Isoperimetric inequalities and capacities on Riemannian manifolds

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Dedicated to V.G.Maz'ya on the occasion of his 60th birthday

Abstract

We discuss extensions of some results of V.G.Maz'ya to Riemannian manifolds. His proofs of the relationships between capacities, isoperimetric inequalities and Sobolev inequalities did not use specific properties of the Euclidean space. His method, transplanted to manifolds, gives a unified approach to such results as parabolicity criteria, eigenvalues estimates, heat kernel estimates, etc.

Contents

1 Introduction

One of the important contributions of Maz'ya is the general understanding of the relationship between isoperimetric inequalities and functional inequalities of the Sobolev type in regions of $\mathbb{R}^n$. However, his method is not linked to specific properties of the Euclidean spaces and can be carried over to Riemannian manifolds. This was fully understood by Maz'ya in the 1960s when he wrote in [24, p.153]

> "Let us mention that, despite here we restrict ourself by considering sets in $\mathbb{R}^n$, the method of proof of criteria of [such] type does not use specific properties of the Euclidean space. The proofs can be carried over to the case of singular Riemannian manifolds; the exponents in the embedding theorems will depend not only on measures μ, ν and on the degree of irregularity of the boundary of [region] D but also on the singularities of the metric of the manifold."

Nowadays it is commonly accepted that embedding theorems and Sobolev type inequalities play the crucial role in Analysis on manifolds. A series of deep investigations undertaken during the last two decades, have revealed the fact that practically all important properties of solutions to second order elliptic and parabolic PDE have certain geometric background. For example, such properties as the mean-value inequality and the Harnack inequality are equivalent to some geometric conditions stated in terms of Sobolev and Poincaré inequalities as well as the volume of geodesic balls. On the other hand, estimates of the eigenvalues, Green kernel and heat kernel are closely related to the isoperimetric inequalities.

Maz'ya was one of the first mathematicians who started looking at the properties of solutions through the geometric properties of the domain. His method was based on the systematic use of the level sets of the functions and the co-area formula. The latter was independently discovered by Kronrod [18] (the case of two variables) and Federer [5] (the general case).

In this note, I would like to sketch some results of Maz'ya as they should look for Riemannian manifolds, and their relations with recent developments. Let M denote a smooth connected Riemannian manifold of dimension n. Let $B(x,r)$ denote the open geodesic ball of radius r centered at $x \in M$. Denote by μ the Riemannian volume and by σ the $(n-1)$-dimensional volume on hypersurfaces (that is, the $(n-1)$-Hausdorff measure). Differential operators such as the gradient ∇, the divergence div and the Laplace operator Δ, are those associated with the Riemannian metric.

Given a compact set $F \subset M$ and an open set $G \subset M$ containing F, we call the couple (F,G) a *capacitor*. Each capacitor has p-capacity (where $p \geq 1$) defined by

$$\operatorname{cap}_p(F,G) := \inf_u \int_{G \setminus F} |\nabla u|^p \, d\mu \,, \tag{1}$$

where the inf is taken over all Lipschitz functions u with compact support in G such that $u = 1$ on F. Clearly, $\operatorname{cap}_p(F,G)$ increases on enlargement of F and on shrinking of Ω. We will write $\operatorname{cap}_p(F)$ for $\operatorname{cap}_p(F,M)$. If F is precompact then we set $\operatorname{cap}_p(F,G) = \operatorname{cap}_p(\overline{F},G)$.

The following are the key ingredients of Maz'ya's method.

(i) **The estimate of the Dirichlet integral via the capacities of level sets.** Let u be a Lipschitz function on M with compact support. Denote

$$U_t := \{x \in M : |u(x)| \geq t\} \,.$$

Then

$$\int_M |\nabla u|^p \, d\mu \geq a_p \int_0^\infty \operatorname{cap}_p(U_t) d(t^p), \tag{2}$$

where

$$a_p = \begin{cases} (p-1)^{p-1} p^{-p}, & if \quad p > 1, \\ 1, & if \quad p = 1. \end{cases} \tag{3}$$

See [25, Theorem 3], [26, Section 2.3.1] or [27, p.150-51].

