ISOPERIMETRIC INEQUALITIES FOR RIEMANNIAN PRODUCTS

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INTRODUCTION

We shall say that the f-isoperimetric inequality holds on the smooth Riemannian manifold M, or, what is the same, that the manifold M has isoperimetric function f, if for any open set $D \subset M$ of finite volume v, having smooth boundary, the measure of codimension 1 of the boundary ∂D is not less than f(v). The need to calculate an isoperimetric function arises, for example, in the investigation of elliptic and parabolic equations on a manifold (cf., e.g., [1-5]). A vast literature is devoted to proofs of various isoperimetric inequalities and Sobolev-type inequalities connected with them. The list of papers [6-8] has a purely illustrative character.

In the present paper we solve the problem of finding an isoperimetric function up to a constant for the direct product $M_1 \times M_2$ of the manifolds M_1 and M_2, if isoperimetric inequalities on M_1 and M_2 are known. For example, it turns out that if the isoperimetric functions of the manifolds M_1 and M_2 are equal to v^{α} and v^{β} respectively, where $0 \leqslant \alpha, \beta < 1$, then on the manifold $M_1 \times M_2$ one has a v^{γ}-isoperimetric inequality, where

$$\frac{1}{1-\gamma} = \frac{1}{1-\alpha} + \frac{1}{1-\beta}.$$

The precise formulation of the basic results is given in Sec. 1. In Sec. 2 the geometric problem of the proof of the isoperimetric inequality is reduced to finding the minimum of a certain functional of functions of one variable. The latter problem is solved (up to multiplication by a constant) in Sec. 3.

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Notation. The symbols $\mu_1, \mu_2, \mu = \mu_1 \times \mu_2$ will denote the measures on the Riemannian manifolds M_1, M_2 and $M = M_1 \times M_2$, induced by the Riemannian metric. If N is a Riemannian manifold of dimension n (for example, a submanifold of M), then its n-dimensional volume will be denoted by $|N|$. The letter c denotes an absolute positive constant.

1. Formulation of the Basic Results

Let $\varphi: (0, +\infty) \to [0, +\infty)$ be a monotone decreasing right continuous function, where $\lim_{t\to\infty} \varphi(t) = 0$. We call the function $\psi(s) = \mathrm{mes}\,\{t > 0 \mid \varphi(t) > s\}$, $s > 0$, the generalized inverse of $\varphi(t)$. It is easy to see that the generalized inverse function ψ also decreases monotonically on $(0, +\infty)$, is right continuous, and $\lim_{s\to\infty} \psi(s) = 0$. Moreover, the function φ itself is the generalized function of ψ, i.e., $\varphi(t) = \mathrm{mes}\,\{s > 0 \mid \psi(s) > t\}$, and one has

$$\int_0^{\infty} \varphi(t)\,dt = \int_0^{\infty} \psi(s)\,ds.$$

THEOREM 1. Let the manifolds M_1 and M_2 have isoperimetric functions f and g, which are continuous on the intervals $(0, |M_1|)$ and $(0, |M_2|)$, respectively. Then on the manifold M the (1/2)h(v)-isoperimetric inequality holds, where

$$h(v) = \inf_{\varphi, \psi}\left[\int_0^{\infty} f(\varphi(t))\,dt + \int_0^{\infty} g(\psi(s))\,ds\right]; \tag{1}$$

here φ and ψ are generalized mutually inverse functions such that $\varphi \leqslant |M_1|$, $\psi \leqslant |M_2|$ and

$$\int_0^{\infty} \varphi(t)\,dt = \int_0^{\infty} \psi(s)\,ds = v, \tag{2}$$

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and the infimum is taken over all such pairs φ, ψ.

Remark. The assertion of Theorem 1 is sharp in the following sense. If the manifolds M_1 and M_2 have one-parameter families of subsets on which the isoperimetric inequalities reduce to equalities (cf. Sec. 2 for a precise formulation), then any isoperimetric function of the manifold M does not exceed h(v). We have been unable to liquidate the margin between (1/2)h(v) and h(v).

Under additional restrictions on f and g one can get estimates for the function h, which are more convenient for applications.

