A Liouville property for Schrödinger operators

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Contents

[*]Supported by the EPSRC Fellowship, B/94/AF/1782

1 Introduction

Let us consider a stationary Schrödinger equation in $\mathbb{R}^d$,

$$\Delta u - u\mu(x) = 0, \tag{1.1}$$

and study the question if the equation (1.1) has a bounded non-vanishing global solution in $\mathbb{R}^d$. Here $\mu(x) \geq 0$ is either a measurable function or, more generally, a measure of a certain class. Our aim is to characterize those μ which admit such solutions.

Let us note that the Laplace operator can be replaced by a more general elliptic or subelliptic operator as well as $\mathbb{R}^d$ can be replaced by a more general harmonic space X. However, for the sake of Introduction, we restrict our attention to the simplest setup and let $X = \mathbb{R}^d$.

We start with description of admissible measures μ. We shall deal with the following three classes of measures (in ascending order):

1. measures of *Kato class* (see exact definition in Section 2);

2. *smooth Radon measures*, i.e., Radon measures which do not charge polar sets;

3. *smooth measures*, i.e., measures which are countable sums of Radon measures, not charging polar sets.

Respectively, the simplest examples of such measures are:

1. Lebesgue measure and any Radon measure with a locally bounded density with respect to Lebesgue measure;

2. a Radon measure with a locally summable density with respect to Lebesgue measure;

3. a measure with a measurable (possibly, infinite) density with respect to Lebesgue measure.

The equation (1.1) is understood in the distribution sense. If μ is a smooth Radon measure, then for any bounded measurable u, the product $u\mu$ is a distribution, and the meaning of (1.1) is straightforward.. If μ is a smooth measure then we have to assume in addition that $u \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$ to ensure that $u\mu$ is a distribution.

The following notion is our main tool:

Definition 1.1 *A measure μ on X is called big if one of the following statements is true (and non-big*[1] *otherwise):*

[1] We reserve the natural antonym "small" for another notion.

(1) *μ is a Kato measure and equation (1.1) has no bounded continuous solution u in X except for the constant zero;*

(2) *μ is a smooth Radon measure and equation (1.1) has no bounded finely continuous*[2] *solution u in X except the constant zero;*

(3) *μ is a smooth measure and the inequality*

$$\Delta u \geq u\mu \tag{1.2}$$

has no positive bounded finely continuous solution $u \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$ on X except the constant zero.

Remarks. 1. As a matter of fact, each of these definitions contains the previous one as a particular case (see below Propositions 2.9 and 2.6). Let us emphasize the differences between (1)-(3):

- (2) differs from (1) by the requirement of *fine* continuity of the solution;
- (3) differs from (2) by the assumption of *positivity* of u and by considering *inequality* (1.2) instead of equation (1.1) (note that in definition (2), we have $u \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$ automatically since u is bounded and the value of μ on any bounded Borel set is finite).

2. It is natural to consider only finely continuous solutions of (1.1). Suppose that $\mu \in \mathcal{L}^1_{\mathrm{loc}}(X)$ and let $u \in \mathcal{L}^\infty_{\mathrm{loc}}(X)$ satisfy (1.1). Fix a ball B with closure in X and let q denote the difference of potentials generated by the density $u\mu$ on B. Then $\Delta(u+q) = \Delta u - u\mu = 0$ on B, hence there exists a harmonic function h on B such that $u + q = h$ a.e. on B. Then $w := h - q$ is finely continuous on B and $\Delta w - w\mu = (u - w)\mu = 0$ (in the sense of distributions). Using the fact that non-empty finely open sets have strictly positive Lebesgue measure we obtain: There exists a unique finely continuous function $\tilde{u}$ on X such that $\tilde{u} = u$ a.e. Moreover, $\tilde{u}$ is a solution of (1.1). (For a stronger statement see [13, Theorem 5.4].) If, on the contrary, $\mu \neq 0$ is supported by a Borel set A having Lebesgue measure zero, then $f := 1_{U\setminus A}$ satisfies $\Delta f = 0 = f\mu$, so f trivially is a bounded solution of (1.1), a solution, which is of little interest. However, the only finely continuous function being equal to f a.e. is the constant 1, which is no solution of (1.1).

By default, all measures considered are smooth. The term "big" with respect to a measure is justified by the following property which will be proved among others in the main body of the paper:

> *If μ and ν are measures and $\mu \geq c\nu$ for some constant $c > 0$ then bigness of ν implies bigness of μ.*

[2] We recall that open sets are finely open, superharmonic functions on open sets are finely continuous, and that the fine topology is the coarsest topology having these properties. Note that superharmonic functions are lower semi-continuous by definition.

Our main purpose is to provide criteria for bigness. In $\mathbb{R}^d$, $d \le 2$, any $\mu \neq 0$ is big. Indeed, if $u \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$ is positive and bounded then $\Delta u \ge u\mu \ge 0$ which together with fine continuity of u implies that u is subharmonic (see [12]). Since any bounded subharmonic function on $\mathbb{R}^d, d \le 2$, is constant then $u \equiv$ const. Since $u\mu \le \Delta u = 0$ then $\mu \neq 0$ implies $u \equiv 0$.

So, let us assume now that $d \ge 3$. One of our principal results is a characterization of a non-big measure in terms of splitting it into two parts each of them being non-big for different reasons. Before we can state it, we describe the two basic classes of non-big measures. Using the Green function G of the Laplace operator,

$$G(x,y) = \frac{c_d}{|x-y|^{d-2}},$$

we define the function G^μ by

$$G^\mu := \int_X G(\cdot, y) d\mu(y). \tag{1.3}$$

Our first claim is:

(i) *If $G^\mu \not\equiv \infty$, then μ is non-big.*

To state the second case of non-bigness, we need another definition.

Definition 1.2 *We say that a set $A \subset X$ is thick if for any positive superharmonic function s on X, $s \ge 1$ on A implies that $s \ge 1$ on X.*

It is easy to see that if $A \subset B$ and A is thick then B is thick as well. Another useful property: If A is thick then $A \backslash K$ is thick as well for any compact K. Thickness has a very natural probabilistic characterization: The set A is thick if and only if Brownian motion starting from an arbitrary point, hits A with probability 1 (see e.g. [4, p. 264]), i.e., if and only if it is recurrent in the sense of [3].

We say that μ is *supported* by a Borel set A if $\mu(X\backslash A) = 0$. Our second claim is:

(ii) *If μ is supported by a non-thick set then μ is non-big.*

It turns out that the two cases of non-bigness of measure μ - finiteness of G^μ and being supported by a non-thick set - and their combination, exhaust all non-big measures as stated by the following main theorem.

Theorem 1.1 *(Criterion of non-bigness) A smooth measure μ on X is non-big if and only if it decomposes into a sum $\mu = \mu_1 + \mu_2$ of two smooth measures such that μ_1 is supported by a non-thick set A and $G^{\mu_2} \not\equiv \infty$.*

The following particular cases were known before:

1. The first author [8] proved that if μ is a continuous function on a Riemannian manifold then $G^\mu < \infty$ implies that μ is non-big.

2. Arendt, Batty and Benilan [2] proved that if μ is a locally integrable function in $\mathbb{R}^d$, $d \geq 3$, and $\mu = \mu_1 + \mu_2$ where μ_1 is supported by a compact and $G^{\mu_2} < \infty$ then μ is non-big (and that the converse holds if μ is radial).

3. Batty [3] gave the characterization of Theorem 1.1 for the case of a measurable function $\mu \geq 0$ on $\mathbb{R}^d$ by means of probabilistic methods, i.e., using Brownian motion and Feynman-Kac formula (see Remark after Proposition 2.9 at the end of Section 2.2).

Let us mention for comparison that our proof is entirely analytic and uses only very basic properties such as the minimum principle for superharmonic functions and the solvability of the Dirichlet problem.

Theorem 1.1 may provide an efficient way of verifying that a given measure is non-big. On the other hand, it seems to be more difficult to use it for proving that a measure is big. We state below a sufficient condition for bigness in the spirit of the Wiener criterion. Let B_k denote a ball of radius 2^k centred at the origin x_0, and $A_k := \overline{B}_k \backslash B_{k-1}$.

Theorem 1.2 *Given a smooth measure μ on $\mathbb{R}^d$, let us denote for any $k = 1, 2, ...$*

$$s_k(x) := \int_{A_k} G(x,y) d\mu(y) \tag{1.4}$$

and

$$\alpha_k := \sup\{s_k(x) : x \in A_k\}.$$

If

$$\sum_{k=1}^{\infty} \frac{s_k(x_0)}{1+\alpha_k} = \infty \tag{1.5}$$

then μ is big.

Although condition (1.5) is not necessary for bigness of μ, it is not very far from that as is shown by the next statement.

Theorem 1.3 *Let a smooth measure μ on X be supported by a set $A \subset X$ and let*

$$\beta_k := \inf\{s_k(x) : x \in A_k \cap A\}. \tag{1.6}$$

where $s_k(x)$ is defined by (1.4). If

$$\sum_{k=1}^{\infty} \frac{s_k(x_0)}{1+\beta_k} < \infty$$

then μ is non-big.

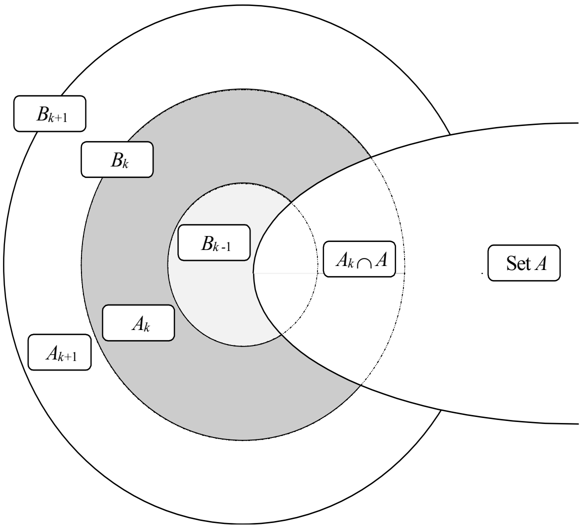

Figure 1 Sets A_k, B_k and A

Let us remark that Theorems 1.1 and 1.3 remain true in a very general setting of harmonic spaces whereas Theorem 1.2 relies on a uniform Harnack inequality which imposes a rather strong restriction on the geometry of the space X. We tried to underline this by denoting the underlying space in Theorems 1.1 and 1.3 by X rather than $\mathbb{R}^d$.

The structure of the paper is the following. In the next three sections, we discuss the problem on Riemannian manifolds. Restricting our attention first to this case allows us to avoid many technical difficulties which appear in the general case of harmonic spaces and to give (hopefully) clear ideas of the proofs. We introduce Liouville functions associated with Kato measures and, more generally, smooth measures. Studying the correspondence between measures and Liouville functions we obtain simple proofs of Theorems 1.1, 1.2, 1.3 for the case when X is a Riemannian manifold. In Section 5 we develop Wiener type criteria. In Section 6, we consider various examples - applications of the above theorems, and some counterexamples to show to what extent the hypotheses of Theorems 1.2 and 1.3 are sharp. Section 7 starts with a brief introduction to the theory of perturbation of harmonic spaces which is an abstract counterpart of the Schrödinger equation. It contains also further examples of differential operators to which our results are applicable. Moreover, we explicitly write down the generalization of the results previously obtained for Riemannian manifolds. Instead of being bounded the functions we consider are supposed to be bounded by multiples of an arbitrary positive harmonic function h (which is of considerable interest also in the special case of a Riemannian manifold). In Section 8 we discuss relations to the minimality of the harmonic function h. Appendix contains the proof of Proposition 2.1 and the relation between smooth measures and Kato measures.

The work was done when the first author enjoyed hospitality of the Bielefeld University. He gratefully acknowledges support of DAAD (Germany) and EPSRC (UK).

2 Schrödinger equations on Riemannian manifolds

Let X be a (connected) Riemannian manifold and let Δ denote the Laplace operator of the Riemannian metric of X. A reader who is not familiar with Riemannian geometry, may safely assume that X is $\mathbb{R}^d$ or an arbitrary domain in $\mathbb{R}^d$. The general case when X is a harmonic space, is more involved and will be considered in the next sections.

Let $\mathcal{O}_r$ denote the set of all regular[3] precompact regions in X, and for every $B \in \mathcal{O}_r$ let us introduce the following notation:

1. $\mathcal{C}_b(B)$ ($\mathcal{B}_b(B)$ resp.) - the set of all bounded continuous (Borel resp.) functions on B;

2. $H_B f$ - the (Perron) solution to the Dirichlet problem in $B \in \mathcal{O}_r$,

$$\begin{cases} \Delta u = 0, \\ u|_{\partial B} = f, \end{cases}$$

   for continuous real functions f on ∂B (for later generalizations, it will be convenient to interpret this as $H_B f$ being a harmonic function on B such that $\lim_{x \to z} H_B f(x) = f(z)$ for every $z \in \partial B$),

3. $G_B(x, y)$ - the Green function of the Laplace operator in B, i.e., for any $y \in B$,

$$\begin{cases} \Delta G_B(\cdot, y) = -\delta_y, \\ G_B(\cdot, y)|_{\partial B} = 0 \end{cases}.$$

4. G_B^μ - (μ being a measure on X) defined by

$$G_B^\mu := \int_B G_B(\cdot, y)\, d\mu(y).$$

We define a function G on $X \times X$ by

$$G(x, y) := \sup_{B \subset\subset X} G_B(x, y) = \lim_{B \uparrow X} G_B(x, y)\ ,$$

where $B \uparrow X$ means an exhaustion of X by an increasing sequence in $\mathcal{O}_r$.

Let us note that the function $G(x, y)$ may happen to be infinite for all x, y or it is finite for all $x \neq y$. The manifold X is called *parabolic* if $G \equiv \infty$ and non-parabolic (or a *Green space*) otherwise and then G is a global Green function. For example, $\mathbb{R}^d$ is parabolic if and only if $d \leq 2$ (see [22] for potential theory of Riemannian manifolds).

A countable sum of Radon measures $\mu \geq 0$ on X is called *smooth* if $\mu(P) = 0$ for every polar set P in X. For example, Lebesgue measure λ^d on $\mathbb{R}^d$ is smooth, and any measure having a density (finite or not) with respect to a smooth measure is a smooth measure.

[3] Alternatively, we may assume that $\mathcal{O}_r$ is the set of all precompact regions with smooth boundary.

Kato measures are particularly nice smooth Radon measures: A measure $\mu \geq 0$ on X is called a (local) *Kato measure* if for any $B \in \mathcal{O}_r$ the function G_B^μ is continuous and real on B.

Clearly, sums and positive multiples of Kato measures are Kato measures. Moreover, any measure $\nu \geq 0$ majorized by a Kato measure μ is a Kato measure: It suffices to note that, for every $B \in \mathcal{O}_r$, we have the equality

$$G_B^\nu + G_B^{\mu-\nu} = G_B^\mu,$$

and the functions G_B^ν and $G_B^{\mu-\nu}$ are lower semi-continuous[4]. This implies that G_B^ν is continuous if G_B^μ is continuous.

In particular, a measure μ with a locally bounded density with respect to the Riemannian measure λ, is a Kato measure. Indeed, the Riemannian measure is Kato, which follows from local integrability of the Green function. Therefore, any measure $\mu \leq \text{const}\lambda$ is Kato as well. Finally, due to the local nature of the definition of Kato measure, it suffices to have $\mu \leq \text{const}\lambda$ on any compact, with const depending on the compact, which is equivalent to μ having a locally bounded density with respect to λ.

It is not difficult to see that a measure $\mu \geq 0$ is smooth if and only if it is the limit of an increasing sequence of Kato measures or, equivalently, if and only if it has a density with respect to a Kato measure (see Proposition 9.1 in the Appendix).

Let us mention that μ being a Kato measure is not only a *sufficient* condition (see [1], [7], [11], and Proposition 2.9 below), but also a *necessary* condition for continuity of finely continuous bounded solutions of (1.1) (see [17], [20]).

We first show that Theorem 1.1 is trivially true if X is parabolic. Since parabolicity is equivalent to the fact that any bounded subharmonic function on X is necessarily constant (see [22]) then we can refer to the argument from Introduction, which shows that any non-big measure on a parabolic manifold is identically zero.

On the other hand, any non-polar set A is thick on a parabolic manifold. Indeed, the only positive superharmonic functions on a parabolic manifold are constants. Hence, Definition 1.2 is trivially satisfied: If $s \geq 1$ q.e. on A then $s \geq 1$ on X. Therefore, the property "μ_1 is supported by a non-thick set" means that μ_1 is supported by a polar set which is equivalent to $\mu_1 = 0$. The property "$G^{\mu_2} \not\equiv \infty$" is equivalent to $\mu_2 = 0$ since $G \equiv \infty$.

We conclude that the statement of Theorem 1.1 holds because all measures involved are zero.

So we may turn to the main case when X is a Green space. We will constantly use the Green function $G(x, y)$ and the following integral operators defined by

$$K^\mu f := G^{f\mu} = \int_B G(\cdot, y) f(y) d\mu(y)$$

and the localized version

$$K_B^\mu f := G_B^{f\mu} = \int_B G_B(\cdot, y) f(y) d\mu(y) \qquad (B \in \mathcal{O}_r).$$

[4]Lower semi-continuity of G^μ for any measure μ follows from Fatou's lemma.

If μ is a Kato measure, $B \in \mathcal{O}_r$, and $f \in \mathcal{B}_b(B)$, then

$$K_B^\mu f = G_B^{f^+\mu} - G_B^{f^-\mu} \in \mathcal{C}_b(B)$$

since $f^+\mu$ and $f^-\mu$ are Kato measures. Moreover, for every $B' \in \mathcal{O}_r$ containing the closure of B,

$$G_B(\cdot, y) = G_{B'}(\cdot, y) - H_B G_{B'}(\cdot, y) \tag{2.1}$$

and hence

$$K_B^\mu f = K_{B'}^\mu f - H_B K_{B'}^\mu f. \tag{2.2}$$

This shows that $K_B^\mu f$ is continuous up to the boundary ∂B and $(K_B^\mu f)|_{\partial B} = 0$.

