Landis' proof of Harnack inequalities

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Harnack inequality in $\mathbb{R}^n$

Let L be an elliptic operator in $\mathbb{R}^n$ of one of the forms

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) \tag{1}$$

or

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}. \tag{2}$$

We say that L satisfies a uniform Harnack inequality (H) if there exists a constant C such that, for any positive solution u of $Lu = 0$ in a ball $B_r(x)$, we have

$$\sup_{B_{r/2}(x)} u \le C \inf_{B_{r/2}(x)} u.$$

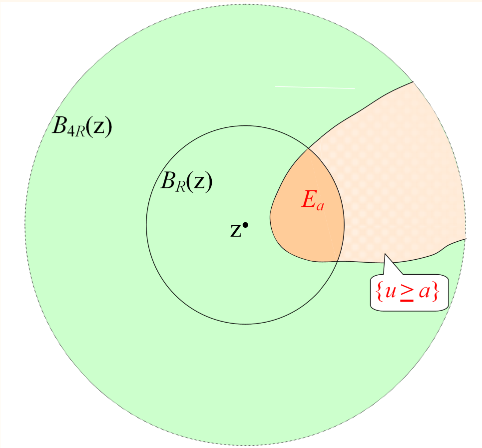
If $\{a_{ij}\}$ is uniformly elliptic then the operator (1) satisfies (H) by a theorem of Moser [7], 1961, while (2) satisfies (H) by a theorem of Krylov and Safonov [5], 1980.

E.M.Landis worked on both problems. He developed an alternative approach to the proof of Moser's theorem. Using this approach, he proved (H) for a class of non-divergent operators L of Cordes type [6], 1971. His ideas were used by Krylov and Safonov [5], 1980.

The approach of Landis has been useful for elliptic and parabolic PDEs on Riemannian manifolds and even on singular metric measure spaces of fractal types ([1], [2], [3], etc.).

Weak Harnack inequality

Fix $z \in \mathbb{R}^n$ and write $B_R = B_R(z)$. Let λ be the ellipticity constant of L. Let u be a positive solution of $Lu = 0$ in some ball B_{4R}.



Denote

$$E_a = \{u \geq a\} \cap B_R.$$

We say that L satisfies the *weak Harnack inequality* (wH) if for any $\theta > 0$ there exists $\delta = \delta(\theta, n, \lambda) > 0$ s.t.

$$|E_a| \geq \theta |B_R| \ \Rightarrow \ \inf_{B_R} u \geq \delta a$$

Clearly, $(H) \Rightarrow (wH)$ because if E_a is non-empty then by (H)

$$\inf_{B_R} u \geq C^{-1} \sup_{B_R} u \geq C^{-1} a.$$

Theorem 1 *(E.M.Landis)* $(wH) \Rightarrow (H)$

This theorem works in a very general setting of metric measure spaces (see [3]) and uses only the following properties of solutions and the underlying space (the operator L is not used explicitly):

(i) if u is a solution then also $u + \text{const}$ is also a solution;

(ii) volume doubling: $|B_{2R}| \leq C\,|B_R|$.

The arguments below follow [4].

Proof of (wH) for L in the divergence form

For simplicity take $L = \Delta$. Let $a = 1$. Consider the function $v = \log \frac{1}{u}$ so that $\Delta v = |\nabla v|^2$. Multiplying this equation by a cutoff function and integrating by parts, we obtain

$$\int_{B_{2R}} |\nabla v|^2 \, d\mu \leq C \frac{|B_{2R}|}{R^2}. \tag{3}$$

Consider the set

$$E = \{u \geq 1\} \cap B_R = \{v \leq 0\} \cap B_R = \{v_+ = 0\} \cap B_R.$$

By a version of the Poincaré inequality

$$\int_{B_{2R}} |\nabla v|^2 \, d\mu \geq c \frac{|E|}{R^2 \, |B_{2R}|} \int_{B_{2R}} v_+^2 d\mu. \tag{4}$$

Since $|E| \geq \theta \, |B_R|$, combining (3) and (4) yields

$$⨍_{B_{2R}} v_+^2 d\mu \leq \frac{\text{const}}{\theta}.$$

On the other hand, since $\Delta v \geq 0$, Moser's mean value inequality for subsolutions yields

$$\sup_{B_R} v_+^2 \leq C ⨍_{B_{2R}} v_+^2 d\mu$$

whence

$$\sup_{B_R} v_+ \leq \frac{\text{const}}{\theta}$$

and

$$\inf_{B_R} u \geq \delta \left(\theta\right) > 0.$$

Preliminaries for (wH) for L in the non-divergence form

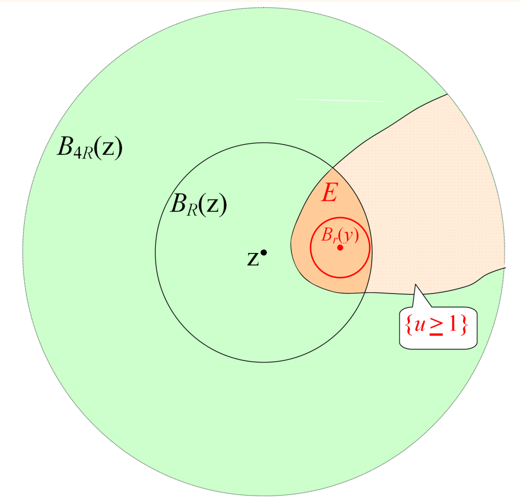
Lemma 2 *Let u be a positive solution of $Lu = 0$ in B_{4R}. If $E = \{u \geq a\} \cap B_R$ contains a ball $B_r(y)$ then*

$$\inf_{B_R} u \geq c\left(\frac{r}{R}\right)^s a \tag{5}$$

for some $c, s > 0$ depending on n and λ.