THEOREM 2. Let the functions f and g be continuous, monotone increasing on the interval $(0, +\infty)$, and the functions f(x)/x and g(y)/y be monotone decreasing. Then

$$h(v) \geqslant (1/3)\, h_0(v), \tag{3}$$

where

$$h_0(v) = \inf_{xy=v} (f(x)\, y + g(y)\, x). \tag{4}$$

Remarks. 1. The expression f(x)y + g(y)x is natural from the geometric point of view. In fact, if $D_1 \subset M_1, D_2 \subset M_2, |D_1| = x, |D_2| = y$, then $|D_1 \times D_2| = xy$ and $|\partial (D_1 \times D_2)| = |\partial D_1|\, y + |\partial D_2|\, x$.

2. If one does not impose any conditions on the functions f and g besides the monotonicity, then h cannot be estimated in terms of h_0: an example is given in Sec. 3 when $h(v) \equiv 0$, and $h_0(v) > 0$.

3. In the case of monotone increasing f(x)/x and g(y)/y, and also in a somewhat more general situation one can prove the estimate $h(v) \geqslant C_\varepsilon h_0((1-\varepsilon)v)$ (where it is impossible to get rid of the $\varepsilon > 0$).

THEOREM 2a. Let the functions f and g be nonnegative, continuous on the intervals $(0, V_1)$ and $(0, V_2)$, and symmetric with respect to the points $V_1/2$ and $V_2/2$ respectively. Also let f and g increase on the intervals $(0, V_1/2)$ and $(0, V_2/2)$ respectively, and functions $f(x)/x, g(y)/y$ be monotone decreasing on these intervals. Then for $v \leqslant (1/2)\, V_1 V_2$ one has

$$h(v) \geqslant c \min \left\{ h_0(v),\ f\left(\frac{v}{V_2}\right) V_2,\ g\left(\frac{v}{V_1}\right) V_1 \right\}, \tag{5}$$

where

$$h_0(v) = \inf_{\substack{xy=cv \\ x \leqslant (1/2)V_1,\ y \leqslant (1/2)V_2}} (f(x)\, y + g(y)\, x).$$

Remarks. 1. For a manifold of finite volume V one can assume that the isoperimetric function is symmetric with respect to the point V/2, since the boundaries of an open set and its complement coincide.

2. All three terms on the right side of (5) are essential. Thus the term $f(v/V_2)V_2$ corresponds to the following geometric situation. If $D_1 \subset M_1$, $D = D_1 \times M_2$, $|D| = v$, then

$$|\partial D| = |\partial D_1| \cdot V_2 \geqslant f(|D_1|)\, V_2 = f(v/V_2)\, V_2.$$

3. If one of the volumes V_i is equal to ∞, then it is necessary to throw the corresponding term in (5) away.

2. Proof of Theorem 1

We need the following auxiliary assertions.

LEMMA 1. Let N be a smooth Riemannian manifold, $\rho \in C^\infty(N)$, $S_t = \{x \in N \mid \rho(x) = t\}$ be a level set of the function ρ (which is a submanifold for almost all t); let ν, ν_t be the measures on N and S_t, generated by the Riemannian metric.

Then if $\eta \in L^1(N, \nu), \eta \geqslant 0$, one has

$$\int_N \eta\, |\nabla \rho|\, d\nu = \int_0^\infty dt \int_{S_t} \eta\, d\nu_t. \tag{6}$$

LEMMA 2. Let D be an open subset of the manifold N, having smooth boundary. Then

$$|\partial D| = \inf_{\{F_n\}} \overline{\lim_{n\to\infty}} \int_N |\nabla F_n| \, d\nu = \inf_{\{F_n\}} \underline{\lim_{n\to\infty}} \int_N |\nabla F_n| \, d\nu,$$

where $\{F_n\}$ is a monotone increasing sequence of functions $F_n \in C^\infty(N)$, converging pointwise to χ_D, the characteristic function of the set D.

Both these lemmas are special cases of more general assertions proved in [10]. In this connection we note that the requirement of smoothness demanded in the present paper of all the functions and boundaries considered is not essential, but merely simplifies the arguments.

Proof of Theorem 1. Let D be an open set in M with smooth boundary, $|D| = v < \infty$. Let $\{F_n\}$ be a monotone increasing sequence of smooth functions on M, where $F_n \to \chi_D$. By Lemma 2 it suffices to prove that

$$\overline{\lim_{n\to\infty}} \int_M |\nabla F_n| \, d\mu \geqslant \frac{1}{2} h(v). \tag{7}$$