By definition of G and using (2.1), we obtain that

$$G_B^\mu + \int_B H_B G(\cdot, y)\, d\mu(y) = \int_B G(\cdot, y)\, d\mu(y) = G^{\mathbf{1}_B\mu} \tag{2.3}$$

where $x \mapsto \int_B H_B G(\cdot, y)\, d\mu(y)$ is harmonic on B for any Radon measure μ on X. Thus a Radon measure $\mu \geq 0$ on X is a Kato measure if and only if, given any $B \in \mathcal{O}_r$, the function $G^{\mathbf{1}_B\mu}$ is continuous and real on B (and hence on X).

In the following proposition, we have collected the technical tools which facilitate considerably working with the Schrödinger equation. The main purpose of them is to construct the solution to the Dirichlet problem for the Schrödinger equation, and to provide the identity (2.5) which will be constantly used.

Proposition 2.1 *Let μ be a Kato measure on X and $B \in \mathcal{O}_r$. Then the following holds:*

(1) K_B^μ is a compact operator on $\mathcal{B}_b(B)$ (with respect to uniform convergence).

(2) If t is a positive superharmonic function on B and $s \in \mathcal{B}_b(B)$ such that $s + K_B^\mu s + t$ is a positive superharmonic function on B, then $s + t \geq 0$.

(3) The operator $I + K_B^\mu$ is invertible on $\mathcal{B}_b(B)$.

(4) For every $f \in \mathcal{C}(\partial B)$, the function

$$H_B^\mu f := (I + K_B^\mu)^{-1} H_B f \tag{2.4}$$

is the unique (Perron) solution to the Dirichlet problem

$$\begin{cases} \Delta u - u\mu = 0, \\ u|_{\partial B} = f. \end{cases}$$

If $f \geq 0$ then $0 \leq H_B^\mu f \leq H_B f$ and, more generally, $H_B^{\mu+\nu} f \leq H_B^\mu f$ for every Kato measure ν on X.

(5) The function $u := H_B^\mu f$ satisfies the following identity in B:

$$u + K_B^\mu u = H_B f\,. \tag{2.5}$$

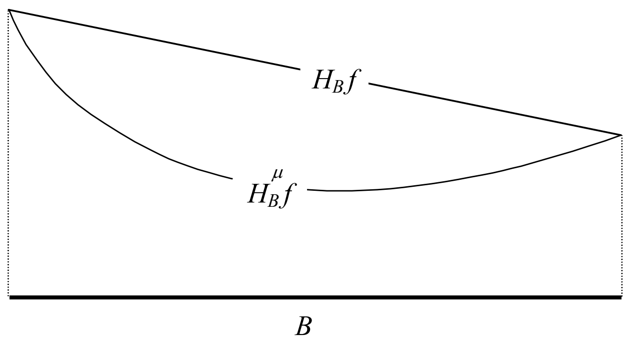

Figure 2 $H_B f$ and $H_B^\mu f$

It is easy to see that a function u satisfying (2.5) is a solution of the Dirichlet problem for $\Delta - \mu$. Indeed, we have $K_B^\mu u = G_B^{u\mu}$ and $\Delta G_B^{u\mu} = -u\mu$ on B, hence (2.5) implies that $\Delta u - u\mu = \Delta H_B f = 0$ on B. Moreover, $K_B^\mu u|_{\partial B} = 0$, hence $u|_{\partial f} = (H_B f)|_{\partial B} = f$. A detailed proof for Proposition 2.1 can be found in [7] and [11] (except for the minor modification of an additional t in (2)). For convenience of the reader, we reproduce the proof in the Appendix at the end of the paper. Of course, the key identity (2.5) is just another form of (2.4).

The main technical tool for proving Theorem 1.1 is a so-called *Liouville function* of a measure μ which will be denoted by L^μ and which is basically equal to the maximal function v such that $0 \le v \le 1$ and $\Delta v \ge \mu v$. A careful definition of L^μ will be given in the following two subsections. Let us highlight the following properties of L^μ which will be proved below among others and which are crucial for Theorem 1.1:

- a $0-1$ law: either $L^\mu \equiv 0$ and μ is big or $\sup L^\mu = 1$ and μ is non-big (Proposition 2.8 and Lemma 3.3);
- monotonicity of bigness: a positive multiple of a big measure is again big (Lemma 3.2, Proposition 3.7), which is based on the inequality $L^\mu + L^\nu \le 1 + L^{\mu+\nu}$ (Lemma 3.5);
- inequality of Corollary 2.7: if μ is supported by a set A then $L^\mu + \widehat{R}^A 1 \ge 1$ where $\widehat{R}^A 1$ is a regularized reduced function of A;
- inequality $L^\mu + G^\mu \ge 1$ (Lemmas 2.2 and 2.4).

2.1 Liouville function for Kato measures

Given a Kato measure μ on our Riemannian manifold X, it is natural to construct a global bounded solution to (1.1) which might serve as function deciding bigness of μ by using the following procedure. Let us choose an exhaustion of X - a sequence (B_k) in $\mathcal{O}_r$ such that $\overline{B}_k \subset B_{k+1}$ and $\bigcup_{k=1}^{\infty} B_k = X$. For example, in $\mathbb{R}^d$ it may be a sequence of concentric balls with radii tending to ∞.

Let us denote by u_k the solution in B_k of the Dirichlet problem for the Schrödinger equation with the boundary function 1 : $u_k := H^\mu_{B_k}1$. Proposition 2.1 implies that $0 \le u_k \le 1$ and the sequence (u_k) is decreasing:

$$u_{k+1}|_{B_k} = H^\mu_{B_k} u_{k+1} \le H^\mu_{B_k} 1 = u_k.$$

Therefore, there exists the limit:

$$L^\mu(x) := \lim_{k\to\infty} u_k(x) \qquad (x \in X). \tag{2.6}$$

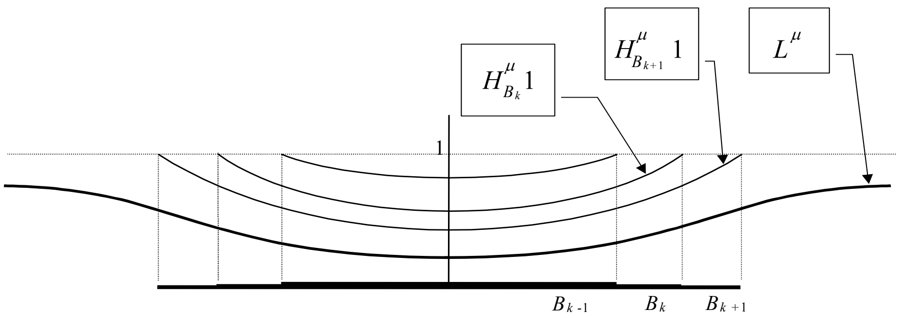

Figure 3 Liouville function L^μ as the limit of $H^\mu_{B_k}1$

Clearly, L^μ does not depend on the choice of the sequence (B_k). The function L^μ will be called *Liouville function* of 1 with respect to μ.

Lemma 2.2 *Let μ be a Kato measure. Then L^μ is a continuous subharmonic function on X, $0 \le L^\mu \le 1$. The function $L^\mu + K^\mu L^\mu$ is harmonic and*

$$L^\mu + K^\mu L^\mu \le 1 \le L^\mu + G^\mu.$$

In particular, L^μ is a solution of (1.1).

Proof. Take (B_k) and (u_k) as above and let $u = L^\mu$. Since each u_k is subharmonic on B_k, the function u is subharmonic on X, and of course $0 \le u \le 1$. For every $k \in \mathbb{N}$,

$$1 = u_k + K^\mu_{B_k} u_k \le u_k + G^\mu_{B_k}.$$

Letting k tend to infinity and applying Fatou's lemma we obtain that

$$u + K^\mu u \le 1 \le u + G^\mu.$$

Now fix $k \in \mathbb{N}$. By Proposition 2.1, for every $m > k$,

$$u_m + K^\mu_{B_k} u_m = u_m + K^\mu_{B_m} u_m - H_{B_k} K^\mu_{B_m} u_m = 1 - H_{B_k} K^\mu_{B_m} u_m,$$

hence $u_m + K^\mu_{B_k} u_m$ is harmonic on B_k. This implies that the infimum $u + K^\mu_{B_k} u$ is harmonic on B_k and u is continuous on B_k.

The sequence $(u + K^\mu_{B_k} u)_{k\in\mathbb{N}}$ is increasing to $u + K^\mu u$ which does not exceed 1. Thus $u + K^\mu u$ is harmonic (which shows again that u is subharmonic). In particular, $\Delta u - u\mu = 0$ (which of course follows as well directly from $\Delta u_k - u_k\mu = 0$ on B_k). ■

Lemma 2.3 *Let (μ_n) be an increasing sequence of Kato measures. Then the sequence (L^{μ_n}) is decreasing. If $\mu = \sup_n \mu_n$ is a Kato measure as well then $L^\mu = \inf_n L^{\mu_n}$.*

Proof. The first statement follows from Proposition 2.1. Let (B_k) be a sequence in $\mathcal{O}_r$ exhausting X and fix $k \in \mathbb{N}$.

Defining

$$u_{k,n} = H_{B_k}^{\mu_n} 1$$

we know by Proposition 2.1 that

$$1 = u_{k,n} + K_{B_k}^{\mu_n} u_{k,n} = u_{k,n} + K_{B_k}^{\mu} u_{k,n} - K_{B_k}^{\mu-\mu_n} u_{k,n}$$

for every $n \in \mathbb{N}$ and that the sequence $(u_{k,n})_{n\in\mathbb{N}}$ is decreasing to a function u_k. Moreover,

$$0 \le K_{B_k}^{\mu-\mu_n} u_{k,n} \le G_{B_k}^{\mu-\mu_n}$$

and

$$\lim_{n\to\infty} G_{B_k}^{\mu-\mu_n} = 0.$$

So we obtain that

$$1 = u_k + K_{B_k}^{\mu} u_k,$$

hence $u_k = H_{B_k} 1$. Thus

$$L^\mu = \inf_k u_k = \inf_k \inf_n u_{k,n} = \inf_n \inf_k u_{k,n} = \inf L^{\mu_n}.$$

■

2.2 Liouville function and bigness for smooth measures

We now extend the definition of L^μ to arbitrary smooth measures: For every smooth measure μ let

$$L^\mu := \inf\{L^\nu : \nu \text{ Kato measure}, \nu \le \mu\}. \tag{2.7}$$

Again, L^μ will be called *Liouville function* of 1 with respect to μ.

Lemma 2.3 implies that

$$L^\mu = \inf L^{\mu_n} \tag{2.8}$$

for any sequence (μ_n) of Kato measures increasing to μ. Indeed, if ν is any Kato measure such that $\nu \le \mu$ then the sequence $(\mu_n \curlywedge \nu)$ is increasing to ν (where $\mu_n \curlywedge \nu$ denotes the largest measure which is smaller than μ_n and ν), hence

$$L^\nu = \inf_n L^{\mu_n \curlywedge \nu} \ge \inf_n L^{\mu_n}.$$

(It is now easy to show that (2.8) holds for any sequence (μ_n) of smooth measures increasing to μ.)

Proposition 2.4 *For every smooth measure μ, $0 \le L^\mu \le 1$, the function $L^\mu + K^\mu L^\mu$ is subharmonic, and*

$$L^\mu + K^\mu L^\mu \le 1 \le L^\mu + G^\mu.$$

In particular, L^μ is subharmonic, $\Delta L^\mu \ge L^\mu \mu$.

If μ is a smooth Radon measure, then $L^\mu + K^\mu L^\mu$ is harmonic, $\Delta L^\mu = L^\mu \mu$.

Proof. Let μ be a smooth measure and fix a sequence (μ_n) of Kato measures such that $\mu_n \uparrow \mu$. Define

$$v := L^\mu, \qquad v_n := L^{\mu_n}.$$

Every v_n is subharmonic and we obtain from (2.8) that the sequence (v_n) is decreasing to v, hence v is subharmonic, $0 \le v \le 1$. Moreover, by Lemma 2.2

$$v + K^{\mu_n} v \le v_n + K^{\mu_n} v_n \le 1 \le v_n + G^{\mu_n} \le v_n + G^\mu$$

for every $n \in \mathbb{N}$, hence

$$v + K^\mu v \le 1 \le v + G^\mu.$$

The function $w := v + K^\mu v$ is finely continuous. For every $n \in \mathbb{N}$, the function $v_n + K^{\mu_n} v_n$ is harmonic by Lemma 2.2 and hence

$$v_n + K^{\mu_n} v = (v_n + K^{\mu_n} v_n) - K^{\mu_n}(v_n - v) \tag{2.9}$$

is subharmonic. This implies that

$$w = \lim_{n\to\infty} (v_n + K^{\mu_n} v)$$

is subharmonic as well. In particular,

$$\Delta v - v\mu = \Delta w \ge 0.$$

Suppose finally that μ is a smooth Radon measure and fix $B \in \mathcal{O}_r$. Then, by (2.9) and (2.3),

$$\begin{aligned} 0 \le H_B w - w &= \lim_{n\to\infty} (K^{\mu_n}(v_n - v) - H_B K^{\mu_n}(v_n - v)) \\ &= \lim_{n\to\infty} K_B^{\mu_n}(v_n - v) \le \lim_{n\to\infty} K_B^{\mu}(v_n - v) = 0 \end{aligned}$$

on $\{G_B^\mu < \infty\}$. Since $\{G_B^\mu = \infty\}$ is a polar set and $H_B w - w$ is finely continuous, we obtain that $H_B w - w = 0$. Thus w is harmonic, $\Delta v - v\mu = \Delta w = 0$. ■

Lemma 2.5 *Let μ be a smooth measure on X and let v be a finely continuous function on X such that $|v| \le 1$. Assume that, for every $B \in \mathcal{O}_r$ and for every Kato measure $\nu \le \mu$, the function $v + K_B^\nu v^+ - K_B^\mu v^-$ is subharmonic on B. Then $v \le L^\mu$.*

Note that the additional assumption on v is satisfied if $K^\mu |v|$ is bounded and $v + K^\mu v$ is subharmonic. Indeed, using (2.3) we obtain that $v + K_B^\mu v$ is subharmonic on B and hence

$$v + K_B^\nu v^+ - K_B^\mu v^- = (v + K_B^\mu v) - K_B^{\mu-\nu} v^+$$

is subharmonic on B.

Proof. Fix a Kato measure ν on X such that $\nu \le \mu$ and let $B \in \mathcal{O}_r$. Define functions s and t on B by

$$s := H_B^\nu 1 - v, \qquad t := 1 - (v + K_B^\nu v^+ - K_B^\mu v^-).$$

By assumption, t is superharmonic and

$$t + K_B^\nu v^+ = (1 - v) + K_B^\mu v^- \geq 0,$$

hence $t \geq 0$. Moreover,

$$s + K_B^\nu s + K_B^{\mu-\nu} v^- = 1 - (v + K_B^\nu v) + K_B^{\mu-\nu} v^- = t.$$

Using (2) of Proposition 2.1 we conclude that

$$s + K_B^{\mu-\nu} v^- \geq 0.$$

If (ν_n) is a sequence of Kato measures increasing to μ, then

$$\lim_{n\to\infty} K_B^{\mu-\nu_n} v^- = 0 \quad \text{on} \quad \{K_B^\mu v^- < \infty\}.$$

So we obtain that

$$\inf_n H_B^{\nu_n} 1 \geq v \qquad \text{on} \quad \{K^\mu v^- < \infty\}.$$

By assumption, $K_B^\mu v^- \not\equiv \infty$, hence the set $\{K_B^\mu v^- = \infty\}$ is polar. The function $\inf_n H_B^{\nu_n} 1$ is subharmonic, hence finely continuous, and we get that $\inf_n H_B^{\nu_n} 1 \geq v$ on X. Thus finally

$$L^\mu = \inf_{B \in \mathcal{O}_r} \inf_n H_B^{\nu_n} 1 \geq v.$$

■

Proposition 2.6 *Let μ be a smooth measure. Then L^μ can be characterized in the following way:*

(1) L^μ is the maximal finely continuous function v on X such that $|v| \leq 1$, $v \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$, and $\Delta v \geq v\mu$.

(2) If $1 \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$ then L^μ is the maximal finely continuous solution v of (1.1) such that $|v| \leq 1$.

Proof. It follows immediately from Proposition 2.4 that L^μ has the desired properties.

Conversely, let v be a finely continuous function on X such that $|v| \leq 1$, $v \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$, and $\Delta v \geq v\mu$. Fix $B \in \mathcal{O}_r$ and a Kato measure $\nu \leq \mu$. Then

$$w := v + K_B^\nu v^+ - K_B^\mu v^-$$

is upper bounded, finely continuous, and

$$\Delta w = \Delta v - v^+ \nu + v^- \mu \geq 0$$

on B, hence w is subharmonic on B. Thus $v \leq L^\mu$ by Lemma 2.5.

If, in addition, $1 \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$ and v is a finely continuous solution of (1.1) with $|v| \leq 1$, then of course $v \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$ and we obtain that $\pm v \leq L^\mu$. ■

Let $\mathcal{S}^+(X)$ denote the set of all positive superharmonic functions on X. For every $A \subset X$ and $s \in \mathcal{S}^+(X)$, we define

$$\begin{aligned} R^A s &= \inf\{v \in \mathcal{S}^+(X) : v \geq s \text{ on } A\} \\ &= \inf\{v \in \mathcal{S}^+(X) : v \leq s \text{ on } X, v = s \text{ on } A\}, \\ \widehat{R}^A s(x) &= \liminf_{y \to x} R^A s(y). \end{aligned}$$

Then $\widehat{R}^A s \in \mathcal{S}^+(X)$ and, of course, $\widehat{R}^A s \leq R^A s \leq s$ on X and $R^A s = s$ on A. Moreover, $\{\widehat{R}^A s < R^A s\}$ is a polar subset of A.

Let us note that thickness of A is equivalent to $R^A 1 \equiv 1$ which is in turn equivalent to $\widehat{R}^A 1 \equiv 1$.

