LOCALIZED UPPER BOUNDS OF HEAT KERNELS FOR DIFFUSIONS VIA A MULTIPLE DYNKIN-HUNT FORMULA

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Abstract. We prove that for a general diffusion process, certain assumptions on its behavior *only within a fixed open subset* of the state space imply the existence and sub-Gaussian type off-diagonal upper bounds of the *global* heat kernel on the fixed open set. The proof is mostly probabilistic and is based on a seemingly new formula, which we call a *multiple Dynkin-Hunt formula*, expressing the transition function of a Hunt process in terms of that of the part process on a given open subset. This result has an application to heat kernel analysis for the *Liouville Brownian motion*, the canonical diffusion in a certain random geometry of the plane induced by a (massive) Gaussian free field.

Contents

1. Introduction

Let (M,d) be a locally compact separable metric space equipped with a σ-finite Borel measure μ and let $X = \bigl(\{X_t\}_{t\in[0,\infty]}, \{\mathbb{P}_x\}_{x\in M_\Delta}\bigr)$ be a diffusion on M, where $M_\Delta := M\cup\{\Delta\}$ denotes the one-point compactification of M. The themes of this paper are existence of the heat kernel $p_t(x,y)$ (the transition density of X with respect to μ) and off-diagonal upper bounds of $p_t(x,y)$ of the form

$$p_t(x,y) \le F_t(x,y)\exp\biggl(-\gamma\Bigl(\frac{d(x,y)^\beta}{t}\Bigr)^{\frac{1}{\beta-1}}\biggr) \tag{1.1}$$

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for some $\gamma \in (0,\infty)$, $\beta \in (1,\infty)$ and a positive function $F_t(x,y)$. In most typical cases, $F_t(x,y)$ is given either by the power function $F_t(x,y) = c_0 t^{-\alpha}$ for some $c_0, \alpha \in (0,\infty)$ or by the volume function

$$F_t(x,y) = c_0 \mu\big(B(x,t^{1/\beta})\big)^{-1/2} \mu\big(B(y,t^{1/\beta})\big)^{-1/2}, \tag{1.2}$$

where β is as in (1.1) and $B(x,r) := \{y \in M \mid d(x,y) < r\}$ for $(x,r) \in M \times (0,\infty)$.

For $\beta = 2$, (1.1) is called a *Gaussian* upper bound and has been extensively studied in the classical setting where M is a complete Riemannian manifold. For example, when M has non-negative Ricci curvature, the Gaussian bound (1.1) under (1.2), together with a matching lower bound, has been proved for the Brownian motion on M by Li and Yau [35] and for uniformly elliptic diffusions on M by Saloff-Coste [40]. It is also known by the results of Grigor'yan [20, 21] and Saloff-Coste [39, 40] that these bounds are characterized or implied by certain scale-invariant functional inequalities, such as Poincaré, local Sobolev and relative Faber-Krahn inequalities, in conjunction with the volume doubling property

$$0 < \mu(B(x,2r)) \leq c_{\mathrm{vd}} \mu(B(x,r)) < \infty. \tag{1.3}$$

Saloff-Coste's proofs have developed from Moser's iteration argument in [37, 38] combined with Davies' method in [14] for making the constant γ in (1.1) arbitrarily close to $\frac{1}{4}$, and have been extended by Sturm [42, 43] to the framework of a general strongly local regular Dirichlet space whose associated intrinsic metric is non-degenerate. This last property basically means that for each relatively compact ball $B(x,r)$ there exists a cutoff function $\varphi = \varphi_{x,r}$ satisfying $\mathbf{1}_{B(x,r)} \leq \varphi \leq \mathbf{1}_{B(x,2r)}$ and "$|\nabla\varphi| \leq r^{-1}$" μ-a.e., which makes it possible to apply the methods developed for Riemannian manifolds to an abstract setting. It should also be noted that such cutoff functions allow us to deduce *localized* Gaussian bounds from *localized* assumptions; for example, a Gaussian upper bound of $p_t(x,y)$ for *given* $x, y \in M$ is implied by a local Sobolev inequality *on two balls* $B(x,r_x)$ *and* $B(y,r_y)$ *alone*. See [15, 23, 41, 42, 43] and references therein for further details of Gaussian bounds.

The values of β *greater than* 2 naturally appear in the study of diffusions on fractals. Barlow and Perkins have proved in their seminal work [11] that the canonical diffusion on the two-dimensional Sierpiński gasket satisfies (1.1) with (1.2) and $\beta = \log_2 5 > 2$ as well as a matching lower bound, which indicate a lower diffusion speed of the heat and are thereby called *sub*-Gaussian bounds. Such two-sided bounds with $\beta > 2$ have been established also for nested fractals by Kumagai [33], affine nested fractals by Fitzsimmons, Hambly and Kumagai [18] and Sierpiński carpets by Barlow and Bass [4, 5] (see also [8]), which in turn have motivated a number of recent studies on characterizing sub-Gaussian bounds, like [7, 10, 24, 27, 28, 32, 34] for two-sided and [1, 22, 25, 27, 31] for upper. A huge technical difficulty in the sub-Gaussian case is that, even though we can construct good cutoff functions similar to the Gaussian case *a posteriori on the basis of sub-Gaussian bounds* as has been done in [1, 7, 27], it is hopeless to have such functions *a priori*; indeed, the natural distance function may well even *not* belong to the domain of the Dirichlet form as proved in [29, Proposition A.3] for the two-dimensional Sierpiński gasket. Therefore in getting sub-Gaussian bounds, practically we cannot use analytic methods developed for Gaussian bounds, and most of the existent researches have made indispensable use of arguments on the diffusion process instead.

While calculations with the diffusion enable us to estimate various analytic quantities through probabilistic considerations, it is not clear whether they admit localized implications similar to the analytic proofs of Gaussian bounds, and there seems to be no result in the literature stating such implications explicitly. In fact, unless the diffusion X has a certain prescribed local regularity property as in the case of Riemannian manifolds and that of resistance forms treated in [32], localizing *existence* results for the heat kernel $p_t(x,y)$ is already highly non-trivial, since its existence on a given subset could be prevented by the possibly very bad behavior of the diffusion outside the subset. These issues of localization have been carefully avoided in the known probabilistic derivations of sub-Gaussian heat kernel bounds, either by assuming as in [31] the ultracontractivity of the heat semigroup and thereby the existence and boundedness of the heat kernel $p_t(x,y)$, or by assuming good situations everywhere in every scale as in [22, 25, 28] and their descendants [24, 27].

The purpose of this paper is to provide a new probabilistic method of obtaining *localized* existence and sub-Gaussian upper bounds of the heat kernel $p_t(x,y)$ of X from *localized* assumptions on X. Now we briefly outline the statements of our main theorems.

The main localized existence theorem for the heat kernel (Theorem 5.4) is proved for a Radon measure μ on M with full support and a μ-symmetric Hunt process $X = \bigl(\{X_t\}_{t\in[0,\infty]}, \{\mathbb{P}_x\}_{x\in M_\Delta}\bigr)$ on M (*not* necessarily with continuous sample paths) whose Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular on $L^2(M,\mu)$. Let U be a non-empty open subset of M, set $\tau_U := \inf\{t\in[0,\infty) \mid X_t \in M_\Delta\setminus U\}$ ($\inf\emptyset := \infty$) and let $\{T^U_t\}_{t\in(0,\infty)}$ denote the Dirichlet heat semigroup on U. Then Theorem 5.4 states that for an interval $I\subset(0,\infty)$ and open subsets V,W of M, a "μ-*almost everywhere* upper bound for $\{T^U_t\}_{t\in(0,\infty)}$ on $I\times V\times W$ by a locally bounded upper semi-continuous kernel $H = H_t(x,y)$" yields a Borel measurable function $p^U = p^U_t(x,y)$ with $0 \le p^U_t(x,y) \le H_t(x,y)$ such that for $\mathcal{E}$-*quasi-every* $x\in V$, for any $t\in I$,

$$\mathbb{P}_x[X_t\in dy,\, t<\tau_U] = p^U_t(x,y)\,d\mu(y) \qquad \text{on } W. \tag{1.4}$$

In fact, the same sort of results along with some additional regularity properties of $p_t(x,y)$ have been obtained for $I=(0,\infty)$ and $U=V=W=M$ in [22, Sections 7 and 8] and [6, Theorem 3.1], but our Theorem 5.4 should suffice for most applications since it already guarantees the expected bound $p^U_t(x,y)\le H_t(x,y)$ without requiring any regularity of the heat kernel $p^U_t(x,y)$.

The proof of Theorem 5.4 is mostly based on potential theory for regular symmetric Dirichlet forms developed in [19, Chapters 2 and 4]; it should not be very difficult to generalize Theorem 5.4 to a wider framework where the same kind of potential theory is still available. As an intermediate step for the proof of Theorem 5.4, we also prove in Proposition 5.6 that "for $\mathcal{E}$-*quasi-every* $x\in V$" in the above statement can be improved to "for *any* $x\in V$" if the inequality $\mathbb{P}_x[X_t\in dy,\, t<\tau_U]\le H_t(x,y)\,d\mu(y)$ holds on W for *any* $(t,x)\in I\times V$.

Next we turn to our second main theorem on localized sub-Gaussian upper bounds of heat kernels (Theorem 6.2). For the reader's convenience, we give here the precise statement of a simplified version of it. For $B\subset M$, set $\tau_B := \inf\{t\in[0,\infty)\mid X_t\in M_\Delta\setminus B\}$ ($\inf\emptyset:=\infty$) and let $\mathcal{B}(B)$ denote its Borel σ-field under the relative topology inherited from M.

Theorem 1.1. *Let (M,d) be a locally compact separable metric space, let μ be a σ-finite Borel measure μ on M and let $X = \bigl(\Omega,\mathcal{M},\{X_t\}_{t\in[0,\infty]},\{\mathbb{P}_x\}_{x\in M_\Delta}\bigr)$ be a Hunt process on $(M,\mathcal{B}(M))$ with life time ζ. Let $N\in\mathcal{B}(M)$ and assume that for any $x\in M\setminus N$,*

$$\mathbb{P}_x\bigl[X_t\in M_\Delta\setminus N \textit{ for any } t\in[0,\infty),\ [0,\zeta)\ni t\mapsto X_t\in M \textit{ is continuous}\bigr]=1 \tag{1.5}$$

(namely, $M\setminus N$ is X-invariant and the restriction $X|_{M\setminus N}$ of X to $M\setminus N$ is a diffusion).

Let $\beta\in(1,\infty)$, let $R\in(0,\infty)$, let U be a non-empty open subset of M with $\operatorname{diam}U\le R$ *and let $F=F_t(x,y) : (0,R^\beta]\times U\times U\to(0,\infty)$ be Borel measurable. Let $c_F,\alpha_F,c,\gamma\in(0,\infty)$ and assume that the following three conditions* $(\mathrm{DB})_\beta$, $(\mathrm{DU})^{U,R}_F$ *and* $(\mathrm{P})^{U,R}_\beta$ *hold:*

$(\mathrm{DB})_\beta$ *For any $(t,x,y),(s,z,w)\in(0,R^\beta]\times U\times U$ with $s\le t$,*

$$\frac{F_s(z,w)}{F_t(x,y)} \le c_F\biggl(\frac{t\vee d(x,z)^\beta\vee d(y,w)^\beta}{s}\biggr)^{\alpha_F}. \tag{1.6}$$

$(\mathrm{DU})_F^{U,R}$ *For any* $(t,x)\in(0,R^\beta)\times(U\setminus N)$ *and any* $A\in\mathcal{B}(U)$,

$$\mathbb{P}_x[X_t\in A,\, t<\tau_U]\leq\int_A F_t(x,y)\,d\mu(y). \tag{1.7}$$

$(\mathrm{P})_\beta^{U,R}$ *For any* $(x,r)\in(U\setminus N)\times(0,R)$ *with* $B(x,r)\subset U$ *and any* $t\in(0,\infty)$,

$$\mathbb{P}_x[\tau_{B(x,r)}\leq t]\leq c\exp\bigl(-\gamma(r^\beta/t)^{\frac{1}{\beta-1}}\bigr). \tag{1.8}$$

Let $\varepsilon\in(0,1)$ *and set* $U^\circ_{\varepsilon R}:=\{x\in M\mid\inf_{y\in M\setminus U}d(x,y)>\varepsilon R\}$ (note that $U^\circ_{\varepsilon R}$ is an open subset of U). *Then there exists a Borel measurable function* $p=p_t(x,y):(0,\infty)\times(M\setminus N)\times U^\circ_{\varepsilon R}\to[0,\infty)$ *such that for any* $(t,x)\in(0,\infty)\times(M\setminus N)$ *the following hold:*

$$\mathbb{P}_x[X_t\in A]=\int_A p_t(x,y)\,d\mu(y)\qquad\text{for any } A\in\mathcal{B}(U^\circ_{\varepsilon R}), \tag{1.9}$$

and furthermore for any $y\in U^\circ_{\varepsilon R}$,

$$p_t(x,y)\leq\begin{cases}c_\varepsilon F_t(x,y)\exp\bigl(-\gamma_\varepsilon(d(x,y)^\beta/t)^{\frac{1}{\beta-1}}\bigr) & \text{if } t<R^\beta \text{ and } x\in U,\\ c_\varepsilon(\inf_{U\times U}F_{(2t)\wedge R^\beta})\exp\bigl(-\gamma_\varepsilon(R^\beta/t)^{\frac{1}{\beta-1}}\bigr) & \text{if } t<R^\beta \text{ and } x\notin U,\\ c_\varepsilon(\inf_{U\times U}F_{R^\beta}) & \text{if } t\geq R^\beta\end{cases} \tag{1.10}$$

for some $c_\varepsilon\in(0,\infty)$ *explicit in* $\beta,c_F,\alpha_F,c,\gamma,\varepsilon$ *and* $\gamma_\varepsilon:=(\frac{1}{5}\varepsilon)^{\frac{\beta}{\beta-1}}\gamma$.

The strength of Theorem 1.1 is that *the conditions* $(\mathrm{DU})_F^{U,R}$ *and* $(\mathrm{P})_\beta^{U,R}$ *are independent of the behavior of* X *after exiting* U *and thereby completely localized within* U but assure nevertheless the existence and an upper bound of the heat kernel $p=p_t(x,y)$ for the *global* transition function $\mathbb{P}_x[X_t\in dy]$.

The power function $F_t(x,y)=c_0t^{-\alpha}$ clearly satisfies $(\mathrm{DB})_\beta$, and it is easy to see that $(\mathrm{DB})_\beta$ holds also for the volume function (1.2) provided (1.3) is satisfied for any $(x,r)\in U\times(0,R)$; see Example 5.10 for some more details. In view of these examples of $F=F_t(x,y)$, $(\mathrm{DU})_F^{U,R}$ amounts to an *on-diagonal* upper bound of the heat kernel $p^U=p^U_t(x,y)$ for $\{T^U_t\}_{t\in(0,\infty)}$, which is known to be implied in the setting of a regular symmetric Dirichlet form by the *local Nash inequality* as shown in [31, Lemma 4.3] and by the *Faber-Krahn inequality* as treated in [25, Subsection 5.2 and (5.48)].

The proof of Theorem 1.1 relies essentially only on two probabilistic iteration arguments based on the strong Markov property of X, where the series in the resulting upper estimates are shown to converge to the desired bounds by making heavy use of the condition $(\mathrm{P})_\beta^{U,R}$. In this sense, $(\mathrm{P})_\beta^{U,R}$ *could be considered as the probabilistic replacement for cutoff functions with well-controlled energy.* One iteration argument involves the behavior of X within U alone and is used in the first step of the proof of Theorem 1.1 to obtain an off-diagonal sub-Gaussian type upper bound of the Dirichlet heat kernel $p^U=p^U_t(x,y)$ on U *without assuming the symmetry of* X (Proposition 6.5). The other iteration is formulated as an equality, which we call a *multiple Dynkin-Hunt formula*, expressing the global transition function $\mathbb{P}_x[X_t\in A]$ in terms of $\mathbb{P}_y[X_s\in A,\, s<\tau_U]$, $(s,y)\in[0,t]\times U$, for each Borel subset A of M with $\overline{A}\subset U$ (Theorem 3.3) and thus enabling us to deduce upper bounds for the former from those for the latter together with $(\mathrm{P})_\beta^{U,R}$ (Proposition 6.6).

Note that the case of bounded (M,d) has been excluded from the main results of [1, 22, 24, 25, 27, 28], mainly due to their construction of the global heat kernel $p_t(x,y)$ as the limit as $U\uparrow M$ of the Dirichlet heat kernel $p^U_t(x,y)$ on U; indeed, taking the limit as $U\uparrow M$ is not allowed for bounded (M,d) since part of their conditions $(\mathrm{FK})_\Psi$ (Faber-Krahn inequality) and $(\mathrm{E})_\Psi$ (mean exit time estimate, see (7.16) and (7.17) in Theorem 7.3 below) must fail when the ball $B(x,r)$ coincides with M. We expect that this difficulty can be overcome by applying the main results of this paper, so that their results should be easily extended to the case of bounded (M,d). In fact, Barlow, Bass, Kumagai and Teplyaev [9] have used an argument very similar to our proof of Theorem 3.3 and Proposition 6.6 in [9, Proof of Proposition 2.12] for the resolvent of the diffusion to extend part of the main results of [24, 28] to the case of bounded (M,d). Our proof of Theorem 1.1 has successfully localized their idea by working directly with the transition function (semigroup) rather than the resolvent.

Finally, we remark that Theorem 1.1 has been recently applied in [2] to prove the continuity and sub-Gaussian off-diagonal upper bounds of the heat kernel of the *Liouville Brownian motion*, the canonical diffusion in a certain random geometry of $\mathbb{R}^2$ induced by a (massive) Gaussian free field. These results in [2] have had to rely strongly on Theorem 1.1 due to the fact that the unboundedness of $\mathbb{R}^2$ precludes any uniform estimates of volumes and exit times over the whole $\mathbb{R}^2$ valid for almost every environment, as opposed to the case of the two-dimensional torus, where the same kind of results have been obtained independently and simultaneously in [36].

The rest of this paper is organized as follows. In Section 2, we collect basic definitions and facts concerning Hunt processes. Section 3 formulates one of our two iteration arguments as a multiple Dynkin-Hunt formula and proves it for an *arbitrary* Hunt process (Theorem 3.3). In Section 4, we recall the notions of the symmetry of a Hunt process, the associated symmetric Dirichlet form and its regularity, together with some basic potential theory that is needed in Section 5 to state and prove our main localized existence theorem for the heat kernel (Theorem 5.4). In Section 6 we state our main theorem on localized sub-Gaussian upper bounds of heat kernels (Theorem 6.2) and a global version of it (Theorem 6.4) and prove them on the basis of our other probabilistic iteration (Proposition 6.5) and the multiple Dynkin-Hunt formula combined with the condition $(\mathrm{P})_\beta^{U,R}$ (Proposition 6.6). Lastly, Section 7 is devoted to providing sufficient conditions for $(\mathrm{P})_\beta^{U,R}$ (Theorems 7.2 and 7.3) as a localized version of the (well-)known results in [3, 22, 25].

Notation. In this paper, we adopt the following notation and conventions.

(0) The symbols $\subset$ and $\supset$ for set inclusion *allow* the case of the equality.

(1) $\mathbb{N} = \{n \in \mathbb{Z} \mid n > 0\}$, i.e., $0 \notin \mathbb{N}$.

(2) We set $\sup \emptyset := 0$ and $\inf \emptyset := \infty$. We write $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$, $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$ for $a, b \in [-\infty, \infty]$, and we use the same notation also for $[-\infty,\infty]$-valued functions and equivalence classes of them. All numerical functions treated in this paper are assumed to be $[-\infty,\infty]$-valued.

(3) Let E be a topological space. The Borel σ-field of E is denoted by $\mathscr{B}(E)$. We set

$$C(E) := \{u \mid u : E \to \mathbb{R},\ u \text{ is continuous}\},$$

$$C_c(E) := \{u \in C(E) \mid \text{the closure of } u^{-1}(\mathbb{R} \setminus \{0\}) \text{ in } E \text{ is compact}\},$$

$$\mathcal{B}(E) := \{u \mid u : E \to [-\infty,\infty],\ u \text{ is Borel measurable (i.e., } \mathscr{B}(E)\text{-measurable)}\},$$

$$\mathcal{B}^+(E) := \{u \in \mathcal{B}(E) \mid u \text{ is } [0,\infty]\text{-valued}\},$$

$$\mathcal{B}_b(E) := \{u \in \mathcal{B}(E) \mid \|u\|_{\sup} < \infty\},$$

where $\|u\|_{\sup} := \|u\|_{\sup,E} := \sup_{x\in E} |u(x)|$ for $u : E \to [-\infty,\infty]$.

2. Basics on Hunt processes

In this section, we introduce our framework of a Hunt process. To keep the main results of this paper accessible to those who are not familiar with the theory of Markov processes, we explain basic definitions and facts in some detail. See [19, Section A.2] and [13, Section A.1] for further details on Hunt processes.

Let M be a locally compact separable metrizable topological space. The interior, closure and boundary of $A \subset M$ in M are denoted by $\operatorname{int} A$, $\overline{A}$ and ∂A, respectively. Each $A \subset M$ is equipped with the relative topology inherited from M, so that its Borel σ-field $\mathcal{B}(A)$ can be expressed as $\mathcal{B}(A) = \{B \cap A \mid B \in \mathcal{B}(M)\}$. Let $M_\Delta := M \cup \{\Delta\}$ denote the one-point compactification of M, which satisfies $\mathcal{B}(M_\Delta) = \mathcal{B}(M) \cup \{A \cup \{\Delta\} \mid A \in \mathcal{B}(M)\}$. In what follows, $[-\infty, \infty]$-valued functions on M are always set to be 0 at Δ unless their values at Δ are already defined: $u(\Delta) := 0$ for $u : M \to [-\infty, \infty]$.