Let ∇_x and ∇_y denote gradients on the manifolds M_1 and M_2 respectively. Then $|\nabla F_n|^2 = |\nabla_x F_n|^2 + |\nabla_y F_n|^2$, so $|\nabla F_n| \geqslant |\nabla_x F_n|$, $|\nabla F_n| \geqslant |\nabla_y F_y|$. Hence instead of (7) it suffices to prove that

$$\overline{\lim_{n\to\infty}} \int_M |\nabla_x F_n| \, d\mu + \overline{\lim_{n\to\infty}} \int_M |\nabla_y F_n| \, d\mu \geqslant h(v). \tag{8}$$

We estimate the second summand in (8) as follows. For each $x \in M_1$ we let $S(x) = \{y \in M_2 \mid (x, y) \in D\}$. It follows from Sard's theorem that for almost all x the section S(x) has smooth boundary. We consider $F_n(x, y)$ as a function on M_2 for fixed x. By Lemma 2 we have

$$\underline{\lim_{n\to\infty}} \int_{M_2} |\nabla_y F_n(x, y)| \, d\mu_2(y) \geqslant |\partial S(x)| \tag{9}$$

(for almost all x). Integrating (9) over M_1 and using Fatou's lemma, we get

$$\overline{\lim_{n\to\infty}} \int_M |\nabla_y F_n| \, d\mu \geqslant \int_{M_1} |\partial S(x)| \, d\mu_1. \tag{10}$$

The geometric meaning of (10) is this. We have estimated the part of the measure of the boundary ∂D, which depends on the "extent" of D along M_1. If $M_1 = M_2 = \mathbf{R}$, D is a domain in $\mathbf{R}^2$, (10) means that the length of the boundary ∂D is not less than the projection of D to M_1.

Now we estimate the first summand in (8). Of course it could be estimated analogously to (10), but it is more convenient to do this as follows. Let $\sigma(x) = \mu_2 S(x)$. Then the part of the measure of the boundary ∂D, which depends on the "extent" along M_2, roughly speaking, is larger the larger the values assumed by σ. For example, if σ assumes only one value, then D can be a cylinder $D_2 \times M_1$, and the whole boundary ∂D is determined by the extent along M_1.

The formal realization of this idea is the following. By Fubini's formula we have

$$\int_M |\nabla_x F_n| \, d\mu = \int_{M_1} d\mu_1 \int_{M_2} |\nabla_x F_n| \, d\mu_2 \geqslant \int_{M_1} d\mu_1 \left| \nabla_x \int_{M_2} F_n(x, y) \, d\mu_2 \right|. \tag{11}$$

Let

$$\sigma_n(x) = \int_{M_2} F_n(x, y) \, d\mu_2(y).$$

Then $\sigma_n(x)$ is a monotone increasing sequence of smooth functions on M_1, converging pointwise to $\sigma(x)$. Using (11), Lemma 1, and the f-isoperimetric inequality, we get

$$\int_M |\nabla_x F_n| \, d\mu \geqslant \int_{M_1} |\nabla_x \sigma_n| \, d\mu_1 =$$

$$= \int_0^\infty dt \int_{\{\sigma_n = t\}} |\nabla_x \sigma_n| \, |\nabla_x \sigma_n|^{-1} \, d\mu_{1t} = \int_0^\infty \mu_{1t}\{\sigma_n = t\} \, dt \geqslant \int_0^\infty f(\mu_1\{\sigma_n > t\}) \, dt$$

set f(0) = 0). Passing to the limit as $n \to \infty$, we get

$$\overline{\lim_{n\to\infty}} \int_M |\nabla_x F_n| \, d\mu \geqslant \int_0^\infty f(\mu_1\{\sigma > t\}) \, dt \tag{12}$$

Combining (10) and (12) now and using the isoperimetric inequality $|\partial S(x)| \geqslant g(\sigma(x))$, we get

$$\overline{\lim_{n\to\infty}} \int_M |\nabla_x F_n| \, d\mu + \overline{\lim_{n\to\infty}} \int_M |\nabla_y F_n| \, d\mu \geqslant \int_{M_1} g(\sigma(x)) \, d\mu_1 + \int_0^\infty f(\mu_1\{\sigma > t\}) \, dt. \tag{13}$$

We set $\varphi(t) = \mu_1\{\sigma > t\}$; let ψ be the generalized inverse function to φ. It follows from the definition of that the functions ψ and σ are equimeasurable. Hence $\varphi \leqslant |M_1|$, $\psi \leqslant |M_2|$,

$$\int_0^\infty \varphi(t) \, dt = \int_0^\infty \psi(s) \, ds = \int_{M_1} \sigma \, d\mu_1 = |D| = v,$$

and the right side of (13) is equal to

$$\int_0^\infty g(\psi(s)) \, ds + \int_0^\infty f(\varphi(t)) \, dt \geqslant h(v).$$

This proves (8) and hence Theorem 1.