The integral of the function u itself can be estimated via the measure of sets U_t. Indeed, for any $\alpha \geq 1$, the following inequality holds

$$\int_M |u|^{\alpha p} \, d\mu \leq \left(\int_0^\infty \mu(U_t)^{\frac{1}{\alpha}} d(t^p) \right)^\alpha . \tag{4}$$

If $\alpha = 1$ then the equality is attained in (4), and it is trivial. See [26, Section 2.3.2] for the proof in the general case.

(ii) **Upper bound of capacity via flux.** Let $G \subset M$ be a precompact open set and $F \subset G$ be compact. Suppose that a Lipschitz function u is defined in $\overline{G \setminus F}$ such that $u = a$ on ∂F and $u = b$ on ∂G where $a < b$ are real constants. For any $t \in [a,b]$, denote $E_t := \{x : u(x) = t\}$ and define the *p-flux* of u through E_t by

$$\operatorname{flux}_p(u, E_t) := \int_{E_t} |\nabla u|^{p-1} \, d\sigma$$

(if E_t is empty then set $\operatorname{flux}_p(u,E_t) = 0$). Then, for all $p > 1$,

$$\operatorname{cap}_p(F,G) \leq \left(\int_a^b \frac{dt}{\operatorname{flux}_p(u,E_t)^{\frac{1}{p-1}}} \right)^{1-p} \tag{5}$$

(this is true also if $a > b$ in which case one should switch the limits in the integral (5)). Furthermore, the equality in (5) is attained for some function u satisfying the above conditions. See [21], [23, Lemma 1], [26, Lemma 2.2.2/1] or [27, p.149]. In fact, the idea of (5) goes back to Pólya and Szegö [29].

(iii) **Lower bound for capacity via the isoperimetric function.** We say that M admits *an isoperimetric function* $I(v)$ if, for any precompact open set $\Omega \subset M$ with smooth boundary,

$$\sigma(\partial\Omega) \geq I(\mu(\Omega)). \tag{6}$$

If M admits a positive continuous isoperimetric function $I(v)$ then, for any capacitor (F, G) and $p > 1$,

$$\operatorname{cap}_p(F, G) \geq \left(\int_{\mu(F)}^{\mu(G)} \frac{dv}{I(v)^{\frac{p}{p-1}}} \right)^{1-p}. \tag{7}$$

If $p = 1$ then, instead of (7), we have

$$\operatorname{cap}_1(F, G) \geq \inf_{\mu(F) \leq v \leq \mu(G)} I(v). \tag{8}$$

See [21], [26, Corollary 2.2.3/2] or [27, p.150-51].

We will use (i)-(iii) as building blocks to show how to obtain a series of important results in a unified way.

2 Capacity of balls

Fix a point $o \in M$ and denote $B_r = B(o, r)$ and $S_r := \sigma(\partial B_r)$. Let us assume that M is geodesically complete so that any ball is precompact. Let us choose in (5) $u = \operatorname{dist}(\cdot, o)$. Then $E_t = \partial B_t$ and $|\nabla u| \leq 1$ which implies $\operatorname{flux}_p(u, E_t) \leq S_t$. Hence, (5) yields, for all $R > r > 0$,

$$\operatorname{cap}_p(B_r, B_R) \leq \left(\int_r^R \frac{dt}{S_t^{\frac{1}{p-1}}} \right)^{1-p} \tag{9}$$

(let us assume for simplicity $p > 1$ whenever it is needed).