Corollary 2.7 *Let μ be a smooth measure which is supported by a (Borel) set A. Then*

$$L^\mu \geq 1 - \widehat{R}^A 1. \tag{2.10}$$

Proof. Since $\widehat{R}^A 1 \in \mathcal{S}^+(X)$, the function $v := 1 - \widehat{R}^A 1$ is subharmonic and $0 \leq v \leq 1$. Moreover, $v = 0$ on $A \setminus \{\widehat{R}^A 1 < R^A 1\}$, and the measure μ does not charge the polar set $A \setminus \{\widehat{R}^A 1 < R^A 1\}$. Therefore $\Delta v \geq 0 = v\mu$. Thus $v \leq L^\mu$ by Proposition 2.6. ■

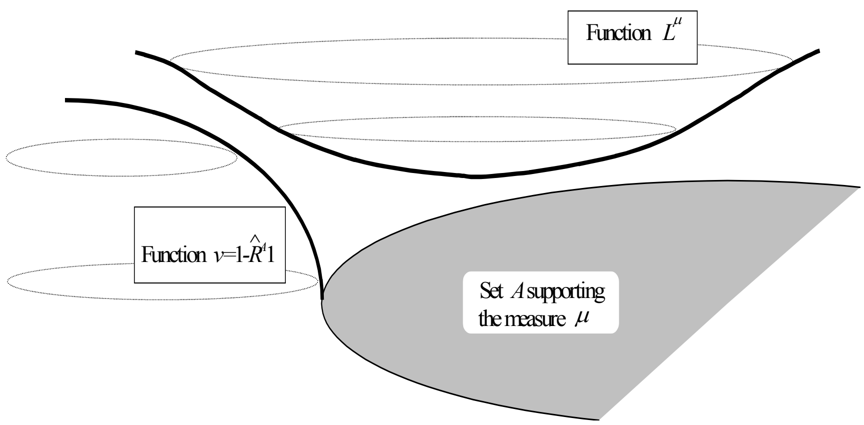

Figure 4 $v = 1 - \widehat{R}^A 1$ is a subsolution: $\Delta v \geq v\mu$, and $0 \leq v \leq 1$ whence $v \leq L^\mu$

Example. Let A be an open subset of X and let $\mu = \infty \mathbf{1}_A \lambda$. Then μ is a smooth measure and

$$L^\mu = 1 - R^A 1. \tag{2.11}$$

Indeed, $L^\mu \geq 1 - R^A 1$ by Corollary 2.7. On the other hand, $K^\mu L^\mu \leq 1$ by Proposition 2.4 and hence $L^\mu = 0$ λ-a.e. on A. Since L^μ is subharmonic, we conclude that $L^\mu = 0$ on A. So $s := 1 - L^\mu \in \mathcal{S}^+(X)$, $s = 1$ on A, hence $s \geq R^A 1$, i.e., $1 - R^A 1 \geq L^\mu$.

Note that $L^\mu = 1 - R^A 1$ is no solution of (1.1) if A is a non-empty open set which is non-thick!

More generally, we get the equality (2.11) in the following situation: ν any Kato measure, $\mu = \infty\nu$, and A a set supporting ν such that $\nu(V) > 0$ for any $x \in A$ and any (Borel) fine neighbourhood V of x.

Proposition 2.6 shows that the definitions for bigness given in the Introduction are consistent and that we have the following characterization:

Proposition 2.8 *(Criterion of bigness) For a smooth measure μ consider the following statements:*

(1) μ is big.

(2) $L^\mu = 0$.

(3) If u is a bounded continuous solution of (1.1), then $u = 0$.

Then: (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). If μ is a Kato measure, then the three statements are equivalent.

For sake of completeness let us add the following:

Proposition 2.9 *If μ is a Kato measure then every bounded finely continuous solution of (1.1) is continuous.*

Proof. Fix a bounded finely continuous solution v of (1.1) and take $B \in_r$. Then

$$\Delta(v + K_B^\mu v) = \Delta v - v\mu = 0 \qquad \text{on } B,$$

hence there exists a harmonic function g on B such that $v + K_B^\mu v = g$ λ-a.e. on B. By assumption on μ, the function $K_B^\mu v$ is continuous, hence the function $v + K_B^\mu v$ is finely continuous. Therefore $v + K_B^\mu v = g$ on B, i.e., $v|_B = g - K_B^\mu v$ is continuous. ∎

Remark: Assuming that $X = \mathbb{R}^d$, $d \geq 3$, and that μ is absolutely continuous with respect to Lebesgue measure, i.e., that $\mu = V\lambda$ with a Borel measurable function $V \geq 0$ on $\mathbb{R}^d$, it is now easily seen that μ is big if and only if the Schrödinger semigroup associated with V is strongly stable on $\mathcal{L}^1(\mathbb{R}^d)$ (see [3, page 464]). Let $(X_t)_{t>0}$ denote Brownian motion on $\mathbb{R}^d$. By Feynman-Kac formula (see e.g. [7, Theorem 6.7]), we know that, for every ball B in $\mathbb{R}^d$ and every $m \in \mathbb{N}$,

$$H_B^{\min(V,m)\lambda} 1(x) = E^x \left(\exp\left(- \int_0^{\tau_B} \min(V(X_t), m)\, dt \right) \right) \qquad (x \in \mathbb{R}^d)$$

(where τ_B denotes the first exit time from B). Hence, by (2.6) and (2.7),

$$L^{V\lambda}(x) = E^x \left(\exp\left(- \int_0^{\infty} V(X_t)\, dt \right) \right) \qquad (x \in \mathbb{R}^d).$$

3 Small and big measures on Riemannian manifolds

Unless stated otherwise, we shall assume in what follows that X is a Riemannian manifold which is a Green space and that μ, ν are smooth measures on X.

For the study of non-big measures it will be useful to introduce the subset of small measures. We begin with the following simple observation: The subharmonic function L^μ has a smallest harmonic majorant P^μ, namely

$$P^\mu = \lim_{B \uparrow X} H_B L^\mu. \tag{3.1}$$

Of course,

$$L^\mu \le P^\mu \le 1,$$

and μ is big if and only if $P^\mu = 0$. Moreover, $P^\mu \le P^\nu$ if $\nu \le \mu$.

Definition 3.1 *We shall say that a smooth measure μ is small, if $P^\mu = 1$.*

Lemma 3.1 *μ is small if and only if $1 - L^\mu$ is a potential*[5]*.*

Proof. The function $s := 1 - L^\mu$ is a positive superharmonic function on X, hence we have a Riesz decomposition $s = g + p$ where g is a positive harmonic function and p is a potential on X. So $L^\mu + p = 1 - g$ is harmonic. Since $\lim_{B\uparrow X} H_B p = 0$, we conclude from (3.1) that $P^\mu = 1 - g$. Thus μ is small if and only if $g = 0$, i.e., if and only if $s = p$. ■

Lemma 3.2 *(Monotonicity of bigness and smallness) If $\nu \le \mu$, then μ is big if ν is big, and ν is small if μ is small.*

Proof. In view of Proposition 2.8 it suffices to recall that $0 \le L^\mu \le L^\nu$ and $P^\mu \le P^\nu \le 1$. ■

Lemma 3.3 *(A $0-1$ law) If $L^\mu \not\equiv 0$ (i.e., if μ is non-big) then* $\sup L^\mu = 1$.

Proof. Indeed, let us denote $u = L^\mu$ and $a = \sup u$. Then $0 < a \le 1$, $0 \le u/a \le 1$. By Proposition 2.6, the function u is the maximal finely continuous function on X satisfying $|u| \le 1$, $u \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$ and $\Delta u \ge u\mu$. Therefore, we obtain that $u/a \le u$, whence $a \ge 1$ and $a = 1$. ■

In some cases every non-big measure is small:

Proposition 3.4 *If every bounded harmonic function on X is constant (i.e., if* 1 *is a minimal harmonic function), then each μ is big or small (and conversely).*

Proof. If μ is non-big, then $\sup L^\mu = 1$ by Lemma 3.3, hence $\sup P^\mu = 1$. This implies that $P^\mu = 1$ if 1 is minimal (for the converse see Corollary 8.4). ■

We note that the assumption of Proposition 3.4 implies as well that each subset is either thick or thin (see Proposition 8.3).

[5]By definition, a potential is a non-negative superharmonic function which admits no non-negative harmonic minorant except for zero.

Lemma 3.5 *We have $L^\mu + L^\nu \le 1 + L^{\mu+\nu}$ and $P^\mu + P^\nu \le 1 + P^{\mu+\nu}$.*

Proof. By (2.7), we may assume that μ, ν are Kato measures. For every $k \in \mathbb{N}$, let

$$u_k = H^\mu_{B_k} 1, \qquad v_k = H^\nu_{B_k} 1, \qquad w_k = H^{\mu+\nu}_{B_k} 1.$$

Then we have by (2.5) and $H_{B_k} 1 = 1$

$$u_k + K^\mu_{B_k} u_k = 1, \quad v_k + K^\nu_{B_k} v_k = 1, \quad w_k + K^{\mu+\nu}_{B_k} w_k = 1.$$

By Proposition 2.1, $w_k \le u_k$ and $w_k \le v_k$, and hence

$$\begin{aligned} (1-u_k) + (1-v_k) &= K^\mu_{B_k} u_k + K^\nu_{B_k} v_k \\ &\ge K^\mu_{B_k} w_k + K^\nu_{B_k} w_k = K^{\mu+\nu}_{B_k} w_k = 1 - w_k. \end{aligned}$$

Letting $k \to \infty$ we obtain that $(1 - L^\mu) + (1 - L^\nu) \ge 1 - L^{\mu+\nu}$ whence the first inequality. The second inequality now follows by (3.1). ■

Lemma 3.6 *If ν is small, then $P^{\mu+\nu} = P^\mu$.*

Proof. We always have $P^{\mu+\nu} \le P^\mu$. If ν is small, i.e., if $P^\nu = 1$, then Lemma 3.5 implies that $P^\mu \le P^{\mu+\nu}$. ■

Proposition 3.7 *(1) If μ is non-big and ν is small, then $\mu + \nu$ is non-big.*

(2) The set of all small measures is a convex cone.

(3) Any strictly positive multiple of a big measure is a big measure.

Property (3) was proved by A.Grigor'yan and N.Nadirashvili [10] in the case when μ is a continuous function. Let us mention that our proof is much shorter and more general than that of [10].

Proof. (1) If μ is non-big and ν is small, then $P^{\mu+\nu} = P^\mu \not\equiv 0$, hence $\mu + \nu$ is non-big.

(2) Suppose that μ and ν are small. Then by Lemma 3.5

$$1 + 1 = P^\mu + P^\nu \le 1 + P^{\mu+\nu},$$

hence $1 \le P^{\mu+\nu}$, i.e., $\mu+\nu$ is small. In particular, 2μ is small if μ is small. And then monotonicity (Lemma 3.2) implies that any multiple of a small measure is small.

(3) By Lemma 3.5

$$2L^\mu \le 1 + L^{2\mu}.$$

So, if 2μ is big, then $2L^\mu \le 1$, hence $L^\mu = 0$ by our 0-1-law (Lemma 3.3), i.e., μ is big. Another application of our monotonicity lemma finishes the proof. ■

If 1 is a minimal harmonic function, then non–big measures form a convex cone by Proposition 3.4 and Proposition 3.7. We note that the converse holds as well (see Proposition 8.4), i.e., if 1 is not minimal then there are non–big measures μ, ν on X such that $\mu + \nu$ is big.

Lemma 3.8 *(First case of smallness) If $G^\mu \not\equiv \infty$, then the measure μ is small.*

Proof. By Proposition 2.4, $1 \leq L^\mu + G^\mu$ whence for any $B \in \mathcal{O}_r$

$$1 = H_B 1 \leq H_B L^\mu + H_B G^\mu.$$

Since $G^\mu \not\equiv \infty$ then $\lim_{B\uparrow X} H_B G^\mu = 0$. Hence

$$1 \leq \lim_{B\uparrow X} H_B L^\mu = P^\mu,$$

and μ is small. ■

We recall that a subset A of X is non-thick if $\widehat{R}^A 1 \not\equiv 1$. It is called 1-*thin* (or *thin at* ∞) if $\widehat{R}^A 1$ is a potential.

Lemma 3.9 *(Second case of non-bigness and smallness) If μ is supported by a set A and A is non-thick (thin resp.) then μ is non-big (small resp.).*

Proof. By Corollary 2.7

$$1 \leq L^\mu + \widehat{R}^A 1.$$

If A is non-thick, then $\widehat{R}^A 1(x) < 1$ for some $x \in X$ and then $L^\mu(x) > 0$, μ is non-big. If A is thin, then $\lim_{B\uparrow X} H_B \widehat{R}^A 1 = 0$, hence $1 \leq P^\mu$, μ is small. ■

It may be interesting to note the following:

Lemma 3.10 *If μ is a smooth Radon measure, then $L^\mu + K^\mu L^\mu = P^\mu$.*

Proof. Let $u = L^\mu$. By Proposition 2.4 the function $g := u + K^\mu u$ is harmonic. Moreover, for every $B \in \mathcal{O}_r$, $g = H_B g = H_B u + H_B K^\mu u$. Since $\lim_{B\uparrow X} H_B K^\mu u = 0$, we obtain that

$$g = \lim_{B\uparrow X} H_B u = P^\mu.$$

■

Moreover, we note that, for every $c > 0$,

$$P^{c\mu} = P^\mu$$

(for a proof see Proposition 7.20). This implies that

$$P^\mu = \lim_{\varepsilon\to 0} L^{\varepsilon\mu}$$

if μ is a Kato measure (see Corollary 7.21).

4 Characterization of small and big measures

It is now easy to obtain a characterization of non-big and small measures which generalizes and improves Theorem 1.1 stated in the Introduction.

Theorem 4.1 *For any Riemannian manifold X and any smooth measure μ on X, the following statements are equivalent:*

(i) μ *is non-big (small resp.).*

(ii) μ *can be represented as a sum of two measures* $\mu = \mu_1 + \mu_2$ *where* μ_1 *is supported by a non-thick set (thin set resp.) and* $G^{\mu_2} \not\equiv \infty$*.*

(iii) There is an open set A *which is non-thick (thin resp.) and such that for the measure* $\mu_2 := \mathbf{1}_{\complement A}\mu$*, the function* G^{μ_2} *is a bounded potential.*

If μ *is a Kato measure then each of (i)-(iii) is equivalent to:*

(iv) There is an open set A *which is non-thick (thin resp.) such that for the measure* $\mu_2 := \mathbf{1}_{\complement A}\mu$*, the function* G^{μ_2} *is a continuous bounded potential.*

Proof. The discussion after the introduction of smooth measures shows that it suffices to consider the case where X is a Green space. Then we have the following:
(iv) $\Longrightarrow$ (iii): $\Longrightarrow$ (ii): Trivial.

(ii) $\Longrightarrow$(i): Let $\mu = \mu_1 + \mu_2$ where μ_1 is supported by a non-thick set (thin set resp.) and $G^{\mu_2} \not\equiv \infty$. The measure μ_1 is non-big (small resp.) by Lemma 3.9, and the measure μ_2 is small by Lemma 3.8, hence μ is non-big (small resp.) by Proposition 3.7.

(i) $\Longrightarrow$ (iii): We have to prove that if μ is non-big (small resp.) then there is an open set A which is non-thick (thin resp.) and such that $G^{\mathbf{1}_{\complement A}\mu}$ bounded. Let $u = L^\mu$, fix a real $\alpha \in]0,1[$, and let

$$A := \{u < \alpha\}.$$

Since u is u.s.c., the set A is open.

Why A is non-thick (thin resp.)? The function $1-u$ is superharmonic on X and $1-u \geq 1-\alpha$ on A, hence by definition of $\widehat{R}^A 1$, we have

$$\widehat{R}^A 1 \leq \frac{1-u}{1-\alpha}.$$

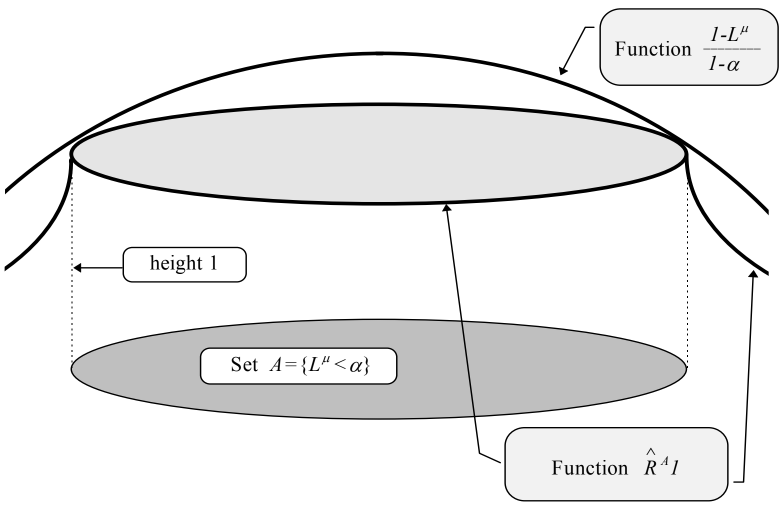

Figure 5 Comparison of $\frac{1-L^\mu}{1-\alpha}$ and $\widehat{R}^A 1$

If μ is non-big, then $\sup u = 1$ by Lemma 3.3, hence $\inf \widehat{R}^A 1 = 0$, A is non-thick. If μ is small, then $1 - u$ is a potential by Lemma 3.1, hence $\widehat{R}^A 1$ is a potential, A is thin.

Why $G^{\mathbf{1}_{\complement A}\mu}$ is bounded? We have $\alpha \le u$ on $\complement A$. By Proposition 2.4, $K^\mu u \le 1$, hence

$$G^{\mathbf{1}_{\complement A}\mu} = K^\mu \mathbf{1}_{\complement A} \le \frac{1}{\alpha} K^\mu u \le \frac{1}{\alpha}.$$

(i) $\Longrightarrow$ (iv): Now we assume that μ is a Kato measure. Let us choose A as above and let $\mu_2 = \mathbf{1}_{\complement A}\mu$. Then G^{μ_2} is even continuous. Indeed, fix $B \in \mathcal{O}_r$. Then

$$G^{\mu_2} = G^{\mathbf{1}_{\complement B}\mu_2} + G^{\mathbf{1}_B \mu_2}$$

where the first term on the right hand side is harmonic on B and the second term is continuous, since μ_2 is a Kato measure. Thus, G^{μ_2} is continuous on B and, since B is arbitrary, on the whole of X. ∎

Another way to prove it is to show directly (iii) $\Longrightarrow$(iv) by using the following property of Kato measures:

Proposition 4.2 *If μ is a Kato measure and $G^\mu \not\equiv \infty$, then G^μ is continuous and real.*

Proof. We note first that G^μ is finite on a dense subset of X. Now fix $B \in \mathcal{O}_r$ and choose an exhaustion (B_n) of X such that $B_1 = B$. Then

$$G^\mu = G^{\mathbf{1}_B \mu} + \sum_{n=2}^{\infty} G^{\mathbf{1}_{B_n \setminus B_{n-1}} \mu}$$

where all potentials on the right hand side are continuous and real and each term in the sum is harmonic on B. Being finite on a dense subset of B the sum is harmonic on B. Thus G^μ is continuous and real on B. ∎

5 Wiener type results on Riemannian manifolds

5.1 Sufficient condition for non-bigness

Let us restate Theorem 1.3 for Riemannian manifolds.