Proof. Let $a = 1$. We use the following barrier function

$$w(x) = \left(\frac{1}{|x-y|^s} - \frac{1}{(3R)^s}\right) r^s$$



It satisfies $w|_{\partial B_r(y)} \leq 1$ and $w|_{\partial B_{4R}(z)} \leq 0$
If s is big enough then $Lw > 0$.
Comparing w and u by the maximum principle, we obtain $u \geq w$ in $B_{4R}(z) \setminus B_r(y)$. Since

$$\inf_{B_R(z)} w(x) \geq \left(\frac{1}{(2R)^s} - \frac{1}{(3R)^s}\right) r^s = c\left(\frac{r}{R}\right)^s$$

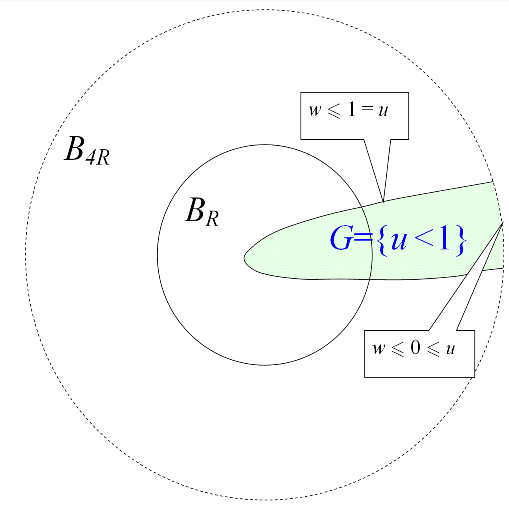
we obtain the same lower bound for u in B_R that is (5).

Lemma 3 (Lemma of growth in a thin domain) *Let u be a non-negative L-harmonic function in a ball B_{4R}. There exists $\varepsilon = \varepsilon(n, \lambda) > 0$ with the following property: if*

$$|\{u < a\} \cap B_{4R}| \le \varepsilon |B_{4R}|$$

then $\inf_{B_R} u \ge \frac{1}{2}a$.

Proof. Let $z = 0$, $a = 1$, $G = \{u < 1\} \cap B_{4R}$. Let us solve in B_{4R} the Dirichlet problem



$$Lv = -1_G, \qquad v|_{\partial B_{4R}} = 0.$$

Then $v \ge 0$ and, by the theorem of Alexandrov and Pucci,

$$\sup_{B_{4R}} v \le CR \|1_G\|_{L^n} = CR|G|^{1/n} \le CR^2\varepsilon^{1/n}.$$

The function $w(x) = 1 - \frac{|x|^2}{(4R)^2} - K\frac{v(x)}{R^2}$ satisfies in G the inequality $Lw \ge 0$ provided K is large enough.

Since $w \leq 1$ and $w|_{\partial B_{4R}} \leq 0$, it follows that $w \leq u$ in G. Hence, for a small enough ε,

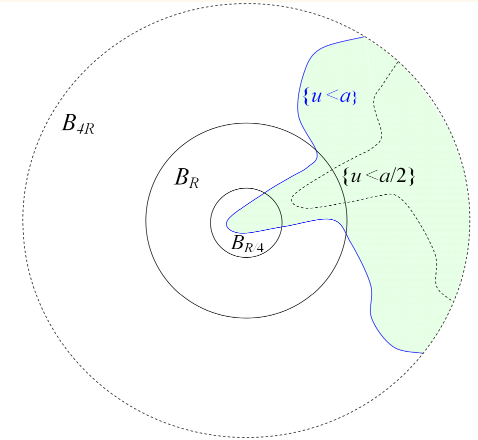
$$\inf_{B_R} u = \inf_{B_R \cap G} u \geq \inf_{B_R \cap G} w \geq \inf_{B_R} w \geq 1 - \frac{1}{16} - KC\varepsilon^{1/n} > \frac{1}{2}.$$

Lemma 4 *Let u be a non-negative L-harmonic function in a ball B_{4R}. If*

$$|\{u < a\} \cap B_R| \leq \varepsilon |B_R|$$

then $\inf_{B_R} u \geq \gamma a$, *where* $\gamma = \gamma(n, \lambda) > 0$.

Proof. Applying Lemma 3 to the ball B_R instead of B_{4R}, we obtain that



$$\inf_{B_{R/4}} u \geq \frac{a}{2}.$$

Hence, the set $\left\{u \geq \frac{a}{2}\right\} \cap B_R$ contains the ball $B_{R/4}$. By Lemma 2, we obtain

$$\inf_{B_R} u \geq c \left(\frac{R/4}{R}\right)^s a = c4^{-s} a,$$

which was to be proved.