Now we give an example confirming the sharpness of the function h(v) we have found. Let us assume that for sufficiently large families of subsets in M_1 and M_2 the f- and g-isoperimetric inequalities are sharp. Namely, let $B_i(t)$, $t > 0$, be a family of open sets in M_i with smooth boundaries, such that: a) $\overline{B_i(a)} \subset B_i(b)$ for $a < b$; b) the union of all the boundaries $\partial B_i(t)$ coincides with the whole manifold M_i without a set of measure zero; c) one has

$$|\partial B_i(t)| = \begin{cases} f(|B_i(t)|), & i = 1, \\ g(|B_i(t)|), & i = 2. \end{cases}$$

Moreover, one can assume that

$$|B_i(t)| = t \tag{14}$$

(otherwise we make a change of parameter t).

We show that for each $v \in (0, |M|)$ and for each positive ε one can find a domain $D \subset M$ such that $|D| = v$ and $|\partial D| \leqslant h(v) + \varepsilon$. We introduce functions ρ_i an M_i: $\rho_i(x) = t$, if $x \in \partial B_i(t)$. We fix $\varepsilon > 0$, $v \in (0, |M|)$ and we choose the functions φ and ψ in (1) so that $\int_0^\infty \varphi(t) \, dt = \int_0^\infty \psi(s) \, ds = v$ and

$$h(v) \geqslant \int_0^\infty f(\varphi(t)) \, dt + \int_0^\infty g(\psi(s)) \, ds - \varepsilon; \tag{15}$$

one can assume that the functions φ and ψ are smooth and mutually inverse in the generalized sense.

We consider the domain

$$D = \left\{(x, y) \in M \,\Big|\, \frac{1}{\varphi(\rho_1(x))} \rho_2(y) < 1\right\}. \tag{16}$$

Let $S_1(x) = \{y \in M_2 \mid (x, y) \in D\}$ and $S_2(y) = \{x \in M_1 \mid (x, y) \in D\}$ be sections of the domain D. Obviously they belong to the families $\{B_i(t)\}$. By (14) and (16) we have

$$|S_1(x)| = \mu_2\{y \in M_2 \mid \rho_2(y) < \varphi(\rho_1(x))\} = \mu_2 B_2(\varphi(\rho_1(x))) = \varphi(\rho_1(x));$$
$$\mu_1\{x \mid |S_1(x)| > s\} = \mu_1\{x \mid \varphi(\rho_1(x)) > s\} = \mu_1\{x \mid \rho_1(x) < \psi(s)\} = \psi(s).$$

Consequently, the function $|S_1(x)|$ is equimeasurable with φ; analogously, $|S_2(y)|$ is equimeasurable with ψ. In particular,

$$|D| = \int_{M_1} |S_1(x)| \, d\mu_1(x) = \int_0^\infty \varphi(t) \, dt = v.$$

We note that

$$|\partial D| \leqslant \int_{M_1} |\partial S_1(x)|\, d\mu_1(x) + \int_{M_2} |\partial S_2(y)|\, d\mu_2(y)$$

(geometrically this inequality is obvious; the rigorous proof uses Lemma 2 for a special sequence $\{F_n\}$ and the triangle inequality $|\nabla F_n| \leqslant |\nabla_x F_n| + |\nabla_y F_n|$).

Since the sections S_i belong to the family $\{B_i(t)\}$, one has $|\partial S_1| = f(|S_1|)$, $|\partial S_2| = g(|S_2|)$. Using the equimeasurability of $|S_1(x)|$ and $|S_2(y)|$ with φ and ψ respectively, and (15), we get

$$|\partial D| \leqslant \int_0^\infty f(\varphi(t))\, dt + \int_0^\infty g(\psi(s))\, ds \leqslant h(v) + \varepsilon.$$

3. Proof of Theorem 2

We show that for any generalized mutually inverse functions φ and ψ, satisfying (2), one has

$$I = \int_0^\infty f(\varphi(t))\, dt + \int_0^\infty g(\psi(s))\, ds \geqslant \frac{1}{3} h_0(v),$$

where f, g, h_0 are the functions from the hypothesis of Theorem 2. Let

$$\Omega = \overline{\{(t, s) \in \mathbf{R}^2 \mid 0 \leqslant s \leqslant \varphi(t)\}}$$