In applications, it is frequently more convenient to use the volume function V_r rather than S_r. Let us show that the following estimate holds

$$\operatorname{cap}_p(B_r, B_R) \leq 2^p \left(\int_r^R \left(\frac{t - r}{V_t - V_r} \right)^{\frac{1}{p-1}} dt \right)^{1-p}. \tag{10}$$

Observe first that $V_t' = S_t$. Hence, (10) will follow from (9) if we prove the following inequality

$$\int_a^b \frac{dt}{(v')^\gamma} \geq 2^{-1-\gamma} \int_a^b \left(\frac{t - a}{v(t) - v(a)} \right)^\gamma dt,$$

for all monotone increasing functions v on $[a, b]$, where $b > a$ and $\gamma > 0$ (it suffices to take $\gamma = \frac{1}{p-1}$). Without loss of generality, we may assume $a = 0$ and $v(a) = 0$. Denote by $s(t)$ the increasing rearrangement of v' on (a, b) and define the function v^* by

$$v^* := \int_0^t s(\tau) d\tau.$$

Then, by the convexity of v^*,

$$s(t) \leq \frac{v^*(2t) - v^*(t)}{t} \leq \frac{v^*(2t)}{t}.$$

By the properties of rearrangements $v^*(t) \leq v(t)$, whence

$$\int_0^b \frac{dt}{(v')^\gamma} = \int_0^b \frac{dt}{s(t)^\gamma} \geq \int_0^b \left(\frac{t}{v^*(2t)} \right)^\gamma dt \geq 2^{-1-\gamma} \int_0^b \left(\frac{t}{v(t)} \right)^\gamma dt,$$

which was to be proved[1].

In the case when M possesses a rotation symmetry with respect to o, it is possible to show that (9) becomes the equality. Such a manifold is called *spherically symmetric* with the pole o and can be described as follows. Topologically, M can be identified with $\mathbb{R}^n$ (assuming M is geodesically complete and non-compact), and the metric of M can be written in the polar coordinates (ρ, θ) (centered at o) as follows

$$ds^2 = d\rho^2 + \psi^2(\rho) d\theta^2.$$

Here $\theta \in \mathbb{S}^{n-1}$ and $d\theta^2$ is the standard metric on $\mathbb{S}^{n-1}$. Let us denote such a manifold by M_ψ. Function ψ is any positive smooth function on $\mathbb{R}_+$ such that $\psi(0) = 0$ and $\psi'(0) = 1$ (see [8]). It is easy to see that

$$S_r = \omega_n \psi(r)^{n-1} \quad \text{and} \quad V_r = \int_0^r S_t dt$$

so that each function S_r, V_r uniquely determines M_ψ.

For example, if $\psi(\rho) = \rho$ then $M_\psi = \mathbb{R}^n$. If $\psi(\rho) = \sinh \rho$ then $M = \mathbb{H}^n$. If $\psi(\rho) = \sin \rho$ then $M = \mathbb{S}^n$ (in this case, ρ varies in $[0, \pi]$ and M is compact).

As was remarked already, capacity of a ball on M_ψ is computed by

$$\operatorname{cap}_p(B_r, B_R) = \left(\int_r^R \frac{dt}{S_t^{\frac{1}{p-1}}} \right)^{1-p}. \tag{11}$$

Taking $R = \infty$, one obtains $\operatorname{cap}_p(B_r)$. For example if $M_\psi = \mathbb{R}^n$ then $\operatorname{cap}_p(B_r) = \operatorname{const} r^{n-p}$ provided $p < n$, and $\operatorname{cap}_p(B_r) = 0$ if $p \geq n$. If $M_\psi = \mathbb{H}^n$ then the exact computation is generally not possible but one easily gets the estimate

$$\operatorname{cap}_p(B_r) \sim \operatorname{const} S_r, \quad r \to \infty.$$

For comparison, in $\mathbb{R}^n$ one has $\operatorname{cap}_p(B_r) = \operatorname{const} S_r r^{1-p}$, if $p < n$.

