Homologies of digraphs and the Künneth formula

Alexander Grigor'yan

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Paths and boundary operator

Let V be a finite set. For any $p \geq 0$, an *elementary* p-*path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V that will be denoted by $i_0...i_p$ or by $e_{i_0...i_p}$. Fix a field $\mathbb{K}$. A p-*path* is any formal $\mathbb{K}$-linear combinations of elementary p-paths, that is, any p-path has a form

$$v = \sum_{i_0, i_1, ..., i_p \in V} v^{i_0 i_1 ... i_p} e_{i_0 i_1 ... i_p}, \quad \text{where} \quad v^{i_0 i_1 ... i_p} \in \mathbb{K}.$$

Denote by $\Lambda_p = \Lambda_p(V)$ the $\mathbb{K}$-linear space of all p-paths. Set $\Lambda_{-1} = \{0\}$.

Definition. For any $p \geq 0$, the *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ is a $\mathbb{K}$-linear operator that acts on elementary paths by

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0...\widehat{i_q}...i_p}, \tag{1}$$

where the hat $\widehat{i_q}$ means omission of the index i_q.

For example, $e_{ij} \in \Lambda_1$, $e_{ijk} \in \Lambda_2$ and

$$\partial e_{ij} = e_j - e_i, \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}.$$

One can show that $\partial^2 v = 0$ for any $v \in \Lambda_p$ and $p \geq 1$. Hence, we obtain a chain complex $\Lambda_* (V)$:

$$0 \leftarrow \Lambda_0 \leftarrow \Lambda_1 \leftarrow ... \leftarrow \Lambda_{p-1} \leftarrow \Lambda_p \leftarrow ...$$

where arrows are given by the boundary operator ∂.

Definition. An elementary p-path $e_{i_0...i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and non-regular otherwise.

For example, e_{iij} is non-regular, while e_{iji} is regular provided $i \neq j$.

Consider the following subspace of Λ_p:

$$\mathcal{R}_p \equiv \mathcal{R}_p (V) := \operatorname{span}_{\mathbb{K}} \left\{ e_{i_0...i_p} : i_0...i_p \text{ is regular} \right\},$$

whose elements are called *regular* p-paths. We would like to consider ∂ on the spaces $\mathcal{R}_p$. However, ∂ is not invariant on $\{\mathcal{R}_p\}$. For example, $e_{iji} \in \mathcal{R}_2$ for $i \neq j$ while $\partial e_{iji} = e_{ji} - e_{ii} + e_{ij}$ contains a non-regular component e_{ii} and, hence, is not in $\mathcal{R}_1$.

To overcome this difficulty, consider the complementary subspace

$$N_p := \operatorname{span}_{\mathbb{K}} \left\{ e_{i_0...i_p} : i_0...i_p \text{ is non-regular} \right\}$$

One can show that $\partial N_p \subset N_{p-1}$ so that the boundary operator ∂ is well-defined on $\{N_p\}$ and hence, on the quotient spaces Λ_p/N_p. Since $\Lambda_p = \mathcal{R}_p \oplus N_p$ and, hence, $\mathcal{R}_p \cong \Lambda_p/N_p$, we can define a *regular boundary operator* $\partial : \mathcal{R}_p \to \mathcal{R}_{p-1}$ as pullback of $\partial : \Lambda_p/N_p \to \Lambda_{p-1}/N_{p-1}$.

For regular ∂, the formula (1)

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0 \ldots \widehat{i_q} \ldots i_p}$$

should be read as follows: all non-regular paths in the right hand side are set to be 0.

For example, for non-regular $\partial : \Lambda_2 \to \Lambda_1$ we have $\partial e_{iji} = e_{ji} - e_{ii} + e_{ij}$ whereas for regular $\partial : \mathcal{R}_2 \to \mathcal{R}_1$ we have $\partial e_{iji} = e_{ji} + e_{ij}$ since e_{ii} is set to be zero.

