Negative eigenvalues of two-dimensional Schrödinger operators

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Abstract

We prove a certain upper bound for the number of negative eigenvalues of the Schrödinger operator $H = -\Delta - V$ in $\mathbb{R}^2$.

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1 Introduction

1.1 Main statement

Given a non-negative L^1_{loc} function $V(x)$ on $\mathbb{R}^n$, consider the Schrödinger type operator

$$H_V = -\Delta - V$$

where $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ is the classical Laplace operator. More precisely, H_V is defined as a form sum of $-\Delta$ and $-V$, so that, under certain assumptions about V, the operator H_V is self-adjoint in $L^2(\mathbb{R}^n)$. Denote by $\operatorname{Neg}(V, \mathbb{R}^n)$ the number of non-positive eigenvalues of H_V counted with multiplicity, assuming that its spectrum in $(-\infty, 0]$ is discrete.

For the operator H_V in $\mathbb{R}^n$ with $n \geq 3$ a celebrated inequality of Cwikel-Lieb-Rozenblum says that

$$\operatorname{Neg}(V, \mathbb{R}^n) \leq C_n \int_{\mathbb{R}^n} V(x)^{n/2} dx. \tag{1.1}$$

This estimate was proved independently by the above named authors in 1972-1977 in [9], [21], and [27], respectively[1].

The estimate (1.1) is not valid in $\mathbb{R}^2$ as one can see on simple examples. On the contrary, in $\mathbb{R}^2$ a similar lower bound holds:

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \geq c \int_{\mathbb{R}^2} V(x)\, dx \tag{1.2}$$

[1]See also [13], [18], [19], [20], [23] for further developments.

that was proved in [12].

Our main result – Theorem 1.1 below, provides an upper bound for $\operatorname{Neg}(V, \mathbb{R}^2)$. To state it, let us introduce some notation. For any $n \in \mathbb{Z}$ define the annuli U_n and W_n in $\mathbb{R}^2$ by

$$U_n = \begin{cases} \{e^{2^{n-1}} < |x| < e^{2^n}\}, & n \geq 1, \\ \{e^{-1} < |x| < e\}, & n = 0, \\ \{e^{-2^{|n|}} < |x| < e^{-2^{|n|-1}}\}, & n \leq -1, \end{cases} \tag{1.3}$$

and

$$W_n = \left\{x \in \mathbb{R}^2 : e^n < |x| < e^{n+1}\right\}. \tag{1.4}$$

Given a potential (=a non-negative L^1_{loc}-function) $V(x)$ on $\mathbb{R}^2$ and $p > 1$, define for any $n \in \mathbb{Z}$ the following quantities:

$$A_n(V) = \int_{U_n} V(x)(1 + |\ln|x||)\, dx \tag{1.5}$$

and

$$B_n(V) = \left(\int_{W_n} V^p(x) |x|^{2(p-1)}\, dx\right)^{1/p}. \tag{1.6}$$

We will write for simplicity A_n and B_n for $A_n(V)$ and $B_n(V)$, respectively, if it is clear from the context to which potential V this refers.

Theorem 1.1 *For any non-negative function $V \in L^1_{loc}(\mathbb{R}^2)$ and $p > 1$, we have*

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \leq 1 + C \sum_{\{n \in \mathbb{Z}: A_n > c\}} \sqrt{A_n} + C \sum_{\{n \in \mathbb{Z}: B_n > c\}} B_n, \tag{1.7}$$

where C, c are some positive constants depending only on p.

The additive term 1 in (1.7) reflects a special feature of $\mathbb{R}^2$: for any non-trivial potential V, the spectrum of H_V has a negative part, no matter how small are the sums in (1.7). In $\mathbb{R}^n$ with $n \geq 3$, $\operatorname{Neg}(V, \mathbb{R}^n)$ can be 0 provided the integral in (1.1) is small enough.

In fact, the quantity $\operatorname{Neg}(V, \mathbb{R}^2)$ is understood in a more general manner using the Morse index of an appropriate energy form, rather than the operator H_V directly (see Section 3) so that $\operatorname{Neg}(V, \mathbb{R}^2)$ always makes sense.

1.2 Discussion and historical remarks

So far the best known upper bound for $\operatorname{Neg}(V, \mathbb{R}^2)$ for a general class of potentials V was due to Solomyak [29] who proved that[2]

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \leq 1 + C \|A\|_{1,\infty} + C \sum_{n \in \mathbb{Z}} B_n, \tag{1.8}$$

[2]In fact, the estimate of [29] is even sharper than (1.8) because B_n are defined in [29] using not the L^p-norm but a certain Orlicz norm. Further improvement of the term B_n can be found in [17].

where A denotes the whole sequence $\{A_n\}_{n\in\mathbb{Z}}$ and $\|A\|_{1,\infty}$ is the weak l^1-norm (the Lorentz norm) defined by

$$\|A\|_{1,\infty} = \sup_{s>0} s\#\{n : A_n > s\}.$$

In particular, the result of Solomyak [29] implies that if the right hand side of (1.8) is finite then the following semi-classical asymptotic holds:

$$\mathrm{Neg}\left(\alpha V, \mathbb{R}^2\right) = O(\alpha) \quad \text{as } \alpha \to \infty, \tag{1.9}$$

as one should expect for "nice" potentials from quantum mechanical considerations.