Proof of (wH) for L in the non-divergence form

Let u be a positive solution to $Lu = 0$ in a ball B_{4R}. Assuming that the set

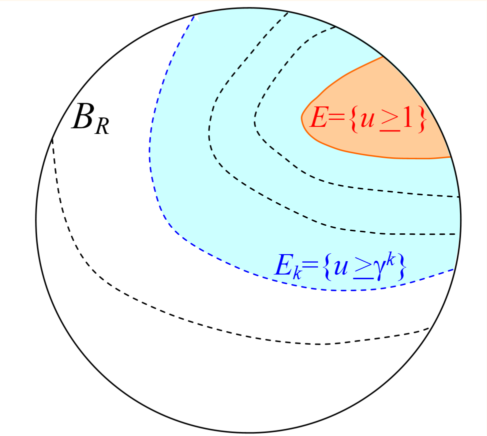
$$E = \{u \geq 1\} \cap B_R$$

satisfies the condition $|E| \geq \theta |B_R|$, we need to prove that $\inf_{B_R} u \geq \delta$ for some $\delta > 0$.

Consider for any non-negative integer k the set

$$E_k = \left\{u \geq \gamma^k\right\} \cap B_R,$$

where $\gamma \in (0,1)$ is the constant from Lemma 4.



Claim. There exist $\beta > 0$ and a positive integer l such that for any $k \geq 0$ the following dichotomy holds:
(i) either $|E_{k+1}| \geq (1+\beta)|E_k|$
(ii) or $E_{k+l} = B_R$

Let (i) hold for $k = 0, ..., N-1$ and does not hold for $k = N$. Then we have

$$|E_N| \geq (1+\beta)|E_{N-1}| \geq ... \geq (1+\beta)^N |E_0|.$$

Since $|E_N| \leq |B_R|$ and $|E_0| = |E| \geq \theta\,|B_R|$, it follows that $\theta\,(1+\beta)^N \leq 1$ whence

$$N \leq \frac{\ln \frac{1}{\theta}}{\ln(1+\beta)}.$$

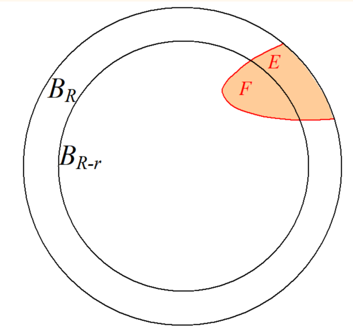
On the other hand, applying (ii) for $k = N$, we obtain $E_{N+l} = B_R$ that is,

$$\inf_{B_R} u = \inf_{E_{N+l}} u \geq \gamma^{N+l} \geq \gamma^{\frac{\ln \frac{1}{\theta}}{\ln(1+\beta)}+l} =: \delta.$$

It suffices to prove Claim for the special case $k = 0$, that is,

(i) either $|E_1| \geq (1+\beta)\,|E_0|$ (ii) or $E_l = B_R$

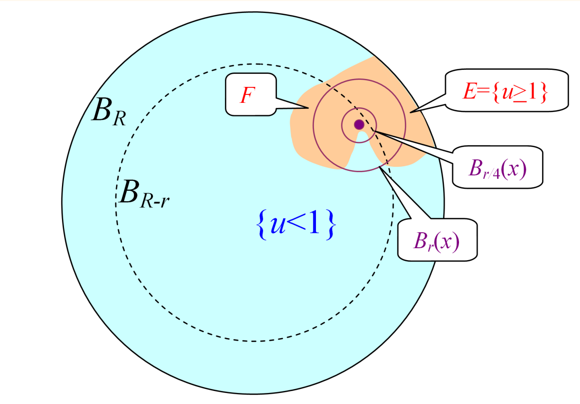
while for a general case apply the special case to u/γ^k.



Choose $0 < r < R$ so that the set

$$F := E \cap B_{R-r} = \{u \geq 1\} \cap B_{R-r}$$

has measure $|F| = \dfrac{1}{2}\,|E|$, and consider two cases.



Case 1. Assume that there exists $x \in F$ such that

$$|\{u < 1\} \cap B_r(x)| \le \varepsilon \, |B_r| \, ,$$

where ε is the constant from Lemma 3. By Lemma 3

$$\inf_{B_{r/4}(x)} u \ge \frac{1}{2}$$

Hence, in B_R there is a ball $B_{r/4}(x)$ where $u \ge \dfrac{1}{2}$.

By Lemma 2, we conclude that

$$\inf_{B_R} u \ge c \left(\frac{r/4}{R} \right)^s \frac{1}{2}.$$

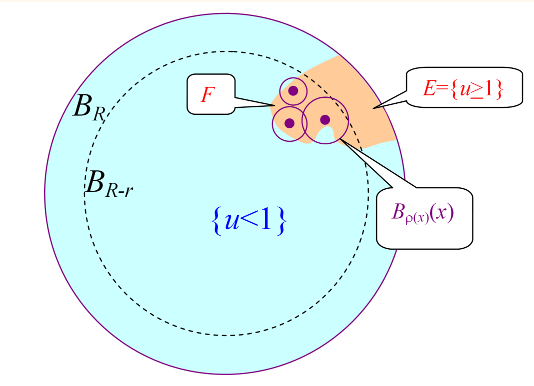
By the choice of r we have $|B_R| - |B_{R-r}| = |B_R \setminus B_{R-r}| \ge |E \setminus F| = \frac{1}{2} |E| \ge \frac{1}{2}\theta \, |B_R|$ which implies after division by $|B_R| = cR^n$ that

$$1 - \left(\frac{R-r}{R} \right)^n \ge \frac{1}{2}\theta.$$

It follows that $\dfrac{r}{R} \geq 1 - \left(1 - \frac{1}{2}\theta\right)^{1/n}$ .and, hence,

$$\inf_{B_R} u \geq \frac{c}{2} 4^{-s} \left(1 - \left(1 - \frac{1}{2}\theta\right)^{1/n}\right)^s =: \delta > 0.$$

Therefore, $E_l = B_R$ for any l such that $\gamma^l \leq \delta$, that is, the alternative (ii) takes places.