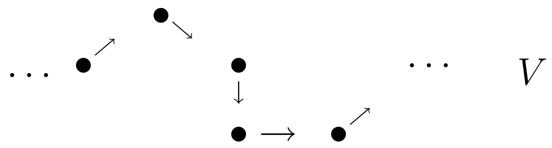
Denote by $\mathcal{R}_* (V)$ the chain complex

$$0 \leftarrow \mathcal{R}_0 \leftarrow \mathcal{R}_1 \leftarrow \ldots \leftarrow \mathcal{R}_{p-1} \leftarrow \mathcal{R}_p \leftarrow \ldots$$

where all the arrows are given by regular operator ∂. Below we use always the regular boundary operator ∂.

Definition. A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and a set $E \subset \{V \times V \setminus \mathrm{diag}\}$ of (directed) *edges*. The fact that $(i, j) \in E$ is also denoted by $i \to j$.

Definition. Let $G = (V, E)$ be a digraph. An elementary p-path $i_0 ... i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, ..., p-1$, and *non-allowed* otherwise.



Consider the following linear space

$$\mathcal{A}_p \equiv \mathcal{A}_p(G) = \mathrm{span}_{\mathbb{K}} \left\{ e_{i_0 ... i_p} : i_0 ... i_p \text{ is allowed} \right\}. \tag{2}$$

Definition. The elements of $\mathcal{A}_p$ are called *allowed* p-paths.

By construction $\mathcal{A}_p \subset \mathcal{R}_p$ but spaces $\mathcal{A}_p$ are in general *not* invariant for ∂. For example, let e_{abc} be allowed, that is, $a \to b \to c$. Then $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab}$ is not allowed if $a \not\to c$.

To fix this problem, consider the following subspace of $\mathcal{A}_p$

$$\Omega_p \equiv \Omega_p(G) := \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}. \tag{3}$$

Definition. The elements of Ω_p are called *∂-invariant* p-paths.

Claim. *If $v \in \Omega_p$ then $\partial v \in \Omega_{p-1}$.*

Indeed, $v \in \Omega_p$ implies $\partial v \in \mathcal{A}_{p-1}$ and $\partial(\partial v) = 0 \in \mathcal{A}_{p-2}$, which implies that $\partial v \in \Omega_{p-1}$.

Hence, we obtain a chain complex $\Omega_* = \Omega_*(G)$:

$$0 \leftarrow \Omega_0 \leftarrow \Omega_1 \leftarrow \ldots \leftarrow \Omega_{p-1} \leftarrow \Omega_p \leftarrow \Omega_{p+1} \leftarrow \ldots$$

Recall that by construction $\Omega_p \subset \mathcal{A}_p \subset \mathcal{R}_p$. Note also that

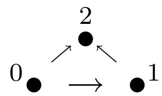
$$\Omega_0 = \mathcal{A}_0 = \mathcal{R}_0 = \operatorname{span}_{\mathbb{K}}\{e_i : i \in V\}, \quad \Omega_1 = \mathcal{A}_1 = \operatorname{span}_{\mathbb{K}}\{e_{ij} : (i,j) \in E\}.$$

Definition. Define the *path homologies* of the digraph G by

$$H_p(G, \mathbb{K}) = H_p(G) := H_p(\Omega_*(G)) = \ker \partial|_{\Omega_p} / \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

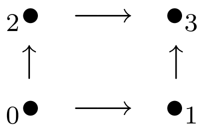
It is easy to show that $H_0(G) \cong \mathbb{K}$ if G is connected, but all other $H_p(G)$ carry non-trivial information about G.

Example. Consider the *triangle* digraph



Then $e_{012} \in \Omega_2$ as $e_{012} \in \mathcal{A}_2$ and $\partial e_{012} = e_{12} - e_{02} + e_{01} \in \mathcal{A}_1$. In fact, $\Omega_2 = \mathcal{A}_2 = \operatorname{span}\{e_{012}\}$, $\Omega_p = \mathcal{A}_p = \{0\}\ \forall p \geq 3$, and $H_p = \{0\}\ \forall p \geq 1$ (the only closed element in Ω_1 is $e_{12} - e_{02} + e_{01}$, which is exact as it is the boundary of e_{012}; hence $H_1 = \{0\}$).