3 Parabolicity of manifolds

A Riemannian manifold M is called *parabolic* if any of the following equivalent properties holds:

(a) any bounded subharmonic function on M is constant;

(b) there is no positive fundamental solution of the Laplace operator on M;

(c) $\operatorname{cap}(F) = 0$ for any compact $F \subset M$;

(d) the Brownian motion on M is recurrent[2].

Let us say that the manifold M is *p-parabolic* if $\operatorname{cap}_p(F) = 0$ for any compact $F \subset M$. This notion is connected to the properties of the p-Laplace operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

Indeed, it is known that the p-parabolicity is equivalent to the fact that any bounded p-subharmonic function is constant and to the non-existence of a positive fundamental solution of Δ_p (see [14]).

Finding convenient geometric conditions for parabolicity and non-parabolicity is an important task of potential theory on manifolds. Clearly, the estimates of capacity can give certain conditions of parabolicity.

Assume for simplicity that M is geodesically complete and non-compact (observe that a compact manifold is always p-parabolic). The upper bound of capacity (9) implies the following parabolicity test: if

$$\int^\infty \frac{dt}{S_t^{\frac{1}{p-1}}} = \infty \tag{12}$$

[1]The upper bound (10) for capacity in the case $p = 2$ was proved by Sturm [30] using a different method. His estimate has the better constant 2 rather than $4 = 2^p$.

[2]It is not quite obvious that (a)–(d) should be equivalent - see [13] for a detailed account of parabolicity and related topics.

then $\text{cap}_p(B_r) = 0$ and, therefore, M is p-parabolic. For example, if $S_r = cr^{n-1}$ then (12) holds provided $n \le p$. Therefore, if $n \le p$ then $\mathbb{R}^n$ is p-parabolic.

The capacity upper bound (10) gives the following parabolicity test in terms of the volume growth function: a geodesically complete manifold M is p-parabolic provided

$$\int^{\infty} \left(\frac{t}{V_t}\right)^{\frac{1}{p-1}} dt = \infty. \tag{13}$$

For example, (13) holds if $V_r \le Cr^p$, for large r, or even if $V_r \le Cr^p \log^{p-1} r$.

The parabolicity condition (12) for the case when $p = 2$ and $n = 2$ goes back to Ahlfors [1] and Nevanlinna [28] who proved it for simply connected Riemann surfaces (in this case, the parabolicity of M is equivalent to the fact that M is conformally equivalent to $\mathbb{R}^2$). For general manifolds, (12) was noticed by many authors, see for example [9], [19].

The sufficient condition $V_r \le Cr^2$ for 2-parabolicity was first proved by Cheng and Yau [3]. The 2-parabolicity under the more general hypothesis (13) was proved by Karp [16], Varopoulos [31] and the author [10]. The case $p = n$ was studied by Keselman and Zorich [17]. The proof for arbitrary $p > 1$ was obtained by Holopainen [15].

As we have seen, the capacity approach allows to obtain all mentioned parabolicity criteria in a few lines.

Let us turn to non-parabolicity tests. Assume that M admits the isoperimetric function $I(v)$ and let for simplicity $\mu(M) = \infty$. We claim that if

$$\int^{\infty} \frac{dv}{I^{\frac{p}{p-1}}(v)} < \infty \tag{14}$$

then M is p-non-parabolic. Indeed, given (14), the capacity estimate (7) implies $\text{cap}_p(F) > 0$ for large enough compact F. For example, let $I(v) = cv^{1-1/n}$. Then (14) holds if $n > p$. In particular, $\mathbb{R}^n$ is p-non-parabolic provided $n > p$.

The non-parabolicity test (14) was obtained by Fernández [7] and by the author [10], for the case $p = 2$. The general case seems to have not been published.

