small and large values of t. Then the estimate for the intermediate values of t follows from the fact that the function $t \mapsto \sup_{x,y\in M} p_t(x,y)$ is decreasing. ■

7. A lower bound of the heat kernel

In order to obtain a lower bound of the heat kernel, we use the following notion. We say that a weighted manifold (M, μ) satisfies an anti-Faber-Krahn inequality if, for any $v > 0$, there is an open set $\Omega_v \subset M$ such that $\mu(\Omega_v) = v$ and

$$\lambda_1(\Omega_v) \le \Lambda(v). \tag{7.1}$$

We shall use the following result from [6].

Theorem 7.1. ([6, Theorem 3.2]) *Let Λ be a function as in Theorem 6.1. Assume that (M, μ) satisfies an anti-Faber-Krahn inequality with the function Λ. Define a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ by the identity*

$$t = \int_0^{\gamma(t)} \frac{dv}{v\Lambda(v)} \tag{7.2}$$

and assume that $\gamma(t)$ satisfies the following property: there exists some constant $c_\gamma > 0$ such that

$$\frac{\gamma'(s)}{\gamma(s)} \ge C_\gamma \frac{\gamma'(t)}{\gamma(t)}, \quad \textit{for all } 0 < t \le s \le 2t. \tag{7.3}$$

Then, for all $t>0$, the heat kernel $p_t(x,y)$ of (M,μ) satisfies

$$\sup_{x\in M} p_t(x,x) \geq \frac{1}{\gamma\left(\frac{2}{c_\gamma}t\right)}. \tag{7.4}$$

To apply Theorem 7.1 we need the following lemma.

Lemma 7.2. *Consider the manifold $M=\mathbb{R}^n\setminus\{0\}$ with measure $d\mu(x)=\mathrm{e}^{-\frac{1}{|x|^\alpha}}dx$ where $\alpha>0$. For any $r>0$ set*

$$B_r := \{x\in\mathbb{R}^n\setminus\{0\}: \ |x|<r\}.$$

There exists some constant $C>0$ such that for all $0<r<1$,

$$\lambda_1(B_r) \leq Cr^{-2(1+\alpha)} \tag{7.5}$$

(in fact $\lambda_1(B_r)\approx r^{-2(1+\alpha)}$) and for all $r\geq 1$

$$\lambda_1(B_r)\leq Cr^{-2} \tag{7.6}$$

(in fact $\lambda_1(B_r)\approx r^{-2}$).

Proof. Let us first prove (7.6). Fix $r\geq 1$ and consider a test function

$$\varphi(x)=\begin{cases} (|x|-r/4)_+, & |x|\leq r/2,\\ \frac{1}{4}r, & r/2<|x|<3r/4,\\ (r-|x|)_+ & |x|\geq 3r/4,\end{cases}$$

that is a Lipschitz function with compact support in B_r. By the variational principle, we have

$$\lambda_1(B_R)\leq \frac{\int_M |\nabla\varphi|^2\,d\mu}{\int_M \varphi^2 d\mu}.$$

Clearly, we have

$$\int_M \varphi^2 d\mu \geq \int_{B_{r/2}\setminus B_{r/4}} \varphi^2 d\mu = \left(\frac{1}{4}r\right)^2 \mu\left(B_{r/2}\setminus B_{r/4}\right) \approx r^2 r^n = r^{n+2},$$

where we use the fact that outside $B_{r/4}$ the measure μ is finitely proportional to the Lebesgue measure. Also, since $|\nabla\varphi|\leq 1$, we have

$$\int_M |\nabla\varphi|^2\,d\mu \leq \mu(B_r)\leq Cr^n.$$

Combining this with the previous line, we obtain (7.6).

Let us now prove (7.5). Set $S(r)=\mu^+(B_r)$ and $V(r)=\mu(B_r)$. By [10, Theorem 2.10] (see also [9]) we have

$$\lambda_1(B_r)\approx \frac{1}{F(r)} \tag{7.7}$$

for all $r>0$, where

$$F(r) := \sup_{0<\xi<r}\left[V(\xi)\int_\xi^r \frac{dt}{S(t)}\right].$$

By definition of μ we have

$$S(r)=\omega_n r^{n-1}\mathrm{e}^{-\frac{1}{r^\alpha}}$$

and

$$V(r)=\int_0^r S(t)\,dt \approx r^{n+\alpha}\mathrm{e}^{-\frac{1}{r^\alpha}}.$$

Let us show that there exists some $c>0$ such that for $0<r<1$

$$V\left(\frac{r}{2}\right)\int_{r/2}^{r}\frac{dt}{S\left(t\right)}\geq cr^{2(1+\alpha)},$$

which would imply $F\left(r\right)\geq cr^{2(1+\alpha)}$ and, hence, (7.5).