Consider the *square* digraph:

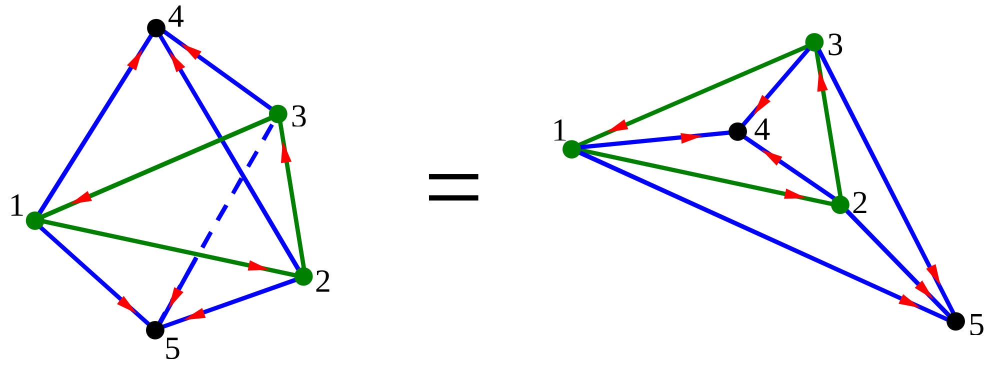


For this digraph $\mathcal{A}_2 = \operatorname{span}\{e_{013}, e_{023}\}$ but neither e_{013} nor e_{023} is ∂-invariant. However, the 2-path $v := e_{013} - e_{023}$ is ∂-invariant as

$$\partial v = (e_{13} - e_{03} + e_{01}) - (e_{23} - e_{03} + e_{02}) = e_{13} + e_{01} - e_{23} - e_{02} \in \mathcal{A}_1,$$

In fact, $\Omega_2 = \operatorname{span}\{v\}$, $\Omega_p = \mathcal{A}_p = \{0\}\ \forall p \geq 3$, and $H_p = \{0\}\ \forall p \geq 1$.

Consider one more example of a digraph G:



A computation shows that $H_1(G) = \{0\}$ and $H_p(G) = \{0\}$ for $p \geq 3$, whereas $\dim H_2(G) = 1$ and

$$H_2(G) = \operatorname{span}\{e_{124} + e_{234} + e_{314} - (e_{125} + e_{235} + e_{315})\}.$$

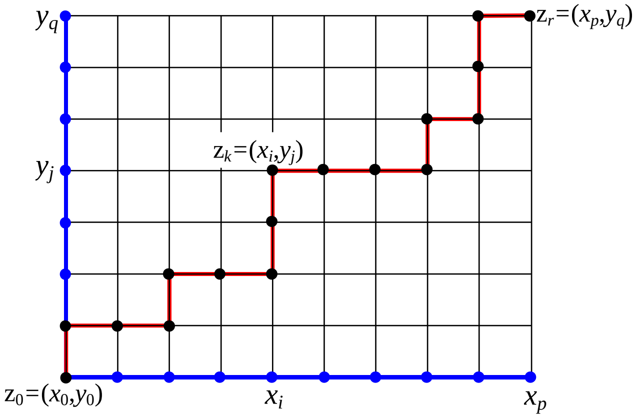
It is interesting to observe that G is a planar graph but nevertheless its second homology group is non-zero.

Cross product of paths

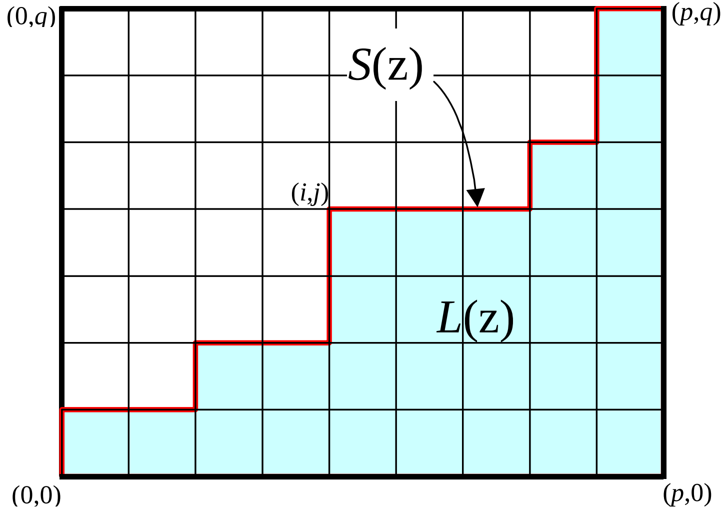
Given two finite sets X, Y, consider their Cartesian product $Z = X \times Y$.