Theorem 5.1 *Let μ be a smooth measure on a non-parabolic Riemannian manifold X, $\mu = \sum_k \mu_k$ where each μ_k is supported by some Borel set A_k. Let us denote*

$$s_k(x) := \int G(x, y)\, d\mu_k(y)$$

and

$$\beta_k := \inf s_k(A_k) \tag{5.1}$$

(with $\inf \emptyset = 0$). If for some point $x_0 \in X$

$$\sum_{k=1}^{\infty} \frac{s_k(x_0)}{1 + \beta_k} < \infty \tag{5.2}$$

then μ is non-big.

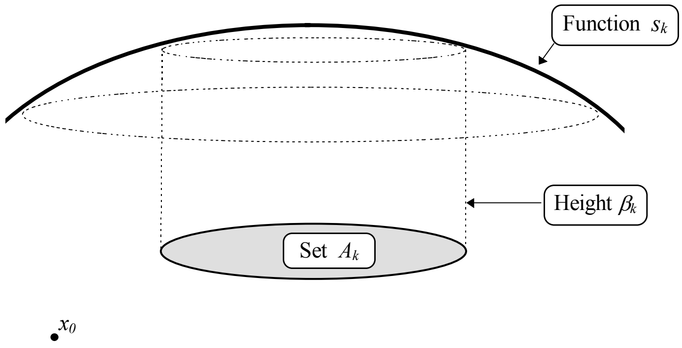

Figure 6 Function s_k

Proof. Let us first remark that $\beta_k < \infty$. Indeed, if $\mu(A_k) = 0$, then $\mu_k = 0$, $s_k = 0$, hence $\beta_k = 0$. If $\mu(A_k) > 0$, then A_k is not polar, hence the potential $s_k = G^{\mu_k}$ is not identically infinite on A_k, and $\beta_k < \infty$.

It follows from (5.2) that for a big enough $K \in \mathbb{N}$,

$$\sum_{k=K}^{\infty} \frac{s_k(x_0)}{1+\beta_k} < \frac{1}{2}.$$

Let us split the measure μ as follows:

$$\mu = \sum_{k=1}^{K-1} \mu_k + \sum_{k=K}^{\infty} \mu_k \tag{5.3}$$

and note that for the first sum we have by (5.2)

$$G^{\mu_1+\mu_2+\ldots+\mu_{K-1}} = s_1 + s_2 + \ldots + s_{K-1} \not\equiv \infty,$$

hence $\mu_1 + \mu_2 + \ldots + \mu_{K-1}$ is small by Lemma 3.8. Therefore, non-bigness of μ will follow by Proposition 3.7 from that of the second sum in (5.3).

This means that we can assume from the very beginning that $K = 1$ and thus

$$\sum_{k=1}^{\infty} \frac{s_k(x_0)}{1+\beta_k} < \frac{1}{2}. \tag{5.4}$$

Let us distinguish two types of sets A_k: when $\beta_k > 1$ and when $\beta_k \le 1$, and split the measure μ into two parts:

$$\nu_1 = \sum_{\beta_k > 1} \mu_k\ , \quad \nu_2 = \sum_{\beta_k \le 1} \mu_k\ .$$

The measure ν_1 is supported by the set $A' := \cup_{\beta_k > 1} A_k$. We claim that A' is non-thick. Indeed, let us prove that the function

$$s := \sum_{\beta_k > 1} \frac{s_k}{\beta_k}$$

is such that $s|_A \geq 1$, $\inf s < 1$ and s is superharmonic. The former is obvious because $s_k \geq \beta_k$ on A_k whereas the latter is implied by (5.4):

$$s(x_0) = \sum_{\beta_k > 1} \frac{s_k(x_0)}{\beta_k} \leq 2 \sum_{\beta_k > 1} \frac{s_k(x_0)}{1+\beta_k} < 1,$$

which also yields $s \in \mathcal{S}^+(X)$. So, A' is non-thick.

The measure ν_2 is small since

$$G^{\nu_2}(x_0) = \sum_{\beta_k \leq 1} s_k(x_0) \leq 2 \sum_{\beta_k \leq 1} \frac{s_k(x_0)}{1+\beta_k} < 1,$$

whence $\mu = \nu_1 + \nu_2$ is non-big by Theorem 4.1. ■

5.2 Sufficient condition for bigness

Let us state now a more general version of Theorem 1.2.

Theorem 5.2 *Let X be a Riemannian manifold X, let (B_k) be an exhaustion of X by sets in $\mathcal{O}_r$, and define $A_1 := \overline{B}_1$, $A_k := \overline{B}_k \backslash B_{k-1}$ for $k = 2, 3, \ldots$. Let μ be a smooth measure on X, $\mu = \sum_k \mu_k$ where each μ_k is supported by A_k. Let us denote*

$$t_k(x) := \int G_{B_{k+1}}(x, y)\, d\mu_k(y)$$

and assume that $\alpha_k := \sup t_k(A_k) < \infty$ for every $k \in \mathbb{N}$.

If

$$\sum_{k=3}^{\infty} \inf_{x \in B_{k-2}} \frac{t_k(x)}{1+\alpha_k} = \infty \tag{5.5}$$

then the measure μ is big.

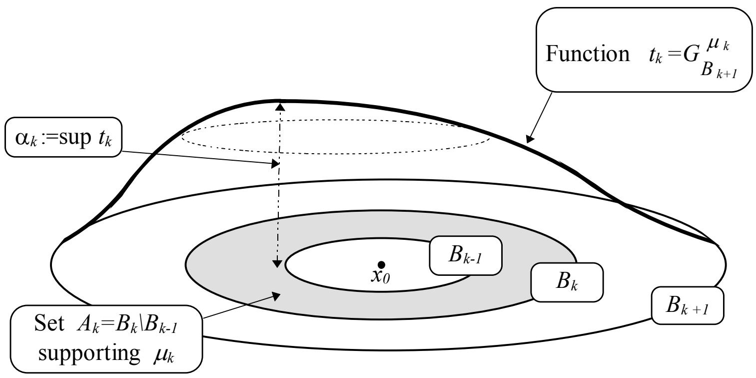

Figure 7 Function t_k

Corollary 5.3 (=Theorem 1.2) *Let in the setting of Theorem 5.2* $X = \mathbb{R}^d$, $d \geq 3$, *and* B_k *be concentric balls of radii* 2^k *centred at a point* $x_0 \in \mathbb{R}^d$. *Let*

$$s_k(x) := \int G(x,y) d\mu_k(y)$$

and $\alpha'_k := \sup s_k(A_k)$.

If

$$\sum_{k=1}^{\infty} \frac{s_k(x_0)}{1+\alpha'_k} = \infty \tag{5.6}$$

then μ *is big.*

Proof of Corollary. The hypothesis (5.6) implies (5.5) because:

1. A_k is separated from the boundary B_{k+1} which implies together with the explicit formula for the Green function in $\mathbb{R}^d$ that on A_k

$$t_k \geq \text{const}\, s_k \ ;$$

2. the function t_k is positive and harmonic in B_{k-1}, whence by the Harnack inequality

$$\inf t_k(B_{k-2}) \geq \text{const}\, t_k(x_0) \ .$$

■

Let us emphasize that Theorem 5.2 does not require the Harnack inequality. On the other hand, Corollary 5.3 is true not only in $\mathbb{R}^d$ but also on any complete Riemannian manifold possessing the uniform Harnack inequality for harmonic functions and a polynomial decay of the Green function $G(x,y)$ with respect to the Riemannian distance $\text{dist}(x,y)$ (one should take the radii of the balls B_k to be C^k for a large enough C). For example, this is the case for a complete Riemannian manifold with non-negative Ricci curvature satisfying the following volume growth condition for some $\varepsilon > 0$:

$$\frac{V(x,R)}{V(x,r)} \geq \text{const} \left(\frac{R}{r}\right)^{2+\varepsilon}, \ \forall x \in M, \ \forall R > r > 0$$

where $V(x,r)$ is the Riemannian volume of the ball of radius r and the centre x. See [19] for the Harnack inequality on such manifolds and estimates of the Green function. For more general conditions for the Harnack inequality, see [9], [21], [23].

Proof of Theorem 5.2. Suppose first that μ is a Kato measure and consider the functions

$$u := L^\mu \ \text{ and } \ v_k := H^{\mu_k}_{B_{k+1}} 1 \, .$$

We have to show that $u = 0$. Taking

$$a_k := \sup u(B_k)$$

we obtain that, on each B_{k+1},

$$a_{k+1} v_k = H^{\mu_k}_{B_{k+1}} a_{k+1} \geq H^{\mu_k}_{B_{k+1}} u \geq H^{\mu}_{B_{k+1}} u = u.$$

Defining

$$b_k := \sup v_k(B_{k-2}), \qquad k = 3, 4, ...$$

we get in particular that

$$a_{k-2} = \sup u(B_{k-2}) \le a_{k+1} \sup v_k(B_{k-2}) = a_{k+1} b_k.$$

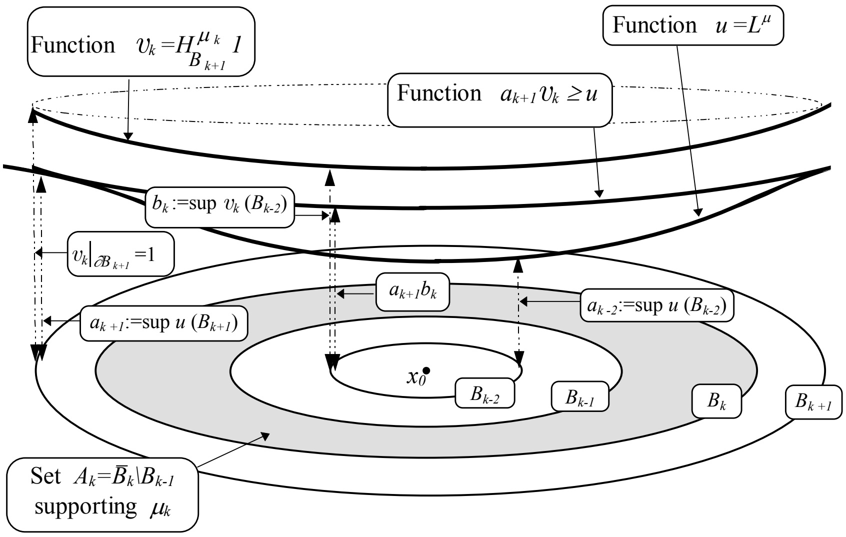

Figure 8 Proof that $a_{k-2} \le a_{k+1} b_k$

By iteration and by $a_k \le 1$, this leads to

$$a_n \le \prod_{i=0}^{\infty} b_{n+2+3i}.$$

for every $n \in \mathbb{N}$. Since clearly $a_n \le a_{n+1} \le a_{n+2}$, this implies that

$$a_n^3 \le a_n a_{n+1} a_{n+2} \le \prod_{k=n+2}^{\infty} b_k. \tag{5.7}$$

In order to deduce that $a_n = 0$ for every n (and hence $u = 0$), it is sufficient to know that

$$\sum_{k=3}^{\infty} (1 - b_k) = \infty. \tag{5.8}$$

Now we concentrate on verifying (5.8). It will follow in turn from (5.5) and from the inequality

$$1 - v_k \ge \frac{t_k}{4\,(1 + \alpha_k)} \tag{5.9}$$

which is true in B_{k+1}. To prove it, let us fix k and consider

$$\nu := \frac{\mu_k}{2(1 + \alpha_k)}, \quad w := H^{\nu}_{B_{k+1}} 1.$$

We obviously have $\nu \le \mu_k$ and therefore

$$w = H^{\nu}_{B_{k+1}} 1 \ge H^{\mu_k}_{B_{k+1}} 1 = v_k.$$

Hence, (5.9) will follow from

$$1 - w \geq \frac{t_k}{4\,(1+\alpha_k)}.$$

Clearly, $t_k \leq \alpha_k$ on B_{k+1}, since $t_k = G^{\mu_k}_{B_{k+1}}$ and μ_k is supported by A_k.We have by the identity (2.5)

$$w + K^{\nu}_{B_{k+1}} w = 1 \tag{5.10}$$

whence

$$1 - w = K^{\nu}_{B_{k+1}} w \leq G^{\nu}_{B_{k+1}} \leq \frac{1}{2(1+\alpha_k)} G^{\mu_k}_{B_{k+1}} = \frac{t_k}{2(1+\alpha_k)} \leq \frac{1}{2}$$

and $w \geq \frac{1}{2}$ in B_{k+1}. By using (5.10) again, we have

$$1 - w = K^{\nu}_{B_{k+1}} w \geq \frac{1}{2} G^{\nu}_{B_{k+1}} = \frac{1}{4\,(1+\alpha_k)} G^{\mu_k}_{B_{k+1}} = \frac{t_k}{4\,(1+\alpha_k)},$$

This finishes the proof if μ is a Kato measure.

Finally, consider the general case of a smooth measure μ, and define the measures μ_k as above. Then there are Kato measures $\tilde{\mu}_k$ such that $\tilde{\mu}_k \leq \mu_k$ and

$$2 \; \frac{\inf_{x\in B_{k-2}} G^{\tilde{\mu}_k}_{B_{k+1}}(x)}{1 + \sup_{x\in A_k} G^{\tilde{\mu}_k}_{B_{k+1}}(x)} \geq \frac{\inf_{x\in B_{k-2}} G^{\mu_k}_{B_{k+1}}(x)}{1 + \sup_{x\in A_k} G^{\mu_k}_{B_{k+1}}(x)}.$$

Hence the Kato measure $\tilde{\mu} := \sum_{k=1}^{\infty} \tilde{\mu}_k$ satisfies the assumptions of Theorem 5.2. Therefore, $\tilde{\mu}$ is big. Since $\tilde{\mu} \leq \mu$ we finally conclude that μ is big as well. ■

5.3 About an axiomatic approach

As we can see from the proofs of Theorems 4.1, 5.1 and 5.2, they do not use much the Laplace and/or Schrödinger equations. Instead, we needed a few properties which can be stated axiomatically, which leads to a similar statement in a general setup of harmonic spaces. It should be clear from the proof above that what we need is basically the following:

1. a notion of harmonic functions in a certain topological space and solvability of the corresponding Dirichlet problem in a wide enough family of open sets;

2. maximum/minimum principle;

3. convergence principle: an increasing sequence of harmonic functions should converge to harmonic function provided the limit is locally bounded;

This will be considered in detail in Section 7.

6 Examples

6.1 Thick and thin sets in $\mathbb{R}^d$

It may be interesting to note that Theorem 1.2 allows us to deduce the following result (which of course is well known):

Proposition 6.1 *(Wiener's criterion at ∞) Let A be a Borel set in $\mathbb{R}^d$, $d > 2$, and let (B_k) denote the sequence of balls of radii 2^k, centred at the origin x_0. Let us denote $A_k = A \cap (\overline{B}_k \setminus B_{k-1})$, $k = 1, 2, \ldots$, and let s_k be the equilibrium potential of A_k. Then A is thick if and only if*

$$\sum_{k=1}^{\infty} s_k(x_0) = \infty. \tag{6.1}$$

Remark: Since any bounded harmonic function on $\mathbb{R}^d$ is constant then by Proposition 3.4, non-thickness of A is equivalent to thinness. In particular, A is thin if and only if the series in (6.1) converges.

Proof. Let us first assume that (6.1) holds. For each $k \in \mathbb{N}$ we may choose a compact subset A'_k of A_k such that the equilibrium potential $s'_k = G^{\mu_k}$ of A'_k satisfies

$$s'_k(x_0) \geq \frac{1}{2} s_k(x_0) \; .$$

Here μ_k denotes the equilibrium measure of A'_k which is supported by A'_k. The set of all $x \in A'_k$ such that $s'_k(x) < 1$ is polar. So $\alpha'_k := \sup s'_k(A'_k) = 1$ if A'_k is non-polar. If, however, A'_k is polar then $s'_k = 0$. Therefore

$$\sum_{k=1}^{\infty} \frac{s'_k(x_0)}{1 + \alpha'_k} \geq \frac{1}{2} \sum_{k=1}^{\infty} s'_k(x_0) = \infty.$$

By Theorem 1.2 we obtain that $\mu = \sum_k \mu_k$ is big. Clearly, μ is supported by A. Thus, A is thick by Lemma 3.9.

Now let us assume that (6.1) is not true, and prove that A is non-thick. We first choose a natural $K > 1$ such that

$$\sum_{k=K}^{\infty} s_k(x_0) < 1 \; ,$$

and denote $A' = \bigcup_{k=K}^{\infty} A_k$. Since the sets A and A' differ by a compact, they are thick or non-thick simultaneously. There exists a positive superharmonic function s_0 on X such that $s_0(x_0) < \frac{1}{2}$ and $s_0 \geq 1$ on the polar set $\bigcup_{k=K}^{\infty} A_k \cap \{s_k < 1\}$. Then the function

$$s := s_0 + \sum_{k=K}^{\infty} s_k$$

is a positive superharmonic function such that $s \geq 1$ on A', but $s(x_0) < 1$. Therefore A' is non-thick. ■

Corollary 6.2 *In the notation of Proposition 6.1, A is thick is and only if*

$$\sum_{k=1}^{\infty} \frac{\mathrm{cap} A_k}{2^{k(d-2)}} = \infty. \tag{6.2}$$

Let now Ω^f be a set of revolution in $\mathbb{R}^d$, $d \geq 3$, as follows:

$$\Omega^f := \left\{ x = (x_1, x_2, ...x_d) \in \mathbb{R}^d \middle| \; x_1 \geq 0, \; \sqrt{x_2^2 + x_3^2 + ... + x_d^2} \leq f(x_1) \right\}$$

where f is a continuous non-negative function on $[0, +\infty)$.