4 Isoperimetric inequality and Sobolev inequality

Assume that M admits the isoperimetric function $I(v) = cv^{1-1/n}$ like $\mathbb{R}^n$. Then (7) and (8) imply the following "isoperimetric inequality" for capacity

$$\text{cap}_p(F) \ge c' \mu(F)^{1-p/n}, \tag{15}$$

whence by (2)

$$\int |\nabla u|^p \, d\mu \ge c' a_p \int_0^{\infty} \mu(U_t)^{1-p/n} d(t^p). \tag{16}$$

Assume $p < n$ and apply (4) with $\alpha = \frac{n}{n-p}$, which yields

$$\left(\int |u|^{\frac{np}{n-p}} d\mu\right)^{\frac{n-p}{n}} \le \int_0^{\infty} \mu(U_t)^{1-p/n} d(t^p).$$

Comparing with (16), we conclude

$$\int |\nabla u|^p \, d\mu \ge c'' \left(\int |u|^{\frac{np}{n-p}} d\mu\right)^{\frac{n-p}{n}}. \tag{17}$$

This inequality in $\mathbb{R}^n$ was proved by S.L.Sobolev for $p > 1$ and by E.Gagliardo for $p = 1$. Maz'ya [20] as well as Federer and Fleming [6], were first to realize that the Sobolev inequality (17) is a consequence of the isoperimetric inequality (6). This has made it possible to prove Sobolev inequalities on manifolds where the method of singular integrals originally applied by Sobolev, would not work.

5 Capacity and the principal frequency

For any precompact region $\Omega \subset M$, let us denote by $\lambda(\Omega)$ the first eigenvalue of the Dirichlet problem in Ω for the Laplace operator. By the variational property,

$$\lambda(\Omega) = \inf_u \frac{\int |\nabla u|^2 \, d\mu}{\int |u|^2 \, d\mu}$$

where the inf is taken over all Lipschitz functions $u \not\equiv 0$ compactly supported in Ω. Analogously, let us define the *principal p-frequency* $\lambda_p(\Omega)$ by

$$\lambda_p(\Omega) := \inf_u \frac{\int |\nabla u|^p \, d\mu}{\int |u|^p \, d\mu} \tag{18}$$

where Ω is any open subset of M. Clearly, $\lambda_p(\Omega)$ is decreasing on enlargement of Ω.

For any open set Ω, let us define *the Maz'ya constant* of Ω as follows:

$$m_p(\Omega) := \inf_{F \subset\subset \Omega} \frac{\operatorname{cap}_p(F, \Omega)}{\mu(F)}. \tag{19}$$

We claim that

$$m_p(\Omega) \geq \lambda_p(\Omega) \geq a_p m_p(\Omega) \tag{20}$$

where a_p is defined by (3). The left hand side inequality in (20) easily follows if we take the test function u in definition (18) of $\lambda_p(\Omega)$ so that u is equal to 1 on a compact set $F \subset \Omega$, and apply definition (1) of capacity. To prove the lower bound of $\lambda_p(\Omega)$ in (20), let us apply (4) with $\alpha = 1$ and compare it with (2), rewritten for Ω instead of M. Then we have

$$\int_\Omega |\nabla u|^p \, d\mu \geq a_p \int_0^\infty \operatorname{cap}_p(U_t, \Omega) d(t^p),$$

and

$$\int_\Omega |u|^p \, d\mu \leq \int_0^\infty \mu(U_t) d(t^p),$$

whence

$$\frac{\int_\Omega |\nabla u|^p \, d\mu}{\int_\Omega |u|^p \, d\mu} \geq a_p m_p(\Omega)$$

which was to be proved.

In fact, both the Sobolev inequality (17) and the eigenvalue estimate (20) are particular cases of a more general theorem of Maz'ya [26, Theorem 2.3.2/1], which covers also the norms of u in Orlicz spaces.