Definition. A regular elementary path $z = z_0 z_1 ... z_r$ on Z is called *step-like* if, for any $k = 1, ..., r$, the vertices z_{k-1} and z_k have the same projections either on X or on Y.

Any step-like r-path z on Z determines by projections regular elementary paths $x = x_0 ... x_p$ and $y =$ on X and $y = y_0 ... y_q$ on Y, where $p + q = r$.



Every vertex (x_i, y_j) of a step-like path z can be represented as a point (i, j) of $\mathbb{Z}^2$ so that the whole path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting the points $(0, 0)$ and (p, q).



Definition. Define the *elevation* $L(z)$ of the path z as the number of the cells in $\mathbb{Z}_+^2$ below the staircase $S(z)$.

By definition, any p-path u on X is given by $u=\sum_x u^x e_x$ where x is any elementary p-paths on X and $u^x \in \mathbb{K}$. Extend the summation to all elementary paths x with arbitrary length, by setting $u^x=0$ if the length of x is not equal to p.

Definition. For any paths $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ with $p,q \geq 0$ define their *cross product* $u \times v$ as a path on Z by the following rule: for any step-like elementary path z on Z, the component $(u \times v)^z$ is defined by

$$(u \times v)^z = (-1)^{L(z)} u^x v^y, \tag{4}$$

where x and y are the projections of z onto X and Y, while for the other paths z set $(u \times v)^z = 0$. It follows that $u \times v \in \mathcal{R}_{p+q}(Z)$.

For any elementary regular p-path x on X and q-path y on Y with $p,q \geq 0$ denote by $\Pi_{x,y}$ the set of all step-like paths z on Z whose projections on X and Y are x and y respectively. It follows from (4) that, for all regular elementary paths x,y,

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z. \tag{5}$$

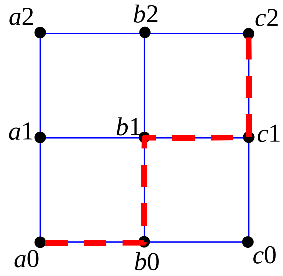
Example. Denote the vertices of X by letters a, b, c etc and the vertices of Y by integers $0, 1, 2$, etc. The vertices of $Z = X \times Y$ will be denoted as $a0, b2$, $c1$, etc, as the fields on the chessboard. For example, we have

$e_a \times e_{01} = e_{a0a1}, \quad e_{ab} \times e_0 = e_{a0b0}$

$e_{ab} \times e_{01} = e_{a0b0b1} - e_{a0a1b1}$

$e_{abc} \times e_{01} = e_{a0b0c0c1} - e_{a0b0b1c1} + e_{a0a1b1c1}$

$e_{abc} \times e_{012} = e_{a0b0c0c1c2} - e_{a0b0b1c1c2} + e_{a0b0b1b2c2} + e_{a0a1b1c1c2} - e_{a0a1b1b2c2} + e_{a0a1a2b2c2}$



Proposition 1 (Product rule for cross product) *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial (u \times v) = (\partial u) \times v + (-1)^p \, u \times (\partial v). \tag{6}$$

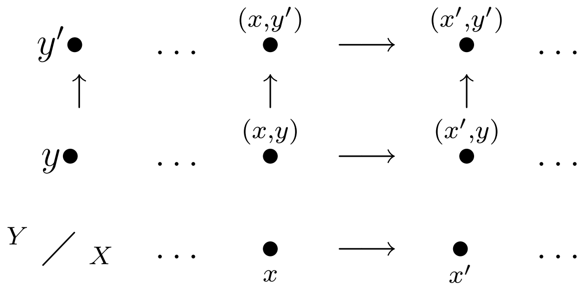
Cartesian product of digraphs

To simplify notation, we denote the set of vertices of a digraph by the same letter as the digraph itself.