Let $X = \bigl(\Omega, \mathcal{M}, \{X_t\}_{t\in[0,\infty]}, \{\mathbb{P}_x\}_{x\in M_\Delta}\bigr)$ be a Hunt process on $(M, \mathcal{B}(M))$ with life time ζ and shift operators $\{\theta_t\}_{t\in[0,\infty]}$. By definition, $(\Omega, \mathcal{M})$ is a measurable space, $\{X_t\}_{t\in[0,\infty]}$ is a family of $\mathcal{M}/\mathcal{B}(M_\Delta)$-measurable maps $X_t : \Omega \to M_\Delta$ such that $X_t(\omega) = \Delta$ for any $t \in [\zeta(\omega), \infty]$ for each $\omega \in \Omega$, where $\zeta(\omega) := \inf\{t \in [0, \infty) \mid X_t(\omega) = \Delta\}$, and $\{\theta_t\}_{t\in[0,\infty]}$ is a family of maps $\theta_t : \Omega \to \Omega$ satisfying $X_s \circ \theta_t = X_{s+t}$ for any $s, t \in [0, \infty]$. It is further assumed that for each $\omega \in \Omega$, $[0, \infty) \ni t \mapsto X_t(\omega) \in M_\Delta$ is right-continuous and the limit $X_{t-}(\omega) := \lim_{s\to t,\, s<t} X_s(\omega)$ exists in M_Δ for any $t \in (0, \infty)$; see [19, Section A.2, (M.6)]. The pair X of such a stochastic process $\bigl(\Omega, \mathcal{M}, \{X_t\}_{t\in[0,\infty]}\bigr)$ and a family $\{\mathbb{P}_x\}_{x\in M_\Delta}$ of probability measures on $(\Omega, \mathcal{M})$ is then called a *Hunt process on* $(M, \mathcal{B}(M))$ if and only if it is a normal Markov process on $(M, \mathcal{B}(M))$ whose minimum completed admissible filtration $\mathcal{F}_* = \{\mathcal{F}_t\}_{t\in[0,\infty]}$ is *right-continuous* and it is *strong Markov* and *quasi-left-continuous* with respect to $\mathcal{F}_*$; see [19, Section A.2, (M.2)–(M.5), the paragraph before Lemma A.2.2, (A.2.3) and (A.2.4)] for the precise definitions of these notions.

For $x \in M_\Delta$, the expectation (that is, the integration on Ω) under the measure $\mathbb{P}_x$ is denoted by $\mathbb{E}_x[(\cdot)]$. We remark that by [13, Exercise A.1.20-(i)], for each $\mathcal{F}_\infty$-measurable random variable $Y : \Omega \to [0, \infty]$ the function $M_\Delta \ni x \mapsto \mathbb{E}_x[Y] \in [0, \infty]$ is *universally measurable*, i.e., measurable with respect to the *universal σ-field* $\mathcal{B}^*(M_\Delta)$ *of* M_Δ defined as $\mathcal{B}^*(M_\Delta) := \bigcap_\nu \mathcal{B}^\nu(M_\Delta)$; here ν runs through the set of probability (or equivalently, σ-finite) measures on $(M_\Delta, \mathcal{B}(M_\Delta))$ and $\mathcal{B}^\nu(M_\Delta)$ denotes the ν-completion of $\mathcal{B}(M_\Delta)$.

The Hunt process X gives rise to a family $\{\mathcal{P}_t\}_{t\in[0,\infty)}$ of Markovian kernels on $(M, \mathcal{B}(M))$ called the *transition function of* X, which is defined by

$$\mathcal{P}_t(x, A) := \mathbb{P}_x[X_t \in A], \qquad t \in [0, \infty),\ x \in M,\ A \in \mathcal{B}(M). \tag{2.1}$$

Then for $t \in [0, \infty)$ and $u \in \mathcal{B}(M)$, we define

$$\mathcal{P}_t u(x) := \int_M u(y)\mathcal{P}_t(x, dy) = \mathbb{E}_x[u(X_t)] \tag{2.2}$$

for $x \in M$ satisfying $\mathbb{E}_x[u^+(X_t)] \wedge \mathbb{E}_x[u^-(X_t)] < \infty$, so that $\mathcal{P}_t(\mathcal{B}^+(M)) \subset \mathcal{B}^+(M)$ and $\mathcal{P}_t(\mathcal{B}_{\mathrm{b}}(M)) \subset \mathcal{B}_{\mathrm{b}}(M)$. Note that our convention of setting $\mathcal{P}_t u(\Delta) := 0$ is consistent with (2.2) for $x = \Delta$ since $\mathbb{E}_\Delta[u(X_t)] = \mathbb{E}_\Delta[u(\Delta)] = 0$ by $\mathbb{P}_\Delta[X_t = \Delta] = 1$. Obviously, if $u \in \mathcal{B}(M)$ is $[0, 1]$-valued then so is $\mathcal{P}_t u$, and the Markov property of X (see [19, (A.2.2)] or [13, (A.1.3)]) easily implies the semigroup property

$$\mathcal{P}_t\mathcal{P}_s u = \mathcal{P}_{t+s} u, \qquad t, s \in [0, \infty),\ u \in \mathcal{B}^+(M) \cup \mathcal{B}_{\mathrm{b}}(M). \tag{2.3}$$

Moreover, it easily follows from the sample path right-continuity of X and the Dynkin class theorem [12, Chapter 0, Theorem 2.2] that

$$[0, \infty) \times M \ni (t, x) \mapsto \mathcal{P}_t u(x) \text{ is Borel measurable for any } u \in \mathcal{B}^+(M) \cup \mathcal{B}_{\mathrm{b}}(M). \tag{2.4}$$

Recall that $\sigma : \Omega \to [0, \infty]$ is called an *$\mathcal{F}_*$-stopping time* if and only if $\{\sigma \le t\} \in \mathcal{F}_t$ for any $t \in [0, \infty)$. For $B \subset M_\Delta$, we define its *entrance time* $\dot\sigma_B$ and *exit time* τ_B *for* X by

$$\dot\sigma_B(\omega) := \inf\{t \in [0, \infty) \mid X_t(\omega) \in B\}, \quad \omega \in \Omega \qquad \text{and} \qquad \tau_B := \dot\sigma_{M_\Delta\setminus B}, \tag{2.5}$$

and we also set $\hat{\sigma}_B(\omega) := \inf\{t \in (0,\infty) \mid X_{t-}(\omega) \in B\}$ for $\omega \in \Omega$. If $B \in \mathcal{B}(M_\Delta)$, then $\dot{\sigma}_B, \tau_B, \hat{\sigma}_B$ are $\mathcal{F}_*$-stopping times and

$$\mathbb{P}_x[\dot{\sigma}_B \leq \hat{\sigma}_B] = 1 \qquad \text{for any } x \in M_\Delta \tag{2.6}$$

by [19, Theorem A.2.3], where the case of $\Delta \in B$ is easily deduced from that of $B \in \mathcal{B}(M)$ by using the equalities $\dot{\sigma}_{B\cup\{\Delta\}} = \dot{\sigma}_B \wedge \zeta$ and $\hat{\sigma}_{B\cup\{\Delta\}} = \hat{\sigma}_B \wedge \hat{\sigma}_{\{\Delta\}}$ for $B \subset M$ and the quasi-left-continuity [19, (A.2.4)] of X (see also [13, Theorem A.1.19 and Exercise A.1.26-(ii)]). Note that if $B \subset M_\Delta$, $t \in [0,\infty]$ and $\omega \in \{\dot{\sigma}_B \geq t\}$ then $\dot{\sigma}_B(\omega) = t + \dot{\sigma}_B(\theta_t(\omega))$.

Next we introduce the part of X on open sets. Let U be a non-empty open subset of M, let $U_\Delta := U \cup \{\Delta_U\}$ denote its one-point compactification and define

$$X_t^U(\omega) := \begin{cases} X_t(\omega) & \text{if } t < \tau_U(\omega), \\ \Delta_U & \text{if } t \geq \tau_U(\omega), \end{cases} \qquad (t,\omega) \in [0,\infty] \times \Omega \tag{2.7}$$

and $\mathbb{P}_{\Delta_U} := \mathbb{P}_\Delta$. Then $X^U := \big(\Omega, \mathcal{M}, \{X_t^U\}_{t\in[0,\infty]}, \{\mathbb{P}_x\}_{x\in U_\Delta}\big)$, called the *part of X on U*, is a Hunt process on $(U, \mathcal{B}(U))$ by [19, Theorem A.2.10]. Its transition function is naturally extended to $(M, \mathcal{B}(M))$ as a family $\{\mathcal{P}_t^U\}_{t\in[0,\infty)}$ of Markovian kernels on $(M, \mathcal{B}(M))$ given by (with the obvious convention that $\Delta_U \notin M$)

$$\mathcal{P}_t^U(x, A) := \mathbb{P}_x[X_t^U \in A] = \mathbb{P}_x[X_t \in A,\, t < \tau_U], \quad t \in [0,\infty),\ x \in M,\ A \in \mathcal{B}(M). \tag{2.8}$$

Also for $t \in [0,\infty)$ and $u \in \mathcal{B}(M)$, similarly to (2.2) we further define

$$\mathcal{P}_t^U u(x) := \int_M u(y)\mathcal{P}_t^U(x,dy) = \int_U u(y)\mathcal{P}_t^U(x,dy) = \mathbb{E}_x[u(X_t)\mathbf{1}_{\{t<\tau_U\}}] \tag{2.9}$$

for $x \in M$ satisfying $\mathbb{E}_x[u^+(X_t)\mathbf{1}_{\{t<\tau_U\}}] \wedge \mathbb{E}_x[u^-(X_t)\mathbf{1}_{\{t<\tau_U\}}] < \infty$, where $\mathbf{1}_A : \Omega \to \{0,1\}$ denotes the indicator function of $A \subset \Omega$ given by $\mathbf{1}_A|_A := 1$ and $\mathbf{1}_A|_{\Omega\setminus A} := 0$. Then $\mathcal{P}_t^U u(x) = 0$ for $x \in M \setminus U$, $\mathcal{P}_t^U(\mathcal{B}^+(M)) \subset \mathcal{B}^+(M)$, $\mathcal{P}_t^U(\mathcal{B}_{\mathrm{b}}(M)) \subset \mathcal{B}_{\mathrm{b}}(M)$, and (2.3) and (2.4) hold with $\{\mathcal{P}_t^U\}_{t\in[0,\infty)}$ in place of $\{\mathcal{P}_t\}_{t\in[0,\infty)}$.

3. A multiple Dynkin-Hunt formula for Hunt processes

As in Section 2, let M be a locally compact separable metrizable topological space and let X be a Hunt process on $(M, \mathcal{B}(M))$ with life time ζ and shift operators $\{\theta_t\}_{t\in[0,\infty]}$. *Throughout the rest of this paper, we fix this setting and follow the notation introduced in Section* 2.

In this section, we state and prove a *multiple Dynkin-Hunt formula* (Theorem 3.3 below) which gives an expression of $\mathcal{P}_t u$ in terms of $\mathcal{P}_s^U u$, $s \in [0,t]$, for a non-empty open subset U of M and functions $u \in \mathcal{B}^+(M) \cup \mathcal{B}_{\mathrm{b}}(M)$ supported in U. It will be used later in Section 6 to deduce upper bounds for $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ from those for $\{\mathcal{P}_t^U\}_{t\in(0,\infty)}$.

The statement of Theorem 3.3 requires the following definition and proposition.

Definition 3.1. For $\sigma : \Omega \to [0,\infty]$ and $B \subset M_\Delta$, the *entrance time* $\dot{\sigma}_{B,\sigma}$ and *exit time* $\tau_{B,\sigma}$ *of B after σ for X* are defined by (with the convention that $[\infty,\infty) := \emptyset$)

$$\dot{\sigma}_{B,\sigma}(\omega) := \inf\{t \in [\sigma(\omega),\infty) \mid X_t(\omega) \in B\}, \quad \omega \in \Omega \quad \text{and} \quad \tau_{B,\sigma} := \dot{\sigma}_{M_\Delta\setminus B,\sigma}, \tag{3.1}$$

so that $\dot{\sigma}_{B,\sigma}(\omega) = \sigma(\omega) + \dot{\sigma}_B(\theta_{\sigma(\omega)}(\omega))$ and $\tau_{B,\sigma}(\omega) = \sigma(\omega) + \tau_B(\theta_{\sigma(\omega)}(\omega))$ for any $\omega \in \Omega$.

Proposition 3.2. *For any $\mathcal{F}_*$-stopping time σ and any $B \in \mathcal{B}(M_\Delta)$, the entrance time $\dot{\sigma}_{B,\sigma}$ and exit time $\tau_{B,\sigma}$ of B after σ for X are $\mathcal{F}_*$-stopping times.*

Proof. This proposition should be well-known, but we give an explicit proof for completeness. We follow [13, Proof of Theorem A.1.19]. For each $t \in (0,\infty)$, the set $\{\dot{\sigma}_{B,\sigma} < t\} = \{\omega \in \Omega \mid \dot{\sigma}_{B,\sigma}(\omega) < t\}$ is equal to the projection on Ω of

$$\{(s,\omega) \in [0,t) \times \Omega \mid \sigma(\omega) \leq s,\ X_s(\omega) \in B\},$$

which is easily shown to belong to the product σ-field $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$ by the sample path right-continuity of X and the assumption that σ is an $\mathcal{F}_*$-stopping time. Therefore [16, Chapter III, 13 and 33] imply that $\{\dot{\sigma}_{B,\sigma} < t\} \in \mathcal{F}_t$, which means that $\dot{\sigma}_{B,\sigma}$, and hence also $\tau_{B,\sigma}$, are $\mathcal{F}_*$-stopping times since $\mathcal{F}_*$ is right-continuous. □

Now we state the main theorem of this section. Recall for $\sigma : \Omega \to [0,\infty]$ that the map $X_\sigma : \Omega \to M_\Delta$ is defined as $X_\sigma(\omega) := X_{\sigma(\omega)}(\omega)$ and that X_σ is $\mathcal{F}_\infty/\mathcal{B}(M_\Delta)$-measurable if σ is $\mathcal{F}_\infty$-measurable by the sample path right-continuity of X.

Theorem 3.3 (A multiple Dynkin-Hunt formula). *Let U be a non-empty open subset of M, let $B \in \mathcal{B}(M)$ satisfy $\overline{B} \subset U$ and define $\mathcal{F}_*$-stopping times τ_n and σ_n, $n \in \mathbb{N}$, by*

$$\tau_1 := \tau_U \qquad \text{and inductively} \qquad \sigma_n := \dot{\sigma}_{B,\tau_n} \quad \text{and} \quad \tau_{n+1} := \tau_{U,\sigma_n}, \quad n \in \mathbb{N}. \tag{3.2}$$

Then for any $u \in \mathcal{B}^+(M) \cup \mathcal{B}_{\mathrm{b}}(M)$ with $u|_{M\setminus B} = 0$ and any $(t,x) \in [0,\infty) \times M$,

$$\mathcal{P}_t u(x) = \mathcal{P}^U_t u(x) + \sum_{n\in\mathbb{N}} \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma_n \le t\}} \mathcal{P}^U_{t-\sigma_n} u(X_{\sigma_n})\bigr]. \tag{3.3}$$

Note that by (2.4) for $\{\mathcal{P}^U_t\}_{t\in[0,\infty)}$, the random variable $\mathbf{1}_{\{\sigma_n \le t\}} \mathcal{P}^U_{t-\sigma_n} u(X_{\sigma_n})$ in (3.3) is $\mathcal{F}_\infty$-measurable for any $u \in \mathcal{B}^+(M) \cup \mathcal{B}_{\mathrm{b}}(M)$, any $t \in [0,\infty)$ and any $n \in \mathbb{N}$.

Recall that the *Dynkin-Hunt formula* refers to (the heat kernel version of) the following equality, which is an easy consequence of Proposition 3.4 below: for any non-empty open subset U of M, any $u \in \mathcal{B}^+(M) \cup \mathcal{B}_{\mathrm{b}}(M)$ and any $(t,x) \in [0,\infty) \times M$,

$$\mathcal{P}_t u(x) = \mathcal{P}^U_t u(x) + \mathbb{E}_x\bigl[\mathbf{1}_{\{\tau_U \le t\}} \mathcal{P}_{t-\tau_U} u(X_{\tau_U})\bigr]. \tag{3.4}$$

(3.3) can be regarded as an indefinite iteration of (3.4) through restarting X at the entrance time $\dot{\sigma}_{B,\tau_U}$ of B after τ_U, which is why we call (3.3) a *multiple Dynkin-Hunt formula*.

For the proof of Theorem 3.3 we need a variation of the strong Markov property of X as in the following proposition. Recall for each $\mathcal{F}_*$-stopping time σ that the collection

$$\mathcal{F}_\sigma := \{A \in \mathcal{F}_\infty \mid A \cap \{\sigma \le t\} \in \mathcal{F}_t \text{ for any } t \in [0,\infty)\} \tag{3.5}$$

is a σ-field in Ω with respect to which σ is measurable, that X_σ is $\mathcal{F}_\sigma/\mathcal{B}^*(M_\Delta)$-measurable by [13, Exercise A.1.20-(ii)], and that the map $\theta_\sigma : \Omega \to \Omega$, $\theta_\sigma(\omega) := \theta_{\sigma(\omega)}(\omega)$, is $\mathcal{F}_\infty/\mathcal{F}_\infty$-measurable by [13, Theorem A.1.21].

Proposition 3.4. *Let σ be an $\mathcal{F}_*$-stopping time, let $\tau : \Omega \to [0,\infty]$ be $\mathcal{F}_\infty$-measurable and let $T : \Omega \to [0,\infty]$ be $\mathcal{F}_\sigma$-measurable and satisfy $\sigma(\omega) \le T(\omega)$ for any $\omega \in \Omega$. Then for any $x \in M_\Delta$ and any $u \in \mathcal{B}_{\mathrm{b}}(M_\Delta)$, it holds that for $\mathbb{P}_x$-a.e. $\omega \in \{\sigma < \infty\}$,*

$$\mathbb{E}_x\bigl[u(X_T)\mathbf{1}_{\{T<\sigma+\tau\circ\theta_\sigma\}} \bigm| \mathcal{F}_\sigma\bigr](\omega) = \mathbb{E}_{X_{\sigma(\omega)}}\bigl[u(X_{T(\omega)-\sigma(\omega)})\mathbf{1}_{\{T(\omega)-\sigma(\omega)<\tau\}}\bigr]. \tag{3.6}$$

Proof. We follow [30, Proofs of Proposition 2.6.17 and Corollary 2.6.18]. For $u \in \mathcal{B}_{\mathrm{b}}(M_\Delta)$, let $Y_u(\omega)$ denote the right-hand side of (3.6) for $\omega \in \{\sigma < \infty\}$ and set $Y_u(\omega) := 0$ for $\omega \in \{\sigma = \infty\}$. Let $x \in M_\Delta$. For the proof of (3.6) it suffices to show that $Y_u : \Omega \to \mathbb{R}$ possesses the following properties:

$$Y_u \text{ is } \mathcal{F}_\sigma\text{-measurable} \quad \text{and} \quad \mathbb{E}_x\bigl[u(X_T)\mathbf{1}_{\{T<\sigma+\tau\circ\theta_\sigma\}}\mathbf{1}_A\bigr] = \mathbb{E}_x[Y_u\mathbf{1}_A] \text{ for any } A \in \mathcal{F}_\sigma. \tag{3.7}$$

We first prove (3.7) for $u \in C(M_\Delta)$. Let $n \in \mathbb{N}$ and define $T_n : \Omega \to [0,\infty]$ by

$$T_n|_{\{\sigma+(k-1)2^{-n}\le T<\sigma+k2^{-n}\}} := \sigma + k2^{-n}, \quad k \in \mathbb{N} \qquad \text{and} \qquad T_n|_{\{T=\infty\}} := \infty, \tag{3.8}$$

so that T_n is $\mathcal{F}_\sigma$-measurable and $T_n - 2^{-n} \le T \le T_n$. Also define $Y_{u,n}$ in the same way as Y_u with T_n in place of T. Then $Y_{u,n}|_{\{T=\infty\}} = 0 = Y_u|_{\{T=\infty\}}$, and $\lim_{n\to\infty} Y_{u,n} = Y_u$ on $\{T < \infty\}$ by $T_n - 2^{-n} \le T \le T_n$, the sample path right-continuity of X and dominated convergence. Also for $k \in \mathbb{N}$, on $\{\sigma + (k-1)2^{-n} \le T < \sigma + k2^{-n}\} \in \mathcal{F}_\sigma$ we have $Y_{u,n} =$

$\mathbb{E}_{X_\sigma}[u(X_{k2^{-n}})\mathbf{1}_{\{k2^{-n}<\tau\}}]$, and since the latter is $\mathcal{F}_\sigma$-measurable by [13, Exercise A.1.20] so are $Y_{u,n}$ and $Y_u = \lim_{n\to\infty} Y_{u,n}$. Now for $A \in \mathcal{F}_\sigma$, thanks to dominated convergence,

$$\begin{aligned}\mathbb{E}_x\bigl[u(X_{T_n})\mathbf{1}_{\{T_n<\sigma+\tau\circ\theta_\sigma\}}\mathbf{1}_A\bigr] &= \sum_{k\in\mathbb{N}} \mathbb{E}_x\bigl[\mathbf{1}_{A\cap\{T_n=\sigma+k2^{-n}<\infty\}}\bigl((u(X_{k2^{-n}})\mathbf{1}_{\{k2^{-n}<\tau\}})\circ\theta_\sigma\bigr)\bigr]\\ &= \sum_{k\in\mathbb{N}} \mathbb{E}_x\bigl[\mathbf{1}_{A\cap\{T_n=\sigma+k2^{-n}<\infty\}}\mathbb{E}_{X_\sigma}[u(X_{k2^{-n}})\mathbf{1}_{\{k2^{-n}<\tau\}}]\bigr]\\ &= \mathbb{E}_x[Y_{u,n}\mathbf{1}_A]\end{aligned}$$

by the strong Markov property [13, Theorem A.1.21] of X at time σ, and we conclude (3.7) by using $T_n - 2^{-n} \le T \le T_n$ and the sample path right-continuity of X to let $n \to \infty$.