Case 2 (main). Assume that, for any $x \in F$, we have

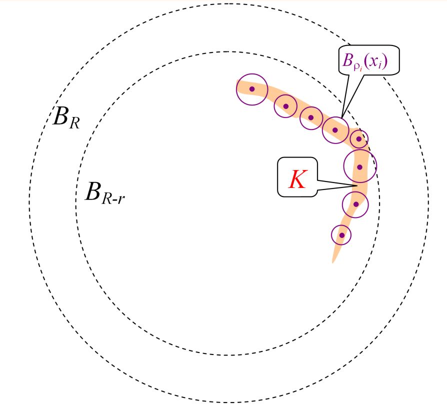
$$|\{u < 1\} \cap B_r(x)| > \varepsilon |B_r|,$$

For any $x \in F$ and $\rho > 0$ consider the quotient:

$$Q(x, \rho) = \frac{|\{u < 1\} \cap B_\rho(x)|}{|B_\rho|}$$

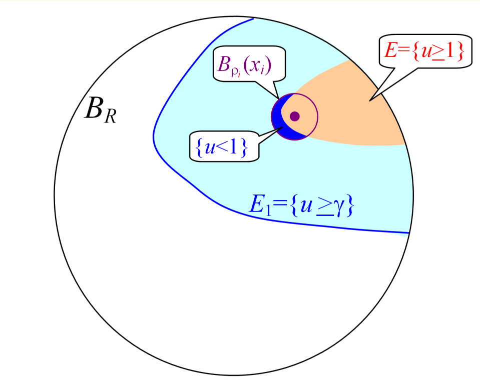
As $\rho \to 0$, $Q(x, \rho) \to 0$ for almost all $x \in F$ because in F we have $u \geq 1$. On the other hand, $Q(x, r) > \varepsilon$ for any $x \in F$. Hence, for almost all $x \in F$, there exists $\rho(x) \in (0, r)$ such that $Q(x, \rho(x)) = \varepsilon$, that is,

$$\left|\{u < 1\} \cap B_{\rho(x)}(x)\right| = \varepsilon \left|B_{\rho(x)}\right|. \tag{6}$$



There is a compact set $K \subset F$ such that $|K| \geq \frac{1}{2}|F| = \frac{1}{4}|E|$ and such that $\rho(x)$ is defined for all $x \in K$.
By a standard ball covering argument, there exists in K a finite sequence $\{x_i\}$ such that the balls $\left\{B_{\rho_i}(x_i)\right\}$ are disjoint while $\left\{B_{3\rho_i}(x_i)\right\}$ cover K, where $\rho_i = \rho(x_i)$.
Since $x_i \in B_{R-r}$ and $\rho_i < r$, it follows that $B_{4\rho_i}(x_i) \subset B_{4R}$.
Using (6) and Lemma 4, we obtain that

$$\inf_{B_{\rho_i}(x_i)} u \geq \gamma.$$



It follows that
$(E_1 \setminus E) \cap B_{\rho_i(x_i)} = \{\gamma \leq u < 1\} \cap B_{\rho_i(x_i)} = \{u < 1\} \cap B_{\rho_i}(x_i)$
whence by (6)

$$\left|(E_1 \setminus E) \cap B_{\rho_i(x_i)}\right| = \varepsilon \left|B_{\rho_i}(x_i)\right|.$$

Hence, $\quad |E_1 \setminus E| \geq \sum_i \varepsilon \left|B_{\rho_i}(x_i)\right| = 3^{-n} \sum_i \varepsilon \left|B_{3\rho_i}(x_i)\right|$

$\geq 3^{-n}\varepsilon |K| \geq \beta |E| \quad$ where $\beta = \frac{1}{4}3^{-n}\varepsilon$,

and $\quad |E_1| \geq (1+\beta)|E|$ so that we have Case (i).

Preliminaries for the proof of $(wH) \Rightarrow (H)$

Lemma 5 (Reiteration of the weak Harnack inequality)

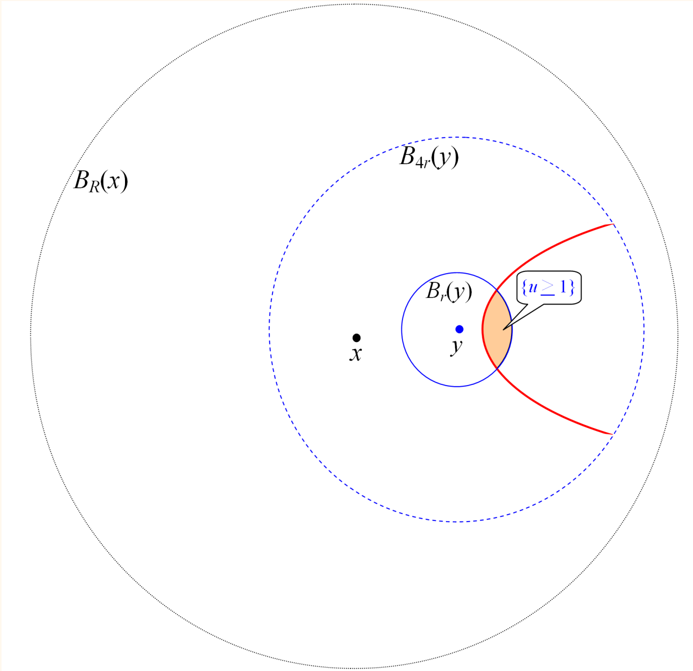
Let u be a non-negative L-harmonic function in some ball $B_R(x)$. Consider a ball $B_r(y)$ where $y \in B_{\frac{1}{9}R}(x)$ and $r \leq \frac{2}{9}R$.
If for some $\theta > 0$

$$|\{u \geq 1\} \cap B_r(y)| \geq \theta |B_r|$$

then

$$u(x) \geq \delta \left(\frac{r}{R}\right)^s$$

where $\delta = \delta(\theta, n, \lambda) > 0$ and $s = s(n, \lambda) > 0$.