Definition. *Cartesian product* $X \Box Y$ of two digraphs X, Y is a digraph Z with the set of vertices $X \times Y = \{(x, y) : x \in X, y \in Y\}$ and with the set of edges as follows: for $x, x' \in X$ and $y, y' \in Y$,

$$(x, y) \to (x', y') \text{ if either } x \to x' \text{ and } y = y' \text{ or } x = x' \text{ and } y \to y'.$$

as is shown on the following diagram:



Clearly, any regular elementary path on $Z = X \Box Y$ is allowed if and only if it is step-like and its projections onto X and Y are allowed.

Proposition 2 *Let* $p, q \geq 0$ *and* $r = p + q$.

(a) If $u \in \mathcal{A}_p(X)$ *and* $v \in \mathcal{A}_q(Y)$ *then* $u \times v \in \mathcal{A}_r(Z)$.

(b) If $u \in \Omega_p(X)$ *and* $v \in \Omega_q(Y)$ *then* $u \times v \in \Omega_r(Z)$. *Moreover, the operation* $u, v \mapsto u \times v$ *extends to that for the homology classes* $u \in H_p(X)$ *and* $v \in H_q(Y)$ *so that* $u \times v \in H_r(Z)$.

Proof. (a) It suffices to prove this for $u = e_x$ and $v = e_y$. By (5) $e_x \times e_y$ is a linear combination of e_z with $z \in \Pi_{x,y}$. If x and y are allowed then any $z \in \Pi_{x,y}$ is allowed, which implies that $e_x \times e_y \in \mathcal{A}_r(Z)$.

(b) We already know that $u \times v$ is allowed. Hence, it suffices to prove that $\partial(u \times v)$ is allowed, which follows from the product rule:

$$\partial(u \times v) = \partial u \times v + (-1)^p u \times \partial v \tag{7}$$

as the right hand side is allowed by (a). For the second claim it suffices to verify two properties. Firstly, if u and v are closed then $u \times v$ is closed, which is obvious from (7). Secondly, if one of u, v is exact then also $u \times v$ is exact: indeed, if, for example, $u = \partial w$ then

$$\partial(w \times v) = \partial w \times v + (-1)^{p+1} w \times \partial v = u \times v$$

so that $u \times v$ is exact. ■

Theorem 3 *Let X, Y be two finite digraphs and $Z = X \Box Y$. Then we have the following isomorphism of the chain complexes:*

$$\Omega_* (Z) \cong \Omega_* (X) \otimes \Omega_* (Y), \tag{8}$$

which is given by the map $u \otimes v \mapsto u \times v$ with $u \in \Omega_* (X)$ and $v \in \Omega_* (Y)$.

The right hand side of (8) is the tensor product of the two chain complexes. More explicitly (8) means that, for any $r \geq 0$,

$$\Omega_r (Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} (\Omega_p (X) \otimes \Omega_q (Y)). \tag{9}$$

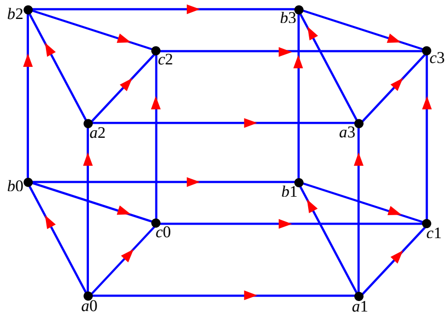
Isomorphism (8) and an abstract theorem of Künneth yield

$$H_* (Z) \cong H_* (X) \otimes H_* (Y). \tag{10}$$

The latter is called the *Künneth formula* for homologies. The Künneth formula is known for simplicial and singular homologies of products. For Cartesian product of digraphs we have a stronger isomorphism (8), which can be referred to as the Künneth formula for chain complexes. It has no analogue in algebraic topology.

Example. Consider the digraph $Z = X \Box Y$, where

$$X = \begin{array}{ccc} & \overset{b}{\bullet} & \\ \nearrow & & \searrow \\ {}^{a}\bullet & \longrightarrow & \bullet^{c} \end{array}, \qquad Y = \begin{array}{ccc} {}^{2}\bullet & \longrightarrow & \bullet_{3} \\ \uparrow & & \uparrow \\ {}^{0}\bullet & \longrightarrow & \bullet_{1} \end{array}.$$

$Z =$ 

For $r = 4$ we obtain from (9) that

$$\Omega_4(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=4\}} \left(\Omega_p(X) \otimes \Omega_q(Y)\right) = \Omega_2(X) \otimes \Omega_2(Y)$$

because on both digraphs X, Y we have $\Omega_p = \{0\}$ for $p \geq 3$.