We consider all rectangles

$$\Pi_{p,q} = \{(t, s) \in \mathbf{R}^2 \mid 0 \leqslant t \leqslant p,\ 0 \leqslant s \leqslant q\},$$

lying entirely in Ω. If for one of them $pq \geqslant (1/3)v$, then

$$\int_0^\infty f(\varphi(t))\, dt \geqslant \int_0^p f(q)\, dt = f(q)p,$$

$$\int_0^\infty g(\psi(s))\, ds \geqslant qg(p),$$

$$I \geqslant f(q)p + g(p)q \geqslant h_0((1/3)v) \geqslant (1/3)h_0(v).$$

The last inequality is valid since the function

$$\frac{h_0(v)}{v} = \inf_{xy=v} \left(\frac{f(x)}{x} + \frac{g(y)}{y} \right)$$

is monotone decreasing.

Let the area of any rectangle $\Pi_{p,q} \subset \Omega$ not exceed (1/3)v. Since the area of Ω is equal to v, one can find a rectangle $\Pi_{p,q}$, dividing the domain Ω into three parts: $\Pi_{p,q}$, $\{t > p\} \cap \Omega$, $\{s > q\} \cap \Omega$, where the areas of the last two parts are not less than (1/3)v. In other words,

$$\int_p^\infty \varphi(t)\, dt \geqslant \frac{1}{3} v, \quad \int_q^\infty \psi(s)\, ds \geqslant \frac{1}{3} v.$$

Then

$$\int_0^\infty f(\varphi(t))\, dt \geqslant \int_p^\infty \frac{f(\varphi(t))}{\varphi(t)} \varphi(t)\, dt \geqslant \frac{f(q)}{q} \int_p^\infty \varphi(t)\, dt \geqslant \frac{1}{3} v \frac{f(q)}{q},$$

$$\int_0^\infty g(\psi(s))\, ds \geqslant \frac{1}{3} v \frac{g(p)}{p} \geqslant \frac{1}{3} v \frac{g\left(\frac{v}{q}\right)}{v/q} = \frac{1}{3} q g\left(\frac{v}{q}\right),$$

$$I \geqslant \frac{1}{3}\left(f(q)\frac{v}{q} + g\left(\frac{v}{q}\right)q\right) \geqslant \frac{1}{3} h_0(v).$$

The proof of Theorem 2a, and also of the generalization mentioned in the remarks on Theorem 2, goes analogously.

We give an example showing that for monotone increasing functions f and g, the function h(v) cannot be bounded below, in general by $h_0(v)$ (obviously $h(v) \leqslant h_0(v)$ always).

For the sake of some simplification, in the following example the functions f and g will be piecewise-constant (in particular, discontinuous). Let $a_0 = 1, a_1, a_2, \ldots$ be a monotone increasing sequence, which we shall make more precise below. For now we shall only assume that $a_{n+1} \geqslant 2a_n$. We define the function f(x) for $x \geqslant 1$ as follows:

$$f|_{[a_0, a_1)} = 1, \quad f|_{[a_n, a_{n+1})} = \sqrt{a_{n-1}a_{n+1}}, \quad n \geqslant 1,$$

for $x \leqslant 1$ we set

$$f(x) = f(1/x)^{-1}. \tag{17}$$

Also let $g \equiv f$. It follows in an obvious way from (4) and (17) that for $v \geqslant 1, h_0(v) \geqslant h_0(1) \geqslant 2$. At the same time it turns out that $h(v) \equiv 0$. To prove this it suffices to give an example of a function φ, having the following properties:

a) φ is its own generalized inverse;

b) $\int_0^\infty \varphi(t)\,dt = \infty$;

c) $\int_0^\infty f(\varphi(t))\,dt < \infty$.

For $t \geqslant 1, \varphi(t)$ is defined as follows:

$$\varphi|_{[a_n, a_{n+1})} = \frac{1}{a_{n+2}},$$

and for $t \leqslant 1, \varphi(t)$ is defined so as to satisfy a).

It is easy to show that b) and c) will hold if

$$\sum_{n=1}^\infty \frac{a_n}{a_{n+1}} = \infty, \quad \sum_{n=1}^\infty \sqrt{\frac{a_n}{a_{n+2}}} < \infty. \tag{18}$$

Relation (18) holds, for example, for the recursively defined sequence:

$$a_{2n+1} = 2a_{2n}, \quad a_{2n+2} = (2n+2)^4 a_{2n+1} \quad (n \geqslant 0).$$

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