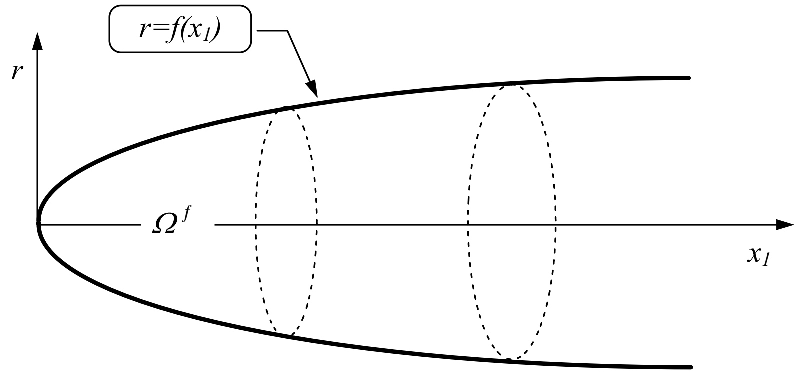

Figure 9 Region Ω^f

Proposition 6.3 *Suppose that the function f satisfies the following regularity condition*

$$C^{-1} \leq \frac{f(t)}{f(\tau)} \leq C, \quad \textit{for all } 1 \leq \tau \leq t \leq 2\tau \tag{6.3}$$

and is linearly bounded:

$$f(t) \leq C(1+t). \tag{6.4}$$

Then the set Ω^f is thick if and only if

$$\int_1^{\infty} \frac{f^{d-3}(t)}{t^{d-2}} dt = \infty, \quad d > 3, \tag{6.5}$$

or

$$\int_1^{\infty} \frac{dt}{t \log\left(1 + \frac{t}{f(t)}\right)} = \infty, \quad d = 3. \tag{6.6}$$

The proof can be easily obtained by Proposition 6.1 (see below the proof of Proposition 6.4 for how to estimate $s_k(x_0)$).

Let us consider some examples. If $d > 3$ then one can see from (6.5) that Ω^f is thick for $f(t) = t$ and non-thick for $f(t) = t^{1-\varepsilon}, \varepsilon > 0$. If $d = 3$ then Ω^f is thick for $f(t) = t^{-N}$ with any $N > 0$ and non-thick for $f(t) = \exp\left(-\log^{1+\varepsilon} t\right)$, $\varepsilon > 0$. In particular, a half-space of $\mathbb{R}^d$ is always thick since it contains a thick Ω^f (if $d \le 2$ then any non-polar set is thick).

Let us show that a hyperplane Σ in $\mathbb{R}^d$ is thick. Let $\Gamma_\pm$ be two half-spaces with the boundary Σ. If $s \ge 0$ is a superharmonic function on $\mathbb{R}^d$ such that $s \ge 1$ on Σ then by taking $\min(s, 1)$, we may assume that $s = 1$ on Σ. Let us define two other functions:

$$s_\pm(x) = \begin{cases} 1, & \text{if} \quad x \in \Gamma_\pm \\ s(x), & \text{otherwise} \end{cases}.$$

Obviously, each function s_+, s_- is superharmonic, and by thickness of the half-space, $s_\pm \ge 1$ on $\mathbb{R}^d$. This implies $s \ge 1$ on $\mathbb{R}^d$ and thickness of Σ.

6.2 Big and small measures in $\mathbb{R}^d$

Let us denote by λ the Lebesgue measure in $\mathbb{R}^d$. The following is a criterion of bigness of a measure which is nearly rotationally invariant with respect to λ and which is restricted to the domain of revolution Ω^f. Note that by Proposition 3.4, non-bigness in $\mathbb{R}^d$ is equivalent to smallness.

Proposition 6.4 *Suppose that a function f satisfies the conditions (6.3), (6.4) and a positive function $q(x) \in \mathcal{L}^1_{\text{loc}}\left(\mathbb{R}^d\right)$ satisfies the condition*

$$C^{-1} \le \frac{q(y)}{q(x)} \le C \quad \textit{whenever } 1 \le |x| \le |y| \le 2|x|. \tag{6.7}$$

Then the measure μ defined by

$$\mu = q \mathbf{1}_{\Omega^f} \lambda$$

is big if and only if

$$\int_1^\infty \frac{f^{d-3}(t)}{t^{d-2}} \min\left(1, q^*(t) f^2(t)\right) dt = \infty, \quad d > 3, \tag{6.8}$$

or

$$\int_1^\infty \frac{1}{t} \frac{dt}{q^*(t)^{-1} f(t)^{-2} + \log\left(1 + \frac{t}{f(t)}\right)} = \infty, \quad d = 3. \tag{6.9}$$

where

$$q^*(t) = q(t, 0, 0, ...0).$$

Examples. 1. Let $q(x) = O\left(|x|^{-2}\right)$ as $x \to \infty$, and $f(t) \sim t$. Then both conditions (6.8), (6.9) transform to

$$\int^\infty t q^*(t) dt = \infty. \tag{6.10}$$

For example, it is satisfied (and the measure μ is big) if

$$q^*(t) \sim \frac{1}{t^2 \log t} \quad \text{as} \quad t \to +\infty$$

and is not satisfied if for some $\varepsilon > 0$

$$q^*(t) \sim \frac{1}{t^2 \log^{1+\varepsilon} t}.$$

The condition (6.10) for bigness of the measure $\mu = q(x)\lambda$ (without restriction to Ω^f) was obtained in [8] and [2]. The situation when μ is supported by Ω^f, is more subtle for proving the bigness, and neither of the previous methods works. This becomes especially transparent if we choose the function $f(t)$ to grow sublinearly as below, so that $\mathbb{R}^d$ cannot be covered by a finite number of congruent copies of Ω^f.

2. Let $d > 3$ and let for large t

$$f(t) \sim \frac{t}{\log^\alpha t}$$

where $0 < \alpha \le \frac{1}{d-3}$ so that (6.5) is satisfied and thus, Ω^f is thick (otherwise, any measure supported by Ω^f, is small). If

$$q^*(t) \sim \frac{\log^{\alpha(d-1)-1} t}{t^2}$$

then it is easy to check that (6.8) holds and thus, $\mu = q(x)\mathbf{1}_{\Omega^f}\lambda$ is big. If for some $\varepsilon > 0$

$$q^*(t) \sim \frac{\log^{\alpha(d-1)-1-\varepsilon} t}{t^2}$$

then μ is small.

3. If $f(t) \sim t^\alpha$ with $\alpha < 1$ then Ω^f is always thin provided $d > 3$, by Proposition 6.3. Let us consider the case $d = 3$ and put

$$f(t) \sim t^{-N}$$

with $N > 0$. By Proposition (6.3), Ω^f is thick. Then for the function

$$q^*(t) \sim \frac{t^{2N}}{\log t},$$

the measure μ is big whereas for the function

$$q^*(t) \sim \frac{t^{2N}}{\log^{1+\varepsilon} t}, \quad \varepsilon > 0$$

the measure μ is small (both follow from the criterion (6.9)).

Proof of Proposition 6.4. To apply Theorems 1.2 and 1.3, we have to estimate the potentials $s_k = G^{\mu_k}$ where $\mu_k = q\mathbf{1}_{\Omega^f_k}\lambda$ and $\Omega^f_k = \Omega^f \cap \left(\overline{B}_k \backslash B_{k-1}\right)$. Let us introduce also the cylinders

$$C_k = \left\{2^{k-1} \le x_1 \le 2^k, \sqrt{x_2^2 + x_3^2 + ... + x_d^2} \le f(2^k)\right\}$$

of height $h_k = 2^{k-1}$ and of radius $r_k = f(2^k)$, and the measures

$$\nu_k = q^*(2^k)\mathbf{1}_{C_k}\lambda.$$

Let $s_k^* := G^{\nu_k}$. It is clear from conditions (6.3), (6.4), and (6.7) that the ratio of $s_k(x)$ and $s_k^*(x)$ is uniformly bounded from above and below for all x and k. Therefore, in the criteria (1.5) and (1.6), we may replace the functions s_k by s_k^*.

First, we have to estimate sup and inf of s_k^* over C_k. Since the measure ν_k has a constant density on C_k, it amounts to computing $G^{\mathbf{1}_{C_k}\lambda}$. It is not difficult to show that under the condition $r_k \le \operatorname{const} h_k$, the function $G^{\mathbf{1}_{C_k}\lambda}$ has comparable maximum and minimum on C_k which are of the order

$$\begin{cases} r_k^2, & d>3 \\ r_k^2 \log\left(1+\frac{h_k}{r_k}\right), & d=3 \end{cases}.$$

Next, we have obviously

$$G^{\mathbf{1}_{C_k}\lambda}(0) \sim \frac{\lambda(C_k)}{2^{k(d-2)}} \sim \frac{h_k r_k^{d-1}}{2^{k(d-2)}}.$$

Thus, by Theorems (1.2) and (1.3), μ is big if and only if

$$\sum_k \frac{q^*(2^k) h_k r_k^{d-1}}{2^{k(d-2)}} \frac{1}{1+q^*(2^k) r_k^2} = \infty, \quad \text{case } d>3,$$

$$\sum_k \frac{q^*(2^k) h_k r_k^2}{2^k} \frac{1}{1+q^*(2^k) r_k^2 \log\left(1+\frac{h_k}{r_k}\right)} = \infty, \quad \text{case } d=3,$$

which easily amounts to

$$\int^\infty \frac{f(t)^{d-3}}{t^{d-2}} \frac{dt}{q^*(t)^{-1} f(t)^{-2}+1} = \infty, \quad \text{case } d>3,$$

$$\int^\infty \frac{1}{t} \frac{dt}{q^*(t)^{-1} f(t)^{-2} + \log\left(1+\frac{t}{f(t)}\right)} = \infty, \quad \text{case } d=3,$$

i.e., (6.8) and 6.9. ■

6.3 A scattered measure in $\mathbb{R}^d$ which is big

The following measure appeared in [16], and its bigness was proved there by using the scaling property of $\mathbb{R}^d$. Let (B_k) be the sequence of balls in $\mathbb{R}^d$ of radii 2^k, centred at the origin $x_0 = 0$. Let E_k be a Borel subset of $A_k := \overline{B}_k \setminus B_{k-1}$, $E := \bigcup_k E_k$ and let

$$q(x) := \frac{1}{1+|x|^2}.$$

The measure μ is defined as

$$\mu = q\,\mathbf{1}_E \lambda.$$

The claim is:

if for some $\varepsilon > 0$ and all $k \ge 1$

$$\lambda(E_k) \ge \varepsilon \lambda(A_k)$$

then μ is big.

Of course, we can assume $d \geq 3$. The difficulty is that the measure μ may be very scattered. However, this situation can be handled by Theorem 1.2. In the notations of Theorem 1.2, we have

$$s_k(x) = \int_{E_k} G(x, y) q(y) d\lambda(y).$$

For any $x \in A_k$, we have

$$s_k(x) \leq \sup q(A_k) \int_{A_k} G(x, y) d\lambda(y) \leq 4^{2-k} \int_{B(x, 2^{k+1})} G(x, y) dy \leq \text{const}.$$

Therefore, $\alpha_k \leq \text{const}$.

Let us estimate $s(x_0)$ from below:

$$\begin{aligned} s_k(x_0) &\geq \text{const} \int_{E_k} |y|^{2-d} |y|^{-2} d\lambda(y) \\ &\geq \text{const}\, \varepsilon \lambda (A_k) 2^{-kd} \\ &\geq \text{const}. \end{aligned}$$

Thus, the series (1.5) is divergent, and μ is big.

In the same way, one can prove bigness of μ for the function

$$q(x) \sim \frac{1}{|x|^2 \log |x|}$$

whereas for the function

$$q(x) \sim \frac{1}{|x|^2 \log^{1+\varepsilon} |x|}, \quad \varepsilon > 0,$$

μ is small.

6.4 Counter-example to Theorem 1.2

We will use here the notations of Theorems 1.2 and 1.3. There is a gap between the condition (1.6)

$$\sum_k \frac{s_k(x_0)}{1 + \beta_k} < \infty$$

for non-bigness of a measure μ and the condition (1.5)

$$\sum_k \frac{s_k(x_0)}{1 + \alpha_k} = \infty$$

which ensures bigness of μ. Recall that β_k is infimum of $s_k(x)$ over $A_k \cap A$ whereas α_k is a supremum of s_k over the same set. In particular, $\beta_k \leq \alpha_k$. We will show here that one cannot replace β_k by α_k in the former statement.

More precisely, we construct in $\mathbb{R}^d$, $d > 2$, a big measure μ such that

$$\sum_k \frac{s_k(x_0)}{1 + \alpha_k} < \infty. \tag{6.11}$$

Moreover, the measure μ will possess a stronger property

$$\sum_k G^{\frac{\mu_k}{1+s_k}}(x_0) < \infty \tag{6.12}$$

where

$$\mu_k = \mathbf{1}_{A_k}\mu.$$

Indeed, (6.12) implies (6.11) as follows:

$$\frac{s_k(x_0)}{1+\alpha_k} = \frac{1}{1+\alpha_k} G^{\mu_k}(x_0) = G^{\frac{\mu_k}{1+\alpha_k}}(x_0) \le G^{\frac{\mu_k}{1+s_k}}(x_0).$$

Let B_k be a sequence of concentric balls of radii 2^k as in Theorems 1.2 and 1.3. Inside any annulus $A_k = \overline{B}_k \backslash B_{k-1}$, choose a very small ball b_k so that the union of all b_k is a non-thick set. Let χ be any big measure on $\mathbb{R}^d$, and let ν be a measure supported by $\bigcup_k b_k$ such that in each b_k it is proportional to the equilibrium measure γ_k of the ball b_k :

$$\nu = \sum_k a_k \gamma_k$$

with some coefficients $a_k > 0$.

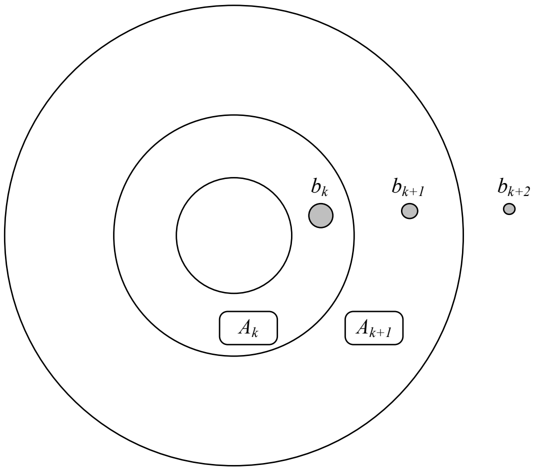

Figure 10 Each ball b_k carries its equilibrium measure

Let $\mu = \chi + \nu$. Regardless of choice of a_k, the measure μ is big because χ is big. Let $\nu_k := \mathbf{1}_{A_k}\nu = a_k\gamma_k$ and $\chi_k := \mathbf{1}_{A_k}\chi$ then

$$\begin{aligned}
\frac{\mu_k}{1+s_k} &= \frac{\chi_k + \nu_k}{1 + G^{\chi_k} + G^{\nu_k}} \\
&\le \frac{\chi_k}{1+G^{\nu_k}} + \frac{\nu_k}{1+G^{\nu_k}} \\
&\le \frac{\chi_k}{a_k G^{\gamma_k}} + \frac{\nu_k}{1+G^{\nu_k}}.
\end{aligned}$$

By choosing a_k large enough, we can ensure

$$\sum_k G^{\frac{\chi_k}{a_k G^{\gamma_k}}}(x_0) < \infty. \tag{6.13}$$

The measure ν is non-big because it is supported by a non-thick set. Thus, by Theorem 1.2, we have

$$\sum_k G^{\frac{\nu_k}{1+\sup G^{\nu_k}}}(x_0) < \infty. \tag{6.14}$$

Since by the choice of γ_k we have $\sup G^{\gamma_k} = 1$ and $G^{\gamma_k} \equiv 1$ on b_k then

$$\frac{\nu_k}{1+G^{\nu_k}} = \frac{\nu_k}{1+\sup G^{\nu_k}},$$

and (6.14) implies

$$\sum_k G^{\frac{\nu_k}{1+G^{\nu_k}}}(x_0) < \infty$$

which together with (6.13) yields (6.12).

6.5 Counter-example to Theorem 5.2

Here we use the notations of Theorem 5.2. We will show that the hypothesis (5.5)

$$\sum_k \inf_{x\in B_{k-2}} \frac{t_k(x)}{1+\alpha_k} = \infty \tag{6.15}$$

cannot in general be relaxed to

$$\sum_k \frac{t_k(x_0)}{1+\alpha_k} = \infty. \tag{6.16}$$

Since, in the setting of Corollary 5.3, (6.16) does imply bigness of μ (note that under hypotheses of Corollary 5.3, $s_k \sim t_k$) then we are looking at a situation with no uniform Harnack inequality. So, let us take $X = \mathbb{H}^2$ - the hyperbolic plane. We will construct a non-big measure μ for which (6.16) holds.