We know that $\Omega_2(X) = \operatorname{span}(e_{abc})$ and $\Omega_2(Y) = \operatorname{span}(e_{013} - e_{023})$, whence it follows that $\Omega_4(Z)$ is spanned by a singe 4-path

$$
\begin{aligned}
& e_{abc} \times (e_{013} - e_{023}) = e_{a0b0c0c1c3} - e_{a0b0b1c1c3} + e_{a0b0b1b3c3} \\
& + e_{a0a1b1c1c3} - e_{a0a1b1b3c3} + e_{a0a1a3b3c3} \\
& - e_{a0b0c0c2c3} + e_{a0b0b2c2c3} - e_{a0b0b2b3c3} \\
& - e_{a0a2b2c2c3} + e_{a0a2b2b3c3} - e_{a0a2a3b3c3}.
\end{aligned}
$$

Similarly one can compute $\Omega_r(Z)$ for other values of r. For example,

$$
\Omega_3(Z) \cong \Omega_1(X) \otimes \Omega_2(Y) \bigoplus \Omega_2(X) \otimes \Omega_1(Y),
$$

which implies $\dim \Omega_3(Z) = 3 \cdot 1 + 1 \cdot 4 = 7$ and the generators of $\Omega_3(Z)$ are

$$
\begin{aligned}
& e_{ab} \times (e_{013} - e_{023}), \ e_{ac} \times (e_{013} - e_{023}), \ e_{bc} \times (e_{013} - e_{023}) \\
& e_{abc} \times e_{01}, \ e_{abc} \times e_{13}, \ e_{abc} \times e_{02}, \ e_{abc} \times e_{23}
\end{aligned}
$$

Since all the homology groups of X, Y are trivial except for H_0, we obtain that the same is true for homologies of Z.

Example. Consider $Z = X \Box Y$ where X, Y are *cyclic* digraphs:

$$X = \begin{array}{ccc} & \overset{b}{\bullet} & \\ \nearrow & & \searrow \\ {}_a\bullet & \longleftarrow & \bullet_c \end{array}, \quad Y = \begin{array}{ccc} {}_1\bullet & \longrightarrow & \bullet_2 \\ \uparrow & & \downarrow \\ {}_0\bullet & \longleftarrow & \bullet_3 \end{array}.$$

Note that X is not a triangle and Y is not a square.

One can show that all homologies $H_p(X)$ and $H_q(Y)$ are trivial for $p, q \geq 2$ whereas

$$\begin{aligned} H_1(X) &= \operatorname{span}(e_{ab} + e_{bc} + e_{ca}) \\ H_1(Y) &= \operatorname{span}(e_{01} + e_{12} + e_{23} + e_{30}). \end{aligned}$$

It follows from (10) that

$$H_2(Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=2\}} (H_p(X) \otimes H_q(Y)) = H_1(X) \otimes H_1(Y),$$

in particular, $\dim H_2(Z) = 1$. The generating element of $H_2(Z)$ is

$$(e_{ab} + e_{bc} + e_{ca}) \times (e_{01} + e_{12} + e_{23} + e_{30}).$$

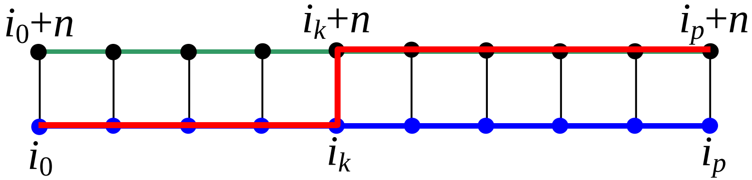
For any digraph X, define the *cylinder* over X by

$$\operatorname{Cyl} X := X \Box Y \text{ with } Y = \left({}^0\bullet \rightarrow \bullet^1 \right).$$

Assuming that the vertices of X are enumerated by $0, 1, ..., n-1$, let us enumerate the vertices of $\operatorname{Cyl} X$ by $0, 1, ..., 2n-1$ as follows: the vertex $(i, 0)$ of $\operatorname{Cyl} X$ receives the number i, while $(i, 1)$ receives $i + n$.