Let us show an example of application of (20). If (15) is valid then

$$\frac{\operatorname{cap}_p(F, \Omega)}{\mu(F)} \geq c' \mu(F)^{-p/n} \geq c' \mu(\Omega)^{-p/n},$$

whence

$$m_p(\Omega) \geq c' \mu(\Omega)^{-p/n}$$

and, by (20),

$$\lambda_p(\Omega) \geq c' a_p \mu(\Omega)^{-p/n}.$$

6 Cheeger's inequality

Inequality (20) provides a powerful tool for estimating principal frequencies. Let us set in the next discussion $p = 2$ and omit p from all notation (say, cap means cap_2 etc.). Then (20) takes the form

$$m(\Omega) \geq \lambda(\Omega) \geq \frac{1}{4} m(\Omega). \tag{21}$$

The Cheeger constant $h(\Omega)$ of the region Ω is defined by

$$h(\Omega) := \inf_{F \subset \Omega} \frac{\sigma(\partial F)}{\mu(F)}$$

where F runs over all compact subsets of Ω with smooth boundary[3]. The Cheeger inequality [2] says

[3] The Cheeger constant $h(\Omega)$ can be regarded as the limiting case of the Maz'ya constant $m_p(\Omega)$ for $p = 1$.

that

$$\lambda(\Omega) \geq \frac{1}{4} h^2(\Omega). \tag{22}$$

Let us deduce (22) from (21). Indeed, given the Cheeger constant $h(\Omega)$, we may say that Ω admits the isoperimetric function $I(v) = h(\Omega)v$. Hence, by (7),

$$\operatorname{cap}(F, \Omega) \geq \left(\int_{\mu(F)}^{\mu(\Omega)} \frac{dv}{I(v)^2} \right)^{-1} \geq h^2(\Omega)\mu(F),$$

whence $m(\Omega) \geq h^2(\Omega)$, which together with (21) finishes the proof of (22).

The ratio of $\lambda(\Omega)/h^2(\Omega)$ can be made arbitrarily large, by choosing "bad" regions Ω. The Maz'ya inequality (20) shows that in such situations it is better to use capacity to estimate $\lambda(\Omega)$. We will show an application of that in Section 7. Note that the estimates (21) were proved by Maz'ya as early as in 1962 (see [22, Theorem 1]), long before the question of estimating eigenvalues on manifolds was even raised.

For arbitrary $p > 1$, one gets similarly

$$m_p(\Omega) \geq (p-1)^{1-p} \, h^p(\Omega),$$

whence, by (20),

$$\lambda_p(\Omega) \geq p^{-p} h^p(\Omega).$$

7 Eigenvalues of balls on spherically symmetric manifolds

We assume in this section that $M = M_\psi$ is a spherically symmetric manifold and follow the notation introduced in Section 2. The question to be discussed here is how to estimate $\lambda(B_R)$ as a function of the radius R?

Let us modify the definition of the Maz'ya constant $m_p(\Omega)$ in the case when $\Omega = B_R$. Namely, in the definition (19), let us assume in addition that F runs only over balls B_r with $r < R$. In the definition (18) of $\lambda_p(B_R)$, the minimum is attained for the radially symmetric function u. Therefore, in the proof of the Maz'ya inequalities (20), if $\Omega = B_R$ then it suffices to use only the radially symmetric test functions. Hence, we obtain

$$m_p(B_R) \geq \lambda_p(B_R) \geq a_p m_p(B_R) \tag{23}$$

where m_p now refers to the modified Maz'ya constant.