Let B_k be the concentric balls of radii 2^k centred at a point $x_0 \in \mathbb{H}^2$. Let ν_k denote the equilibrium measure of $A_k := \overline{B}_k \backslash B_{k-1}$ with respect to B_{k+1}. This means, in particular, that ν_k is supported by A_k,

$$\sup G^{\nu_k}_{B_{k+1}} = 1$$

and

$$G^{\nu_k}_{B_{k+1}} \equiv 1 \text{ on } A_k.$$

Since $G^{\nu_k}_{B_{k+1}}$ is continuous and harmonic in B_{k-1} then necessarily $G^{\nu_k}_{B_{k+1}} \equiv 1$ on B_{k-1}, too. In particular,

$$G^{\nu_k}_{B_{k+1}}(x_0) = 1.$$

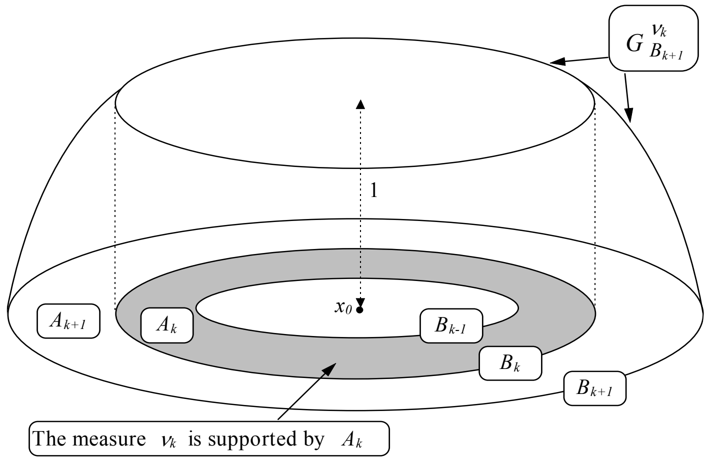

Figure 11 ν_k is the equilibrium measure of A_k

Now let Γ be a half-space in $\mathbb{H}^2$ containing x_0 on its boundary. Let us define

$$\mu = \mathbf{1}_\Gamma \sum_k \nu_k.$$

Since

$$t_k = G^{\mathbf{1}_\Gamma \nu_k}_{B_{k+1}} \le G^{\nu_k}_{B_{k+1}} \le 1,$$

then $\alpha_k = \sup t_k \le 1$ and $1 + \alpha_k \le 2$. By symmetry, we have

$$t_k(x_0) = \frac{1}{2} G^{\nu_k}_{B_{k+1}}(x_0) = \frac{1}{2}$$

whence (6.16) holds.

At the same time, the measure μ is non-big because it is supported by a half-space of $\mathbb{H}^2$ which is non-thick. The easiest way to see that is to map conformally $\mathbb{H}^2$ onto the unit disk $\mathbb{D}$ in $\mathbb{R}^2$. A two-dimensional conformal mapping preserves superharmonic functions and thus, thickness. A half-disk is obviously non-thick in $\mathbb{D}$ - one can construct its reduced function by solving the corresponding Dirichlet problem in its complement. Therefore, Γ is non-thick either.

Incidentally, we have now an example of two non-thick sets whose union is thick - two half-spaces in $\mathbb{H}^2$ which cover the whole space.

6.6 Harmonic functions with the third boundary condition

Let Γ be the half-space $\{x_1 > 0\}$ in $\mathbb{R}^d$, and let Σ denote its boundary $\{x_1 = 0\}$. Given a non-negative continuous function q on Σ, we define the third boundary value problem in Γ as follows:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Gamma \\ \frac{\partial u}{\partial x_1} - qu = 0 & \text{on} \quad \Sigma \end{cases}. \tag{6.17}$$

It is assumed that the function u is continuous in $\Gamma \cup \Sigma$. The boundary condition is understood in a weak sense. The latter means the following: Let $\Sigma_t := \{x_1 = t\}$

and let $y = (x_2, x_3, ...x_d)$, then for any $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$\lim_{t\to 0+} \int\limits_{\Sigma_t} \left(\frac{\partial u}{\partial x_1} - qu \right) \phi dy = 0. \tag{6.18}$$

We denote by σ the $(d-1)$-dimensional Lebesgue measure on Σ considered as a measure on $\mathbb{R}^d$.

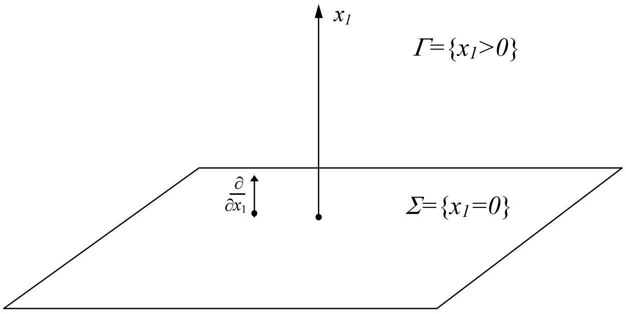

Figure 12 Sets Γ and Σ

Proposition 6.5 *The following statements are equivalent:*

(1) There exists a non-zero bounded solution to (6.17);

(2) The measure $q\sigma$ is non-big on $\mathbb{R}^d$.

Moreover, the function $L^{2q\sigma}|_{\Gamma\cup\Sigma}$ is a solution to (6.17), and for any other solution u of (6.17) satisfying the restriction $|u| \le 1$, we have $|u| \le L^{2q\sigma}$.

Remark: The idea to consider the third boundary value problem was suggested to us by V.A.Kondratiev.

Proof. The idea of the proof is to extend the solution u *evenly* in x_1 to the whole space $\mathbb{R}^d$ and to observe that the extended function satisfies the Schrödinger equation (1.1) with $\mu = 2q\sigma$. Let us note that $2q\sigma$ is a Kato measure on $\mathbb{R}^d$ which follows from the fact that the singularity on the Green function $G(x,y)$ is of order $|x-y|^{2-d}$ which is locally uniformly σ-integrable.

Let u be a bounded smooth function on Γ, continuous on $\Gamma \cup \Sigma$. Let us define the function w on $\mathbb{R}^d$ as follows

$$w(x_1, y) := u(|x_1|, y).$$

We claim that u is a solution to (6.17) if and only if

$$\Delta w = 2qw\sigma. \tag{6.19}$$

Let us note that in both cases: "if" and "only if", u is harmonic in Γ and thus, the function w is harmonic off Σ. Indeed, either u is harmonic in Γ as a solution to (6.17) or w is harmonic off Σ by (6.19) because the measure $2q\sigma$ is supported by Σ.

We are left to show that the boundary condition (6.18) is equivalent to (6.19). Fix a test function $\phi \in C_0^\infty(\mathbb{R}^d)$. Applying Green's formula, harmonicity of w off Σ and the fact that w is even and continuous, we obtain

$$\begin{aligned}
\langle \Delta w, \phi \rangle &= \langle w, \Delta\phi \rangle = \int\limits_{\mathbb{R}^d} w\Delta\phi dx \\
&= \lim_{t\to 0+} \left\{ \int\limits_{\{x_1>t\}} w\Delta\phi dx + \int\limits_{\{x_1<-t\}} w\Delta\phi dx \right\} \\
&= \lim_{t\to 0+} \int\limits_{\Sigma_t} \left[\frac{\partial w}{\partial x_1}\phi - \frac{\partial \phi}{\partial x_1} w \right] dy - \lim_{t\to 0+} \int\limits_{\Sigma_{-t}} \left[\frac{\partial w}{\partial x_1}\phi - \frac{\partial \phi}{\partial x_1} w \right] dy \\
&= \lim_{t\to 0+} \int\limits_{\Sigma_t} \frac{\partial w}{\partial x_1}\phi dy - \int\limits_{\Sigma} \frac{\partial \phi}{\partial x_1} w d\sigma - \lim_{t\to 0+} \int\limits_{\Sigma_{-t}} \frac{\partial w}{\partial x_1}\phi dy + \int\limits_{\Sigma} \frac{\partial \phi}{\partial x_1} w d\sigma \\
&= 2 \lim_{t\to 0+} \int\limits_{\Sigma_t} \frac{\partial u}{\partial x_1}\phi dy \\
&= 2 \lim_{t\to 0+} \int\limits_{\Sigma_t} \left[\frac{\partial u}{\partial x_1} - qu \right] \phi dy + 2 \int\limits_{\Sigma} qw\phi d\sigma \;, \qquad (6.20)
\end{aligned}$$

whence we see that the boundary condition (6.18) is equivalent to

$$\langle \Delta w, \phi \rangle = 2\langle qw\sigma, \phi \rangle$$

i.e. to (6.19).

The above argument shows that (1) $\Longrightarrow$ (2). Indeed, existence of a bounded non-trivial solution u of (6.17) implies that equation (6.19) has a non-zero bounded continuous solution w. Therefore, the measure $2q\sigma$ is non-big, and so is $q\sigma$ by Lemma 3.7.

Let us verify (2) $\Longrightarrow$ (1). Since $2q\sigma$ is non-big and Kato, the function $w := L^{2q\sigma}$ is a bounded continuous non-zero solution to (6.19) satisfying in addition $0 \le w \le 1$. We claim that w is even in x_1. Indeed, the function $\widetilde{w}(x_1, y) := w(-x_1, y)$ is also a solution of (6.19) with the same additional properties, and since by Lemma 2.5 w is the maximal such solution, then $\widetilde{w} \le w$. We conclude $\widetilde{w} = w$ and w is even. Let us set $u := w|_{\Gamma\cup\Sigma}$, then u is continuous in $\Gamma \cup \Sigma$ and harmonic in Γ. Moreover, we see by (6.20) and (6.19) that the boundary condition (6.18) is satisfied, too.

We are left to prove the last statement of Proposition 6.5. The fact that $u_0 := L^{2q\sigma}|_{\Gamma\cup\Sigma}$ is a solution to (6.17), was just shown above. Let u be another solution of (6.17) such that $|u| \le 1$. We will verify that $|u| \le u_0$. Indeed, if w is the even extension of u to $\mathbb{R}^d$, then w satisfies (6.19) and $|w| \le 1$. By Lemma 2.5, $w \le w_0 := L^{2q\sigma}$. Therefore, $u \le u_0$. In the same way, $-u \le u_0$ whence $|u| \le u_0$. ∎

Depending on q, the potential $G^{2q\sigma}$ may be both finite or infinite. The set Σ is thick as we saw above so we cannot conclude a priori non-bigness of $2q\sigma$. Both existence and non-existence of a non-trivial bounded solution to (6.17) may actually occur depending on the choice of the function q. Theorems 1.2 and 1.3 provide efficient tools to verify that.

7 Results for harmonic spaces

7.1 Harmonic spaces

A unifying concept in potential theory is the notion of a harmonic space. E.g., passing from solutions of $\Delta u = 0$ to solutions of $\Delta u - u\mu = 0$ (μ being a Kato measure) means passing from the classical harmonic space $(X, \mathcal{H}_\Delta)$ to the harmonic space $(X, \mathcal{H}_{\Delta-\mu})$.

For sake of simplicity we shall restrict our attention to Bauer spaces (but of course everything holds for general harmonic spaces and, in fact, could even be done for balayage spaces or H-cones).

In the following let X be an arbitrary locally compact space with countable base. For every open subset U of X, let $\mathcal{B}(U)$ (resp. $\mathcal{C}(U)$) be the set of all Borel measurable (resp. continuous real) functions on U. As usual, given a set $\mathcal{F}$ of functions on U, $\mathcal{F}_b$ (resp. $\mathcal{F}^+$) will be the set of all bounded (resp. positive) functions in $\mathcal{F}$. Let $\mathcal{U}_c$ denote the family of all open relatively compact subsets of X.

A *harmonic sheaf* on X is a map $\mathcal{H}$ which to every open subset U of X assigns a linear subspace $\mathcal{H}(U)$ of $\mathcal{C}(U)$ such that the following properties hold:

(S_1) *For any two open subsets U, V of X such that $U \subset V$, $\mathcal{H}(U) \subset \mathcal{H}(V)$.*

(S_2) *For any family $(U_i)_{i\in I}$ of open subsets and any numerical function h on $U = \cup_{i\in I} U_i$, $h \in \mathcal{H}(U)$ if $h_{|U_i} \in \mathcal{H}(U_i)$ for every $i \in I$.*

The elements of $\mathcal{H}(U)$ are called *harmonic functions* on U.

A set $V \in \mathcal{U}_c$ is called *regular* if every $f \in \mathcal{C}(\partial V)$ possesses a unique continuous extension $H_V f$ on $\overline{V}$ such that $H_V f$ is harmonic on V and $H_V f \geq 0$ if $f \geq 0$.

The pair $(X, \mathcal{H})$ is called a *Bauer space* if $\mathcal{H}$ has the following properties:

(B_1) *For every $x \in X$, there exists a harmonic function h defined in a neighbourhood of x such that $h(x) \neq 0$.*

(B_2) *For every $x \in X$, there exists a base $\mathcal{V}$ of regular sets such that $U \cap V \in \mathcal{V}$ for any $U, V \in \mathcal{V}$.*

(B_3) (Convergence property of Bauer) *For any increasing sequence (h_n) of positive harmonic functions on an open set U, $h = \sup h_n \in \mathcal{H}(U)$ if h is locally bounded.*

For each regular subset V of X, the map $f \mapsto H_V f$, $f \in \mathcal{C}(\partial V)$, defines a kernel on X which again is denoted by H_V (of course we take $H_V(x, \cdot) = \delta_x$ for $x \in X \setminus V$).

For every open subset U of X, a lower semicontinuous function $s : U \to]-\infty, +\infty]$ is called *hyperharmonic on U* provided that $H_V s \leq s$ for every regular $V \in \mathcal{U}_c$. It is *superharmonic on U* if in addition the functions $H_V s$ are locally bounded on V. A superharmonic function $s \geq 0$ on U is called *potential on U* if 0 is the largest harmonic minorant of s on U. We write $\mathcal{S}(U)$ (resp. $\mathcal{P}(U)$) for the set of all superharmonic functions (resp. potentials) on U. A function t on U is called subharmonic if $-t \in \mathcal{S}(U)$. Every function $s \in \mathcal{S}^+(U)$ admits a unique decomposition $s = h + p$ such that $h \in \mathcal{H}^+(U)$ and $p \in \mathcal{P}(U)$ (Riesz decomposition).

An open subset U of X is a $\mathcal{P}$–set if there exists a strictly positive $p \in \mathcal{P}(U)$. Any subset of a $\mathcal{P}$–set is a $\mathcal{P}$–set. It can be shown that X admits a covering by $\mathcal{P}$–sets. $(X, \mathcal{H})$ is called *$\mathcal{P}$–harmonic* if X itself is a $\mathcal{P}$–set.

Examples. Let X denote an open subset of $\mathbb{R}^d$, $d \geq 1$.

1. Let

$$L = \frac{1}{2} \sum_{k=1}^{r} A_k^2 + A_0 \tag{7.1}$$

where $A_0, A_1, ..., A_r$ are $\mathcal{C}^\infty$-vector fields on X and every $\mathcal{C}^\infty$-vector field $A = (\alpha_1, ..., \alpha_d) : X \to \mathbb{R}^d$ is identified with the differential operator $\sum_{i=1}^d \alpha_i \frac{\partial}{\partial x_i}$. The corresponding sheaf is defined by

$$\mathcal{H}_L(U) = \{h \in \mathcal{C}^\infty(U) : Lh = 0\} \qquad (U \text{ open } \subset X).$$

Let $\mathcal{L}(A_0, A_1, ..., A_r)$ denote the (Lie) algebra generated by $A_0, A_1, ..., A_r$ using the (Lie) product $[A, B] = AB - BA$. Then $(X, \mathcal{H}_L)$ is a Bauer space provided that, for each $x \in X$,

$$\{Z(x) : Z \in \mathcal{L}(A_0, A_1, ..., A_r)\} = \mathbb{R}^d \tag{7.2}$$

([5],[6]). A complete proof for this result using Hörmanders hypoellipticity theorem can be found in [4]. In fact, a stronger convergence axiom holds (*Doob's convergence axiom*): If (h_n) is an increasing sequence of harmonic functions on an open set U such that $h := \sup h_n$ is finite on a dense subset of U, then h is harmonic. If 7.2 holds omitting A_0, then even *Brelot's convergence axiom* is satisfied: If (h_n) is an increasing sequence of harmonic functions on a domain U such that $h := \sup h_n$ is not identically ∞, then h is harmonic.

Note that we get $L = \frac{1}{2}\Delta$ taking $r = d$, $A_0 = 0$, $A_k = \frac{\partial}{\partial x_k}$ for $1 \leq k \leq d$, and $L = \frac{1}{2}\sum_{i=1}^{d-1} \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial x_d}$ (operator of the heat equation, $d \geq 2$) taking $r = d-1$, $A_0 = -\frac{\partial}{\partial x_d}$, $A_k = \frac{\partial}{\partial x_k}$ for $1 \leq k \leq d-1$. We point out that L can be fairly degenerate, e.g., $L = \frac{\partial^2}{\partial x_1^2} + x_1 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}$ on $\mathbb{R}^3$ with $r = 1$, $A_0 = x_1 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}$ and $A_1 = \frac{\partial}{\partial x_1}$. For sublaplacians see e.g.[14].

2. Using probabilistic techniques (martingales, limit theorem) and Krylov's Harnack inequality it is possible to consider elliptic and parabolic operators where the coefficients need not be differentiable (see [18]): Let a_{ij}, b_i, c be continuous real functions on X such that $c \leq 0$. Consider the operator

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c \tag{7.3}$$

on X assuming that the matrix $(a_{ij}(x))_{i,j=1}^d$ is positive definite for every $x \in X$ (elliptic case) or the operator (for $d \geq 2$)

$$L = \frac{1}{2} \sum_{i,j=1}^{d-1} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d-1} b_i \frac{\partial}{\partial x_i} + c - \frac{\partial}{\partial x_d}$$

assuming that the matrix $(a_{ij}(x))_{i,j=1}^{d-1}$ is positive definite for every $x \in X$ (parabolic case). In both cases, the pair $(X, \mathcal{H}_L)$ is a Bauer space if we define

$$\mathcal{H}_L(U) = \{h \in \mathcal{C}(U) : Lh = 0\}.$$

In fact, in the parabolic case, $(X, \mathcal{H}_L)$ is $\mathcal{P}$-harmonic and satisfies Doob's convergence axiom. And, in the elliptic case, $(X, \mathcal{H}_L)$ always satisfies Brelot's convergence axiom and we have a $\mathcal{P}$-harmonic space if e.g. one of the following conditions is satisfied:

(i) c does not vanish identically on any connected component of X.

(ii) X is not dense in $\mathbb{R}^d$ and the functions b_i are bounded on bounded subsets of $\mathbb{R}^d$.

3. It is easy to check that, except for $\mathcal{P}$-harmonicity which has to be investigated separately, nothing changes if X is a Riemannian manifold and L is a differential operator on X which is locally of type 7.1 or 7.3.

In the following let $(X, \mathcal{H})$ be a Bauer space. For simplicity let us assume that $(X, \mathcal{H})$ is *$\mathcal{P}$-harmonic*.

The ($\mathcal{H}-$)fine topology on X is the coarsest topology such that every function $s \in \mathcal{S}^+(X)$ is continuous. Every open subset of X is finely open.