For any regular path v on X, the *lifted* path $\widehat{v}$ on $\operatorname{Cyl} X$ by $\widehat{v} = v \times e_{01}$. For example, if $v = e_{i_0 \ldots i_p}$ then

$$\widehat{v} = e_{i_0 \ldots i_p} \times e_{01} = \sum_{k=0}^{p} (-1)^{p-k} \, e_{i_0 \ldots i_k (i_k + n) \ldots (i_p + n)}. \tag{11}$$



Since $e_{01} \in \Omega_1(Y)$, we see that if $v \in \Omega_p(X)$ then $\widehat{v} \in \Omega_{p+1}(\operatorname{Cyl} X)$.

Example. Let us define the digraph Cube_n inductively: $\operatorname{Cube}_0 = \{0\}$ and

$$\operatorname{Cube}_n = \operatorname{Cyl}\operatorname{Cube}_{n-1}\,.$$

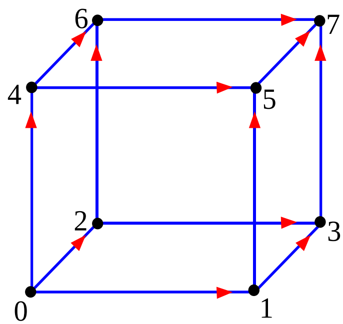
For example, Cube_1 is

$$^0\bullet \longrightarrow \bullet^1$$

Cube_2 is a square

$$\begin{array}{ccc} ^2\bullet & \longrightarrow & \bullet^3 \\ \uparrow & & \uparrow \\ ^0\bullet & \longrightarrow & \bullet^1 \end{array}$$

and Cube_3 is shown here:



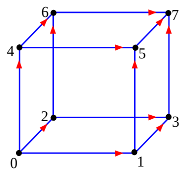
Since $\mathrm{Cube}_n = \mathrm{Cube}_{n-1} \times Y$, where $\Omega_q(Y)$ is non-trivial only for $q = 0, 1$, and $\Omega_n(\mathrm{Cube}_{n-1}) = \{0\}$, we obtain from (9)

$$\Omega_n(\mathrm{Cube}_n) \cong \Omega_{n-1}(\mathrm{Cube}_{n-1}) \otimes \Omega_1(Y).$$

Since $\Omega_1(Y)$ is generated by a single element $v_1 = e_{01}$, we obtain by induction that $\dim \Omega_n(\mathrm{Cube}_n) = 1$. A generating element v_n of $\Omega_n(\mathrm{Cube}_n)$ can be computed inductively by

$$v_n = v_{n-1} \times e_{01} = \widehat{v_{n-1}}.$$

By (11) we obtain successively



$$v_2 = \widehat{v_1} = e_{013} - e_{023},$$
$$v_3 = \widehat{v_2} = e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237},$$
$$\ldots$$
$$\ldots$$

In general, v_n is an alternating sum of $n!$ elementary paths that correspond to partitioning of a solid n-cube into $n!$ simplexes.

By (10) all homology groups of Cube_n are trivial except for H_0.

Sketch of proof of Theorem 3. The main difficulty is to show that each ∂-invariant path w on $Z = X \Box Y$ can be represented as a linear combination of the products $u \times v$ where u is ∂-invariant on X and v is ∂-invariant on Y.

For any $r \geq 0$ consider the space

$$\widetilde{\Omega}_r(Z) = \operatorname{span}\{u \times v : u \in \Omega_p(X), v \in \Omega_q(Y), p + q = r\}$$

By Proposition 2 we have $\widetilde{\Omega}_r(Z) \subset \Omega_r(Z)$, but we have to prove the opposite inclusion. It suffices to prove that

$$\dim \Omega_r(Z) \leq \dim \widetilde{\Omega}_r(Z).$$

In the next argument we take $\mathbb{K} = \mathbb{R}$ (a general field $\mathbb{K}$ requires a more complicated argument). Consider the space

$$\widetilde{\mathcal{A}}_r(Z) = \operatorname{span}\{u \times v : u \in \mathcal{A}_p(X), v \in \mathcal{A}_q(Y), p + q = r\}.$$

By Proposition 2 we have $\widetilde{\mathcal{A}}_r(Z) \subset \mathcal{A}_r(Z)$.