Let us assume in the sequel $p = 2$ and suppress p from the notation. Since the capacity $\operatorname{cap}(B_r, B_R)$ is explicitly given by (11), that is,

$$\operatorname{cap}(B_r, B_R) = \left(\int_r^R \frac{dt}{S_t} \right)^{-1},$$

the modified Maz'ya constant can be computed as follows:

$$m(B_R) = \frac{1}{\Phi(R)} \tag{24}$$

where

$$\Phi(R) := \sup_{r \leq R} \left[V_r \int_r^R \frac{dt}{S_t} \right]. \tag{25}$$

Respectively, we obtain the eigenvalue estimates

$$\frac{1}{\Phi(R)} \geq \lambda(B_R) \geq \frac{1}{4\Phi(R)}. \tag{26}$$

For comparison, the Cheeger inequality implies another estimate

$$\lambda(B_R) \geq \frac{1}{4} \left(\inf_{r \leq R} \frac{S_r}{V_r} \right)^2 \tag{27}$$

(see [4, Lemma 8.2]), which is generally weaker than (27), but in some situation is substantially weaker.

Let us show some examples.

Examples: 1. Let $S_r \asymp r^{N-1}$ for large r, where $N > 0$. Respectively, the volume function is $V_r \asymp r^N$. Simple computation shows that $\Phi(R) \asymp R^2$ for large R whence $\lambda(B_R) \asymp R^{-2}$.

2. Let us take $S_r \asymp r^{-1}$ so that $V_r \asymp \log r$. Then the optimal r in (25) is of the order $R/\sqrt{\log R}$, and we find

$$\Phi(R) \asymp R^2 \log R \quad \text{and} \quad \lambda(B_R) \asymp \frac{1}{R^2 \log R}.$$

Note that (27) gives the weaker estimate

$$\lambda(B_R) \geq \frac{c}{R^2 \log^2 R}.$$

3. If $S_r \asymp (r \log r)^{-1}$ and, therefore, $V_r \asymp \log\log r$ then

$$\Phi(R) \asymp R^2 \log R \log\log R \quad \text{and} \quad \lambda(B_R) \asymp \frac{1}{R^2 \log R \log\log R},$$

which is again better than (27).

8 Heat kernel on spherically symmetric manifolds

By definition, the heat kernel on a Riemannian manifold M is the smallest positive fundamental solution to the heat equation $u_t = \Delta u$ on $\mathbb{R}_+ \times M$. We denote the heat kernel by $p(t, x, y)$ where $t > 0$ and $x, y \in M$. One of the most interesting questions about the heat kernel is obtaining its long time estimates. This question has been attracted much attention in the past decade, and we refer the reader to the surveys [11] and [12] for general overview.

Here, we assume that $M = M_\psi$ is spherically symmetric manifold as above and try to estimate $p(t, o, o)$ as a function of t, as $t \to \infty$. Suppose that we are given a positive non-increasing function $\Lambda(v)$ such that, for each ball B_r,

$$\lambda(B_r) \geq \Lambda(V_r).$$

In other word, Λ provides the lower bound for $\lambda(B_r)$ via the volume of B_r. Then we have, for all $t > 0$,

$$p(t, o, o) \leq \frac{4}{P(t/2)} \tag{28}$$

where the function P is defined by

$$t = \int_0^{P(t)} \frac{dv}{v\Lambda(v)}. \tag{29}$$

(see [4, Eq. (8.19)]). Hence, the question of estimating $p(t, o, o)$ amounts to obtaining sharp lower bounds for $\lambda(B_r)$. For example, if $M_\psi = \mathbb{R}^n$ then $\Lambda(v) = cv^{-2/n}$ whence $P(t) = c' t^{n/2}$ and $p(t, o, o) \leq Ct^{-n/2}$.

For general M_ψ, the function Λ can be obtained from (26). Namely, let us define Λ by

$$\Lambda(V_R) = \frac{1}{4\Phi(R)} \tag{30}$$

where Φ is defined by (25). Then, the formulas (25), (30), (29) and (28) give *implicitly* an upper bound for $p(t, o, o)$ via the volume function V_r. This approach is similar to one adopted in [4], but the novelty here is the usage the Maz'ya estimate (26) of the eigenvalues rather than (27) that was used in [4].