Set $\xi=r/2$ and observe that

$$\int_{\xi}^{r}\frac{1}{S(t)}\,dt=\frac{1}{\omega_n}\int_{\xi}^{r}t^{1-n}\exp\left(\frac{1}{t^{\alpha}}\right)dt\approx r^{1-n}\int_{\xi}^{r}\exp\left(\frac{1}{t^{\alpha}}\right)dt,$$

whence

$$V(\xi)\int_{\xi}^{r}\frac{1}{S(t)}\,dt\approx r^{1+\alpha}\exp\left(-\frac{1}{\xi^{\alpha}}\right)\int_{\xi}^{r}\exp\left(\frac{1}{t^{\alpha}}\right)dt. \tag{7.8}$$

Next let us verify that

$$\exp\left(\frac{1}{\left(\xi+\xi^{1+\alpha}\right)^{\alpha}}\right)\geq C^{-1}\exp\left(\frac{1}{\xi^{\alpha}}\right) \tag{7.9}$$

for some $C>0$. Indeed,

$$\frac{\exp\left(\frac{1}{\xi^{\alpha}}\right)}{\exp\left(\frac{1}{\left(\xi+\xi^{1+\alpha}\right)^{\alpha}}\right)}=\exp\left(\frac{1}{\xi^{\alpha}}-\frac{1}{\xi^{\alpha}\left(1+\xi^{\alpha}\right)^{\alpha}}\right)=\exp\left(\frac{\left(1+\xi^{\alpha}\right)^{\alpha}-1}{\xi^{\alpha}\left(1+\xi^{\alpha}\right)^{\alpha}}\right).$$

Since the function $x\mapsto\frac{(1+x)^{\alpha}-1}{x}$ is bounded for $x\in(0,1)$, say by a constant C, we obtain

$$\frac{\exp\left(\frac{1}{\xi^{\alpha}}\right)}{\exp\left(\frac{1}{\left(\xi+\xi^{1+\alpha}\right)^{\alpha}}\right)}\leq\exp\left(\frac{C}{\left(1+\xi^{\alpha}\right)^{\alpha}}\right)\leq\exp\left(C\right),$$

which proves (7.9). Since $r\geq\xi+\xi^{1+\alpha}$, it follows that

$$\int_{\xi}^{r}\exp\left(\frac{1}{t^{\alpha}}\right)dt\geq\int_{\xi}^{\xi+\xi^{1+\alpha}}\exp\left(\frac{1}{t^{\alpha}}\right)dt\geq\xi^{1+\alpha}\exp\left(\frac{1}{\left(\xi+\xi^{1+\alpha}\right)^{\alpha}}\right)\geq C^{-1}\xi^{1+\alpha}\exp\left(\frac{1}{\xi^{\alpha}}\right).$$

Substituting the estimate above into (7.8) we obtain that, for some constant $C_1>0$,

$$V(\xi)\int_{\xi}^{r}\frac{1}{S(t)}\,dt\geq C_1r^{1+\alpha}\exp\left(-\frac{1}{\xi^{\alpha}}\right)C^{-1}\xi^{1+\alpha}\exp\left(\frac{1}{\xi^{\alpha}}\right)\approx r^{2(1+\alpha)},$$

which finishes the proof of (7.5). ∎

Finally we can prove a lower bound of the heat kernel.

Theorem 7.3. *For the manifold $M=\mathbb{R}^n\setminus\{0\}$ with measure $d\mu(x)=\mathrm{e}^{-\frac{1}{|x|^{\alpha}}}dx$, there exist constants $c,c_0>0$ depending on n and α, such that the heat kernel $p_t\left(x,y\right)$ of $\left(M,\mu\right)$ satisfies the following estimate*

$$\sup_{x\in M}p_t(x,x)\geq c\begin{cases}\exp\left(\frac{c_0}{t^{\frac{\alpha}{2+\alpha}}}\right), & 0<t<1,\\ t^{-\frac{n}{2}}, & t\geq 1.\end{cases} \tag{7.10}$$

Proof. For any $v>0$, take $\Omega_v=B_r$ where r is chosen so that $\mu(B_r)=v$. If v is small enough then by (3.9) we have

$$r^{-1}\approx\left(\log\frac{1}{v}\right)^{\frac{1}{\alpha}}.$$

Hence by Lemma 7.2 we obtain

$$\lambda_1(\Omega_v) \le Cr^{-2(1+\alpha)} \le C' \left(\log \frac{1}{v}\right)^{\frac{2(1+\alpha)}{\alpha}} := \Lambda(v).$$

As in the proof of Theorem 6.2, the function γ from (7.2) has the expression

$$\gamma(t) = \exp\left(-\frac{C_0}{t^{\frac{\alpha}{2+\alpha}}}\right)$$

for some $C_0 > 0$. It is easy to verify that this function γ satisfies property (7.3). Hence by Theorem 7.1 we obtain the lower bound (7.4) for small values of t. The case of large values of t is treated similarly. ∎

Acknowledgments. We would like to thank Yuri Kondratiev for stimulating discussions that motivated this work, and the anonymous referee for his/her careful reading and useful suggestions, leading to some improvement and simplification.

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