We prove separately, that any element from $\Omega_r(Z)$ is a linear combination of $e_x \times e_y$ with allowed x, y, which implies

$$\Omega_r(Z) \subset \widetilde{\mathcal{A}}_r(Z). \tag{12}$$

If digraphs X, Y are such that $\Omega_p(X) = \mathcal{A}_p(X)$ and $\Omega_q(Y) = \mathcal{A}_q(Y)$ for all $p, q \geq 0$ then also $\widetilde{\Omega}_r(Z) = \widetilde{\mathcal{A}}_r(Z)$. Substitution into (12) yields $\Omega_r(Z) \subset \widetilde{\Omega}_r(Z)$, which finishes the proof in this case. However, the main difficulty lies in the fact that in general $\Omega_p \subsetneqq \mathcal{A}_p$.

In the general case we use the inner product for regular paths u, v on a digraph:

$$[u, v] = \sum_x u^x v^x,$$

for which we need $\mathbb{K} = \mathbb{R}$. We prove that if u, u' are allowed paths on X and v, v' are allowed paths on Y then

$$[u \times v, u' \times v'] = C[u, u'][v, v'], \tag{13}$$

where C is a constant depending on the lengths of the paths.

Define the following subspaces:

$\Omega_p^\perp(X)$ – the orthogonal complement of $\Omega_p(X)$ in $\mathcal{A}_p(X)$.

$\Omega_q^\perp(Y)$ – the orthogonal complement of $\Omega_q(Y)$ in $\mathcal{A}_q(Y)$.

$\Omega_r^\perp(Z)$ – the orthogonal complement of $\Omega_r(Z)$ in $\widetilde{\mathcal{A}}_r(Z)$.

We use (13) in order to prove that, for $p+q=r$,

$$\begin{array}{ll} u\in\Omega_p^{\perp}\left(X\right), \quad v\in\mathcal{A}_q\left(Y\right) & \Rightarrow u\times v\in\Omega_r^{\perp}\left(Z\right), \\ u\in\mathcal{A}_p\left(X\right), \quad v\in\Omega_q^{\perp}\left(Y\right) & \Rightarrow u\times v\in\Omega_r^{\perp}\left(Z\right), \end{array} \tag{14}$$

Since

$$\mathcal{A}_p\left(X\right)=\Omega_p\left(X\right)\oplus\Omega_p^{\perp}\left(X\right),$$

any $u\in\mathcal{A}_p\left(X\right)$ admits a decomposition $u=u_{\Omega}+u_{\perp}$ where $u_{\Omega}\in\Omega_p\left(X\right)$ and $u_{\perp}\in\Omega_p^{\perp}\left(X\right)$. Using also a similar decomposition $v=v_{\Omega}+v_{\perp}$ for $v\in\mathcal{A}_q\left(Y\right)$, we obtain

$$u\times v=u_{\Omega}\times v_{\Omega}+u_{\Omega}\times v_{\perp}+u_{\perp}\times v_{\Omega}+u_{\perp}\times v_{\perp}.$$

where $u_{\Omega}\times v_{\Omega}\in\widetilde{\Omega}_r\left(Z\right)$, while by (14) all other terms in the right hand side belong to $\Omega_r^{\perp}\left(Z\right)$. It follows that

$$u\times v\in\widetilde{\Omega}_r\left(Z\right)+\Omega_r^{\perp}\left(Z\right).$$

Since $\widetilde{\mathcal{A}}_r\left(Z\right)$ is spanned by the products $u\times v$ where u,v are allowed, we obtain that

$$\widetilde{\mathcal{A}}_r\left(Z\right)\subset\widetilde{\Omega}_r\left(Z\right)+\Omega_r^{\perp}\left(Z\right).$$

Comparing with the decomposition

$$\widetilde{\mathcal{A}}_r\left(Z\right) = \Omega_r\left(Z\right) \oplus \Omega_r^{\perp}\left(Z\right),$$

we obtain $\dim \Omega_r\left(Z\right) \leq \dim \widetilde{\Omega}_r\left(Z\right)$, which was to be proved. ■