Substituting (30) into (29), we obtain

$$t = 4 \int_0^{V^{-1}(P(t))} \Phi(r) \frac{S_r dr}{V_r}$$

where V^{-1} is the inverse function to $V_{(\cdot)}$. Denote $R = R(t) = V^{-1}(P(t))$. Alternatively, $R(t)$ is defined by

$$t = 4 \int_0^{R(t)} \Phi(r) \frac{S_r dr}{V_r}. \tag{31}$$

Then (28) implies

$$p(t,o,o) \leq \frac{4}{V_{R(t/2)}} \tag{32}$$

Let us try and estimate $R(t)$ more explicitly. Since the function $\Phi(r)$ is monotone increasing, (31) implies, for all $t > t_0 > 0$,

$$t - t_0 \leq 4\Phi(R) \log \frac{V_R}{V_{R_0}} \tag{33}$$

where $R_0 = R(t_0)$. This inequality can be turned into a lower bound for $R(t)$ which can be then used in (32).

For example, let $V_r = \exp(r^\alpha)$ for large r, where $\alpha \in (0,1)$. Then (25) yields, for large R,

$$\Phi(R) \asymp R^{2(1-\alpha)}$$

and (33) implies, for large t,

$$R(t) \geq ct^{\frac{1}{2-\alpha}}.$$

Respectively, the heat kernel admits the following upper bound

$$p(t,o,o) \leq C \exp\left(-ct^{\frac{\alpha}{2-\alpha}}\right).$$

On the other hand, one can show that there is a matching lower bound here - see [4, p.42].

If V_r grows polynomially in r then the estimate (33) for $R(t)$ is too rough. To get a better estimate, let us assume that

(a) V_r satisfies the doubling property: for all $r > 0$ and some constant C,

$$V_{2r} \leq CV_r \tag{34}$$

(b) and, for all $r > 0$ and some $c > 0$,

$$\frac{S_r}{V_r} \geq \frac{c}{r}. \tag{35}$$

We claim that, under (a) and (b), the following estimate holds

$$p(t,o,o) \asymp \frac{1}{V_{\sqrt{t}}}. \tag{36}$$

Previously, this estimate was known provided the doubling property (a) holds and S_r/V_r is non-increasing function - see [4, Corollary 8.5]. On the contrary, the hypothesis (35) does not restrict S_r from above. For example, the graph of S_r against r may have arbitrarily high peeks on small intervals as long as this does not violate the doubling property (34).

To prove (36), let first observe that (25) and (35) imply, for any $R > 0$ and some $r \in (0,R)$,

$$\Phi(R) = V_r \int_r^R \frac{dt}{S_t} \leq \int_r^R \frac{V_t dt}{S_t} \leq \frac{1}{2c}\left(R^2 - r^2\right) < \frac{1}{2c} R^2. \tag{37}$$

Then the integral in (31) can be estimated as follows:

$$\begin{aligned}
\int_0^R \Phi(r) \frac{S_r dr}{V_r} &= \sum_{k=0}^{\infty} \int_{R2^{-(k+1)}}^{R2^{-k}} \Phi(r) \frac{S_r dr}{V_r} \\
&\leq \sum_{k=0}^{\infty} \Phi(R2^{-k}) \int_{R2^{-(k+1)}}^{R2^{-k}} \frac{S_r dr}{V_r} \\
&\leq \sum_{k=0}^{\infty} \Phi(R2^{-k}) \log \frac{V(R2^{-k})}{V(R2^{-(k+1)})} \\
&\leq C \sum_{k=0}^{\infty} \Phi(R2^{-k}) \\
&\leq C' R^2
\end{aligned}$$

where we have used (34) and (37).

Hence, (31) implies

$$R(t) \geq c'\sqrt{t},$$

which together with (32) and (34) yields

$$p(t,o,o) \leq \frac{C''}{V_{\sqrt{t}}}. \tag{38}$$

We are left to note that, by [4, Theorem 7.2], the upper bound (38) and the doubling property (34) imply the lower bound in (36).

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