Note that for $u \in \mathcal{B}_{\mathrm{b}}(M_\Delta)$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathrm{b}}(M_\Delta)$ such that $\sup_{n\in\mathbb{N}} \|u_n\|_{\sup} < \infty$ and $\lim_{n\to\infty} u_n(y) = u(y)$ for any $y \in M_\Delta$, if u_n satisfies (3.7) for any $n \in \mathbb{N}$ then so does u by dominated convergence. Therefore it follows from the previous paragraph that (3.7) holds for $u = \mathbf{1}_B$ with $B \subset M_\Delta$ closed in M_Δ, hence also with $B \in \mathcal{B}(M_\Delta)$ by the Dynkin class theorem [12, Chapter 0, Theorem 2.2], and thus for any $u \in \mathcal{B}_{\mathrm{b}}(M_\Delta)$. □

Proof of Theorem 3.3. For $n \in \mathbb{N}$, $\tau_n \le \sigma_n \le \tau_{n+1}$ by (3.1) and (3.2), and the sample path right-continuity of X implies that $X_{\tau_n} \in M\setminus U$ and $\tau_n < \sigma_n$ on $\{\tau_n < \zeta\}$ and that $X_{\sigma_n} \in \overline{B}$ and $\sigma_n < \tau_{n+1} \wedge \zeta$ on $\{\sigma_n < \infty\}$. Moreover, setting $\tau := \lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \sigma_n$, we see from the quasi-left-continuity [19, (A.2.4)] of X that for any $x \in M$,

$$\mathbb{P}_x[\tau < \zeta] = \mathbb{P}_x[\tau < \zeta,\ \lim_{n\to\infty} X_{\tau_n} = X_\tau = \lim_{n\to\infty} X_{\sigma_n}] = \mathbb{P}_x[\emptyset] = 0. \tag{3.9}$$

Let $(t, x) \in [0,\infty)\times M$. Then for each $\omega \in \{X_t \in B,\ \zeta \le \tau\}$, $t < \zeta(\omega) \le \tau(\omega)$ and hence either $t < \tau_1(\omega)$, or $\tau_n(\omega) \le t < \tau_{n+1}(\omega)$ for some $n \in \mathbb{N}$, whence $\sigma_n(\omega) \le t < \tau_{n+1}(\omega)$ by $X_t(\omega) \in B$; namely $\{X_t \in B,\ \zeta \le \tau\} \subset \{t < \tau_1\} \cup \bigcup_{n\in\mathbb{N}}\{\sigma_n \le t < \tau_{n+1}\}$, and this union is disjoint. Therefore for any $u \in \mathcal{B}_{\mathrm{b}}(M)$ with $u|_{M\setminus B} = 0$, noting that $\tau_{n+1} = \sigma_n + \tau_U \circ \theta_{\sigma_n}$ for any $n \in \mathbb{N}$ and using (3.9), dominated convergence and Proposition 3.4, we obtain

$$\begin{aligned}\mathcal{P}_t u(x) &= \mathbb{E}_x[u(X_t)] = \mathbb{E}_x[u(X_t)\mathbf{1}_{\{X_t\in B,\,\zeta\le\tau\}}]\\ &= \mathbb{E}_x\Biggl[u(X_t)\mathbf{1}_{\{X_t\in B,\,\zeta\le\tau\}}\Biggl(\mathbf{1}_{\{t<\tau_1\}} + \sum_{n\in\mathbb{N}}\mathbf{1}_{\{\sigma_n\le t<\tau_{n+1}\}}\Biggr)\Biggr]\\ &= \mathbb{E}_x[u(X_t)\mathbf{1}_{\{t<\tau_U\}}] + \sum_{n\in\mathbb{N}}\mathbb{E}_x[u(X_t)\mathbf{1}_{\{\sigma_n\le t<\sigma_n+\tau_U\circ\theta_{\sigma_n}\}}]\\ &= \mathcal{P}^U_t u(x) + \sum_{n\in\mathbb{N}}\mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma_n\le t\}}\mathbb{E}_x\bigl[u(X_t)\mathbf{1}_{\{t<\sigma_n\wedge t+\tau_U\circ\theta_{\sigma_n\wedge t}\}}\bigm|\mathcal{F}_{\sigma_n\wedge t}\bigr]\bigr]\\ &= \mathcal{P}^U_t u(x) + \sum_{n\in\mathbb{N}}\int_{\{\sigma_n\le t\}}\mathbb{E}_{X_{\sigma_n\wedge t}(\omega)}\bigl[u(X_{t-\sigma_n(\omega)\wedge t})\mathbf{1}_{\{t-\sigma_n(\omega)\wedge t<\tau_U\}}\bigr]\,d\mathbb{P}_x(\omega)\\ &= \mathcal{P}^U_t u(x) + \sum_{n\in\mathbb{N}}\mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma_n\le t\}}\mathcal{P}^U_{t-\sigma_n}u(X_{\sigma_n})\bigr],\end{aligned}$$

where the equality in the fourth line holds since $\{\sigma_n \le t\} \in \mathcal{F}_{\sigma_n\wedge t}$ by [30, Lemma 1.2.16]. Thus we have proved (3.3) for $u \in \mathcal{B}_{\mathrm{b}}(M)$ with $u|_{M\setminus B} = 0$, which easily implies (3.3) for $u \in \mathcal{B}^+(M)$ with $u|_{M\setminus B} = 0$ by monotone convergence. □

4. Symmetry of a Hunt process and the associated Dirichlet form

In this section, assuming the symmetry of our Hunt process X, we first recall that such X naturally gives rise to a symmetric Dirichlet form, and then introduce related potential theoretic notions. We refer the reader to [19, 13] for further details.

4.1. **The Dirichlet form of a symmetric Hunt process.** In the rest of this paper, we fix a metric d on M compatible with the topology of M, and a Radon measure μ on M with full support, i.e., a Borel measure on M such that $\mu(K) < \infty$ for any $K \subset M$ compact and $\mu(U) > 0$ for any $U \subset M$ non-empty open. We set $B(x,r) := \{y \in M \mid d(x,y) < r\}$ for $(x,r) \in M \times (0,\infty)$ and $\operatorname{diam} A := \sup_{x,y\in A} d(x,y)$ for $A \subset M$. For $q \in [1,\infty)$, we set $\|u\|_q := (\int_M |u|^q d\mu)^{1/q}$ for $u \in \mathcal{B}(M)$ and $\mathcal{B}L^q(M,\mu) := \{u \in \mathcal{B}(M) \mid \|u\|_q < \infty\}$, and we also set $\langle u,v\rangle := \int_M uv\,d\mu$ for $u,v \in \mathcal{B}^+(M)$ and for $u,v \in \mathcal{B}(M)$ with $\|uv\|_1 < \infty$. For $\|\cdot\|_q$ and $\langle\cdot,\cdot\rangle$, we use the same notation for μ-equivalence classes of functions as well.

Now we assume that X is *μ-symmetric*, i.e., $\langle \mathcal{P}_t u, v\rangle = \langle u, \mathcal{P}_t v\rangle$ for any $t \in (0,\infty)$ and any $u,v \in \mathcal{B}^+(M)$. Then for each $t \in (0,\infty)$, as in [19, (1.4.13)] we can easily verify that $\|\mathcal{P}_t u\|_2 \le \|u\|_2$ for any $u \in \mathcal{B}^+(M)$, so that $\mathcal{P}_t u$ is defined μ-a.e. and determines an element $T_t u$ of $L^2(M,\mu)$ for each $u \in L^2(M,\mu)$ independently of a particular choice of a μ-version of u. Thus the transition function $\{\mathcal{P}_t\}_{t\in[0,\infty)}$ of X canonically induces a symmetric contraction semigroup $\{T_t\}_{t\in(0,\infty)}$ on $L^2(M,\mu)$ which is also *Markovian*, i.e., $0 \le T_t u \le 1$ μ-a.e. for any $t \in (0,\infty)$ and any $u \in L^2(M,\mu)$ with $0 \le u \le 1$ μ-a.e. This semigroup $\{T_t\}_{t\in(0,\infty)}$ is in fact strongly continuous thanks to the sample path right-continuity of X as shown in [19, Lemma 1.4.3-(i)] and hence determines a symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(M,\mu)$ by [19, Lemma 1.3.4-(i) and Theorem 1.4.1]. Namely, we have a dense linear subspace $\mathcal{F}$ of $L^2(M,\mu)$ and a non-negative definite symmetric bilinear form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ given by

$$\begin{aligned} \mathcal{F} &:= \Bigl\{ u \in L^2(M,\mu) \Bigm| \lim_{t\downarrow 0} t^{-1}\langle u - T_t u, u\rangle < \infty \Bigr\}, \\ \mathcal{E}(u,v) &:= \lim_{t\downarrow 0} t^{-1}\langle u - T_t u, v\rangle, \qquad u,v \in \mathcal{F}, \end{aligned} \tag{4.1}$$

respectively, and $(\mathcal{E},\mathcal{F})$ is *closed* (i.e., $\mathcal{F}$ forms a Hilbert space with inner product $\mathcal{E}_1 := \mathcal{E} + \langle\cdot,\cdot\rangle$) and *Markovian* (i.e., $u^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \le \mathcal{E}(u,u)$ for any $u \in \mathcal{F}$). $(\mathcal{E},\mathcal{F})$ is called the *Dirichlet form of the μ-symmetric Hunt process* X. Note that by [19, Lemma 1.3.3-(i)],

$$T_t(L^2(M,\mu)) \subset \mathcal{F} \qquad \text{for any } t \in (0,\infty). \tag{4.2}$$

In what follows we further assume that the Dirichlet form $(\mathcal{E},\mathcal{F})$ of X is *regular on* $L^2(M,\mu)$, i.e., that $\mathcal{F} \cap C_c(M)$ is dense both in $(\mathcal{F},\mathcal{E}_1)$ and in $(C_c(M), \|\cdot\|_{\sup})$. Note that this framework actually contains any regular symmetric Dirichlet form on any locally compact separable metric space (M,d) equipped with a Radon measure μ with full support, since any such form can be realized as the Dirichlet form of some μ-symmetric Hunt process on $(M,\mathcal{B}(M))$ by the fundamental result [19, Theorem 7.2.1] from Dirichlet form theory.

4.2. **Capacity, quasi-continuity and exceptional sets.** The following potential theoretic notions are adopted from [19, Section 2.1] and [13, Sections 1.2 and 1.3].

Definition 4.1. (1) We define the *1-capacity* Cap_1 *associated with* $(M,\mu,\mathcal{E},\mathcal{F})$ by

$$\begin{aligned} \mathrm{cap}_1(U) &:= \inf\{\mathcal{E}_1(u,u) \mid u \in \mathcal{F},\ u \ge 1 \ \mu\text{-a.e. on } U\} && \text{for } U \subset M \text{ open in } M, \\ \mathrm{Cap}_1(A) &:= \inf\{\mathrm{cap}_1(U) \mid U \subset M \text{ open in } M,\ A \subset U\} && \text{for } A \subset M \end{aligned} \tag{4.3}$$

(recall $\mathcal{E}_1 := \mathcal{E} + \langle\cdot,\cdot\rangle$). Clearly, Cap_1 extends cap_1 and $\mu(A) \le \mathrm{Cap}_1(A)$ for $A \in \mathcal{B}(M)$.

(2) A subset N of M is called *$\mathcal{E}$-polar* if and only if $\mathrm{Cap}_1(N) = 0$. Moreover, if $A \subset M$ and $\mathcal{S}(x)$ is a statement in $x \in A$, then we say that *$\mathcal{S}$ holds $\mathcal{E}$-q.e. on A* if and only if $\{x \in A \mid \mathcal{S}(x) \text{ fails}\}$ is $\mathcal{E}$-polar. When $A = M$ we simply say *"$\mathcal{S}$ holds $\mathcal{E}$-q.e."* instead.

(3) Let $U \subset M$ be open in M. A function $u : U \setminus N \to [-\infty,\infty]$, with $N \subset M$ $\mathcal{E}$-polar, is called *$\mathcal{E}$-quasi-continuous on U* if and only if for any $\varepsilon \in (0,\infty)$ there exists an open

subset V of M with $U \cap N \subset V$ and $\mathrm{Cap}_1(V) < \varepsilon$ such that $u|_{U \setminus V}$ is $\mathbb{R}$-valued and continuous. When $U = M$, such u is simply called *$\mathcal{E}$-quasi-continuous* instead.

Remark 4.2. There are several equivalent ways of defining the notions of $\mathcal{E}$-polar sets and $\mathcal{E}$-quasi-continuous functions. See [13, Section 1.2 and Theorem 1.3.14] in this connection.

Note that Cap_1 is countably subadditive by [19, Lemma 2.1.2 and Theorem A.1.2].

Let $U \subset M$ be open in M. By [19, Lemma 2.1.4], if u, v are $\mathcal{E}$-quasi-continuous functions on U and $u \leq v$ μ-a.e. on U, then $u \leq v$ $\mathcal{E}$-q.e. on U. In particular, for each $u \in L^2(M, \mu)$, an $\mathcal{E}$-quasi-continuous μ-version of u, if it exists, is unique up to $\mathcal{E}$-q.e. By [19, Theorem 2.1.3], each $u \in \mathcal{F}$ admits an $\mathcal{E}$-quasi-continuous μ-version, which is denoted as $\widetilde{u}$.

For each $t \in (0, \infty)$, while $T_t u = \mathcal{P}_t u$ μ-a.e. for any $u \in L^2(M, \mu)$ by the definition of T_t, more strongly it actually holds by [19, Theorem 4.2.3-(i)] that for any $u \in \mathcal{B}L^2(M, \mu)$,

$$\mathcal{P}_t u \text{ is an } \mathcal{E}\text{-quasi-continuous } \mu\text{-version of } T_t u. \tag{4.4}$$

The following definition gives a probabilistic counterpart of the notion of $\mathcal{E}$-polar sets.

Definition 4.3. A Borel set $N \in \mathcal{B}(M)$ is called *properly exceptional for X* if and only if $\mu(N) = 0$ and for any $x \in M \setminus N$, $\mathbb{P}_x[\dot{\sigma}_N \wedge \hat{\sigma}_N = \infty] = 1$ or, by (2.6), equivalently

$$\mathbb{P}_x[\dot{\sigma}_N = \infty] = 1. \tag{4.5}$$

Note that $\{\dot{\sigma}_N \wedge \hat{\sigma}_N = \infty\} = \{X_0, X_t, X_{t-} \in M_\Delta \setminus N \text{ for any } t \in (0, \infty)\} \in \mathcal{F}_\infty$ and that $\{\dot{\sigma}_N = \infty\} = \{X_t \in M_\Delta \setminus N \text{ for any } t \in [0, \infty)\} \in \mathcal{F}_\infty$. Every properly exceptional set for X is $\mathcal{E}$-polar by [19, Theorem 4.2.1-(ii)], and conversely any $\mathcal{E}$-polar set is included in a Borel properly exceptional set for X by [19, Theorem 4.1.1].

4.3. **The Dirichlet form of the part process on open sets.** Let U be a non-empty open subset of M and set $\mu|_U := \mu|_{\mathcal{B}(U)}$. Recall that the part X^U of X on U is a Hunt process on $(U, \mathcal{B}(U))$ defined in (2.7) and that its transition function naturally extends to $(M, \mathcal{B}(M))$ as a family $\{\mathcal{P}_t^U\}_{t \in [0,\infty)}$ of Markovian kernels on $(M, \mathcal{B}(M))$ given by (2.8). In the present situation, the assumed μ-symmetry of X implies that X^U is $\mu|_U$-symmetric. More precisely, for any $t \in (0, \infty)$ and any $u, v \in \mathcal{B}^+(M)$, we have $\langle \mathcal{P}_t^U u, v \rangle = \langle u, \mathcal{P}_t^U v \rangle$ by [19, Lemma 4.1.3] and hence also $\|\mathcal{P}_t^U u\|_2 \leq \|u\|_2$ as in [19, (1.4.13)]. Thus we obtain a Markovian symmetric contraction semigroup $\{T_t^U\}_{t \in (0,\infty)}$ on $L^2(M, \mu)$ canonically induced by $\{\mathcal{P}_t^U\}_{t \in (0,\infty)}$ in the same way as for $\{\mathcal{P}_t\}_{t \in (0,\infty)}$. Moreover, under the natural identification of $L^2(U, \mu|_U)$ with the closed linear subspace $\{u \in L^2(M, \mu) \mid u = 0 \ \mu\text{-a.e. on } M \setminus U\}$ of $L^2(M, \mu)$, the strongly continuous Markovian semigroup on $L^2(U, \mu|_U)$ induced by the transition function of X^U is easily shown to be given by $\{T_t^U|_{L^2(U,\mu|_U)}\}_{t \in (0,\infty)}$, and hence (4.1) with T_t^U in place of T_t gives the Dirichlet form $(\mathcal{E}^U, \mathcal{F}_U)$ of X^U. In fact,

$$\mathcal{F}_U = \{u \in \mathcal{F} \mid \widetilde{u} = 0 \ \mathcal{E}\text{-q.e. on } M \setminus U\} \qquad \text{and} \qquad \mathcal{E}^U = \mathcal{E}|_{\mathcal{F}_U \times \mathcal{F}_U} \tag{4.6}$$

by [19, Theorem 4.4.2] and $(\mathcal{E}^U, \mathcal{F}_U)$ is regular on $L^2(U, \mu|_U)$ by [19, Lemma 1.4.2-(ii) and Corollary 2.3.1]. $(\mathcal{E}^U, \mathcal{F}_U)$ is called the *part of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on U*.

For $t \in (0, \infty)$ and $u \in \mathcal{B}L^2(M, \mu)$, while $T_t^U u = \mathcal{P}_t^U u$ μ-a.e. by definition, more strongly

$$\mathcal{P}_t^U u \text{ is an } \mathcal{E}\text{-quasi-continuous } \mu\text{-version of } T_t^U u, \tag{4.7}$$

similarly to (4.4). Indeed, since $v := T_t^U u \in \mathcal{F}_U \subset \mathcal{F}$ by (4.2) and (4.6), v admits an $\mathcal{E}$-quasi-continuous μ-version $\widetilde{v}$ and then $\widetilde{v} = 0 = \mathcal{P}_t^U u$ $\mathcal{E}$-q.e. on $M \setminus U$ by (4.6). On the other hand, $(\mathcal{P}_t^U u)|_U$ is a μ-version of $v|_U$ which is $\mathcal{E}$-quasi-continuous on U by [19, Theorem 4.4.3] and therefore $(\mathcal{P}_t^U u)|_U = \widetilde{v}|_U$ $\mathcal{E}$-q.e. on U by [19, Lemma 2.1.4]. Thus $\mathcal{P}_t^U u = \widetilde{v}$ $\mathcal{E}$-q.e., which together with the $\mathcal{E}$-quasi-continuity of $\widetilde{v}$ yields (4.7).

5. Localized quasi-everywhere existence of the heat kernel

As in Section 4, let (M, d) be a locally compact separable metric space equipped with a Radon measure μ with full support, and let X be a μ-symmetric Hunt process on $(M, \mathcal{B}(M))$ whose Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular on $L^2(M, \mu)$. *Throughout the rest of this paper, we fix this setting and follow the notation introduced in Section* 4 *in addition to that from Sections* 2 *and* 3.

The purpose of this section is to prove Theorem 5.4 below on the existence of the heat kernel $p^U = p_t^U(x, y)$ for $\{\mathcal{P}_t^U\}_{t\in(0,\infty)}$ on a given subset of $(0, \infty) \times M \times M$ under a suitable upper bound on the Markovian semigroup $\{T_t^U\}_{t\in(0,\infty)}$ *which is assumed only on the given subset.* In the case where the subset is the whole $(0, \infty) \times M \times M$, similar results have been obtained, e.g., in [22, Sections 7 and 8] and [6, Theorem 3.1].

Remark 5.1. The μ-symmetry of X and the regularity of its Dirichlet form are assumed mostly for the sake of simplicity of the framework. In fact, we need these assumptions *only in order to use potential theoretic results from* [19, *Chapters* 2 *and* 4] *in the proof of Theorem* 5.4; it should be possible to extend Theorem 5.4 to a more general framework where the same kind of potential theory remains available, and the reader is referred to Remarks 5.5, 6.1 and 7.1 for the precise settings actually required for the (other) results in Sections 5, 6 and 7, respectively.

For $A \subset M$, let $\mathbf{1}_A : M \to \{0, 1\}$ denote its indicator function given by $\mathbf{1}_A|_A := 1$ and $\mathbf{1}_A|_{M\setminus A} := 0$. In what follows we allow an interval $I \subset \mathbb{R}$ to be a one-point set.

Definition 5.2. Let $I \subset (0, \infty)$ be an interval, V an open subset of M and $W \in \mathcal{B}(M)$. A Borel measurable function $H = H_t(x, y) : I \times V \times W \to [0, \infty]$ is called a *μ-upper bound function on $I \times V \times W$* if and only if the following three conditions are satisfied:

(UB1) $\limsup_{s\downarrow t} H_s(x, y) \le H_t(x, y)$ for any $(t, x, y) \in I \times V \times W$ with $t < \sup I$.

(UB2) $H_t(\cdot, y) : V \to [0, \infty]$ is upper semi-continuous for any $(t, y) \in I \times W$.

(UB3) There exist $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{B}^+(M)$ and non-decreasing sequences $\{I_n\}_{n\in\mathbb{N}}$ of open subsets of I, $\{V_n\}_{n\in\mathbb{N}}$ of open subsets of V and $\{W_n\}_{n\in\mathbb{N}}$ of Borel subsets of W with $I = \bigcup_{n\in\mathbb{N}} I_n$, $V = \bigcup_{n\in\mathbb{N}} V_n$ and $W = \bigcup_{n\in\mathbb{N}} W_n$ such that for any $n \in \mathbb{N}$,

$$\int_{W_n} h_n \, d\mu < \infty \quad \text{and} \quad H_t(x, y) \le h_n(y) \text{ for any } (t, x, y) \in I_n \times V_n \times W_n. \tag{5.1}$$

Remark 5.3. (1) In (UB3) we may assume that $\mu(V_n \cup W_n) < \infty$ for any $n \in \mathbb{N}$, by taking a non-decreasing sequence $\{M_n\}_{n\in\mathbb{N}}$ of open subsets of M with $\overline{M_n}$ compact and $M = \bigcup_{n\in\mathbb{N}} M_n$ and replacing V_n and W_n with $V_n \cap M_n$ and $W_n \cap M_n$, respectively.
(2) It is easy to see that the condition (UB3) in Definition 5.2 is satisfied if W is open in M and $\|H\|_{\sup,K} = \sup_{(t,x,y)\in K} H_t(x, y) < \infty$ for any compact subset K of $I \times V \times W$.