For every $A \subset X$ and $s \in \mathcal{S}^+(X)$, we define

$$\begin{aligned} R^A s &= \inf\{v \in \mathcal{S}^+(X) : v \geq s \text{ on } A\} \\ &= \inf\{v \in \mathcal{S}^+(X) : v \leq s \text{ on } X, v = s \text{ on } A\}, \\ \widehat{R}^A s(x) &= \liminf_{y \to x} R^A s(y). \end{aligned}$$

Then $R_s^S \in \mathcal{S}^+(X)$ and, of course, $\widehat{R}_s^A \leq R_s^A \leq s$.

If $V \in \mathcal{U}_c$ is regular, then

$$R_s^{\complement V} = H_V s \tag{7.4}$$

for every $s \in \mathcal{S}^+(X)$. For an arbitrary $V \in \mathcal{U}_c$, the equation (7.4) is used to *define* a kernel H_V (uniqueness is already assured by having (7.4) for all continuous real potentials). And then, for every hyperharmonic function s on an open subset U of X, we have

$$H_V s \leq s$$

for every $V \in \mathcal{U}_c$ such that $\overline{V} \subset U$. Moreover, $H_V s$ is harmonic on V if s is superharmonic.

A sequence (U_n) in $\mathcal{U}_c$ is called an *exhaustion* of X if $\bigcup_{n=1}^\infty U_n = X$ and $\overline{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$. We note that X may not admit an exhaustion by regular sets (e.g., if X is a W–shaped region in $\mathbb{R}^d \times \mathbb{R}$ and $\mathcal{H}$ is the sheaf of solutions of the heat equation), and this is the reason why we need kernels H_V for arbitrary $V \in \mathcal{U}_c$.

We state two elementary, but useful facts.

Proposition 7.1 *1. Let $s \in \mathcal{S}^+(X)$ and let (U_n) be an exhaustion of X. Then the sequence $(H_{U_n} s)$ decreases to the harmonic part in the Riesz decomposition of s.*

2. If $s \in \mathcal{S}(X)$ and $p \in \mathcal{P}(X)$ such that $s + p \geq 0$ then $s \geq 0$.

We shall say that a sequence (x_n) in U converging to a point $z \in U$ is *regular* if

$$\lim_{n \to \infty} H_U f(x_n) = f(z) \tag{7.5}$$

for every $f \in \mathcal{C}(X)$. Of course, a point $z \in \partial U$ is regular if every sequence in U which converges to z is regular. And U is regular if every boundary point of U is regular. The following minimum principle is very useful (see e.g. [4, p. 107]):

Proposition 7.2 *(Minimum Principle) Let $U \in \mathcal{U}_c$ and $s \in \mathcal{S}(U)$ such that s is lower bounded and* $\liminf_{n\to\infty} s(x_n) \geq 0$ *for every regular sequence (x_n) in U. Then $s \geq 0$.*

A potential $p \in \mathcal{C}(X)$ is *strict* if, for each $x \in X$, the Dirac measure at x is the only measure $\rho \geq 0$ on X such that $\int p\, d\rho = p(x)$ and $\int q\, d\rho \leq q(x)$ for every $q \in \mathcal{P}(X)$. If p is strict, then of course, $H_U p < p$ on U for every $U \in \mathcal{U}_c$. Our assumption that $(X, \mathcal{H})$ is $\mathcal{P}$-harmonic implies the existence of strict potentials.

Our harmonic space $(X, \mathcal{H})$ may (or may not) possess a Green function. A Borel function $G_X : X \times X \to [0, \infty]$ is called a (measurable) Green function for $(X, \mathcal{H})$ provided the following hold:

(i) For every $y \in X$, $G_X(\cdot, y)$ is a potential on X which is harmonic on $\complement\{y\}$.

(ii) For every $p \in \mathcal{P}(X) \cap \mathcal{C}(X)$ which is harmonic outside a compact set, there exists a measure $\mu \geq 0$ on X such that

$$p = \int G_X(\cdot, y)\, \mu(dy).$$

7.2 Perturbations of harmonic spaces

Let us now proceed to perturbation of $(X, \mathcal{H})$. For our purpose it will be most convenient to describe it in terms of the associated kernels.

We shall say that $M = (K_U^M)_{U \in \mathcal{U}_c}$ is a *Kato family of potential kernels on X* if every K_U^M, $U \in \mathcal{U}_c$, is a kernel on U such that, for every $f \in \mathcal{B}_b^+(U)$, the function $K_U^M f$ is a continuous bounded potential on U which is harmonic outside the support of f and such that $K_U^M 1 - K_V^M 1$ is harmonic on $U \cap V$ for all $U, V \in \mathcal{U}_c$.

In the case of a Riemannian manifold X a Kato family $M = (K_U^M)_{U \in \mathcal{U}_c}$ of potential kernels can be identified with a Kato measure μ by

$$K_U^M = K_U^\mu \qquad (U \in \mathcal{U}_c)$$

(of course, we have an analogous identification if our harmonic space $(X, \mathcal{H})$ admits a Green function).

Defining sums and positive multiples of Kato families of potential kernels in the obvious way we obtain a convex cone. We even have a multiplication by locally bounded functions $\varphi \in \mathcal{B}^+(X)$ given by

$$K_U^{\varphi M} f := K_U^M(\varphi f) \qquad (U \in \mathcal{U}_c).$$

For every $p \in \mathcal{P}(X) \cap \mathcal{C}(X)$ there exists a unique Kato family $M(p)$ of potential kernels which is connected with p by

$$K_U^{M(p)} 1 = p - H_U p \qquad (U \in \mathcal{U}_c).$$

Now fix a Kato family $M = (K_U^M)_{U \in \mathcal{U}_c}$. The compatibility condition $K_U^M 1 - K_V^M 1 \in \mathcal{H}(U \cap U)$ implies that

$$K_V^M = K_W^M - H_V K_W^M \tag{7.6}$$

if $V, W \in \mathcal{U}_c$, $\overline{V} \subset W$. This shows that

$$\lim_{n\to\infty} K_V^M f(x_n) = 0 \tag{7.7}$$

for every $f \in \mathcal{B}_b(X)$ and every regular sequence (x_n) in V. Defining

$$K^M := \sup_{V\in\mathcal{U}_c} K_V^M$$

we obtain a kernel K^M on X such that, for every $f \in \mathcal{B}^+(X)$, the function $K^M f$ is hyperharmonic on X and, for every $U \in \mathcal{U}_c$,

$$K_U^M + H_U K^M = K^M$$

(see (2.3)).

As is the case of a Riemannian manifold, each K_V^M, $V \in \mathcal{U}_c$, is a compact operator on $\mathcal{B}_b(V)$, $I + K_V^M$ is invertible, and $(I + K_V^M)^{-1} s \geq 0$ for every $s \in \mathcal{S}_b^+(V)$ ([7, p. 103]). Hence the definition

$$H_V^M := (I + K_V^M)^{-1} H_V \tag{7.8}$$

yields a positive kernel on X. Define

$$\mathcal{H}^M = \{\mathcal{H}^M(U) : U \text{ open } \subset X\}$$

where

$$\mathcal{H}^M(U) = \{u \in \mathcal{C}(U) : H_V^M u = u \text{ for every } V \in \mathcal{U}_c \text{ such that } \overline{V} \subset U\}.$$

It is not difficult to prove the following (see [7]):

Theorem 7.3 *$(X, \mathcal{H}^M)$ is a $\mathcal{P}$–harmonic Bauer space.*

The functions in $\mathcal{H}^M(U)$ are called M–harmonic on U. The corresponding set of M-superharmonic functions (M-potentials resp.) on U is denoted be ${}^M\mathcal{S}(U)$ (${}^M\mathcal{P}(U)$ resp.) and it is easy to show that, for every $U \in \mathcal{U}_c$,

$$\begin{aligned} {}^M\mathcal{H}_b(U) &= (I + K_U^M)^{-1}\mathcal{H}_b(U), \\ {}^M\mathcal{S}_b(U) &= (I + K_U^M)^{-1}\mathcal{S}_b(U), \\ {}^M\mathcal{P}_b(U) &= (I + K_U^M)^{-1}\mathcal{P}_b(U). \end{aligned}$$

If we are in the case of a Riemannian manifold where M corresponds to a Kato measure μ, then a finely continuous locally bounded function u on an open set U is M-harmonic (M-superharmonic, M-subharmonic resp.) if and only if $\Delta u = u\mu$ ($\Delta u \leq u\mu$, $\Delta u \geq u\mu$ resp.).

It is useful to note that the fine topology of $(X, \mathcal{H})$ is the fine topology of $(X, \mathcal{H}^M)$. Moreover, it follows immediately from (7.6) and (7.7) that any regular sequence (x_n) in $U \in \mathcal{U}_c$ (regular with respect to $(X, \mathcal{H})$) is also regular with respect to $(X, \mathcal{H}^M)$ (and conversely).

7.3 h-small and h-big families of potential kernels

We now extend what we did before on Riemannian manifolds to our more general setup. In most cases it will be clear how to get the corresponding general proof and then we shall simply omit it without further reference.

In the sequel, h always denotes a positive harmonic function on X. We stress the fact that even for connected X the function h may vanish on a substantial part of X without being identically zero! (It happens already for solutions of the heat equation.)

Given a Kato family $M = (K_U^M)_{U\in\mathcal{U}_c}$ we define a corresponding Liouville function $L^M h$ by

$$L^M h = \inf_{U\in\mathcal{U}_c} H_U^M h.$$

If $(U_k) \subset \mathcal{U}_c$ is any exhaustion of X then

$$L^M h = \lim_{k\to\infty} H_{U_k}^M h.$$

Proposition 7.4 *If M is a Kato family of potential kernels on X then $L^M h$ is a continuous subharmonic function on X, $0 \le L^M h \le h$. The function $L^M h + K^M L^M h$ is harmonic, $L^M + K^M L^M h \le h \le L^M h + K^M h$. In particular, $L^M h$ is M-harmonic.*

If v is any M-(sub)harmonic function such that $|v| \le h$, then $v \le Lh$.

Since Kato families of potential kernels form a convex cone, we have a *specific order*: $M' \prec M$ if and only if there exists M'' such that $M' + M'' = M$.

Lemma 7.5 *Let (M_n) be a sequence of Kato families of potential kernels which is specifically increasing. If $(\sup_n K_U^{M_n})_{U\in\mathcal{U}_c}$ is a Kato family M of potential kernels, then* $\inf L^{M_n} h = L^M h$.

Definition 7.1 *A family $M = (K_U^M)_{U\in\mathcal{U}_c}$ of kernels K_U^M on U will be called a* smooth family of potential kernels on X *if there exists a sequence (M_n) of Kato families of potential kernels on X such that*

$$K_U^M = \sum_{n=1}^{\infty} K_U^{M_n} \qquad (U \in \mathcal{U}_c).$$

It will be called proper *if $K_U^M 1 \in \mathcal{P}(U)$ for every $U \in \mathcal{U}_c$.*

If we have a corresponding Green function then such a smooth family M will correspond to a smooth measure μ and M will be proper if μ is a Radon measure.

Let us mention that a perturbation by smooth proper families of potential kernels (even by differences of such families) has already been studied in [15].

The set $\mathcal{M}^+(\mathcal{H})$ of all smooth families of potential kernels is a convex cone and its specific order extends the specific order introduced for Kato families of potential kernels on X (since $M', M'', M \in \mathcal{M}^+(\mathcal{H})$, $M' + M'' = M$, M a Kato family, implies that M', M'' are Kato families!). It should be clear by now how to define φM for $\varphi \in \mathcal{B}^+(X)$ and $M \in \mathcal{M}^+(X)$. And then, for any $M \in \mathcal{M}^+(X)$, there exists a

Kato family of potential kernels on X and $\varphi \in \mathcal{B}^+(X)$ such that $\varphi N = M$ (in fact, $M = \varphi M(p)$ for some $p \in \mathcal{P}(X) \cap \mathcal{C}(X)$ and $\varphi \in \mathcal{B}^+(X)$).

Given $M \in \mathcal{M}^+(\mathcal{H})$, we define the global (potential) kernel K^M by

$$K^M := \sup_{U \in \mathcal{U}_c} K_U^M = \lim_{k \to \infty} K_{U_k}^M$$

where (U_k) is any exhaustion of X (recall that $K_U^M \leq K_V^M$ if $U, V \in \mathcal{U}_c$, $\overline{U} \subset V$). We might note that $M = M(p)$ for some $p \in \mathcal{P}(X) \cap \mathcal{C}(X)$ if and only if $K^M 1 \in \mathcal{P}(X) \cap \mathcal{C}(X)$ and then of course $p = K^M 1$.

Proposition 7.6 *For every $f \in \mathcal{B}^+(X)$ the following holds:*

(i) If M is a Kato family and f is bounded with compact support, then $K^M f$ is a continuous bounded potential which is harmonic on $\complement \operatorname{supp}(f)$.

(ii) $K^M f$ is a positive hyperharmonic function. It is a potential if and only if it is bounded by a superharmonic function.

Proof. (i) Choose $U \in \mathcal{U}_c$ containing the support of f. Then $q := K_U^M f$ is a continuous real potential on U, harmonic on $U \setminus \operatorname{supp}(f)$. There exists a (unique) $p \in \mathcal{P}(X) \cap \mathcal{C}(X)$ ("Hervé–lifting") such that p is harmonic on $X \setminus \operatorname{supp}(f)$ and $p - q$ is harmonic on U. Fix an exhaustion (U_k) of X such that $\overline{U} \subset U_1$. Then it is easily seen that $K_{U_k}^M f = p - H_{U_k} p$ for every $k \in \mathbb{N}$. Since $\lim_{k \to \infty} H_{U_k} p = 0$, sequence obtain that $K^M f = p$.

(ii) The function f is a countable sum of positive bounded functions with compact support and M is a countable sum of Kato families of potential kernels. Therefore, by (i), $K^M f$ is a countable sum of continuous real potentials which are harmonic on $X \setminus \operatorname{supp}(f)$. This implies (ii) (see e.g. [4]). ■

Given $M \in \mathcal{M}^+(\mathcal{H})$, we define a corresponding Liouville function $L^M h$ by

$$L^M h = \inf\{L^N h : N \in \mathcal{M}^+(\mathcal{H}), N \text{ Kato family }\}. \tag{7.9}$$

Lemma 7.5 implies that

$$L^M h = \inf_n L^{M_n} h \tag{7.10}$$

for any sequence (M_n) of Kato families of potential kernels on X which is specifically increasing to M.

Lemma 7.7 *For every $M \in \mathcal{M}^+(\mathcal{H})$, $0 \leq L^M h \leq h$, the function $L^M h + K^M L^M h$ is subharmonic, and $L^M h + K^M L^M h \leq h \leq L^M h + K^M h$. In particular, $L^M h$ is subharmonic.*

If M is proper, then $L^M h + K^M L^M h$ is harmonic.

Lemma 7.8 *Let $M \in \mathcal{M}^+(\mathcal{H})$ and let v be a finely continuous function on X such that $|v| \leq h$. Assume that, for every $U \in \mathcal{U}_c$ and for every Kato family $N \in \mathcal{M}^+(\mathcal{H})$ such that $N \prec M$, the function $v + K_U^N v^+ - K_U^M v^-$ is subharmonic on U. Then $v \leq L^M h$.*

Proposition 7.9 *Let $M \in \mathcal{M}^+(\mathcal{H})$, $h \in \mathcal{H}^+(X)$. Then $L^M h$ can be characterized in the following way:*

(1) *$L^M h$ is the maximal finely continuous function v on X such that $|v| \le h$ and, for every $U \in \mathcal{U}_c$, the function $K_U^M|v|$ is bounded and $v + K_U^M v$ is subharmonic on U.*

(1a) *$L^M h$ is the maximal finely continuous function v on X such that $|v| \le h$, $K^M|v|$ is h-bounded and $v + K^M v$ is subharmonic.*

(2) *If M is proper, then $L^M h$ is the maximal finely continuous function v such that $|v| \le h$ and, for every $U \in \mathcal{U}_c$, the function $K_U^M|v|$ is bounded and $v + K_U^M v$ is harmonic on U.*

(2a) *If M is proper, then $L^M h$ is the maximal finely continuous function v such that $|v| \le h$, $K^M|v|$ is h-bounded, and $v + K^M v$ is harmonic.*

Given $h \in \mathcal{H}^+(X)$, the subharmonic function $L^M h$ has a smallest harmonic majorant $P^M h$, namely

$$P^M h = \lim_{U \uparrow X} H_U L^M h, . \tag{7.11}$$

Of course,

$$L^M h \le P^M h \le h,$$

and M is h-big if and only if $P^M h = 0$.

Proposition 7.10 *We have:*

(1) *For every $h \in \mathcal{H}^+(X)$, $L^M h + K^M L^M h \le P^M h$.*

(2) *If M is proper, then $L^M + K^M L^M = P^M$.*

(3) *The operator P^M on $\mathcal{H}^+(X)$ is idempotent.*

Proof. (1) By Lemma 7.7, we know that $L^M h + K^M L^M h$ is a subharmonic minorant of h. In particular, $K^M L^M h$ is a potential by Proposition 7.6. By (7.11), this implies that $L^M h + K^M L^M h \le \lim_{U\uparrow X} H_U(L^M h + K^M L^M h) = P^M h$.

(2) If M is proper, we know in addition that $L^M h + K^M L^M h$ is harmonic, hence $P^M h \le L^M h + K^M L^M h$.

(3) By Proposition 7.9 and (1), we conclude that $L^M h \le L^M(P^M h)$. Therefore, by (7.11), $P^M h \le P^M(P^M h)$. The converse inequality is obvious. ■

Definition 7.2 *We shall say that M is h–*small*, if $P^M h = h$.*

Lemma 7.11 *M is h–small if and only if $h - L^M h$ is a potential.*

Lemma 7.12 *(Monotonicity of bigness and smallness) If $N \prec M$, then M is h–big if N is big, and N is h–small if M is h–small.*

Lemma 7.13 *(A $0-1$ law) If $L^M h \not\equiv 0$ (i.e., if M is non-h–big) then*

$$\sup\{\frac{L^M h}{h}(x) : x \in X,\ h(x) > 0\} = 1.$$

Lemma 7.14 *We have $L^M + L^N \leq I + L^{M+N}$ and $P^M + P^N \leq I + P^{M+N}$.*

Lemma 7.15 *If N is h–small, then $P^{M+N}h = P^M h$.*

Proposition 7.16 *The following is true:*

(1) If M is non-h–big and N is h–small, then $M + N$ is non-h–big.