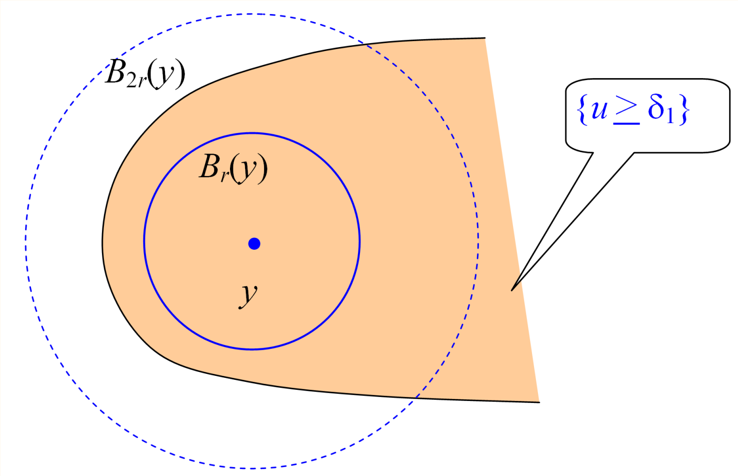
Proof. Note that $B_{4r}(y) \subset B_R(x)$ because $|x-y| + 4r < \frac{1}{9}R + \frac{8}{9}R = R$.

Applying the weak Harnack inequality in $B_r(y)$, we obtain that

$$\inf_{B_r(y)} u \geq \delta_1 := \delta(\theta, n, \lambda).$$

It follows that
$|\{u \geq \delta_1\} \cap B_{2r}(y)| \geq |B_r| = 2^{-n}|B_{2r}|$

If $B_{8r}(y) \subset B_R(x)$ then applying the weak Harnack inequality in $B_{2r}(y)$ we obtain that
$\inf_{B_{2r}(y)} u \geq \delta_1 \delta(2^{-n}, n, \lambda) = \varepsilon\delta_1$
where $\varepsilon = \delta(2^{-n}, n, \lambda)$.



Continuing by induction we obtain the following statement for any positive integer k:

$$\text{if } B_{2^{k+2}r}(y) \subset B_R(x) \text{ then } \inf_{B_{2^k r}} u \geq \varepsilon^k \delta_1. \qquad (7)$$

Let k be the maximal integer such that

$$B_{2^{k+2}r}(y) \subset B_R(x).$$

Then

$$2^{k+2}r + |x-y| \leq R$$

while

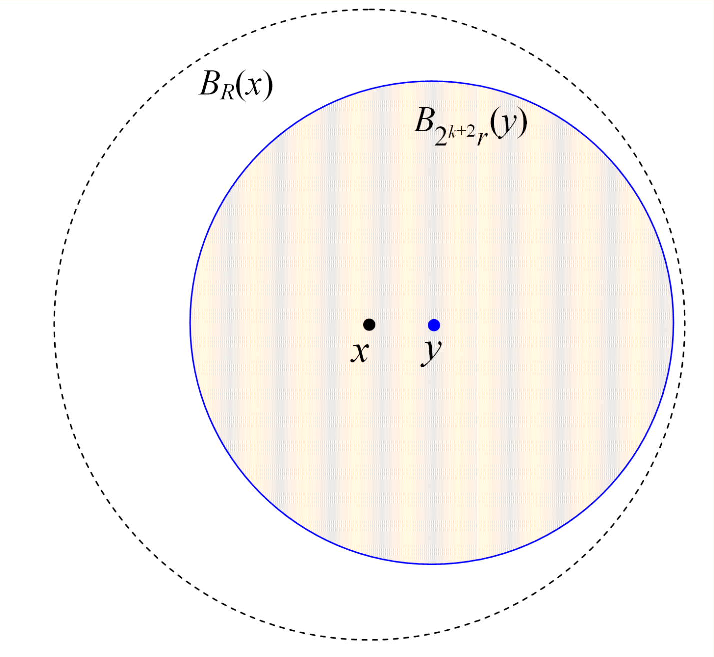
$$2^{k+3}r + |x-y| > R.$$

Since $R > 9\,|x-y|$, it follows that

$$2^k r > \frac{1}{8}\left(R - |x-y|\right) \geq |x-y|$$

and $x \in B_{2^k r}(y)$. By (7) we have

$$u(x) \geq \varepsilon^k \delta_1.$$



On the other hand, $2^k r < R$ whence $k \leq \log_2 \frac{R}{r}$. It follows that

$$u(x) \geq \varepsilon^{\log_2 \frac{R}{r}} \delta_1 = \delta_1 \left(\frac{R}{r}\right)^{\log_2 \varepsilon} = \delta_1 \left(\frac{r}{R}\right)^s.$$