Theorem 5.4. *Let $I \subset (0, \infty)$ be an interval, V an open subset of M, $W \in \mathcal{B}(M)$ and let $H = H_t(x, y)$ be a μ-upper bound function on $I \times V \times W$. Let U be a non-empty open subset of M. Then for each countable dense subset J of I satisfying* $\max I \in J$ *if* $\max I$ *exists, the following three conditions are equivalent:*

(1) *For any $t \in J$ and any $v, w \in L^2(M, \mu)$ with $(v\mathbf{1}_V) \wedge (w\mathbf{1}_W) \ge 0$ μ-a.e.,*

$$\langle v\mathbf{1}_V, T_t^U(w\mathbf{1}_W)\rangle \le \int_{V\times W} v(x)H_t(x, y)w(y)\, d(\mu \times \mu)(x, y). \tag{5.2}$$

(2) *For each $t \in J$ and each $w \in L^2(M, \mu)$ with $w\mathbf{1}_W \ge 0$ μ-a.e.,*

$$T_t^U(w\mathbf{1}_W)(x) \le \int_W H_t(x, y)w(y)\, d\mu(y) \quad \textit{for } \mu\textit{-a.e. } x \in V. \tag{5.3}$$

(3) *There exist a properly exceptional set $N \in \mathcal{B}(M)$ for X and a Borel measurable function $p^U = p^U_t(x,y) : I \times (V \setminus N) \times W \to [0,\infty]$ such that for any $(t,x) \in I \times (V \setminus N)$,*

$$\mathcal{P}^U_t(x,A) = \int_A p^U_t(x,y)\, d\mu(y) \qquad \text{for any } A \in \mathcal{B}(W), \tag{5.4}$$

$$p^U_t(x,y) \le H_t(x,y) \qquad \text{for any } y \in W. \tag{5.5}$$

We first show the following proposition, which is of independent interest and will be used in the proof of the implication (2)⇒(3) of Theorem 5.4 and also in the proof of Theorems 6.2 and 6.4 in the next section.

Remark 5.5. In fact, *Proposition* 5.6 *below applies, without any changes in the proof, to any locally compact separable metrizable topological space M, any σ-finite Borel measure μ on M and any Hunt process X on $(M, \mathcal{B}(M))$.*

Proposition 5.6. *Let $I \in \mathcal{B}([0,\infty))$, let $V, W \in \mathcal{B}(M)$ and let $H = H_t(x,y) : I \times V \times W \to [0,\infty]$ be Borel measurable. Let U be a non-empty open subset of M. Then the following two conditions are equivalent:*

(1) *For any $(t,x) \in I \times V$ and any $A \in \mathcal{B}(W)$,*

$$\mathcal{P}^U_t(x,A) \le \int_A H_t(x,y)\, d\mu(y). \tag{5.6}$$

(2) *There exists a Borel measurable function $p^U = p^U_t(x,y) : I \times V \times W \to [0,\infty]$ such that* (5.4) *and* (5.5) *hold for any $(t,x) \in I \times V$.*

Proof. Since the implication (2)⇒(1) is immediate, it suffices to show the converse (1)⇒(2). By the σ-finiteness of μ, we can choose $\{W_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(W)$ with $W = \bigcup_{n\in\mathbb{N}} W_n$ so that $W_n \subset W_{n+1}$ and $\mu(W_n) < \infty$ for any $n \in \mathbb{N}$. We will construct for each $n \in \mathbb{N}$ a function $p^{U,n} = p^{U,n}_t(x,y) : I \times V \times W_n \to [0,\infty]$ possessing the required properties with W_n in place of W. If $\mu(W_n) = 0$ then it suffices to set $p^{U,n} := 0$ in view of (5.6), and therefore we may assume $\mu(W_n) > 0$. Let $\mathcal{U} = \{A_k\}_{k\in\mathbb{N}}$ be a countable open base for the topology of M, set $A^0_k := M \setminus A_k$ and $A^1_k := A_k$ for $k \in \mathbb{N}$, and define

$$\mathcal{A}_k := \{\textstyle\bigcup_{\alpha\in\mathcal{I}} A^\alpha_k \mid \mathcal{I} \subset \{0,1\}^k\}, \quad k \in \mathbb{N}, \tag{5.7}$$

where $A^\alpha_k := \bigcap_{i=1}^k A^{\alpha_i}_i$ for $\alpha = (\alpha_i)_{i=1}^k \in \{0,1\}^k$, so that $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ is a non-decreasing sequence of σ-fields in M with $\bigcup_{k\in\mathbb{N}} \mathcal{A}_k$ generating $\mathcal{B}(M)$. For $k \in \mathbb{N}$, noting that $M = \bigcup_{\alpha\in\{0,1\}^k} A^\alpha_k$ and that $A^\alpha_k \cap A^\beta_k = \emptyset$ for $\alpha, \beta \in \{0,1\}^k$ with $\alpha \ne \beta$, define $p^{U,n,k} = p^{U,n,k}_t(x,y) : I \times V \times M \to [0,\infty)$ by, for $\alpha \in \{0,1\}^k$ and $(t,x,y) \in I \times V \times A^\alpha_k$,

$$p^{U,n,k}_t(x,y) := \begin{cases} \mu(A^\alpha_k \cap W_n)^{-1} \mathcal{P}^U_t \mathbf{1}_{A^\alpha_k \cap W_n}(x) & \text{if } \mu(A^\alpha_k \cap W_n) > 0, \\ 0 & \text{if } \mu(A^\alpha_k \cap W_n) = 0. \end{cases} \tag{5.8}$$

Then $p^{U,n,k}$ is Borel measurable by (2.4) for $\{\mathcal{P}^U_t\}_{t\in[0,\infty)}$. Furthermore for each $(t,x) \in I \times V$, since $\mathcal{P}^U_t(x, (\cdot)\cap W_n)$ is absolutely continuous with respect to $\mu((\cdot)\cap W_n)$ and $f^{t,x}_n := \frac{d\mathcal{P}^U_t(x,(\cdot)\cap W_n)}{d\mu((\cdot)\cap W_n)} \le H_t(x,\cdot)$ μ-a.e. on W_n by (5.6), $p^{U,n,k}_t(x,\cdot)$ is a version of the $\mathcal{A}_k$-conditional $\frac{\mu((\cdot)\cap W_n)}{\mu(W_n)}$-expectation of $f^{t,x}_n$ and hence $\lim_{k\to\infty} p^{U,n,k}_t(x,y) = f^{t,x}_n(y) \le H_t(x,y)$ for μ-a.e. $y \in W_n$ by the martingale convergence theorem [17, Theorem 10.5.1]. Therefore the function $p^{U,n}_t(x,y) := H_t(x,y) \wedge \liminf_{k\to\infty} p^{U,n,k}_t(x,y)$, $(t,x,y) \in I \times V \times W_n$, has the desired properties. Now the proof of (2) is completed by setting $p^U_t(x,y) := p^{U,n}_t(x,y)$ for $n \in \mathbb{N}$ and $(t,x,y) \in I \times V \times (W_n \setminus W_{n-1})$ ($W_0 := \emptyset$) and using monotone convergence. □

Proof of Theorem 5.4. The implication (2)⇒(1) is immediate, and it is easy to see from (UB3) of Definition 5.2 and Remark 5.3-(1) that (1) implies (2). The implication (3)⇒(2) also follows easily since $T^U_t u = \mathcal{P}^U_t u$ μ-a.e. for any $t \in (0,\infty)$ and any $u \in \mathcal{B}L^2(M,\mu)$.

Therefore it remains to prove (2)⇒(3). Let $\{h_n\}_{n\in\mathbb{N}}, \{I_n\}_{n\in\mathbb{N}}, \{V_n\}_{n\in\mathbb{N}}, \{W_n\}_{n\in\mathbb{N}}$ be as in (UB3) with $\mu(W_n) < \infty$ for any $n \in \mathbb{N}$ as noted in Remark 5.3-(1). Let $\mathcal{A}_k$ be as in (5.7) for each $k \in \mathbb{N}$ and set $\mathcal{A} := \bigcup_{k\in\mathbb{N}} \mathcal{A}_k$, so that $\mathcal{A}$ is countable, generates $\mathcal{B}(M)$ and satisfies $\emptyset \in \mathcal{A}$, $M \setminus A \in \mathcal{A}$ for any $A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$. By (4.7) and [19, Theorem 2.1.2-(i)], there exists a non-decreasing sequence $\{F_k\}_{k\in\mathbb{N}}$ of closed subsets of M such that $\lim_{k\to\infty} \mathrm{Cap}_1(M \setminus F_k) = 0$ and for each $k \in \mathbb{N}$, $\mu(G \cap F_k) > 0$ for any open subset G of M with $G \cap F_k \neq \emptyset$ and $\{\mathcal{P}^U_t \mathbf{1}_{A\cap W_n}|_{F_k} \mid n \in \mathbb{N},\, t \in J,\, A \in \mathcal{A}\} \subset C(F_k)$. Moreover, since

$$\mathcal{P}^U_t \mathbf{1}_{A\cap W_n}(x) = \mathcal{P}^U_{t-l^{-1}}(\mathcal{P}^U_{l^{-1}} \mathbf{1}_{A\cap W_n})(x) = \mathbb{E}_x\bigl[\mathcal{P}^U_{l^{-1}} \mathbf{1}_{A\cap W_n}(X_{t-l^{-1}})\mathbf{1}_{\{t-l^{-1}<\tau_U\}}\bigr]$$

for $l \in \mathbb{N}$ and $t \in [l^{-1},\infty)$ by (2.3) for $\{\mathcal{P}^U_t\}_{t\in[0,\infty)}$, an application of (4.7) and [19, Theorem 4.2.2] to $\mathcal{P}^U_{l^{-1}} \mathbf{1}_{A\cap W_n}$ with $l, n \in \mathbb{N}$ and $A \in \mathcal{A}$ yields an $\mathcal{E}$-polar set $N_0 \in \mathcal{B}(M)$ such that $(0,\infty) \ni t \mapsto \mathcal{P}^U_t \mathbf{1}_{A\cap W_n}(x) \in \mathbb{R}$ is right-continuous for any $x \in M \setminus N_0$, any $n \in \mathbb{N}$ and any $A \in \mathcal{A}$. Then $(M \setminus \bigcup_{k\in\mathbb{N}} F_k) \cup N_0$ is $\mathcal{E}$-polar and therefore by [19, Theorem 4.1.1] we can take a properly exceptional set $N \in \mathcal{B}(M)$ for X satisfying $(M \setminus \bigcup_{k\in\mathbb{N}} F_k) \cup N_0 \subset N$.

Let $n \in \mathbb{N}$ and $(t,x) \in I \times (V \setminus N)$. We claim that for any $A \in \mathcal{B}(M)$,

$$\mathcal{P}^U_t \mathbf{1}_{A\cap W_n}(x) \leq \int_{A\cap W_n} H_t(x,y)\, d\mu(y), \tag{5.9}$$

whose limit as $n \to \infty$ results in (5.6) with $V \setminus N$ in place of V by monotone convergence, thereby proving (2)⇒(3) by virtue of Proposition 5.6. Thus it remains to show (5.9). To this end, let $A \in \mathcal{A}$ and choose $k \in \mathbb{N}$ with $k \geq n$ so that $t \in I_k$ and $x \in V_k \cap F_k$.

First we assume $t \in J$. Then $\mathcal{P}^U_t \mathbf{1}_{A\cap W_n} \leq \int_{A\cap W_n} H_t(\cdot,y)\, d\mu(y)$ μ-a.e. on V by (2), and since $\mu(G \cap V_k \cap F_k) > 0$ for any open subset G of M with $x \in G$ we can take $\{x_l\}_{l\in\mathbb{N}} \subset V_k \cap F_k$ such that $\lim_{l\to\infty} x_l = x$ in M and $\mathcal{P}^U_t \mathbf{1}_{A\cap W_n}(x_l) \leq \int_{A\cap W_n} H_t(x_l,y)\, d\mu(y)$ for any $l \in \mathbb{N}$. Now (5.9) follows by utilizing $\mathcal{P}^U_t \mathbf{1}_{A\cap W_n}|_{F_k} \in C(F_k)$, Fatou's lemma and (UB2) to let $l \to \infty$, where the use of Fatou's lemma is justified by (5.1) with k in place of n.

Next for $t \in I \setminus J$, with $k \in \mathbb{N}$ as above, we can take a strictly decreasing sequence $\{t_l\}_{l\in\mathbb{N}} \subset I_k \cap J$ satisfying $\lim_{l\to\infty} t_l = t$, and then $\mathcal{P}^U_{t_l} \mathbf{1}_{A\cap W_n}(x) \leq \int_{A\cap W_n} H_{t_l}(x,y)\, d\mu(y)$ for any $l \in \mathbb{N}$ by the previous paragraph. Now letting $l \to \infty$ yields (5.9) for this case by the right-continuity of $\mathcal{P}^U_{(\cdot)} \mathbf{1}_{A\cap W_n}(x)$, Fatou's lemma and (UB1), where (5.1) with k in place of n is used again to verify the applicability of Fatou's lemma to the right-hand side.

Thus (5.9) has been proved for any $A \in \mathcal{A}$. Further, we easily see from (5.1) with k in place of n and the dominated convergence theorem that $\{A \in \mathcal{B}(M) \mid A \text{ satisfies (5.9)}\}$ is closed under monotone countable unions and intersections, and hence the monotone class theorem [17, Theorem 4.4.2] implies that (5.9) holds for any $A \in \mathcal{B}(M)$. □

The rest of this section is devoted to presenting examples of μ-upper bound functions. We start with a lemma which is mostly due to [28, Subsection 3.4].

Lemma 5.7. *Let $\Psi : [0,\infty) \to [0,\infty)$ be a homeomorphism satisfying*

$$c_\Psi^{-1}\Bigl(\frac{R}{r}\Bigr)^{\beta_1} \leq \frac{\Psi(R)}{\Psi(r)} \leq c_\Psi\Bigl(\frac{R}{r}\Bigr)^{\beta_2} \qquad \text{for any } r, R \in (0,\infty) \text{ with } r \leq R \tag{5.10}$$

for some $c_\Psi, \beta_1, \beta_2 \in (0,\infty)$ with $1 < \beta_1 \leq \beta_2$, and for $(R,t) \in [0,\infty) \times (0,\infty)$ define

$$\Phi(R,t) := \Phi_\Psi(R,t) := \sup_{r\in(0,\infty)} \Bigl\{\frac{R}{r} - \frac{t}{\Psi(r)}\Bigr\} = \sup_{\lambda\in(0,\infty)} \Bigl\{\frac{R}{\Psi^{-1}(\lambda^{-1})} - \lambda t\Bigr\}. \tag{5.11}$$

Then $\Phi = \Phi_\Psi$ is a $[0,\infty)$-valued lower semi-continuous function such that for any $R, t \in (0,\infty)$, $\Phi(\cdot, t)$ is non-decreasing, $\Phi(R, \cdot)$ is non-increasing, $\Phi(0,t) = 0 < \Phi(R,t)$,

$$a\Phi(R,t) \le \Phi(aR,t) \quad \text{for any } a \in [1,\infty), \tag{5.12}$$

$$(c_\Psi 2^{\beta_1})^{-\frac{1}{\beta_1-1}} \min_{k\in\{1,2\}} \Bigl(\frac{\Psi(R)}{t}\Bigr)^{\frac{1}{\beta_k-1}} \le \Phi(R,t) \le c_\Psi^{\frac{1}{\beta_1-1}} \max_{k\in\{1,2\}} \Bigl(\frac{\Psi(R)}{t}\Bigr)^{\frac{1}{\beta_k-1}}. \tag{5.13}$$

Proof. The lower semi-continuity of $\Phi = \Phi_\Psi$ is clear from (5.11), and the other assertions except the upper inequality in (5.13) have been verified in [28, Remark 3.16 and Lemma 3.19]. To see the upper inequality in (5.13), let $R, t, r \in (0,\infty)$ and set $a := R\Psi(r)/(rt)$. Noting that $R/r - t/\Psi(r) = (a-1)t/\Psi(r) \le 0$ if $a \le 1$, we assume $a > 1$, and set $\beta := \beta_1$ if $r \le R$ and $\beta := \beta_2$ if $r > R$. Then $at/\Psi(r) = R/r \le (c_\Psi \Psi(R)/\Psi(r))^{1/\beta}$ by (5.10), hence $\Psi(r) \ge at(c_\Psi^{-1} at/\Psi(R))^{\frac{1}{\beta-1}}$, and therefore

$$\frac{R}{r} - \frac{t}{\Psi(r)} \le \frac{R}{r} = \frac{at}{\Psi(r)} \le \Bigl(\frac{c_\Psi \Psi(R)}{at}\Bigr)^{\frac{1}{\beta-1}} \le c_\Psi^{\frac{1}{\beta_1-1}} \max_{k\in\{1,2\}} \Bigl(\frac{\Psi(R)}{t}\Bigr)^{\frac{1}{\beta_k-1}},$$

where the last inequality follows by $a \ge 1$, $1 < \beta_1 \le \beta_2$ and the fact that $c_\Psi \ge 1$ by (5.10). Now taking the supremum in $r \in (0,\infty)$ yields the desired inequality. □

Example 5.8. An important special case of Lemma 5.7 is that of $\Psi(r) = r^\beta$ for some $\beta \in (1,\infty)$ treated in [28, Example 3.17], where $\Phi = \Phi_\Psi$ is easily evaluated as

$$\Phi(R,t) = \beta^{-\frac{\beta}{\beta-1}}(\beta-1)\Bigl(\frac{R^\beta}{t}\Bigr)^{\frac{1}{\beta-1}}. \tag{5.14}$$

The following lemma provides a class of typical μ-upper bound functions, which has essentially appeared in [26, (6.10)]. Note that *Lemma* 5.9 *and Example* 5.10 *below, as well as Remark* 5.3 *above, apply to any locally compact separable metric space* (M, d) *and any Radon measure* μ *on* M (i.e., any Borel measure on M that is finite on compact sets).

Lemma 5.9. *Let Ψ and $\Phi = \Phi_\Psi$ be as in Lemma 5.7. Let $I \subset (0,\infty)$ be an interval, let V, W be open subsets of M and let $F = F_t(x,y) : I \times V \times W \to (0,\infty)$ be a Borel measurable function satisfying* (UB1) *and* (UB2) *of Definition* 5.2 *and the following* Ψ-doubling condition $(\mathrm{DB})_\Psi$*:*

$(\mathrm{DB})_\Psi$ *There exist $\alpha_F, c_F \in (0,\infty)$ such that for any $(t,x,y), (s,z,w) \in I \times V \times W$ with $s \le t$,*

$$\frac{F_s(z,w)}{F_t(x,y)} \le c_F \Bigl(\frac{t \vee \Psi(d(x,z)) \vee \Psi(d(y,w))}{s}\Bigr)^{\alpha_F}. \tag{5.15}$$

Also let $c_1, c_2 \in (0,\infty)$ and define $H = H_t(x,y) : I \times V \times W \to (0,\infty)$ by

$$H_t(x,y) := F_t(x,y) \exp\bigl(-c_1 \Phi(c_2 d(x,y), t)\bigr). \tag{5.16}$$

Then $F = F_t(x,y)$ and $H = H_t(x,y)$ are μ-upper bound functions on $I \times V \times W$.

Proof. It is immediate to see that $H = H_t(x,y)$ is Borel measurable and satisfies (UB1) and (UB2), from the corresponding properties of $F = F_t(x,y)$ and the lower semi-continuity of Φ. Also $(\mathrm{DB})_\Psi$ easily implies that $F = F_t(x,y)$ and hence $H = H_t(x,y)$ are bounded on each compact subset of $I \times V \times W$, so that they satisfy (UB3) by Remark 5.3-(2). □

Example 5.10. Let Ψ be as in Lemma 5.7.

(1) A continuous function $F = F_t(x,y) : (0,\infty) \times M \times M \to (0,\infty)$ of the form

$$F_t(x,y) = c_3 t^{-\alpha_1}\bigl(\log(2+t^{-1})\bigr)^{\alpha_2}\bigl(\log(2+t)\bigr)^{\alpha_3} \tag{5.17}$$

for some $c_3, \alpha_1 \in (0,\infty)$ and $\alpha_2, \alpha_3 \in \mathbb{R}$ clearly satisfies (UB1), (UB2) and $(\mathrm{DB})_\Psi$.

(2) Let $R \in (0,\infty]$, let V, W be open subsets of M with $(\operatorname{diam} V) \vee (\operatorname{diam} W) \le R$ and let ν be a Borel measure on M satisfying the *volume doubling property*

$$0 < \nu(B(x,2r)) \le c_{\mathrm{vd}} \nu(B(x,r)) < \infty \tag{5.18}$$

for any $(x,r) \in (V \cup W) \times (0,R)$ for some $c_{\mathrm{vd}} \in (0,\infty)$. Then for each $c_4 \in (0,\infty)$, the function $F = F_t(x,y) : (0,\Psi(R)] \times V \times W \to (0,\infty)$ ($(0,\infty)$ in place of $(0,\Psi(R)]$ for $R = \infty$) defined by

$$F_t(x,y) := c_4 \nu\bigl(B(x,\Psi^{-1}(t))\bigr)^{-1/2} \nu\bigl(B(y,\Psi^{-1}(t))\bigr)^{-1/2} \tag{5.19}$$

is easily proved to be upper semi-continuous and satisfy $(\mathrm{DB})_\Psi$ thanks to (5.18) and (5.10), and in particular it is Borel measurable and satisfies (UB1) and (UB2).

6. Localized upper bounds of heat kernels for diffusions

In this section, we state and prove the main theorem of this paper on deducing heat kernel upper bounds for $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ from those for $\{\mathcal{P}^U_t\}_{t\in(0,\infty)}$ (Theorem 6.2 below). The arguments heavily rely on the decay estimate (6.3) for the exit probabilities $\mathbb{P}_x[\tau_{B(x,r)} \le t]$, for which reasonable sufficient conditions will be presented in the next section. *In the rest of this paper, we fix a homeomorphism $\Psi : [0,\infty) \to [0,\infty)$ and $c_\Psi, \beta_1, \beta_2 \in (0,\infty)$ with $1 < \beta_1 \le \beta_2$ satisfying* (5.10)*, and $\Phi = \Phi_\Psi$ denotes the function given by* (5.11)*.*

Throughout this section, we fix an arbitrary properly exceptional set $N \in \mathcal{B}(M)$ for X such that for any $x \in M \setminus N$,

$$\mathbb{P}_x\bigl[[0,\zeta) \ni t \mapsto X_t \in M \text{ is continuous}\bigr] = 1, \tag{6.1}$$

where $\{[0,\zeta) \ni t \mapsto X_t \in M \text{ is continuous}\} \in \mathcal{F}_\infty$ by [16, Chapter III, 13 and 33]. According to [19, Theorem 4.5.1], such N exists if and only if $(\mathcal{E},\mathcal{F})$ is *local*, i.e., $\mathcal{E}(u,v) = 0$ for any $u, v \in \mathcal{F}$ with $\operatorname{supp}_\mu[u], \operatorname{supp}_\mu[v]$ compact and $\operatorname{supp}_\mu[u] \cap \operatorname{supp}_\mu[v] = \emptyset$. Here for $u \in \mathcal{B}(M)$ or its μ-equivalence class, $\operatorname{supp}_\mu[u]$ denotes its *μ-support* defined as the smallest closed subset of M such that $u = 0$ μ-a.e. on $M \setminus \operatorname{supp}_\mu[u]$, which exists since M has a countable open base for its topology. Note that $\operatorname{supp}_\mu[u] = \overline{u^{-1}(\mathbb{R} \setminus \{0\})}$ for $u \in C(M)$.