(2) The set of all h–small $M \in \mathcal{M}^+(\mathcal{H})$ is a convex cone.

(3) Any strictly positive multiple of an h–big $M \in \mathcal{M}^+(\mathcal{H})$ is h–big.

Lemma 7.17 *(First case of h–smallness) If $K^M h$ is a potential, then M is h–small.*

We shall say that $M \in \mathcal{M}^+(\mathcal{H})$ is *supported* by a (Borel) set A if $K^M \mathbf{1}_{\complement A} = 0$.

Lemma 7.18 *(Second case of non-h–bigness and h–smallness) If M is supported by a set A and A is non-h–thick (h–thin resp.) then M is non-h–big (h–small resp.).*

Theorem 7.19 *For any $\mathcal{P}$-harmonic space $(X, \mathcal{H})$ and any $M \in \mathcal{M}^+(\mathcal{H})$, the following statements are equivalent:*

(i) M is non-h–big (h-small resp.).

(ii) M can be represented as a sum $M = M_1 + M_2$ of two smooth families of potential kernels where M_1 is supported by a non–h–thick set (h-thin set resp.) and $K^{M_2}h$ is a potential.

(iii) There is an open set A which is non-h–thick (h-thin resp.) and such that for $M_2 := \mathbf{1}_{\complement A}M$, the function $K^{M_2}h$ is an h–bounded potential.

If M is a Kato family then each of (i)-(iii) is equivalent to:

(iv) There is an open set A which is non–h–thick (h-thin resp.) such that for $M_2 := \mathbf{1}_{\complement A}M$, the function $K^{M_2}h$ is a continuous h–bounded potential.

Moreover, we have the following result:

Proposition 7.20 *For every $c > 0$, $P^{cM} = P^M$.*

Proof. By monotonicity, it suffices to show that $P^M h \leq P^{2M} h$ for every $h \in \mathcal{H}^+(X)$. Applying Lemma 7.14 to the function $P^M h$, we obtain by Proposition 7.16 that

$$2P^M h = 2P^M(P^M)h \leq P^M h + P^{2M}(P^M h) \leq P^M h + P^{2M} h,$$

hence $P^M h \leq P^{2M} h$. ■

Corollary 7.21 *If M is a Kato family of potential kernels, then $\lim_{\varepsilon \downarrow 0} L^{\varepsilon M} h = P^M h$.*

Proof. Since

$$L^{\varepsilon M} h + \varepsilon K^M L^{\varepsilon M} h = P^{\varepsilon M} h = P^M h$$

for every $\varepsilon > 0$, the (εM)–harmonic functions $L^{\varepsilon M} h$ are increasing to a harmonic function $u \leq P^M h$ as ε decreases to zero. Clearly, $L^M h \leq u$, hence $P^M h \leq u$. Thus $P^M h = u$. ■

8 Relations to minimality of h

Let us first consider a simple sufficient condition for h-bigness:

Proposition 8.1 *Let M be a Kato family of potential kernels on X, let (U_n) be an exhaustion of X, and let (α_n) be a sequence in $[0,1]$ such that*

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

and

$$H^M_{U_{n+1}} h \le (1-\alpha_n)h \ \text{ on } U_n$$

for every $n \in \mathbb{N}$. Then M is h–big.

Proof. Fix $n \in \mathbb{N}$. Our assumption implies that

$$H^M_{U_{m+1}} h \le (1-\alpha_m)h \qquad \text{on } \partial U_m$$

for every $m \in \mathbb{N}$. Induction on k yields that, for every $k \in \mathbb{N}$,

$$H^M_{U_{n+1}} H^M_{U_{n+2}} \dots H^M_{U_{n+k+1}} h \le (1-\alpha_n)(1-\alpha_{n+1}) \dots (1-\alpha_{n+k})h \qquad \text{on } U_n.$$

Since $L^M h$ is M–harmonic and $L^M h \le h$, we know that, for every $k \in \mathbb{N}$,

$$\begin{aligned} L^M h &= H^M_{U_{n+1}} H^M_{U_{n+2}} \dots H^M_{U_{n+k+1}} L^M h \\ &\le H^M_{U_{n+1}} H^M_{U_{n+1}} \dots H^M_{U_{n+k+1}} h \end{aligned}$$

and hence

$$L^M h \le \prod_{m=n}^{n+k} (1-\alpha_m) h \qquad \text{on } U_n.$$

Letting k tend to infinity we obtain that $L^M h = 0$ on U_n. Thus $L^M h = 0$ on X, i.e., M is h–big. ■

$M \in \mathcal{M}^+(\mathcal{H})$ is called *strictly positive*, if $K^M_U 1_V \neq 0$ for every $U \in \mathcal{U}_c$ and every finely open non-empty subset V of U, i.e. if M is not supported by a finely closed proper subset of X. In the case of a Riemannian manifold X where M can be identified with a smooth measure μ by

$$K^M_U = K^\mu_U \qquad (U \in \mathcal{U}_c),$$

M is strict if and only if $\mu(V) > 0$ for every finely open $V \neq \emptyset$.

Clearly, $M(p)$ is strict if $p \in \mathcal{P}(X) \cap \mathcal{C}(X)$ is a strict potential. Conversely, it is not difficult to see the following: If $M \in \mathcal{M}^+(\mathcal{H})$ is strict, then there exists a Kato family N of potential kernels on X such that N is strict and $N \prec M$, and we may even get $N = M(p)$ for some strict $p \in \mathcal{P}(X) \cap \mathcal{C}(X)$.

The following result allows us to find sections which are h–big.

Proposition 8.2 *Suppose that M is strictly positive. Then there exists a locally bounded function $\varphi \in \mathcal{B}^+(X)$ such that φM is h–big.*

Proof. By our previous considerations we may assume that M is a Kato family. Let (U_n) be an exhaustion of X and define

$$W_n = U_{2n+1} \setminus \overline{U}_{2n-1}.$$

Then there exists a sequence (k_n) in $\mathbb{N}$ such that, for every $n \in \mathbb{N}$,

$$H_{W_n}^{k_n M} h < \frac{1}{2} h \quad \text{on } \partial U_{2n}.$$

Indeed (see [14, p. 144]), fix $n \in \mathbb{N}$ and define

$$f_k = H_{W_n}^{kM} h, \quad k \in \mathbb{N}.$$

The sequence (f_k) is decreasing. Let

$$f = \inf f_k.$$

Since $f_k + kK_{W_n}^M f_k = h$, we obtain that

$$s := \sup_k kK_{W_n}^M f_k$$

is superharmonic on W_n and $f + s = h$. Therefore f is finely continuous. Moreover,

$$kK_{W_n}^M f \le kK_{W_n}^M f_k \le h$$

for every $k \in \mathbb{N}$, hence $K_{W_n}^M f = 0$. Since M is strictly positive, this implies that $f = 0$ on W_n. In particular, an application of Dini's lemma yields the existence of a natural number k_n such that $f_{k_n} < \frac{1}{2} h$ on ∂U_{2n}.

Define

$$\varphi = \sum_{n=1}^{\infty} k_n 1_{W_n}.$$

Then, for every $n \in \mathbb{N}$, $\varphi \le \max(k_1, ..., k_n)$ on U_{2n+1} and

$$H_{U_{2n+1}}^{\varphi M} h \le H_{W_n}^{\varphi M} h = H_{W_n}^{k_n M} h < \frac{1}{2} h \text{ on } \partial U_{2n},$$

hence by the minimum principle

$$H_{U_{2n+1}}^{\varphi M} h \le \frac{1}{2} h \quad \text{on } U_{2n}.$$

By Proposition 8.1 (looking at the sequence (U_{2n-1})) we conclude that φM is h–big. ∎

We intend to show that h is minimal if and only if each M is either h-small or h-big. To that end we have to look more closely at h-thick and h-thin sets.

Of course, every subset of an h-thin set is h-thin, and every subset of X containing an h-thick set is h-thick. Given $A \subset X$, there always exist a Borel (even a G_δ-) set $A' \subset X$ such that $A \subset A'$ and $\widehat{R}^{A'} h = \widehat{R}^A h$ (see e.g. [4, p. 250]). In particular, every h-thin set A is contained in a Borel set A' which is h-thin. Now suppose for a moment that $A \subset X$ is not h-thick, i.e., suppose that there exists a superharmonic function $s \geq 0$ on X such that $s \geq h$ on A and $s(x) < h(x)$ for some $x \in X$. Let s_0 be a strictly positive real superharmonic function on X . If $x \in X$ and $s(x) < h(x)$ then $s(x) + \varepsilon s_0(x) < h(x)$ if $\varepsilon > 0$ is sufficiently small and $s \geq h$ on A implies that $s + \varepsilon s_0 > h$ on A, hence on an open neighbourhood of A. So a subset A of X is h-thick if and only if every open neighbourhood of A is h-thick.

Since $\widehat{R}^{A\cup B}h \leq \widehat{R}^A h + \widehat{R}^B h$, the union of two h-thin sets is always h-thin. In general, however, the union of two sets which are not h-thick may be h-thick: In Example 1 the intervals $]0, \frac{1}{2}]$ and $[\frac{1}{2}, 1[$ are not 1-thick, but of course $]0, 1[$ is 1-thick. The following equivalences (which of course are partly known) show that this cannot happen if h is minimal.

Proposition 8.3 *The following statements are equivalent:*

(i) h is minimal.

(ii) Every subset A of X is either h-thin or h-thick.

(iii) For all $A, B \subset X$ which are not h-thick the union $A \cup B$ is not h-thick.

(iv) For every (Borel) subset A of X the set A or the complement of A is h-thick.

Proof. *(i)* $\Longrightarrow$ *(ii)*: Suppose that A is not h-thick. Choose an open neighbourhood U of A which is not h-thick. Then $R^U h$ is a positive superharmonic function. There exist $g \in \mathcal{H}^+(X)$ and $p \in \mathcal{P}(X)$ such that $R^U h = g + p$, but $g \neq h$ since $R^U h \neq h$. So $g = \alpha h$ for some $0 \leq \alpha < 1$. Since $R^U h = h$ on U, we obtain that $p = (1 - \alpha)h$ on U, hence

$$\frac{1}{1-\alpha} p \geq R^U h \geq g.$$

So $g = 0$, $R^U h = p$.

(ii) $\Longrightarrow$ *(iii)*: Trivial since the union of two h-thin sets is h-thin.

(iii) $\Longrightarrow$ *(iv)*: Trivial since X is h-thick.

(iii) $\Longrightarrow$ *(i)*: Suppose that h is not minimal. Then there exist h_1, $h_2 \in \mathcal{H}^+(X) \setminus \{0\}$ such that $h_1 + h_2 = h$ and h_1 is not a multiple of h_2. We may choose $x_1, x_2 \in X$ and $0 < \alpha < \infty$ such that $h_1(x_1) > \alpha h_2(x_1)$ and $h_1(x_2) < \alpha h_2(x_2)$. Define

$$A = \{h_1 \leq \alpha h_2\}.$$

Then $R^A h_1 \leq \alpha h_2$, hence

$$\widehat{R}^A h_1(x_1) < h_1(x_1).$$

So A is not h_1–thick. Moreover,

$$\widehat{R}^{\complement A} h_2 \leq \frac{1}{\alpha} h_1,$$

hence

$$\widehat{R}^{\complement A} h_2(x_2) < h_2(x_2).$$

So $\complement A$ is not h_2–thick. Since $\widehat{R}^B h = \widehat{R}^B h_1 + \widehat{R}^B h_2$ for every $B \subset X$, we finally obtain that $\widehat{R}^A h \neq h$ and $\widehat{R}^{\complement A} h \neq h$, i.e., A and $\complement A$ are not h-thick. ■

Corollary 8.4 *The following statements are equivalent:*

(i) h is minimal.

(ii) Every M is h-small or h-big.

(iii) For all M, N which are not h-big, the sum M + N is not h-big.

Proof
$(i) \Longrightarrow (ii)$: Proposition 7.13.
$(ii) \Longrightarrow (iii)$: Trivial by Proposition 7.16.
$(iii) \Longrightarrow (i)$: Suppose that h is not minimal. By Proposition 8.3, there is a Borel subset A of X such that A and $\complement A$ are not h-thick. Using Proposition 8.2 choose $M \in \mathcal{M}^+(X)$ such that h is M–big. Then $M = 1_A M + 1_{\complement A} M$ and $1_A M$ and $1_{\complement A} M$ are not h-big by Lemma 7.18. ■

9 Appendix

9.1 Proof of Proposition 2.1

(1) Fix a sequence $\{f_n\}$ in $\mathcal{B}_b(V)$ such that $0 \le f_n \le 1$ for every $n \in \mathbb{N}$ and choose $B' \in \mathcal{O}_r$ such that $B \subset B'$. Then the sequence $\{K^\mu_{B'} f_n\}$ (extend f_n by 0) in $\mathcal{C}_b(B')$ is equicontinuous, since $G^\mu_{B'}$ is a continuous real potential on B'. So there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $(K^\mu_{B'} g_n)$ converges uniformly on $\overline{B}$. This implies that the sequence $\{K^\mu_B g_n\} = \{K^\mu_{B'} g_n - H_B K^\mu_{B'} g_n\}$ converges uniformly on B.

(2) Let $w = s + K^\mu_B s + t$ and let C be a compact subset of $\{s^+ > 0\} = \{s > 0\}$. Then $w + K^\mu_B s^-$ is a positive superharmonic function on B and

$$w + K^\mu_B s^- = s + K^\mu_B s^+ + t \ge K^\mu_B s^+ \ge K^\mu_B(\mathbf{1}_C s^+) \qquad \text{on } C.$$

Since $K^\mu_B(\mathbf{1}_C s^+)$ is harmonic on $B \setminus C$ and vanishes on ∂B, the minimum principle implies that

$$w + K^\mu_B s^- \ge K^\mu_B(\mathbf{1}_C s^+) \qquad \text{on } B \setminus C.$$

Since C is an arbitrary compact subset of $\{s^+ > 0\}$, we thus conclude that $w + K^\mu_B s^- \ge K^\mu_B s^+$, i.e., $s + t = w - K^\mu_B s \ge 0$.

(3) It suffices to note that $I + K^\mu_B$ is injective by (2), hence surjective as well since K^μ_B is compact.

(4) *and* (5). Let $u = H^\mu_B f = (I + K^\mu_B)^{-1} H_B f$. Then

$$u + K^\mu_B u = H_B f.$$

In particular, u is continuous on B and tends to f at the boundary of B. Moreover,

$$\Delta u - u\mu = \Delta(u + K^\mu_B u) = \Delta H_B f = 0.$$

Suppose that v is any continuous real function on B such that $\Delta v - v\mu = 0$ and v tends to f at ∂B. Then $s := u - v$ tends to zero at ∂B and satisfies

$$\Delta(s + K^\mu_B s) = \Delta s - s\mu = 0,$$

i.e., $s + K_B^\mu s$ is harmonic. Hence $s + K_B^\mu s = 0$, and $s = 0$ by item (2).

Now assume that $f \geq 0$. Then $u \geq 0$ by item (2), and $u \leq u + K_B^\mu u = H_B f$. Finally, fix a Kato measure ν on X and let $w = H_B^{\mu+\nu} f$. Then the continuous function $s = u - w$ tends to zero at ∂B and

$$s + K_B^{\mu+\nu} s = (u + K_B^\mu u) + K_B^\nu u - (w + K_B^{\mu+\nu} w) = K_B^\nu u$$

is a positive superharmonic function, hence $s \geq 0$ by item (2).

9.2 Kato measures and smooth measures

Proposition 9.1 *Let $\mu \geq 0$ be a measure on $(X, \mathcal{B}(X))$. Then the following statements are equivalent:*

(1) μ is smooth.

(2) μ is the limit of an increasing sequence of Kato measures.

(3) μ has a density with respect to a Kato measure.

Proof. (1) $\Longrightarrow$ (2): Since finite sums of Kato measures are Kato measures, we may assume that μ is a smooth Radon measure with compact support. Then G^μ is a potential on $\mathbb{R}^d$. In particular, the set $\{G^\mu = \infty\}$ is a polar set and hence $\mu(\{G^\mu = \infty\}) = 0$. Therefore, defining

$$A_n = \{n - 1 \leq G^\mu < n\}, \qquad \nu_n = \mathbf{1}_{A_n} \mu,$$

we have $\mu = \sum_{n=1}^\infty \nu_n$ where $G^{\nu_n} \leq n$ since $G^{\nu_n} \leq G^\mu \leq n$ on A_n. Being bounded each G^{ν_n} is the sum of continuous potentials $G^{\nu_{nm}}$, $m \in \mathbb{N}$. It now suffices to take

$$\mu_n = \sum_{i,j=1}^n \nu_{ij}.$$

(2) $\Longrightarrow$ (3): Let (μ_n) be a sequence of Kato measures which is increasing to μ and let $(U_n) \subset \mathcal{U}_c$ be an exhaustion of X. Then $\nu_n := \mathbf{1}_{U_n} \mu_n \uparrow \mu$ as $n \to \infty$ and $G^{\nu_n} \in \mathcal{C}_b(X)$ for every $n \in \mathbb{N}$. There exist $\alpha_n > 0$ such that $\nu := \sum_{n=1}^\infty \alpha_n \nu_n$ satisfies $G^\nu \in \mathcal{C}_b(X)$. Then ν is a Kato measure. Clearly, each ν_n possesses a density with respect to ν and so does μ.

(3) $\Longrightarrow$ (2): Let ν be a Kato measure and $\mu = \varphi\nu$, $\varphi \in \mathcal{B}^+(X)$. Then every $\mu_n := \mathbf{1}_{\{\varphi \leq n\}} \nu$, $n \in \mathbb{N}$, is a Kato measure and $\mu_n \uparrow \mu$.

(2) $\Longrightarrow$ (1): It suffices to recall that a Kato measure does not charge polar sets (the minimum principle implies that G^ν cannot be a continuous real potential if ν is supported by a polar set).

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