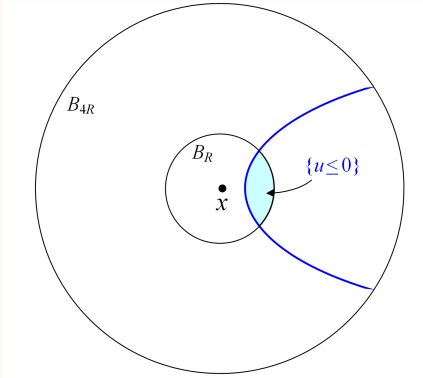
Lemma 6 (Alternative form of the weak Harnack inequality)

Let u be an L-harmonic function in some ball $B_{4R}(x)$
If for some $\theta > 0$

$$|\{u \le 0\} \cap B_R(x)| \ge \theta \, |B_R| \, ,$$

then

$$\sup_{B_{4R}(x)} u \ge (1+\delta) \, u(x)$$



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where $\delta = \delta\,(\theta, n, \lambda) > 0$ is the same as in (wH).

Proof. If $u(x) \le 0$ then there is nothing to prove. Assume that $u(x) > 0$. By rescaling, we can assume also that

$$\sup_{B_{4R}(x)} u = 1.$$

Consider the function $v = 1 - u$ that is a non-negative L-harmonic function in $B_{4R}(x)$. Observe also, that

$$u \le 0 \Leftrightarrow v \ge 1.$$

Hence, we obtain that

$$|\{v \geq 1\} \cap B_R(x)| \geq \theta\,|BR|\,.$$

By the weak Harnack inequality, we conclude that

$$\inf_{B_R(x)} v \geq \delta,$$

where $\delta = \delta\,(n, \lambda, \theta) > 0$. It follows that $v(x) \geq \delta$ and, hence

$$u(x) \leq 1 - \delta < \frac{1}{1+\delta} = \frac{1}{1+\delta} \sup_{B_{4R}} u,$$

which was to be proved.

Lemma 7 (Lemma of growth in a thin domain) *There exists $\varepsilon = \varepsilon\,(n, \lambda) > 0$ such that the following is true: if u is an L-harmonic function in a ball $B_R(x)$ and if*

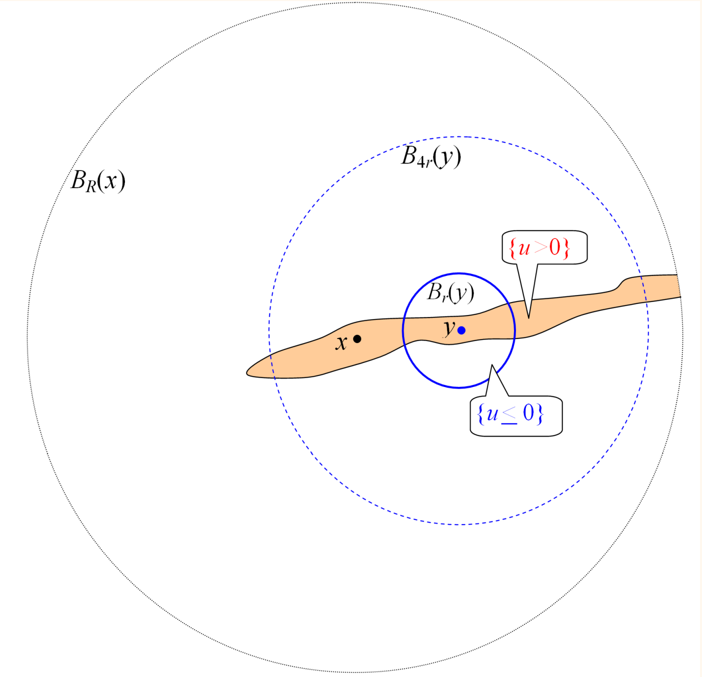
$$|\{u > 0\} \cap B_R(x)| \leq \varepsilon\,|B_R|$$

then

$$\sup_{B_R(x)} u \geq 4u(x).$$

Proof. Fix $\varepsilon > 0$ that will be specified later.

Consider any ball $B_r(y) \subset B_R(x)$
of radius $r = (2\varepsilon)^{\frac{1}{n}} R$ so that $|B_r| = 2\varepsilon |B_R|$.



Then

$$|\{u > 0\} \cap B_r(y)| \leq \varepsilon |B_r| \frac{|B_R|}{|B_r|} \leq \varepsilon \frac{1}{2\varepsilon} = \frac{1}{2}$$

whence

$$|\{u \leq 0\} \cap B_r(y)| \geq \tfrac{1}{2} |B_r|.$$

If $B_{4r}(y) \subset B_R(x)$ then by Lemma 6

$$\sup_{B_{4r}(y)} u \geq (1+\delta)\, u(y)$$

where $\delta = \delta\left(n, \lambda, \frac{1}{2}\right) > 0$. By slightly reducing δ, we obtain the following claim.

Claim. *If $B_{4r}(y) \subset B_R(x)$ and $r = (2\varepsilon)^{1/n} R$ then there exists $y' \in B_{4r}(y)$ such that*

$$u(y') \geq (1+\delta)\, u(y),$$

where $\delta > 0$ depends on n, λ.