Remark 6.1. In fact, *Theorems* 6.2*,* 6.4*, Propositions* 6.5 *and* 6.6 *below apply, without any changes in the proofs, to any locally compact separable metric space (M,d), any σ-finite Borel measure μ on M, any Hunt process X on $(M,\mathcal{B}(M))$ and any $N \in \mathcal{B}(M)$ satisfying* (4.5) *and* (6.1) *for any $x \in M \setminus N$.*

Theorem 6.2. *Let $R \in (0,\infty)$, let U be a non-empty open subset of M with $\operatorname{diam} U \le R$ and let $F = F_t(x,y) : (0,\Psi(R)] \times U \times U \to (0,\infty)$ be a Borel measurable function satisfying $(\mathrm{DB})_\Psi$ of Lemma* 5.9 *with $I = (0,\Psi(R)]$ and $V = W = U$. Let $c, \gamma \in (0,\infty)$ and assume that the following two conditions $(\mathrm{DU})^{U,R}_F$ and $(\mathrm{P})^{U,R}_\Psi$ are fulfilled:*

$(\mathrm{DU})^{U,R}_F$ *For any $(t,x) \in (0,\Psi(R)) \times (U \setminus N)$ and any $A \in \mathcal{B}(U)$,*

$$\mathcal{P}^U_t(x,A) \le \int_A F_t(x,y)\, d\mu(y). \tag{6.2}$$

$(\mathrm{P})^{U,R}_\Psi$ *For any $(x,r) \in (U \setminus N) \times (0,R)$ with $B(x,r) \subset U$ and any $t \in (0,\infty)$,*

$$\mathbb{P}_x[\tau_{B(x,r)} \le t] \le c \exp(-\Phi(\gamma r, t)). \tag{6.3}$$

Let $\varepsilon \in (0,1)$ and set $U^\circ_{\varepsilon R} := \{x \in M \mid \inf_{y \in M\setminus U} d(x,y) > \varepsilon R\}$ (note that $U^\circ_{\varepsilon R}$ is an open subset of U)*. Then there exists a Borel measurable function $p = p_t(x,y) : (0,\infty) \times (M \setminus N) \times U^\circ_{\varepsilon R} \to [0,\infty)$ such that for any $(t,x) \in (0,\infty) \times (M \setminus N)$ the following hold:*

$$\mathcal{P}_t(x,A) = \int_A p_t(x,y)\, d\mu(y) \qquad \textit{for any } A \in \mathcal{B}(U^\circ_{\varepsilon R}), \tag{6.4}$$

and furthermore for any $y \in U^{\circ}_{\varepsilon R}$,

$$p_t(x,y) \le \begin{cases} c_\varepsilon F_t(x,y)\exp\bigl(-\Phi(\gamma_\varepsilon d(x,y),t)\bigr) & \text{if } t<\Psi(R) \text{ and } x\in U,\\ c_\varepsilon(\inf_{U\times U} F_{(2t)\wedge\Psi(R)})\exp(-\Phi(\gamma_\varepsilon R,t)) & \text{if } t<\Psi(R) \text{ and } x\notin U,\\ c_\varepsilon(\inf_{U\times U} F_{\Psi(R)}) & \text{if } t\ge\Psi(R) \end{cases} \tag{6.5}$$

for some $c_\varepsilon \in (0,\infty)$ *explicit in* $c_\Psi, \beta_1, \beta_2, c_F, \alpha_F, c, \gamma, \varepsilon$ *and* $\gamma_\varepsilon := \frac{1}{5}\varepsilon\gamma$.

In light of the equivalence stated in Proposition 5.6 and the examples of Borel measurable functions $F = F_t(x,y)$ satisfying $(\mathrm{DB})_\Psi$ in Example 5.10, $(\mathrm{DU})^{U,R}_F$ of Theorem 6.2 amounts to an *on-diagonal* upper bound of the heat kernel $p^U = p^U_t(x,y)$ for $\{\mathcal{P}^U_t\}_{t\in(0,\infty)}$. Note that *the two conditions* $(\mathrm{DU})^{U,R}_F$ *and* $(\mathrm{P})^{U,R}_\Psi$ *involve only the part* X^U *of* X *on* U *and hence are independent of the behavior of* X *after exiting* U, on account of (2.8) and the obvious fact that $\tau_B(\omega) = \inf\{t\in[0,\infty) \mid X^U_t(\omega)\in U_\Delta\setminus B\}$ for $B\subset U$ and $\omega\in\Omega$.

Remark 6.3. Theorem 5.4 tells us that $(\mathrm{DU})^{U,R}_F$ of Theorem 6.2 is implied, at the price of replacing N with a larger properly exceptional set for X, by its *"μ-a.e."* counterpart for the Markovian semigroup $\{T^U_t\}_{t\in(0,\infty)}$ provided $F = F_t(x,y)$ is a μ-upper bound function on $(0,\Psi(R))\times U\times U$. Remember, though, that *we have proved Theorem* 5.4 *only for a Radon measure* μ *on* M *with full support and a* μ*-symmetric Hunt process* X *on* $(M,\mathcal{B}(M))$ *whose Dirichlet form* $(\mathcal{E},\mathcal{F})$ *is regular on* $L^2(M,\mu)$; recall Remark 5.1 in this connection.

We also have a global version of Theorem 6.2 for the case where its assumptions are valid on $B(y_0,\frac{R'}{2})$ for any $(y_0,R')\in M\times(0,\infty)$ with $R'\le R$, as follows. Set $\Psi(\infty):=\infty$.

Theorem 6.4. *Let* $\delta\in(0,1]$, *let* $R\in(0,\infty]$ *satisfy* $R\ge\delta\operatorname{diam}M$ *and let* $F=F_t(x,y):(0,\Psi(R)]\times M\times M\to(0,\infty)$ *be a Borel measurable function satisfying* $(\mathrm{DB})_\Psi$ *of Lemma* 5.9 *with* $I=(0,\Psi(R)]$ *and* $V=W=M$ (($0,\infty$) in place of $(0,\Psi(R)]$ for $R=\infty$). *Let* $c,\gamma\in(0,\infty)$ *and assume that the two conditions* $(\mathrm{DU})^{B(y_0,R'/2),R'}_F$ *and* $(\mathrm{P})^{B(y_0,R'/2),R'}_\Psi$ *from Theorem* 6.2 *are fulfilled for any* $(y_0,R')\in M\times(0,\infty)$ *with* $R'\le R$. *Then there exists a Borel measurable function* $p=p_t(x,y):(0,\infty)\times(M\setminus N)\times M\to[0,\infty)$ *such that for any* $(t,x)\in(0,\infty)\times(M\setminus N)$, (6.4) *with* $\mathcal{B}(M)$ *in place of* $\mathcal{B}(U^\circ_{\varepsilon R})$ *holds and*

$$p_t(x,y) \le \begin{cases} c'\delta^{-\beta_2\alpha_F}F_t(x,y)\exp\bigl(-\Phi(\gamma'_\delta d(x,y),t)\bigr) & \text{if } t<\Psi(R),\\ c'\delta^{-\beta_2\alpha_F}(\inf_{M\times M}F_{\Psi(R)}) & \text{if } t\ge\Psi(R) \end{cases} \tag{6.6}$$

for any $y\in M$ *for some* $c'\in(0,\infty)$ *explicit in* $c_\Psi,\beta_1,\beta_2,c_F,\alpha_F,c,\gamma$ *and* $\gamma'_\delta := \frac{1}{40}\delta\gamma$.

The rest of this section is devoted to the proof of Theorems 6.2 and 6.4. We start with the proof of the following proposition, which, in view of Proposition 5.6, can be considered as a localized version of [26, Theorem 6.3]. Its proof in [26] is based on a general comparison inequality [26, Theorem 5.1] among the heat kernels on different open sets *which heavily relies on the symmetry of the Markovian semigroups* $\{T^U_t\}_{t\in(0,\infty)}$; see also [22, Theorem 10.4] for an alternative probabilistic proof of the same comparison inequality. Here we give a new proof *which does not require the* μ*-symmetry of* X.

Proposition 6.5. *Under the same assumptions as those of Theorem* 6.2, *there exists* $c'_\varepsilon\in(0,\infty)$ *explicit in* $c_\Psi,\beta_1,\beta_2,c_F,\alpha_F,c,\gamma,\varepsilon$ *such that, with* $\gamma_\varepsilon:=\frac{1}{5}\varepsilon\gamma$, *for any* $(t,x)\in(0,\Psi(R))\times(U\setminus N)$ *and any* $A\in\mathcal{B}(U^\circ_{\varepsilon R})$,

$$\mathcal{P}^U_t(x,A) \le \int_A c'_\varepsilon F_t(x,y)\exp\bigl(-\Phi(\gamma_\varepsilon d(x,y),t)\bigr)\,d\mu(y). \tag{6.7}$$

Proof. Let $(t,x)\in(0,\Psi(R))\times(U\setminus N)$. Let $y_0\in U^\circ_{\varepsilon R}\setminus\{x\}$, set $r:=\frac{1}{4}\varepsilon d(x,y_0)\in(0,\frac{1}{4}\varepsilon R]$ and let $A\in\mathcal{B}(B(y_0,r))$. We first verify (6.7) for such A. Since $A\subset B(y_0,r)\subset B(y_0,4r)\subset$

U by $y_0 \in U^{\circ}_{\varepsilon R}$ and $r \in (0, \frac{1}{4}\varepsilon R]$, if $t \geq \Psi((2+4/\varepsilon)r)$ then (6.7) is immediate from $(\mathrm{DU})^{U,R}_F$ and the upper inequality in (5.13), and therefore we may assume $t < \Psi((2+4/\varepsilon)r)$. We set $r_n := r + 2^{-n/(2\beta_2)}r$ and $\sigma_n := \dot{\sigma}_{B(y_0,r_n)}$ for $n \in \mathbb{N}$, so that $B(y_0, r) \subset B(y_0, r_n) \subset B(y_0, r_k)$ and hence $\sigma_k \leq \sigma_n \leq \dot{\sigma}_{B(y_0,r)}$ for any $k \in \{1, \dots, n\}$.

Let $\omega \in \{[0, \zeta) \ni s \mapsto X_s \in M$ is continuous$\}$. It is easy to see that for $B \subset M$,

$$\text{if} \quad X_0(\omega) \notin \operatorname{int} B \quad \text{and} \quad \dot{\sigma}_B(\omega) < \infty \qquad \text{then} \qquad X_{\dot{\sigma}_B}(\omega) \in \partial B. \tag{6.8}$$

Assume further that $\omega \in \{X_t \in B(y_0, r),\, X_0 = x\}$. Then since $X_0(\omega) = x \notin B(y_0, 4r)$ by $d(x, y_0) > \varepsilon d(x, y_0) = 4r$ and $\dot{\sigma}_{B(y_0,r)}(\omega) \leq t$ by $X_t(\omega) \in B(y_0, r)$, it follows from (6.8) that $X_{\dot{\sigma}_{B(y_0,r)}}(\omega) \in \partial B(y_0, r)$ and hence that

$$\sigma_n(\omega) \leq \dot{\sigma}_{B(y_0,r)}(\omega) < t \qquad \text{for any } n \in \mathbb{N}. \tag{6.9}$$

In particular, $\sigma_{n+1}(\omega) \leq \frac{1}{2}(\sigma_n(\omega) + t)$ for some $n \in \mathbb{N}$; indeed, otherwise for any $n \in \mathbb{N}$ we would have $\sigma_{n+1}(\omega) \geq \frac{1}{2}(\sigma_n(\omega)+t)$, or equivalently $t - \sigma_{n+1}(\omega) \leq \frac{1}{2}(t - \sigma_n(\omega))$, and hence $0 < t - \dot{\sigma}_{B(y_0,r)}(\omega) \leq t - \sigma_n(\omega) \leq 2^{1-n}(t - \sigma_1(\omega))$ by (6.9), contradicting $\lim_{n\to\infty} 2^{-n} = 0$. Thus, setting $\Omega_1 := \Omega$ and

$$\Omega_n := \{\sigma_{k+1} > \tfrac{1}{2}(\sigma_k + t) \text{ for any } k \in \{1, \dots, n-1\}\}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{6.10}$$

we obtain

$$\begin{aligned} &\{X_t \in B(y_0, r),\, X_0 = x,\, [0, \zeta) \ni s \mapsto X_s \in M \text{ is continuous}\} \\ &\qquad \subset \bigcup\nolimits_{n\in\mathbb{N}} \bigl(\Omega_n \cap \{\sigma_{n+1} \leq \tfrac{1}{2}(\sigma_n + t)\}\bigr), \quad \text{where the union is disjoint.} \end{aligned} \tag{6.11}$$

Note that $\Omega_n \in \mathcal{F}_{\sigma_n}$ for any $n \in \mathbb{N}$ since $\mathcal{F}_{\sigma_k} \subset \mathcal{F}_{\sigma_n}$ by $\sigma_k \leq \sigma_n$ and [30, Lemma 1.2.15] for any $k \in \{1, \dots, n\}$. Now by (6.11) along with $A \subset B(y_0, r)$, $\mathbb{P}_x[X_0 = x] = 1$ and (6.1),

$$\begin{aligned} \mathcal{P}^U_t(x, A) &= \mathbb{P}_x[X_t \in A,\, t < \tau_U] \\ &= \mathbb{P}_x\Bigl[\{X_t \in A,\, t < \tau_U\} \cap \bigcup\nolimits_{n\in\mathbb{N}} \bigl(\Omega_n \cap \{\sigma_{n+1} \leq \tfrac{1}{2}(\sigma_n + t)\}\bigr)\Bigr] \\ &= \sum_{n\in\mathbb{N}} \mathbb{P}_x\bigl[\{X_t \in A,\, t < \tau_U\} \cap \bigl(\Omega_n \cap \{\sigma_{n+1} \leq \tfrac{1}{2}(\sigma_n + t)\}\bigr)\bigr]. \end{aligned} \tag{6.12}$$

Let $n \in \mathbb{N}$, set $\sigma_{n,t} := \sigma_n \wedge t$ and $\Omega'_n := \Omega_n \cap \{\sigma_n \leq t,\, \sigma_{n,t} \leq \tau_U\}$, so that $\Omega'_n \in \mathcal{F}_{\sigma_{n,t}}$ by $\Omega_n \in \mathcal{F}_{\sigma_n}$ and [30, Lemma 1.2.16]. Then $\{X_t \in A\} \subset \{\sigma_n \leq t\}$ by $A \subset B(y_0, r_n)$, clearly $\tau_U = \sigma_{n,t} + \tau_U \circ \theta_{\sigma_{n,t}}$ on $\{\sigma_{n,t} \leq \tau_U\}$, and by $\sigma_n \leq \sigma_{n+1}$ we also have $\sigma_{n+1} = \sigma_n + \sigma_{n+1} \circ \theta_{\sigma_n}$, which easily implies that $\{\sigma_n \leq t,\, \sigma_{n+1} \leq \frac{1}{2}(\sigma_n + t)\} = \{\sigma_n + 2\sigma_{n+1} \circ \theta_{\sigma_n} \leq t\}$. Therefore,

$$\begin{aligned} &\{X_t \in A,\, t < \tau_U\} \cap \bigl(\Omega_n \cap \{\sigma_{n+1} \leq \tfrac{1}{2}(\sigma_n + t)\} \\ &= \{X_t \in A\} \cap \Omega_n \cap \{\sigma_n \leq t < \tau_U,\, \sigma_{n+1} \leq \tfrac{1}{2}(\sigma_n + t)\} \\ &= \{X_t \in A\} \cap \Omega_n \cap \{\sigma_n \leq t < \tau_U,\, \sigma_{n,t} \leq \tau_U,\, \sigma_n + 2\sigma_{n+1} \circ \theta_{\sigma_n} \leq t\} \\ &= \{X_t \in A\} \cap \Omega'_n \cap \{\sigma_{n,t} + 2\sigma_{n+1} \circ \theta_{\sigma_{n,t}} \leq t < \sigma_{n,t} + \tau_U \circ \theta_{\sigma_{n,t}}\}. \end{aligned} \tag{6.13}$$

Noting that $(\sigma_{n,t}+2\sigma_{n+1}\circ\theta_{\sigma_{n,t}})\wedge(\sigma_{n,t}+\tau_U\circ\theta_{\sigma_{n,t}})=\sigma_{n,t}+((2\sigma_{n+1})\wedge\tau_U)\circ\theta_{\sigma_{n,t}}$, we see from (6.13), $\Omega'_n\in\mathcal{F}_{\sigma_{n,t}}$ and Proposition 3.4 that

$$\begin{aligned}
&\mathbb{P}_x\bigl[\{X_t\in A,\,t<\tau_U\}\cap\bigl(\Omega_n\cap\{\sigma_{n+1}\leq\tfrac{1}{2}(\sigma_n+t)\}\bigr)\bigr]\\
&=\mathbb{E}_x\bigl[\mathbf{1}_A(X_t)\mathbf{1}_{\Omega'_n\cap\{\sigma_{n,t}+2\sigma_{n+1}\circ\theta_{\sigma_{n,t}}\leq t<\sigma_{n,t}+\tau_U\circ\theta_{\sigma_{n,t}}\}}\bigr]\\
&=\mathbb{E}_x\bigl[\mathbf{1}_{\Omega'_n}\mathbf{1}_A(X_t)\bigl(\mathbf{1}_{\{t<\sigma_{n,t}+\tau_U\circ\theta_{\sigma_{n,t}}\}}-\mathbf{1}_{\{t<\sigma_{n,t}+((2\sigma_{n+1})\wedge\tau_U)\circ\theta_{\sigma_{n,t}}\}}\bigr)\bigr]\\
&=\mathbb{E}_x\bigl[\mathbf{1}_{\Omega'_n}\mathbb{E}_x\bigl[\mathbf{1}_A(X_t)\mathbf{1}_{\{t<\sigma_{n,t}+\tau_U\circ\theta_{\sigma_{n,t}}\}}\bigm|\mathcal{F}_{\sigma_{n,t}}\bigr]\bigr]\\
&\qquad-\mathbb{E}_x\bigl[\mathbf{1}_{\Omega'_n}\mathbb{E}_x\bigl[\mathbf{1}_A(X_t)\mathbf{1}_{\{t<\sigma_{n,t}+((2\sigma_{n+1})\wedge\tau_U)\circ\theta_{\sigma_{n,t}}\}}\bigm|\mathcal{F}_{\sigma_{n,t}}\bigr]\bigr]\\
&=\int_{\Omega'_n}\mathbb{E}_{X_{\sigma_{n,t}}(\omega)}\bigl[\mathbf{1}_A(X_{t-\sigma_{n,t}(\omega)})\mathbf{1}_{\{t-\sigma_{n,t}(\omega)<\tau_U\}}\bigr]\,d\mathbb{P}_x(\omega)\\
&\qquad-\int_{\Omega'_n}\mathbb{E}_{X_{\sigma_{n,t}}(\omega)}\bigl[\mathbf{1}_A(X_{t-\sigma_{n,t}(\omega)})\mathbf{1}_{\{t-\sigma_{n,t}(\omega)<(2\sigma_{n+1})\wedge\tau_U\}}\bigr]\,d\mathbb{P}_x(\omega)\\
&=\int_{\Omega'_n}\mathbb{E}_{X_{\sigma_{n,t}}(\omega)}\bigl[\mathbf{1}_A(X_{t-\sigma_{n,t}(\omega)})\bigl(\mathbf{1}_{\{t-\sigma_{n,t}(\omega)<\tau_U\}}-\mathbf{1}_{\{t-\sigma_{n,t}(\omega)<(2\sigma_{n+1})\wedge\tau_U\}}\bigr)\bigr]\,d\mathbb{P}_x(\omega)\\
&=\int_{\Omega'_n}\mathbb{E}_{X_{\sigma_{n,t}}(\omega)}\bigl[\mathbf{1}_A(X_{t-\sigma_{n,t}(\omega)})\mathbf{1}_{\{2\sigma_{n+1}\leq t-\sigma_{n,t}(\omega)<\tau_U\}}\bigr]\,d\mathbb{P}_x(\omega)\\
&=\int_{\Omega'_n\cap\{X_{\sigma_n}\in(\partial B(y_0,r_n))\setminus N\}}\mathbb{E}_{X_{\sigma_{n,t}}(\omega)}\bigl[\mathbf{1}_A(X_{t-\sigma_{n,t}(\omega)})\mathbf{1}_{\{2\sigma_{n+1}\leq t-\sigma_{n,t}(\omega)<\tau_U\}}\bigr]\,d\mathbb{P}_x(\omega),
\end{aligned}\tag{6.14}$$

where the equality in the last line follows since $\mathbf{1}_{\{\sigma_n\leq t\}}=\mathbf{1}_{\{\sigma_n\leq t,\,X_{\sigma_n}\in(\partial B(y_0,r_n))\setminus N\}}$ $\mathbb{P}_x$-a.s. by $x\in M\setminus(N\cup B(y_0,4r))$, $\mathbb{P}_x[X_0=x]=1$, (6.1), (6.8) and (4.5).

Let $\omega\in\Omega'_n\cap\{X_{\sigma_n}\in(\partial B(y_0,r_n))\setminus N\}$, set $s:=t-\sigma_{n,t}(\omega)$ and $z:=X_{\sigma_{n,t}}(\omega)$, so that $\sigma_{n,t}(\omega)=\sigma_n(\omega)$, $s=t-\sigma_n(\omega)\in[0,t]$ and $z=X_{\sigma_n}(\omega)\in(\partial B(y_0,r_n))\setminus N$ by $\sigma_n(\omega)\leq t$. The integrand in (6.14) is $\mathbb{E}_z[\mathbf{1}_A(X_s)\mathbf{1}_{\{2\sigma_{n+1}\leq s<\tau_U\}}]$, which is 0 if $s=0$ by $\mathbb{P}_z[X_0=z]=1$ and $z\notin B(y_0,r_n)\supset B(y_0,r)\supset A$. Assume $s>0$ and set $\sigma_{n+1,s}:=\sigma_{n+1}\wedge s$. Noting that $\tau_U=\sigma_{n+1,s}+\tau_U\circ\theta_{\sigma_{n+1,s}}$ on $\{\sigma_{n+1,s}\leq\tau_U\}$ and that $\{\sigma_{n+1}\leq\frac{s}{2},\,\sigma_{n+1,s}\leq\tau_U\}=\{\sigma_{n+1}\leq\tau_U\wedge\frac{s}{2}\}\in\mathcal{F}_{\sigma_{n+1,s}}$ by [30, Lemma 1.2.16], we see from Proposition 3.4 that

$$\begin{aligned}
&\mathbb{E}_z[\mathbf{1}_A(X_s)\mathbf{1}_{\{2\sigma_{n+1}\leq s<\tau_U\}}]\\
&=\mathbb{E}_z\bigl[\mathbf{1}_A(X_s)\mathbf{1}_{\{s<\tau_U\}}\mathbf{1}_{\{\sigma_{n+1}\leq s/2,\,\sigma_{n+1,s}\leq\tau_U\}}\bigr]\\
&=\mathbb{E}_z\bigl[\mathbf{1}_A(X_s)\mathbf{1}_{\{s<\sigma_{n+1,s}+\tau_U\circ\theta_{\sigma_{n+1,s}}\}}\mathbf{1}_{\{\sigma_{n+1}\leq\tau_U\wedge(s/2)\}}\bigr]\\
&=\mathbb{E}_z\bigl[\mathbf{1}_{\{\sigma_{n+1}\leq\tau_U\wedge(s/2)\}}\mathbb{E}_z\bigl[\mathbf{1}_A(X_s)\mathbf{1}_{\{s<\sigma_{n+1,s}+\tau_U\circ\theta_{\sigma_{n+1,s}}\}}\bigm|\mathcal{F}_{\sigma_{n+1,s}}\bigr]\bigr]\\
&=\int_{\{\sigma_{n+1}\leq\tau_U\wedge(s/2)\}}\mathbb{E}_{X_{\sigma_{n+1,s}}(\omega')}\bigl[\mathbf{1}_A(X_{s-\sigma_{n+1,s}(\omega')})\mathbf{1}_{\{s-\sigma_{n+1,s}(\omega')<\tau_U\}}\bigr]\,d\mathbb{P}_z(\omega')\\
&=\mathbb{E}_z\bigl[\mathbf{1}_{\{\sigma_{n+1}\leq\tau_U\wedge(s/2),\,X_{\sigma_{n+1}}\in(\partial B(y_0,r_{n+1}))\setminus N\}}\mathcal{P}^U_{s-\sigma_{n+1}}(X_{\sigma_{n+1}},A)\bigr],
\end{aligned}\tag{6.15}$$

where again the last equality follows since $\mathbf{1}_{\{\sigma_{n+1}\leq s/2\}}=\mathbf{1}_{\{\sigma_{n+1}\leq s/2,\,X_{\sigma_{n+1}}\in(\partial B(y_0,r_{n+1}))\setminus N\}}$ $\mathbb{P}_z$-a.s. by $z\in M\setminus(N\cup B(y_0,r_n))$, $\mathbb{P}_z[X_0=z]=1$, (6.1), (6.8) and (4.5).

Further let $\omega'\in\{\sigma_{n+1}\leq\tau_U\wedge\frac{s}{2},\,X_{\sigma_{n+1}}\in(\partial B(y_0,r_{n+1}))\setminus N\}$, set $u:=s-\sigma_{n+1}(\omega')$ and $w:=X_{\sigma_{n+1}}(\omega')$, so that $0<\frac{s}{2}\leq u\leq s\leq t<\Psi(R)$, $w\in(\partial B(y_0,r_{n+1}))\setminus N\subset U\setminus N$ and $d(w,x)\leq d(w,y_0)+d(y_0,x)=r_{n+1}+4r/\varepsilon<(2+4/\varepsilon)r$. Then by $(\mathrm{DB})_\Psi$ and the assumption that $t<\Psi((2+4/\varepsilon)r)$,

$$\frac{F_u(w,y)}{F_t(x,y)}\leq c_F\Bigl(\frac{t\vee\Psi(d(w,x))}{u}\Bigr)^{\alpha_F}\leq c_F\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{s/2}\Bigr)^{\alpha_F}$$

for any $y \in U$, which together with $A \subset U$ and $(\mathrm{DU})_F^{U,R}$ yields

$$\begin{aligned}\mathcal{P}^U_{s-\sigma_{n+1}(\omega')}(X_{\sigma_{n+1}}(\omega'), A) = \mathcal{P}^U_u(w, A) &\leq \int_A F_u(w, y)\, d\mu(y)\\ &\leq c_F\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{s/2}\Bigr)^{\alpha_F}\int_A F_t(x,y)\,d\mu(y). \quad (6.16)\end{aligned}$$

Recalling that $z = X_{\sigma_n}(\omega) \in (\partial B(y_0, r_n)) \setminus N$, we have $(z, r_n - r_{n+1}) \in (U \setminus N) \times (0, R)$, $B(z, r_n - r_{n+1}) \subset U$, and $\tau_{B(z,r_n-r_{n+1})} \leq \sigma_{n+1}$ by $B(y_0, r_{n+1}) \subset M \setminus B(z, r_n - r_{n+1})$. Therefore it follows from (6.15), (6.16) and $(\mathrm{P})_\Psi^{U,R}$ for $(z, r_n - r_{n+1})$ that

$$\begin{aligned}&\mathbb{E}_z[\mathbf{1}_A(X_s)\mathbf{1}_{\{2\sigma_{n+1}\leq s<\tau_U\}}]\\ &\leq c_F\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{s/2}\Bigr)^{\alpha_F}\mathbb{P}_z[\sigma_{n+1}\leq \tau_U\wedge \tfrac{s}{2}]\int_A F_t(x,y)\,d\mu(y)\\ &\leq c_F\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{s/2}\Bigr)^{\alpha_F}\mathbb{P}_z[\tau_{B(z,r_n-r_{n+1})}\leq \tfrac{s}{2}]\int_A F_t(x,y)\,d\mu(y)\\ &\leq cc_F\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{s/2}\Bigr)^{\alpha_F}\exp\bigl(-\Phi(\gamma(r_n-r_{n+1}),\tfrac{s}{2})\bigr)\int_A F_t(x,y)\,d\mu(y). \qquad (6.17)\end{aligned}$$

We easily see from $\omega \in \Omega'_n \subset \Omega_n$ and (6.10) that $0 < s = t - \sigma_n(\omega) \leq 2^{1-n}t$, and then by $t < \Psi((2+4/\varepsilon)r)$ we have $\Psi((2+4/\varepsilon)r)/(2^{n/2}s/2) \geq 2^{n/2} > 1$, which together with (5.13), $r_n - r_{n+1} = (1-2^{-1/(2\beta_2)})2^{-n/(2\beta_2)}r$, (5.10) and $1 < \beta_1 \leq \beta_2$ implies that

$$\begin{aligned}\Phi(\gamma(r_n-r_{n+1}),\tfrac{s}{2}) &\geq (c_\Psi 2^{\beta_1})^{-\frac{1}{\beta_1-1}}\min_{k\in\{1,2\}}\Bigl(\frac{\Psi(\gamma(r_n-r_{n+1}))}{s/2}\Bigr)^{\frac{1}{\beta_k-1}}\\ &\geq c_{\varepsilon,1}\min_{k\in\{1,2\}}\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{2^{n/2}s/2}\Bigr)^{\frac{1}{\beta_k-1}}\\ &= c_{\varepsilon,1}\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{2^{n/2}s/2}\Bigr)^{\frac{1}{\beta_2-1}} \geq c_{\varepsilon,1}2^{\frac{n}{2(\beta_2-1)}} \qquad (6.18)\end{aligned}$$

for some $c_{\varepsilon,1} \in (0,\infty)$ explicit in $c_\Psi, \beta_1, \beta_2, \gamma, \varepsilon$. (6.18) in turn yields

$$\begin{aligned}&\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{s/2}\Bigr)^{\alpha_F}\exp\bigl(-\Phi(\gamma(r_n-r_{n+1}),\tfrac{s}{2})\bigr)\\ &\leq \Bigl(\frac{\Psi((2+4/\varepsilon)r)}{s/2}\Bigr)^{\alpha_F}\exp\biggl(-c_{\varepsilon,1}\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{2^{n/2}s/2}\Bigr)^{\frac{1}{\beta_2-1}}\biggr)\\ &\leq \Bigl(\frac{\Psi((2+4/\varepsilon)r)}{2^{n/2}s/2}\Bigr)^{\alpha_F}2^{\alpha_F n/2}\exp\biggl(-\frac{c_{\varepsilon,1}}{2}\Bigl(\frac{\Psi((2+4/\varepsilon)r)}{2^{n/2}s/2}\Bigr)^{\frac{1}{\beta_2-1}}-\frac{c_{\varepsilon,1}}{2}2^{\frac{n}{2(\beta_2-1)}}\biggr)\\ &\leq c_{\varepsilon,2}2^{-\alpha_F n/2}, \qquad (6.19)\end{aligned}$$

where $c_{\varepsilon,2} := 2^{5\alpha_F(\beta_2-1)}\bigl(\alpha_F(\beta_2-1)/(ec_{\varepsilon,1})\bigr)^{3\alpha_F(\beta_2-1)}$. By (6.17) and (6.19),

$$\mathbb{E}_z[\mathbf{1}_A(X_s)\mathbf{1}_{\{2\sigma_{n+1}\leq s<\tau_U\}}] \leq \frac{cc_F c_{\varepsilon,2}}{2^{\alpha_F n/2}}\int_A F_t(x,y)\,d\mu(y) \qquad (6.20)$$

for $s = t - \sigma_{n,t}(\omega)$ and $z = X_{\sigma_{n,t}}(\omega)$ for any $\omega \in \Omega'_n \cap \{X_{\sigma_n} \in (\partial B(y_0, r_n)) \setminus N\}$, and therefore from (6.14), (6.20) and $\Omega'_n \subset \{\sigma_n \leq t\}$ we obtain

$$\begin{aligned}&\mathbb{P}_x\bigl[\{X_t\in A,\, t<\tau_U\}\cap\bigl(\Omega_n\cap\{\sigma_{n+1}\leq \tfrac{1}{2}(\sigma_n+t)\}\bigr)\bigr]\\ &\qquad\leq \frac{cc_F c_{\varepsilon,2}}{2^{\alpha_F n/2}}\mathbb{P}_x[\sigma_n\leq t]\int_A F_t(x,y)\,d\mu(y). \qquad (6.21)\end{aligned}$$

To conclude (6.7) from (6.12) and (6.21), we show that

$$\mathbb{P}_x[\sigma_n \leq t] \leq c\exp(-\Phi(\gamma r, t)). \tag{6.22}$$

Indeed, setting $\sigma := \dot{\sigma}_{B(y_0,3r)}$, we have $\sigma \leq \sigma_n$ by $B(y_0, r_n) \subset B(y_0, 3r)$ and hence $\sigma_n = \sigma + \sigma_n \circ \theta_\sigma$. Therefore $\{\sigma_n \leq t\} \subset \{\sigma \leq t,\, \sigma_n \circ \theta_\sigma \leq t\}$, and then by the strong Markov property [13, Theorem A.1.21] of X at time σ,

$$\begin{aligned}\mathbb{P}_x[\sigma_n \leq t] \leq \mathbb{P}_x[\sigma \leq t,\, \sigma_n \circ \theta_\sigma \leq t] &= \mathbb{E}_x[\mathbf{1}_{\{\sigma\leq t\}}(\mathbf{1}_{\{\sigma_n\leq t\}} \circ \theta_\sigma)] \\ &= \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma\leq t\}}\mathbb{E}_{X_\sigma}[\mathbf{1}_{\{\sigma_n\leq t\}}]\bigr] \\ &= \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma\leq t,\, X_\sigma\in(\partial B(y_0,3r))\setminus N\}}\mathbb{P}_{X_\sigma}[\sigma_n \leq t]\bigr],\end{aligned} \tag{6.23}$$

where the last equality follows since $\mathbf{1}_{\{\sigma\leq t\}} = \mathbf{1}_{\{\sigma\leq t,\, X_\sigma\in(\partial B(y_0,3r))\setminus N\}}$ $\mathbb{P}_x$-a.s. by $x \in M \setminus (N \cup B(y_0, 4r))$, $\mathbb{P}_x[X_0 = x] = 1$, (6.1), (6.8) and (4.5). Moreover, for $z \in (\partial B(y_0, 3r)) \setminus N$, $B(y_0, r_n) \subset M \setminus B(z, r)$ by $r_n < 2r$, hence $\sigma_n \geq \tau_{B(z,r)}$, and therefore noting that $(z, r) \in (U \setminus N) \times (0, R)$ and that $B(z, r) \subset B(y_0, 4r) \subset U$, we see from $(\mathrm{P})^{U,R}_{\Psi}$ for (z, r) that $\mathbb{P}_z[\sigma_n \leq t] \leq \mathbb{P}_z[\tau_{B(z,r)} \leq t] \leq c\exp(-\Phi(\gamma r, t))$, which together with (6.23) yields (6.22).

Now (6.7) with $c'_\varepsilon := c^2 c_F c_{\varepsilon,2}/(2^{\alpha_F/2} - 1)$ is immediate from (6.12), (6.21), (6.22) and the fact that $d(x, y) \leq d(x, y_0) + d(y_0, y) < 4r/\varepsilon + r < 5r/\varepsilon$ for any $y \in A$ by $A \subset B(y_0, r)$.

Finally, we prove (6.7) for general $A \in \mathcal{B}(U^\circ_{\varepsilon R})$. Note that (6.7) holds for $A = \{x\}$ by $(\mathrm{DU})^{U,R}_F$. In particular, (6.7) is valid for any $A \in \mathcal{B}(U^\circ_{\varepsilon R})$ if $U^\circ_{\varepsilon R} \setminus \{x\} = \emptyset$, and thus we may assume $U^\circ_{\varepsilon R} \setminus \{x\} \neq \emptyset$. Let $\{y_k\}_{k\in\mathbb{N}}$ be a countable dense subset of $U^\circ_{\varepsilon R} \setminus \{x\}$ and set $B_0 := U^\circ_{\varepsilon R} \cap \{x\}$, $B_1 := B(y_1, \frac{1}{4}\varepsilon d(x, y_1))$ and $B_k := B(y_k, \frac{1}{4}\varepsilon d(x, y_k)) \setminus \bigcup_{j=1}^{k-1} B(y_j, \frac{1}{4}\varepsilon d(x, y_j))$ for $k \in \mathbb{N} \setminus \{1\}$. Then $\{B_k\}_{k\in\mathbb{N}\cup\{0\}} \subset \mathcal{B}(U)$, and it is easy to see that $U^\circ_{\varepsilon R} \subset \bigcup_{k\in\mathbb{N}\cup\{0\}} B_k$, where the union is disjoint. Now for any $A \in \mathcal{B}(U^\circ_{\varepsilon R})$, since $A \cap B_0 \in \{\emptyset, \{x\}\}$ and $A \cap B_k \in \mathcal{B}\bigl(B(y_k, \frac{1}{4}\varepsilon d(x, y_k))\bigr)$ for $k \in \mathbb{N}$, we have already proved (6.7) with $A \cap B_k$ in place of A for any $k \in \mathbb{N} \cup \{0\}$, and therefore

$$\begin{aligned}\mathcal{P}^U_t(x, A) = \mathcal{P}^U_t\Bigl(x, \bigcup\nolimits_{k\in\mathbb{N}\cup\{0\}}(A \cap B_k)\Bigr) &= \sum_{k\in\mathbb{N}\cup\{0\}} \mathcal{P}^U_t(x, A \cap B_k) \\ &\leq \sum_{k\in\mathbb{N}\cup\{0\}} \int_{A\cap B_k} c'_\varepsilon F_t(x, y)\exp\bigl(-\Phi(\gamma_\varepsilon d(x, y), t)\bigr)\, d\mu(y) \\ &= \int_A c'_\varepsilon F_t(x, y)\exp\bigl(-\Phi(\gamma_\varepsilon d(x, y), t)\bigr)\, d\mu(y)\end{aligned} \tag{6.24}$$

by monotone convergence, completing the proof of Proposition 6.5. □

Theorems 6.2 and 6.4 are easy consequences of Propositions 5.6, 6.5, and 6.6 below.

Proposition 6.6. *Under the same assumptions as those of Theorem 6.2, there exists $c''_\varepsilon \in (0, \infty)$ explicit in $c_\Psi, \beta_1, \beta_2, c_F, \alpha_F, c, \gamma, \varepsilon$ such that, with $\gamma_\varepsilon := \frac{1}{5}\varepsilon\gamma$, for any $(t, x) \in (0, \Psi(R)) \times (M \setminus N)$ and any $A \in \mathcal{B}(U^\circ_{\varepsilon R})$,*

$$\mathcal{P}_t(x, A) \leq \mathcal{P}^U_t(x, A) + c''_\varepsilon(\inf\nolimits_{U\times U} F_t)\exp(-\Phi(\gamma_\varepsilon R, t))\mu(A). \tag{6.25}$$

Proof. If $U = M$, then (6.25) is trivially valid since $X_t = X^U_t$ and hence $\mathcal{P}_t = \mathcal{P}^U_t$ for any $t \in [0, \infty)$. Therefore we may assume $U \neq M$. Set $B := U^\circ_{(\varepsilon/2)R}$, so that B is open in M and $\overline{B} \subset U$, and define $\mathcal{F}_*$-stopping times τ_n and σ_n, $n \in \mathbb{N}$, by (3.2). For each $n \in \mathbb{N}$, as noted at the beginning of the proof of Theorem 3.3, on $\{\sigma_n < \infty\}$ we have $X_{\sigma_n} \in \overline{B} \subset U$, $\tau_n \leq \sigma_n < \zeta$, hence $X_{\tau_n} \in M \setminus U$ and $\tau_n < \sigma_n$ by the sample path right-continuity of X, and we also easily see that

$$X_{\sigma_n} \in (\partial B) \setminus N \quad \text{on} \quad \{\sigma_n < \infty = \dot{\sigma}_N,\, [0, \zeta) \ni t \mapsto X_t \in M \text{ is continuous}\}. \tag{6.26}$$

Let $(t,x) \in (0,\Psi(R)) \times (M \setminus N)$ and $A \in \mathcal{B}(U^{\circ}_{\varepsilon R})$. Since $\mathbf{1}_A|_{M\setminus B} = 0$ by $A \subset U^{\circ}_{\varepsilon R} \subset B$, from Theorem 3.3 with $u = \mathbf{1}_A$ we obtain

$$\mathcal{P}_t(x,A) = \mathcal{P}^U_t(x,A) + \sum_{n\in\mathbb{N}} \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma_n \le t\}} \mathcal{P}^U_{t-\sigma_n}(X_{\sigma_n}, A)\bigr]. \tag{6.27}$$

Noting (6.26), to estimate each term of the series in (6.27) let $s \in [0,t]$, $z \in (\partial B) \setminus N$ and let $c'_\varepsilon \in (0,\infty)$ and $\gamma_\varepsilon = \frac{1}{5}\varepsilon\gamma$ be as in Proposition 6.5. We claim that

$$\mathcal{P}^U_s(z,A) \le c'_\varepsilon c_F c_{\varepsilon,3} (\inf\nolimits_{U\times U} F_t) \mu(A) \tag{6.28}$$

for some $c_{\varepsilon,3} \in (0,\infty)$ explicit in $c_\Psi, \beta_1, \beta_2, \alpha_F, \gamma, \varepsilon$. Indeed, (6.28) trivially holds for $s = 0$ since $\mathcal{P}^U_0(z,A) = \mathbb{P}_z[X_0 \in A,\, 0 < \tau_U] = 0$ by $\mathbb{P}_z[X_0 = z] = 1$ and $z \notin B \supset A$, and thus we may assume $s \in (0,t]$. Then $s \in (0,\Psi(R))$, $z \in U \setminus N$ by $\overline{B} \subset U$, and hence an application of Proposition 6.5 yields (6.7) with (s,z) in place of (t,x). Let $y \in U^{\circ}_{\varepsilon R}$ and $x_0, y_0 \in U$. By $(\mathrm{DB})_\Psi$, $0 < s \le t < \Psi(R)$ and $\operatorname{diam} U \le R$ we have $F_s(z,y) \le c_F(\Psi(R)/s)^{\alpha_F} F_t(x_0,y_0)$, and furthermore we easily see from $z \in \partial B = \partial U^{\circ}_{(\varepsilon/2)R}$ that $d(z,y) > \frac{1}{2}\varepsilon R$, so that $\exp\bigl(-\Phi(\gamma_\varepsilon d(z,y), s)\bigr) \le \exp\bigl(-\Phi(\frac{1}{2}\varepsilon\gamma_\varepsilon R, s)\bigr)$ by the monotonicity of $\Phi(\cdot, s)$. These facts, (5.13) and (5.10) together imply that

$$\begin{aligned}
&F_s(z,y)\exp\bigl(-\Phi(\gamma_\varepsilon d(z,y), s)\bigr)\\
&\le c_F\Bigl(\frac{\Psi(R)}{s}\Bigr)^{\alpha_F} F_t(x_0,y_0)\exp\bigl(-\Phi(\tfrac{1}{2}\varepsilon\gamma_\varepsilon R, s)\bigr)\\
&\le c_F\Bigl(\frac{\Psi(R)}{s}\Bigr)^{\alpha_F} F_t(x_0,y_0)\exp\biggl(-(c_\Psi 2^{\beta_1})^{-\frac{1}{\beta_1-1}} \min_{k\in\{1,2\}}\Bigl(\frac{\Psi(\frac{1}{2}\varepsilon\gamma_\varepsilon R)}{s}\Bigr)^{\frac{1}{\beta_k-1}}\biggr)\\
&\le c_F c_{\varepsilon,3} F_t(x_0,y_0),
\end{aligned}$$

and taking the infimum in $(x_0,y_0) \in U \times U$ shows that for any $y \in U^{\circ}_{\varepsilon R}$,

$$F_s(z,y)\exp\bigl(-\Phi(\gamma_\varepsilon d(z,y), s)\bigr) \le c_F c_{\varepsilon,3}(\inf\nolimits_{U\times U} F_t). \tag{6.29}$$

Then (6.28) is immediate from (6.7) with (s,z) in place of (t,x), $A \subset U^{\circ}_{\varepsilon R}$ and (6.29).