Applying this Claim with $y = x$ and with $(2\varepsilon)^{1/n} < \frac{1}{4}$ so that $r < R/4$ and, hence, $B_{4r}(x) \subset B_R(x)$, we obtain a point $x_1 \in B_{4r}(x)$ such that

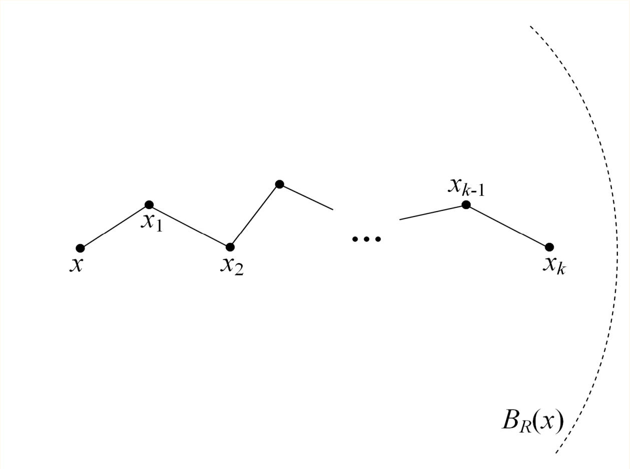
$$u(x_1) \geq (1+\delta)\, u(x).$$

If $B_{4r}(x_1) \subset B_R(x)$ then applying Claim again we obtain a point $x_2 \in B_{4r}(x_1)$ such that

$$u(x_2) \geq (1+\delta)\, u(x_1).$$

We continue construction of the sequence $\{x_k\}$ by induction: as long as $B_{4r}(x_k) \subset B_R(x)$, we obtain $x_{k+1} \in B_{4r}(x_k)$ such that

$$u(x_{k+1}) \geq (1+\delta)\, u(x_k).$$



We stop construction if, for some k, $B_{4r}(x_k)$ is not contained in $B_R(x)$. Hence, if x_k exists then $x_k \in B_R(x)$ and

$$u(x_k) \geq (1+\delta)^k u(x). \tag{8}$$

Besides, we have

$$|x_{l+1} - x_l| < 4r \quad \text{for all } l \leq k-1,$$

which implies that

$$|x_k - x| < 4kr.$$

It is easy to see that if $4kr < R$ then x_k exists. Choose the maximal integer k with $4kr < R$. Then we have

$$4(k+1)r \geq R$$

and, hence,

$$k \geq \frac{R}{4r} - 1 = \frac{1}{4(2\varepsilon)^{1/n}} - 1.$$

It follows from (8) that

$$u(x_k) \geq (1+\delta)^{\frac{1}{4(2\varepsilon)^{1/n}}-1} u(x).$$

Finally, choosing ε small enough, we obtain

$$\sup_{B_R(x)} u \geq u(x_k) \geq 4u(x).$$

Corollary 8 *Let u be an L-harmonic function in a ball $B_R(x)$. If for some $a \in \mathbb{R}$*

$$|\{u > a\} \cap B_R(x)| \leq \varepsilon |B_R|,$$

where $\varepsilon = \varepsilon(n, \lambda)$ is as above, then

$$\sup_{B_R(x)} u \geq a + 4(u(x) - a).$$

Proof. Indeed, just apply Lemma 7 to the L-harmonic function $v = u - a$.

Proof of $(wH) \Rightarrow (H)$

It suffices to prove the following: if u is a non-negative L-harmonic function on a ball $B_{KR}(x)$ (where $K = 18$) and

$$\sup_{B_R(x)} u = 2, \tag{9}$$

then

$$u(x) \geq c = c(n, \lambda) > 0. \tag{10}$$

We construct a sequence $\{x_k\}_{k \geq 1}$ of points such that

$$x_k \in B_{2R}(x) \quad \text{and} \quad u(x_k) = 2^k. \tag{11}$$

A point x_1 with $u(x_1) = 2$ exists in $\overline{B}_R(x)$ by (9). Assume that x_k satisfying (11) is already constructed. Then, for small enough $r > 0$, we have

$$\sup_{B_r(x_k)} u \leq 2^{k+1}.$$

Set

$$r_k = \sup \left\{ r \in (0, R] : \sup_{B_r(x_k)} u \leq 2^{k+1} \right\}.$$

If $r_k = R$ then we stop the process without constructing x_{k+1}. If $r < R$ then we necessarily have

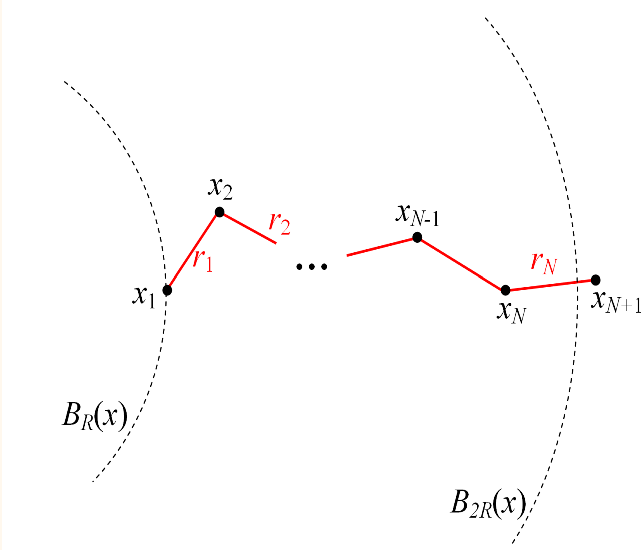
$$\sup_{B_r(x_k)} u = 2^{k+1}.$$

Therefore, there exists $x_{k+1} \in \overline{B}_{r_k}(x_k)$ such that $u(x_{k+1}) = 2^{k+1}$. If $x_{k+1} \in B_{2R}(x)$ then we keep x_{k+1} and go to the next step. If $x_{k+1} \notin B_{2R}(x)$ then we discard x_{k+1} and stop the process.