Let $n \in \mathbb{N}$. By (4.5), (6.1), (6.26) and (6.28),

$$\begin{aligned}
\mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma_n \le t\}}\mathcal{P}^U_{t-\sigma_n}(X_{\sigma_n}, A)\bigr] &= \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma_n \le t,\, X_{\sigma_n} \in (\partial B)\setminus N\}}\mathcal{P}^U_{t-\sigma_n}(X_{\sigma_n}, A)\bigr]\\
&\le c'_\varepsilon c_F c_{\varepsilon,3}(\inf\nolimits_{U\times U} F_t)\mu(A)\mathbb{P}_x[\sigma_n \le t],
\end{aligned} \tag{6.30}$$

and we need to estimate $\mathbb{P}_x[\sigma_n \le t]$. Recall that $\tau_n \le \sigma_n \le \tau_{n+1} = \sigma_n + \tau_U \circ \theta_{\sigma_n}$ as mentioned in the proof of Theorem 3.3. Assume $n \ge 2$. For each $\omega \in \{\sigma_n \le t\}$, since $0 \le \sigma_k(\omega) \le \sigma_n(\omega) \le t$ for any $k \in \{1,\ldots,n\}$, we have $t \ge \sigma_n(\omega) \ge \sigma_n(\omega) - \sigma_1(\omega) = \sum_{k=1}^{n-1}(\sigma_{k+1}(\omega) - \sigma_k(\omega))$ and therefore $\tau_U \circ \theta_{\sigma_k}(\omega) = \tau_{k+1}(\omega) - \sigma_k(\omega) \le \sigma_{k+1}(\omega) - \sigma_k(\omega) \le \frac{t}{n-1}$ for some $k \in \{1,\ldots,n-1\}$. Thus $\{\sigma_n \le t\} \subset \bigcup_{k=1}^{n-1}\{\sigma_k \le t,\, \tau_U \circ \theta_{\sigma_k} \le \frac{t}{n-1}\}$ and hence

$$\mathbb{P}_x[\sigma_n \le t] \le \mathbb{P}_x\Biggl[\bigcup_{k=1}^{n-1}\{\sigma_k \le t,\, \tau_U\circ\theta_{\sigma_k} \le \tfrac{t}{n-1}\}\Biggr] \le \sum_{k=1}^{n-1}\mathbb{P}_x[\sigma_k \le t,\, \tau_U\circ\theta_{\sigma_k} \le \tfrac{t}{n-1}]. \tag{6.31}$$

Furthermore by using first the strong Markov property [13, Theorem A.1.21] of X at time σ_k and then (4.5), (6.1) and (6.26) we see that for any $k \in \{1,\ldots,n-1\}$,

$$\begin{aligned}
\mathbb{P}_x[\sigma_k \le t,\, \tau_U\circ\theta_{\sigma_k} \le \tfrac{t}{n-1}] &= \mathbb{E}_x[\mathbf{1}_{\{\sigma_k \le t\}}(\mathbf{1}_{\{\tau_U \le t/(n-1)\}}\circ\theta_{\sigma_k})]\\
&= \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma_k \le t\}}\mathbb{P}_{X_{\sigma_k}}[\tau_U \le \tfrac{t}{n-1}]\bigr]\\
&= \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma_k \le t,\, X_{\sigma_k}\in(\partial B)\setminus N\}}\mathbb{P}_{X_{\sigma_k}}[\tau_U \le \tfrac{t}{n-1}]\bigr]\\
&\le c\exp\bigl(-\Phi(\tfrac{1}{2}\varepsilon\gamma R, \tfrac{t}{n-1})\bigr);
\end{aligned} \tag{6.32}$$

here the last inequality follows from the fact that for any $z \in (\partial B) \setminus N$, $\tau_{B(z,\varepsilon R/2)} \leq \tau_U$ by $B(z, \frac{1}{2}\varepsilon R) \subset U$ and hence $\mathbb{P}_z[\tau_U \leq \frac{t}{n-1}] \leq \mathbb{P}_z[\tau_{B(z,\varepsilon R/2)} \leq \frac{t}{n-1}] \leq c\exp\bigl(-\Phi(\frac{1}{2}\varepsilon\gamma R, \frac{t}{n-1})\bigr)$ by $(\mathrm{P})_\Psi^{U,R}$ for $(z, \frac{1}{2}\varepsilon R)$. Also, by the monotonicity of $\Phi(\cdot, \frac{t}{n-1})$ and $\Phi(\gamma_\varepsilon R, \cdot)$, (5.12), (5.13), (5.10), $1 < \beta_1 \leq \beta_2$ and $t < \Psi(R)$, for some $c_{\varepsilon,4} \in (0,\infty)$ explicit in $c_\Psi, \beta_1, \beta_2, \gamma_\varepsilon$,

$$\begin{aligned}\Phi(\tfrac{1}{2}\varepsilon\gamma R, \tfrac{t}{n-1}) &\geq \Phi(2\gamma_\varepsilon R, \tfrac{t}{n-1}) \geq 2\Phi(\gamma_\varepsilon R, \tfrac{t}{n-1}) \\ &\geq \Phi(\gamma_\varepsilon R, t) + (c_\Psi 2^{\beta_1})^{-\frac{1}{\beta_1-1}} \min_{k\in\{1,2\}} \Bigl(\frac{\Psi(\gamma_\varepsilon R)}{t/(n-1)}\Bigr)^{\frac{1}{\beta_k-1}} \\ &\geq \Phi(\gamma_\varepsilon R, t) + c_{\varepsilon,4}(n-1)^{\frac{1}{\beta_2-1}}. \end{aligned} \tag{6.33}$$

From (6.31), (6.32) and (6.33) we conclude that for any $n \in \mathbb{N} \setminus \{1\}$,

$$\begin{aligned}\mathbb{P}_x[\sigma_n \leq t] &\leq c(n-1)e^{-c_{\varepsilon,4}(n-1)^{\frac{1}{\beta_2-1}}} \exp(-\Phi(\gamma_\varepsilon R, t)) \\ &\leq cc_{\varepsilon,5}(n-1)^{-2}\exp(-\Phi(\gamma_\varepsilon R, t)),\end{aligned} \tag{6.34}$$

where $c_{\varepsilon,5} := \bigl(3(\beta_2-1)/(ec_{\varepsilon,4})\bigr)^{3(\beta_2-1)}$. For $\mathbb{P}_x[\sigma_1 \leq t]$, set $B' := U^\circ_{(\varepsilon/4)R}$ and $\sigma := \dot\sigma_{B',\tau_U} = \dot\sigma_{B',\tau_1}$ (recall Definition 3.1), so that we have (6.26) with B' and σ in place of B and σ_n, respectively, by substituting $\frac{1}{2}\varepsilon$ for ε. Noting that $\sigma_1 = \sigma + \dot\sigma_B \circ \theta_\sigma$ by $B \subset B'$ and thus that $\{\sigma_1 \leq t\} \subset \{\sigma \leq t,\, \dot\sigma_B \circ \theta_\sigma \leq t\}$, from the strong Markov property [13, Theorem A.1.21] of X at time σ, (4.5), (6.1) and (6.26) we obtain

$$\begin{aligned}\mathbb{P}_x[\sigma_1 \leq t] \leq \mathbb{P}_x[\sigma \leq t,\, \dot\sigma_B \circ \theta_\sigma \leq t] &= \mathbb{E}_x[\mathbf{1}_{\{\sigma\leq t\}}(\mathbf{1}_{\{\dot\sigma_B \leq t\}} \circ \theta_\sigma)] \\ &= \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma\leq t\}}\mathbb{P}_{X_\sigma}[\dot\sigma_B \leq t]\bigr] \\ &= \mathbb{E}_x\bigl[\mathbf{1}_{\{\sigma\leq t,\, X_\sigma \in (\partial B')\setminus N\}}\mathbb{P}_{X_\sigma}[\dot\sigma_B \leq t]\bigr] \\ &\leq c\exp(-\Phi(\gamma_\varepsilon R, t));\end{aligned} \tag{6.35}$$

here, similarly to (6.32), the last inequality holds since for any $z \in (\partial B') \setminus N$, $\tau_{B(z,\varepsilon R/4)} \leq \dot\sigma_B$ by $B(z, \frac{1}{4}\varepsilon R) \subset U \setminus B$ and hence $\mathbb{P}_z[\dot\sigma_B \leq t] \leq \mathbb{P}_z[\tau_{B(z,\varepsilon R/4)} \leq t] \leq c\exp(-\Phi(\gamma_\varepsilon R, t))$ by $(\mathrm{P})_\Psi^{U,R}$ for $(z, \frac{1}{4}\varepsilon R)$ and the monotonicity of $\Phi(\cdot, t)$.

Now (6.25) with $c''_\varepsilon := cc'_\varepsilon c_F c_{\varepsilon,3}(2c_{\varepsilon,5}+1)$ is immediate from (6.27), (6.30), (6.34) and (6.35), completing the proof of Proposition 6.6. □

Proof of Theorem 6.2. Let $c'_\varepsilon, c''_\varepsilon \in (0,\infty)$ be as in Propositions 6.5 and 6.6, respectively, and let $\gamma_\varepsilon := \frac{1}{5}\varepsilon\gamma$. We show that Theorem 6.2 can be concluded from Proposition 5.6 applied to $I = (0,\infty)$, $V = M \setminus N$, $W = U^\circ_{\varepsilon R}$, M in place of U, and $H = H_t(x,y) : (0,\infty) \times (M \setminus N) \times U^\circ_{\varepsilon R} \to [0,\infty)$ given by

$$H_t(x,y) := \begin{cases} (c'_\varepsilon + c''_\varepsilon)F_t(x,y)\exp\bigl(-\Phi(\gamma_\varepsilon d(x,y), t)\bigr) & \text{if } t < \Psi(R) \text{ and } x \in U, \\ c''_\varepsilon c_F 2^{\alpha_F}(\inf_{U\times U} F_{\Psi(R)/2^n})\exp(-\Phi(\gamma_\varepsilon R, t)) & \text{if } \frac{\Psi(R)}{2^{n+1}} \leq t < \frac{\Psi(R)}{2^n} \text{ and } x \notin U, \\ c_\varepsilon(\inf_{U\times U} F_{\Psi(R)}) & \text{if } t \geq \Psi(R), \end{cases} \tag{6.36}$$

where $n \in \mathbb{N} \cup \{0\}$ in the second line and $c_\varepsilon := ((c'_\varepsilon + c''_\varepsilon)c_F 2^{\alpha_F}) \vee (c''_\varepsilon c_F^2 2^{2\alpha_F})$. Obviously $H = H_t(x,y)$ is Borel measurable, and by using $(\mathrm{DB})_\Psi$ it is easily seen to be less than or equal to the right-hand side of (6.5), so that it remains to verify that

$$\mathcal{P}_t(x,A) \leq \int_A H_t(x,y)\,d\mu(y) \tag{6.37}$$

for any $(t,x) \in (0,\infty)\times(M\setminus N)$ and any $A \in \mathcal{B}(U^\circ_{\varepsilon R})$. Note that by $(\mathrm{DB})_\Psi$ and $\operatorname{diam} U \leq R$ we also have

$$H_{\Psi(R)/2}(z,y) \leq c_\varepsilon(\inf_{U\times U} F_{\Psi(R)}) \qquad \text{for any } (z,y) \in (M \setminus N) \times U^\circ_{\varepsilon R}. \tag{6.38}$$

Let $(t,x) \in (0,\infty)\times(M\setminus N)$ and $A \in \mathcal{B}(U^{\circ}_{\varepsilon R})$. If $t < \Psi(R)$ and $x \in U$, then (6.37) easily follows from Propositions 6.5 and 6.6 in view of the fact that $\Phi(\gamma_\varepsilon R, t) \geq \Phi(\gamma_\varepsilon d(x,y), t)$ for any $y \in U^{\circ}_{\varepsilon R}$ by $\operatorname{diam} U \leq R$ and the monotonicity of $\Phi(\cdot, t)$. If $t < \Psi(R)$ and $x \notin U$, then we see from $(\mathrm{DB})_\Psi$ that $c''_\varepsilon(\inf_{U\times U} F_t)\exp(-\Phi(\gamma_\varepsilon R, t)) \leq H_t(x,y)$ for any $y \in U^{\circ}_{\varepsilon R}$, which together with Proposition 6.6 and $\mathcal{P}^U_t(x,A) = 0$ immediately implies (6.37).

Now assume $t \geq \Psi(R)$. Since $\mathcal{P}_{t-\Psi(R)/2}(x,N) = \mathbb{P}_x[X_{t-\Psi(R)/2} \in N] = 0$ by (4.5) and

$$\mathcal{P}_{\Psi(R)/2}(z,A) \leq \int_A H_{\Psi(R)/2}(z,y)\,d\mu(y) \leq \int_A H_t(x,y)\,d\mu(y)$$

for any $z \in M \setminus N$ by the previous paragraph, (6.38) and (6.36), from (2.3) we get

$$\begin{aligned}\mathcal{P}_t(x,A) = \mathcal{P}_{t-\Psi(R)/2}(\mathcal{P}_{\Psi(R)/2}\mathbf{1}_A)(x) &= \int_{M\setminus N} \mathcal{P}_{\Psi(R)/2}(z,A)\mathcal{P}_{t-\Psi(R)/2}(x,dz)\\ &\leq \int_{M\setminus N}\Bigl(\int_A H_t(x,y)\,d\mu(y)\Bigr)\mathcal{P}_{t-\Psi(R)/2}(x,dz)\\ &\leq \int_A H_t(x,y)\,d\mu(y).\end{aligned}$$

Thus (6.37) has been proved and hence Theorem 6.2 follows from Proposition 5.6. □

Proof of Theorem 6.4. Define $H = H_t(x,y) : (0,\infty)\times(M\setminus N)\times M \to [0,\infty)$ by the right-hand side of (6.6), so that it is clearly Borel measurable. Thanks to Proposition 5.6, it suffices to show (6.37) for any $(t,x) \in (0,\infty)\times(M\setminus N)$ and any $A \in \mathcal{B}(M)$ for some $c' \in (0,\infty)$ explicit in $c_\Psi, \beta_1, \beta_2, c_F, \alpha_F, c, \gamma$. For applications of Theorem 6.2 and Proposition 6.6, we remark that for any $(y_0, R') \in M \times (0,\infty)$,

$$\text{if we set}\quad U := B(y_0, \tfrac{R'}{2}) \quad\text{then}\quad \operatorname{diam} U \leq R' \quad\text{and}\quad B(y_0, \tfrac{R'}{4}) \subset U^{\circ}_{(1/4)R'}. \tag{6.39}$$

Let $(t,x) \in (0,\infty)\times(M\setminus N)$. If $R = \infty$, then for any $A \in \mathcal{B}(M)$ and any $n \in \mathbb{N}$ with $n > \Psi^{-1}(t)$, in view of (6.39) we can apply Theorem 6.2 with $n, B(x,\frac{n}{2}), \frac{1}{4}$ in place of R, U, ε respectively and $A \cap B(x,\frac{n}{4})$ in place of A in (6.4) and obtain

$$\mathcal{P}_t(x, A\cap B(x,\tfrac{n}{4})) \leq \int_{A\cap B(x,n/4)} H_t(x,y)\,d\mu(y)$$

with $c' = c_{1/4}$, which yields (6.37) by using monotone convergence to let $n \to \infty$.

Thus we may assume $R < \infty$. Let $y_0 \in M$ and $A \in \mathcal{B}(B(y_0,\frac{R}{4}))$. We claim that (6.37) holds for such A. Indeed, setting $U := B(y_0, \frac{R}{2})$, we have (6.37) with $H_t(x,y)$ replaced by

$$\widetilde{H}_t(x,y) := \begin{cases} c_{1/4}F_t(x,y)\exp\bigl(-\Phi(\frac{1}{20}\gamma d(x,y),t)\bigr) & \text{if } t < \Psi(R) \text{ and } x \in U,\\ c''_{1/4}(\inf_{U\times U} F_t)\exp(-\Phi(\frac{1}{20}\gamma R, t)) & \text{if } t < \Psi(R) \text{ and } x \notin U,\\ c_{1/4}(\inf_{U\times U} F_{\Psi(R)}) & \text{if } t \geq \Psi(R)\end{cases} \tag{6.40}$$

since Theorem 6.2 and Proposition 6.6 with $\varepsilon = \frac{1}{4}$ are applicable by (6.39) and $\mathcal{P}^U_t(x,A) = 0$ if $x \notin U$. Moreover, if $t \leq \Psi(R)$ then for any $y \in M$ and any $z, w \in U$,

$$\begin{aligned}\inf\nolimits_{U\times U} F_t \leq F_t(z,w) &\leq c_F\Bigl(\frac{t \vee \Psi(d(x,z)) \vee \Psi(d(y,w))}{t}\Bigr)^{\alpha_F} F_t(x,y)\\ &\leq c_F\Bigl(\frac{\Psi(\delta^{-1}R)}{t}\Bigr)^{\alpha_F} F_t(x,y)\\ &\leq c_F c_\Psi^{\alpha_F}\delta^{-\beta_2\alpha_F}\Bigl(\frac{\Psi(R)}{t}\Bigr)^{\alpha_F} F_t(x,y)\end{aligned} \tag{6.41}$$

by $(\mathrm{DB})_\Psi$, $\operatorname{diam} M \leq \delta^{-1}R$ and (5.10) (even if $x \in N$), hence

$$c_{1/4}(\inf\nolimits_{U\times U} F_{\Psi(R)}) \leq c_{1/4}c_F c_\Psi^{\alpha_F}\delta^{-\beta_2\alpha_F}(\inf\nolimits_{M\times M} F_{\Psi(R)}), \tag{6.42}$$

and we also easily see from (6.41), (5.12), (5.13), (5.10) and $R \geq \delta \operatorname{diam} M$ that

$$\begin{aligned}
&c''_{1/4}(\inf_{U\times U} F_t)\exp(-\Phi(\tfrac{1}{20}\gamma R,t)) \\
&\leq c''_{1/4}c_F c_\Psi^{\alpha_F}\delta^{-\beta_2\alpha_F}\Bigl(\frac{\Psi(R)}{t}\Bigr)^{\alpha_F}F_t(x,y)\exp(-2\Phi(\tfrac{1}{40}\gamma R,t)) \\
&\leq c''\delta^{-\beta_2\alpha_F}F_t(x,y)\exp\bigl(-\Phi(\gamma'_\delta d(x,y),t)\bigr) \qquad (6.43)
\end{aligned}$$

for any $y \in M$ for some $c'' \in (0,\infty)$ explicit in $c_\Psi, \beta_1, \beta_2, c_F, \alpha_F, c, \gamma$. Therefore putting $c' := c'' \vee (c_{1/4}c_F c_\Psi^{\alpha_F})$, we have $\widetilde{H}_t(x,y) \leq H_t(x,y)$ for any $y \in M$ by (6.40), (6.42) and (6.43), and thus the inequality (6.37) follows from that with $\widetilde{H}_t(x,y)$ in place of $H_t(x,y)$.

Now let $\{y_k\}_{k\in\mathbb{N}}$ be a countable dense subset of M and set $B_1 := B(y_1, \frac{R}{4})$ and $B_k := B(y_k, \frac{R}{4}) \setminus \bigcup_{j=1}^{k-1} B(y_j, \frac{R}{4})$ for $k \in \mathbb{N}\setminus\{1\}$, so that $\{B_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(M)$ and $M = \bigcup_{k\in\mathbb{N}} B_k$, where the union is disjoint. Then for any $A \in \mathcal{B}(M)$, for each $k \in \mathbb{N}$ we have (6.37) with $A \cap B_k$ in place of A by the previous paragraph and $A \cap B_k \in \mathcal{B}(B(y_k, \frac{R}{4}))$, from which (6.37) follows in exactly the same way as (6.24), completing the proof of Theorem 6.4. □

7. Exit probability estimates for diffusions

As already mentioned at the beginning of Section 6, the purpose of this section is to provide reasonable sufficient conditions for the exit probability estimate $(\mathrm{P})_\Psi^{U,R}$ of Theorem 6.2. Recall that since Section 6 we have fixed $\Psi, c_\Psi, \beta_1, \beta_2$ and $\Phi = \Phi_\Psi$ as in Lemma 5.7.

In this section, we fix an arbitrary properly exceptional set $N \in \mathcal{B}(M)$ for X satisfying

$$\textit{both} \quad (6.1) \quad \textit{and} \quad \mathbb{P}_x[\zeta < \infty,\, X_{\zeta-} \in M] = 0 \qquad (7.1)$$

for any $x \in M \setminus N$, where $X_{\zeta-}(\omega) := X_{\zeta(\omega)-}(\omega)$ ($X_{0-} := X_0$, $X_{\infty-} := \Delta = X_\infty$), so that $X_{\zeta-} : \Omega \to M_\Delta$ is $\mathcal{F}_\infty/\mathcal{B}(M_\Delta)$-measurable by the left-continuity of $[0,\infty) \ni t \mapsto X_{t-}(\omega) \in M_\Delta$. By [19, Theorem 4.5.3], such N exists if and only if $(\mathcal{E},\mathcal{F})$ is *strongly local*, i.e., $\mathcal{E}(u,v) = 0$ for any $u, v \in \mathcal{F}$ with $\operatorname{supp}_\mu[u], \operatorname{supp}_\mu[v]$ compact and $u = c$ μ-a.e. on a neighborhood of $\operatorname{supp}_\mu[v]$ for some $c \in \mathbb{R}$.

Below we will also consider the situation where the set N fixed above satisfies

$$\text{both} \quad (6.1) \quad \text{and} \quad \mathbb{P}_x[\zeta < \infty] = 0 \qquad (7.2)$$

for any $x \in M \setminus N$, more strongly than (7.1). By [19, Theorem 4.5.1 and Exercise 4.5.1], such a properly exceptional set $N \in \mathcal{B}(M)$ for X exists if and only if $(\mathcal{E},\mathcal{F})$ is local and *conservative*, i.e., $T_t\mathbf{1} = \mathbf{1}$ μ-a.e. for any (or equivalently, for some) $t \in (0,\infty)$, where $\mathbf{1} := \mathbf{1}_M$; recall (see, e.g., [13, (1.1.9) and (1.1.11)]) that for a Markovian bounded linear operator $T : L^2(M,\mu) \to L^2(M,\mu)$, $T|_{L^2(M,\mu)\cap L^\infty(M,\mu)}$ can be uniquely extended to a linear operator $T : L^\infty(M,\mu) \to L^\infty(M,\mu)$ such that $\lim_{n\to\infty} Tu_n = Tu$ μ-a.e. for any $u \in L^\infty(M,\mu)$ and any $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(M,\mu)$ with $u_n \leq u_{n+1}$ μ-a.e. for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} u_n = u$ μ-a.e.

Remark 7.1. In fact, *Theorems* 7.2 *and* 7.3 *below apply, without any changes in the proofs, to any locally compact separable metric space (M,d), any Hunt process X on $(M,\mathcal{B}(M))$ and any $N \in \mathcal{B}(M)$ satisfying* (4.5) *and* (7.1) *for any $x \in M \setminus N$.*

The main result of this section is the following theorem, which is a localized version of an unpublished result [22, Theorem 9.1] by the first named author. We refer the reader to [25, Subsection 5.4] for an alternative analytic approach. We set $e^{-\infty} := 0$.

Theorem 7.2. *Let U be a non-empty open subset of M and let $R \in (0,\infty]$. Then among the following seven conditions, the latter six* (2)–(7) *are equivalent and imply* (1)*:*

(1) *There exist* $\varepsilon \in (0, \frac{1}{2})$ *and* $\delta \in (0, \infty)$ *such that for any* $(x, r) \in (U \setminus N) \times (0, R)$ *with* $B(x, r) \subset U$ *and* $\overline{B(x, r)}$ *compact and for any* $t \in (0, \delta\Psi(r)]$,

$$\mathbb{P}_x[X_t \in M_\Delta \setminus B(x, r)] \leq \varepsilon. \tag{7.3}$$

(2) *There exist* $\varepsilon \in (0, 1)$ *and* $\delta \in (0, \infty)$ *such that for any* $(x, r) \in (U \setminus N) \times (0, R)$ *with* $B(x, r) \subset U$ *and* $\overline{B(x, r)}$ *compact,*

$$\mathbb{P}_x[\tau_{B(x,r)} \leq \delta\Psi(r)] \leq \varepsilon. \tag{7.4}$$

(3) *There exists* $\varepsilon \in (0, \infty)$ *such that for any* $(x, r) \in (U \setminus N) \times (0, R)$ *with* $B(x, r) \subset U$ *and* $\overline{B(x, r)}$ *compact,*

$$\mathbb{E}_x[\tau_{B(x,r)} \wedge \Psi(r)] \geq \varepsilon\Psi(r). \tag{7.5}$$

(4) *There exist* $\varepsilon \in (0, 1)$ *and* $\delta \in (0, \infty)$ *such that for any* $(x, r) \in (U \setminus N) \times (0, R)$ *with* $B(x, r) \subset U$ *and* $\overline{B(x, r)}$ *compact,*

$$\mathbb{E}_x\Bigl[\exp\Bigl(-\frac{\tau_{B(x,r)}}{\delta\Psi(r)}\Bigr)\Bigr] \leq \varepsilon. \tag{7.6}$$

(5) *There exist* $c, \gamma \in (0, \infty)$ *such that for any* $(x, r) \in (U \setminus N) \times (0, R)$ *with* $B(x, r) \subset U$ *and* $\overline{B(x, r)}$ *compact and for any* $\lambda \in (0, \infty)$,

$$\mathbb{E}_x[e^{-\lambda\tau_{B(x,r)}}] \leq c\exp\Bigl(-\frac{\gamma r}{\Psi^{-1}(\lambda^{-1})}\Bigr). \tag{7.7}$$

(6) *There exist* $c, \gamma \in (0, \infty)$ *such that for any* $(x, r) \in (U \setminus N) \times (0, R)$ *with* $B(x, r) \subset U$ *and* $\overline{B(x, r)}$ *compact and for any* $t \in (0, \infty)$,

$$\mathbb{P}_x[\tau_{B(x,r)} \leq t] \leq c\exp(-\Phi(\gamma r, t)). \tag{7.8}$$

(7) *There exist* $c, \gamma \in (0, \infty)$ *such that for any* $(x, r) \in (U \setminus N) \times (0, R)$ *with* $B(x, r) \subset U$ *and* $\overline{B(x, r)}$ *compact and for any* $t \in (0, \infty)$,

$$\mathbb{P}_x[\tau_{B(x,r)} \leq t] \leq c\exp\biggl(-\gamma\Bigl(\frac{\Psi(r)}{t}\Bigr)^{\frac{1}{\beta_2 - 1}}\biggr). \tag{7.9}$$

Moreover, if N *satisfies* (7.2) *for any* $x \in M \setminus N$*, then with "and* $\overline{B(x, r)}$ *compact" all removed, still the conditions* (2)–(7) *are equivalent, imply* (1) *and are implied by the following condition* (1)′*:*

(1)′ *There exist* $\varepsilon \in (0, \frac{1}{2})$ *and* $\delta \in (0, \infty)$ *such that for any* $(x, r) \in (\overline{U} \setminus N) \times (0, \frac{R}{2})$ *and any* $t \in (0, \delta\Psi(r)]$,

$$\mathbb{P}_x[X_t \in M \setminus B(x, r)] \leq \varepsilon. \tag{7.10}$$

Proof. We follow [22, Proof of Theorem 9.1]; for the implications (4)⇒(5)⇒(6)⇒(7) see also [28, Proofs of Lemma 3.14, Theorem 3.15 and Corollary 3.20].

We treat the two cases simultaneously, one with "and $\overline{B(x, r)}$ compact" kept and without (7.2) and the other with "and $\overline{B(x, r)}$ compact" removed and (7.2) assumed. Let $(x, r) \in (U \setminus N) \times (0, R)$ satisfy $B(x, r) \subset U$ and set $\tau := \tau_{B(x,r)}$. We assume in the former case that $\overline{B(x, r)}$ is compact, while not in the latter case. It easily follows either from (7.1) and the compactness of $\overline{B(x, r)}$ or from (7.2), together with $\mathbb{P}_x[X_0 = x] = 1$ and (4.5), that

$$\mathbb{P}_x\bigl[\tau_{B(x,\rho)} < \infty,\, X_{\tau_{B(x,\rho)}} \notin (\partial B(x, \rho)) \setminus N\bigr] = 0 \qquad \text{for any } \rho \in (0, r]. \tag{7.11}$$

(2)⇒(3): Since $\mathbb{P}_x[\tau > \delta\Psi(r)] = 1 - \mathbb{P}_x[\tau \leq \delta\Psi(r)] \geq 1 - \varepsilon$ by (7.4),

$$\mathbb{E}_x[\tau \wedge \Psi(r)] \geq \mathbb{E}_x[(\tau \wedge \Psi(r))\mathbf{1}_{\{\tau > \delta\Psi(r)\}}] \geq (\delta \wedge 1)\Psi(r)\mathbb{P}_x[\tau > \delta\Psi(r)] \geq (1 - \varepsilon)(\delta \wedge 1)\Psi(r).$$

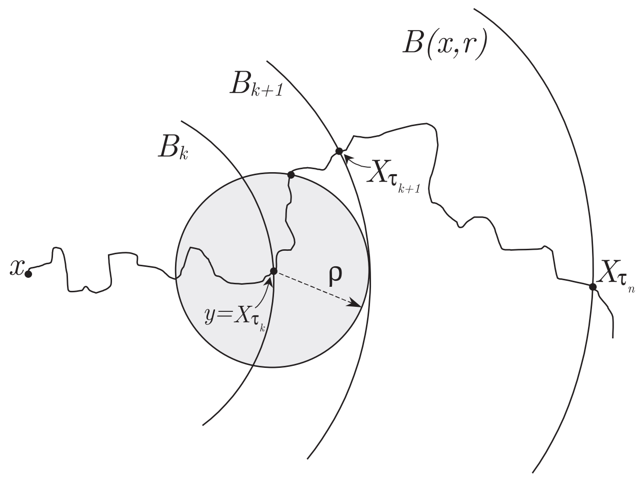

FIGURE 1. Proof of (4)⇒(5): the exit times τ_k, τ_{k+1} and $\tau_{B(y,\rho)}$

(3)⇒(4): For $\lambda, t \in (0,\infty)$, by considering the case of $\tau \geq t$ and that of $\tau \leq t$ separately we easily see that $e^{-\lambda\tau} \leq 1 - \lambda e^{-\lambda t}(\tau \wedge t)$, and therefore for *any* $\delta \in (0,\infty)$, setting $\lambda := (\delta\Psi(r))^{-1}$ and $t := \Psi(r)$, taking $\mathbb{E}_x[(\cdot)]$ and applying (7.5), we obtain

$$\mathbb{E}_x\Big[\exp\Big(-\frac{\tau}{\delta\Psi(r)}\Big)\Big] \leq 1 - \frac{1}{\delta\Psi(r)}e^{-1/\delta}\mathbb{E}_x[\tau \wedge \Psi(r)] \leq 1 - \frac{\varepsilon}{\delta}e^{-1/\delta}.$$

(4)⇒(5): Let $\lambda \in [(\delta\Psi(r))^{-1},\infty)$, set $n := \max\{k \in \mathbb{N} \mid \lambda\delta\Psi(r/k) \geq 1\}$ and $\rho := r/n$. Also set $B_k := B(x,k\rho)$ and $\tau_k := \tau_{B_k}$ for $k \in \{1,\ldots,n\}$. We claim that

$$\mathbb{E}_x[e^{-\lambda\tau_{k+1}}] \leq \varepsilon\mathbb{E}_x[e^{-\lambda\tau_k}] \qquad \text{for any } k \in \{1,\ldots,n\} \text{ with } k < n. \tag{7.12}$$

To see (7.12), let $k \in \{1,\ldots,n\}$ satisfy $k < n$ and let $y \in (\partial B_k) \setminus N$. Then obviously $(y,\rho) \in (U \setminus N) \times (0,R)$, $B(y,\rho) \subset B_{k+1} \subset B(x,r) \subset U$, hence $\tau_{B(y,\rho)} \leq \tau_{k+1}$, and $\overline{B(y,\rho)}$ is compact if $\overline{B(x,r)}$ is. Thus (4) applies to (y,ρ), so that from (7.6) we obtain

$$\mathbb{E}_y[e^{-\lambda\tau_{k+1}}] \leq \mathbb{E}_y[e^{-\lambda\tau_{B(y,\rho)}}] \leq \mathbb{E}_y\Big[\exp\Big(-\frac{\tau_{B(y,\rho)}}{\delta\Psi(\rho)}\Big)\Big] \leq \varepsilon, \tag{7.13}$$

noting that $\lambda \geq (\delta\Psi(\rho))^{-1}$ by the choice of n. Now since $\tau_{k+1} = \tau_k + \tau_{k+1}\circ\theta_{\tau_k}$, it follows by the strong Markov property [13, Theorem A.1.21] of X, (7.11) and (7.13) that

$$\begin{aligned}\mathbb{E}_x[e^{-\lambda\tau_{k+1}}] = \mathbb{E}_x[e^{-\lambda\tau_k}(e^{-\lambda\tau_{k+1}}\circ\theta_{\tau_k})] &= \mathbb{E}_x\big[e^{-\lambda\tau_k}\mathbb{E}_x[e^{-\lambda\tau_{k+1}}\circ\theta_{\tau_k} \mid \mathcal{F}_{\tau_k}]\big]\\ &= \mathbb{E}_x\big[e^{-\lambda\tau_k}\mathbb{E}_{X_{\tau_k}}[e^{-\lambda\tau_{k+1}}]\big]\\ &= \mathbb{E}_x\big[\mathbf{1}_{\{X_{\tau_k}\in(\partial B_k)\setminus N\}}e^{-\lambda\tau_k}\mathbb{E}_{X_{\tau_k}}[e^{-\lambda\tau_{k+1}}]\big]\\ &\leq \varepsilon\mathbb{E}_x[e^{-\lambda\tau_k}].\end{aligned}$$

Thus we have proved (7.12), which together with $\tau = \tau_n$ and (7.6) for (x,ρ) yields

$$\mathbb{E}_x[e^{-\lambda\tau}] \leq \varepsilon^{n-1}\mathbb{E}_x[e^{-\lambda\tau_1}] \leq \varepsilon^{n-1}\mathbb{E}_x\Big[\exp\Big(-\frac{\tau_1}{\delta\Psi(\rho)}\Big)\Big] \leq \varepsilon^n < \varepsilon^{-1}\exp\Big(-\frac{\gamma r}{\Psi^{-1}(\lambda^{-1})}\Big),$$

where $\gamma := \eta\log(\varepsilon^{-1})$ with $\eta := (\delta/c_\Psi)^{1/\beta_1} \wedge 1$ and the last inequality follows since $1 > \lambda\delta\Psi(\frac{r}{n+1}) \geq \lambda\Psi(\frac{\eta r}{n+1})$ by the choice of n and (5.10) and hence $n+1 > \eta r/\Psi^{-1}(\lambda^{-1})$.

(7.7) therefore holds for $\lambda \in [(\delta\Psi(r))^{-1},\infty)$. On the other hand, for $\lambda \in (0,(\delta\Psi(r))^{-1})$, since $\lambda^{-1} > \delta\Psi(r) \geq \Psi(\eta r)$ by (5.10) and hence $\Psi^{-1}(\lambda^{-1}) > \eta r$, we have $\mathbb{E}_x[e^{-\lambda\tau}] \leq 1 < e^{\gamma/\eta}\exp\big(-\gamma r/\Psi^{-1}(\lambda^{-1})\big)$, completing the proof of (4)⇒(5).

(5)⇒(6): For any $t, \lambda \in (0,\infty)$, we see from (7.7) that

$$\mathbb{P}_x[\tau \le t] = \mathbb{P}_x[e^{-\lambda t} \le e^{-\lambda\tau}] \le e^{\lambda t}\mathbb{E}_x[e^{-\lambda\tau}] \le c\exp\Bigl(\lambda t - \frac{\gamma r}{\Psi^{-1}(\lambda^{-1})}\Bigr),$$

and taking the infimum of the right-hand side in $\lambda \in (0,\infty)$ shows (7.8) in view of (5.11).

(6)⇒(7): Since $\Psi(\gamma r) \ge c_\Psi^{-1}(\gamma^{\beta_2}\wedge 1)\Psi(r)$ with $\gamma \in (0,\infty)$ as in (6) by (5.10), if $\Psi(\gamma r) \ge t$ then (7.9) is immediate from (7.8) and the lower inequality in (5.13), whereas if $\Psi(\gamma r) < t$ then we have $\mathbb{P}_x[\tau \le t] \le 1 \le c'\exp\bigl(-\gamma'(\Psi(r)/t)^{\frac{1}{\beta_2-1}}\bigr)$ for some $c', \gamma' \in (0,\infty)$ explicit in $c_\Psi, \beta_1, \beta_2, \gamma$.

(7)⇒(2),(1): For *any* $\varepsilon \in (0, c \wedge \frac{1}{2})$, setting $\delta := \bigl(\gamma/\log(c/\varepsilon)\bigr)^{\beta_2-1} \in (0,\infty)$, for any $t \in (0, \delta\Psi(r)]$ we see from $\{X_t \in M_\Delta \setminus B(x,r)\} \subset \{\tau \le \delta\Psi(r)\}$ and (7.9) that

$$\mathbb{P}_x[X_t \in M_\Delta \setminus B(x,r)] \le \mathbb{P}_x[\tau \le \delta\Psi(r)] \le c\exp\bigl(-\gamma\delta^{-\frac{1}{\beta_2-1}}\bigr) = \varepsilon.$$

(1)′⇒(2) *under* (7.2): Note that (7.10) is valid also for $t = 0$ since $\mathbb{P}_y[X_0 = y] = 1$ for $y \in M$. Let $t := c_\Psi^{-1}2^{-\beta_2}\delta\Psi(r)$, so that $t \in (0, \delta\Psi(\frac{r}{2})]$ by (5.10). We first show that

$$\mathbb{P}_x[\tau \le t,\, X_t \in B(x,\tfrac{r}{2})] \le \varepsilon. \tag{7.14}$$

Indeed, if $y \in (\partial B(x,r)) \setminus N$, then clearly $B(x,\frac{r}{2}) \subset M \setminus B(y,\frac{r}{2})$, $(y,\frac{r}{2}) \in (\overline{U} \setminus N) \times (0,\frac{R}{2})$ by $B(x,r) \subset U$ and hence (1)′ applies to $(y,\frac{r}{2})$, so that (7.10) yields

$$\mathbb{P}_y[X_s \in B(x,\tfrac{r}{2})] \le \mathbb{P}_y[X_s \in M \setminus B(y,\tfrac{r}{2})] \le \varepsilon \tag{7.15}$$

for any $y \in (\partial B(x,r)) \setminus N$ and any $s \in [0,t] \subset [0, \delta\Psi(\frac{r}{2})]$. Then since $\{\tau \le t\} \in \mathcal{F}_{\tau\wedge t}$ by [30, Lemma 1.2.16], it follows from Proposition 3.4, (7.11) and (7.15) that

$$\begin{aligned}
\mathbb{P}_x[\tau \le t,\, X_t \in B(x,\tfrac{r}{2})] &= \mathbb{E}_x[\mathbf{1}_{\{\tau\le t\}}\mathbf{1}_{B(x,r/2)}(X_t)] \\
&= \mathbb{E}_x\bigl[\mathbf{1}_{\{\tau\le t\}}\mathbb{E}_x[\mathbf{1}_{B(x,r/2)}(X_t) \mid \mathcal{F}_{\tau\wedge t}]\bigr] \\
&= \int_{\{\tau\le t\}} \mathbb{E}_{X_{\tau\wedge t}(\omega)}[\mathbf{1}_{B(x,r/2)}(X_{t-\tau(\omega)\wedge t})]\, d\mathbb{P}_x(\omega) \\
&= \int_{\{\tau\le t,\, X_\tau \in (\partial B(x,r))\setminus N\}} \mathbb{P}_{X_\tau(\omega)}[X_{t-\tau(\omega)} \in B(x,\tfrac{r}{2})]\, d\mathbb{P}_x(\omega) \\
&\le \varepsilon\mathbb{P}_x[\tau \le t,\, X_\tau \in (\partial B(x,r)) \setminus N] \le \varepsilon.
\end{aligned}$$

Now noting that $\mathbb{P}_x[X_t = \Delta] = 0$ by (7.2), from (7.10) for $(x,\frac{r}{2})$ and (7.14) we obtain

$$\begin{aligned}
\mathbb{P}_x[\tau \le t] &= \mathbb{P}_x[\tau \le t,\, X_t = \Delta] + \mathbb{P}_x[\tau \le t,\, X_t \in M \setminus B(x,\tfrac{r}{2})] + \mathbb{P}_x[\tau \le t,\, X_t \in B(x,\tfrac{r}{2})] \\
&\le \mathbb{P}_x[X_t = \Delta] + \mathbb{P}_x[X_t \in M \setminus B(x,\tfrac{r}{2})] + \varepsilon \\
&\le 2\varepsilon < 1,
\end{aligned}$$

which, in view of $t = c_\Psi^{-1}2^{-\beta_2}\delta\Psi(r)$, completes the proof of (1)′⇒(2) under (7.2). □

At the last of this paper, as an application of Theorem 7.2 we state and prove a localized version of the well-known fact that the comparability of the mean exit time $\mathbb{E}_x[\tau_{B(x,r)}]$ to $\Psi(r)$ implies the exit probability estimate (7.8). This fact was first observed by M. T. Barlow as treated in [3, Proof of Theorem 3.11], and the proof below is also based on an idea of his in [3].

Theorem 7.3. *Let U be a non-empty open subset of M, let $R \in (0,\infty]$, and assume that there exists $c_{\mathrm{E}} \in (0,\infty)$ such that for any $(x,r) \in (U \setminus N) \times (0,2R)$,*

$$\mathbb{E}_x[\tau_{B(x,r)}] \le c_{\mathrm{E}}\Psi(r), \tag{7.16}$$

and for any $(x,r) \in (U \setminus N) \times (0,R)$ with $B(x,r) \subset U$ and $\overline{B(x,r)}$ compact,

$$\mathbb{E}_x[\tau_{B(x,r)}] \ge c_{\mathrm{E}}^{-1}\Psi(r). \tag{7.17}$$

Then Theorem 7.2-(6) *holds.*

Moreover, additionally if N satisfies (7.2) *for any $x \in M \setminus N$ and if* (7.17) *holds for any $(x,r) \in (U \setminus N) \times (0,R)$ with $B(x,r) \subset U$, then Theorem* 7.2-(6) *with "and $\overline{B(x,r)}$ compact" removed holds.*

Proof. As in the proof of Theorem 7.2, we treat the two cases simultaneously, one with "and $\overline{B(x,r)}$ compact" kept and without (7.2) and the other with "and $\overline{B(x,r)}$ compact" removed and (7.2) assumed. Let $(x,r) \in (U \setminus N) \times (0,R)$ satisfy $B(x,r) \subset U$. We assume in the former case that $\overline{B(x,r)}$ is compact, while not in the latter case. We claim that

$$\mathbb{P}_x[\tau_{B(x,r)} \le \tfrac{1}{2} c_{\mathrm{E}}^{-1} \Psi(r)] \le 1 - (c_{\mathrm{E}}^2 c_\Psi 2^{\beta_2+1})^{-1}, \tag{7.18}$$

which together with the implication (2)⇒(6) of Theorem 7.2 shows the assertions.

To see (7.18) we follow [3, Proof of Lemma 3.16]. Set $\tau := \tau_{B(x,r)}$ and $t := \frac{1}{2} c_{\mathrm{E}}^{-1} \Psi(r)$. By using (7.17), the obvious relation $\tau \le t + (\tau - t)\mathbf{1}_{\{t<\tau\}} = t + (\tau \circ \theta_t)\mathbf{1}_{\{t<\tau\}}$ and the Markov property [13, Theorem A.1.21] of X at time t, we have

$$2t = c_{\mathrm{E}}^{-1}\Psi(r) \le \mathbb{E}_x[\tau] \le t + \mathbb{E}_x[(\tau \circ \theta_t)\mathbf{1}_{\{t<\tau\}}] = t + \mathbb{E}_x\big[\mathbf{1}_{\{t<\tau\}} \mathbb{E}_{X_t}[\tau]\big]. \tag{7.19}$$

Note that $X_t \in B(x,r)$ on $\{t < \tau\}$, that $\mathbb{P}_x[X_t \in N] = 0$ by (4.5), and that for any $y \in B(x,r) \setminus N$, $\tau \le \tau_{B(y,2r)}$ by $B(x,r) \subset B(y,2r)$ and hence $\mathbb{E}_y[\tau] \le \mathbb{E}_y[\tau_{B(y,2r)}] \le c_{\mathrm{E}}\Psi(2r) \le c_{\mathrm{E}} c_\Psi 2^{\beta_2}\Psi(r)$ by (7.16) and (5.10). It follows from (7.19) and these facts that

$$t \le \mathbb{E}_x\big[\mathbf{1}_{\{t<\tau\}} \mathbb{E}_{X_t}[\tau]\big] = \mathbb{E}_x\big[\mathbf{1}_{\{t<\tau,\, X_t \in B(x,r)\setminus N\}} \mathbb{E}_{X_t}[\tau]\big] \le c_{\mathrm{E}} c_\Psi 2^{\beta_2}\Psi(r)(1 - \mathbb{P}_x[\tau \le t]),$$

which immediately implies (7.18). □

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