Hence, we obtain a sequence of balls $\{B_{r_k}(x_k)\}$ such that

$$r_k \leq R, \quad x_k \in B_{2R}(x), \quad u(x_k) = 2^k, \quad \sup_{B_{r_k}(x_k)} u \leq 2^{k+1}. \tag{12}$$

Moreover, we have also $|x_{k+1} - x_k| \le r_k$.
The sequence $\{x_k\}$ cannot be infinite
as $u\left(x_k\right) \to \infty$, while u is bounded in $\overline{B}_{2R}(x)$.



Let N be the largest value of k in this sequence.
Then:

either $r_N = R$ or $r_N < R$ and $x_{N+1} \notin B_{2R}(x)$,

where x_{N+1} is the discarded point.

In the both cases we clearly have

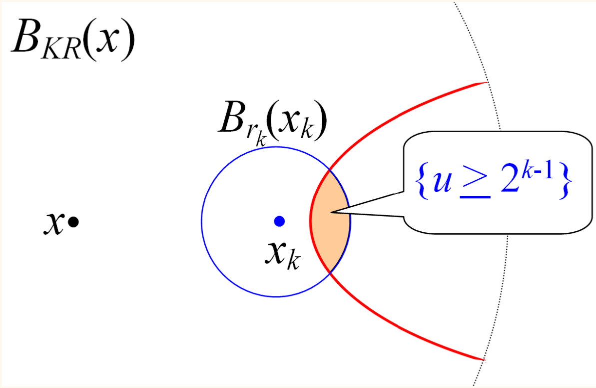
$$r_1 + ... + r_N \ge R. \tag{13}$$

In any ball $B_{r_k}\left(x_k\right)$ we have by (12)

$$\sup_{B_{r_k}(x_k)} u \le 2^{k+1} < 2^{k-1} + 4\left(2^k - 2^{k-1}\right) = 2^{k-1} + 4\left(u\left(x_k\right) - 2^{k-1}\right).$$

By Corollary 8 with $a = 2^{k-1}$ we obtain

$$\left|\left\{u \geq 2^{k-1}\right\} \cap B_{r_k}(x_k)\right| \geq \varepsilon \left|B_{r_k}\right|$$



We apply Lemma 5 with $B_r(y) = B_{r_k}(x_k)$.
Since u is non-negative and L-harmonic in $B_{KR}(x)$, the following conditions need to be satisfied:

$$r_k \leq \tfrac{2}{9}KR \quad \text{and} \quad |x_k - x| \leq \tfrac{1}{9}KR$$

Since $r_k \leq R$ and $|x_k - x| \leq 2R$, the both conditions are satisfied if $K = 18$.

By Lemma 5, we obtain that

$$u(x) \geq \left(\frac{r_k}{R}\right)^s \delta 2^{k-1}, \tag{14}$$

where $\delta = \delta(\varepsilon, n, \lambda) > 0$ and $s = s(n, \lambda) > 0$.

The question remains how to estimate

$$\left(\frac{r_k}{R}\right)^s 2^{k-1}$$

from below, given the fact that we do not know much about the sequence $\{r_k\}$: the only available information is (13). The following trick was invented by Landis.

Since $r_1 + r_2 + ... + r_N \geq R$ and

$$\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{\pi^2}{12} < 1,$$

there exists $k \leq N$ such that

$$r_k \geq \frac{R}{2k^2}.$$

For this k we obtain from (14) that

$$u(x) \geq \delta \left(\frac{r_k}{R} \right)^s 2^{k-1} \geq \delta \frac{2^{k-1}}{(2k^2)^s}.$$

Finally, since

$$m := \inf_{k \geq 1} \frac{2^{k-1}}{(2k^2)^s} > 0,$$

we conclude that

$$u(x) \geq \delta m =: c,$$

which finishes the proof of (10).

References

[1] Grigor'yan A., *The heat equation on non-compact Riemannian manifolds* (in Russian), Matem. Sbornik **182** (1991), 55–87. English translation: Math. USSR Sb. **72** (1992), 47–77.

[2] Grigor'yan A., Hu Eryan, Hu Jiaxin, *Two-sided estimates of heat kernels of jump type Dirichlet forms*, Adv. Math. **330** (2018), 433–515.

[3] Grigor'yan A., Hu Jiaxin, Lau K. -S., *Generalized capacity, Harnack inequality and heat kernels on metric spaces*, J. Math. Soc. Japan **67** (2015), 1485–1549.

[4] Kondratiev V. A, Landis E. M., *Qualitative theory of linear second-order partial differential equations* (in Russian), Itogi Nauki i Techniki, serija Sovremennye Problemy Matematiki, Fundamental'nye Napravlenija **32**, VINITI, Moscow, 1988, 99–215. English translation: Partial Differential Equations III, Encyclopedia of Math. Sci. **32**, Springer, 1990.

[5] Krylov N. V., Safonov M. V. *A certain property of solutions of parabolic equations with measurable coefficients* (in Russian), Izv. Akad. Nauk SSSR **44** (1980), 81–98. English translation: Math. USSR Izvestija **16** (1981), 151–164.

[6] Landis E. M., *The second order equations of elliptic and parabolic types* (in Russian), Nauka, Moscow, 1971. English translation: Transl. of Mathematical Monographs **171**, AMS publications, 1999.

[7] Moser J. *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577-591.