Let us show that (1.8) follows from our estimate (1.7). Indeed, it is easy to verify that

$$\|A\|_{1,\infty} \le \sup_{s>0} s^{1/2} \sum_{\{A_n>s\}} \sqrt{A_n} \le 4\|A\|_{1,\infty}.$$

In particular, we have

$$\sum_{\{A_n>c\}} \sqrt{A_n} \le 4c^{-1/2}\|A\|_{1,\infty},$$

so that (1.7) implies (1.8). In Section 2 will see that our estimate (1.7) provides for certain potentials strictly better results than (1.8).

A simpler (and coarser) version of (1.7) and (1.8) is

$$\mathrm{Neg}\left(V, \mathbb{R}^2\right) \le 1 + C\int_{\mathbb{R}^2} V(x)\left(1 + |\ln|x||\right) dx + C\sum_{n\in\mathbb{Z}} B_n, \tag{1.10}$$

that follows from (1.8) using $\|A\|_{1,\infty} \le \|A\|_1$. In the case when $V(x)$ is a radial function, that is, $V(x) = V(|x|)$, the following estimate was proved by Chadan, Khuri, Martin and Wu [8], [14]:

$$\mathrm{Neg}\left(V, \mathbb{R}^2\right) \le 1 + \int_{\mathbb{R}^2} V(x)\left(1 + |\ln|x||\right) dx. \tag{1.11}$$

Although this estimate is sharper than (1.10), we will see that our main estimate (1.7) gives for certain radial potentials strictly better results than even (1.11).

Laptev and Solomyak [17] improved (1.10) for general potentials by modifying the definition of B_n so that all the terms B_n vanish for radial potentials thus yielding (1.11) (cf. also [15]). Furthermore, they obtained in [16] a necessary and sufficient condition for radial potentials to satisfy the semi-classical asymptotic (1.9).

Another known estimate for $\mathrm{Neg}\left(V, \mathbb{R}^2\right)$ is due to Molchanov and Vainberg [24]:

$$\mathrm{Neg}\left(V, \mathbb{R}^2\right) \le 1 + C\int_{\mathbb{R}^2} V(x)\ln\langle x\rangle\, dx + C\int_{\mathbb{R}^2} V(x)\ln\left(2 + V(x)\langle x\rangle^2\right) dx, \tag{1.12}$$

where $\langle x\rangle = e + |x|$. However, due to the logarithmic term in the second integral, this estimate never leads to (1.9).

The main novelty (and strength) of our estimate (1.7) lies in using of the truncated sum $\sum_{\{A_n>c\}} \sqrt{A_n}$ and $\sum_{\{B_n>c\}} B_n$. For example, it follows from (1.7) that if

$A_n \to 0$ and $B_n \to 0$ then the both sums in (1.7) and, hence, $\mathrm{Neg}\,(V, \mathbb{R}^2)$ are finite, which does not follow from any of the previously known results. For example, this is the case for a potential V such that

$$V\,(x) = o\left(\frac{1}{|x|^2 \ln^2 |x|}\right) \quad \text{as } x \to \infty.$$

The fact that the right hand side of (1.7) in non-linear in α when V is replaced by αV, allows to obtain non-linear in α estimates for $\mathrm{Neg}\,(\alpha V, \mathbb{R}^2)$ for quite simple potentials V. We discuss these and many other examples in Section 2.

The nature of the terms $\sqrt{A_n}$ and B_n in (1.7) can be explained as follows. Different parts of the potential V contribute differently to $\mathrm{Neg}\,(V, \mathbb{R}^2)$. The high values of V concentrated on relatively small areas contribute to $\mathrm{Neg}\,(V, \mathbb{R}^2)$ via the terms B_n, while the low values of V scattered over large areas, contribute via the terms $\sqrt{A_n}$. Since we integrate V over long annuli, the long range effect of V becomes similar to that of an one-dimensional potential. In $\mathbb{R}^1$ one expects

$$\mathrm{Neg}\left(\alpha V, \mathbb{R}^1\right) = O\left(\sqrt{\alpha}\right) \quad \text{as } \alpha \to \infty,$$

which explains the appearance of the square root in (1.7). Another explanation of the role of the terms A_n comes from the following result of [2] and [29]: the condition $\|A\|_{1,\infty} < \infty$ is necessary and sufficient for the semi-classical asymptotic for the operator that comes from the restriction of the corresponding quadratic form to the subspace of radial functions. Loosely speaking, the terms A_n are responsible for the negative spectrum in the radial direction.

An exhaustive account of upper bounds in one-dimensional case can be found in [3], [7], [25], [26]. In particular, the following estimate was proved by Birman and Solomyak [7]:

$$\mathrm{Neg}\left(V, \mathbb{R}^1_+\right) \le 1 + C \sum_{n=0}^{\infty} \sqrt{a_n}, \tag{1.13}$$

where

$$a_n = \int_{I_n} V\,(x)\,(1 + |x|)\,dx$$

and $I_n = [2^{n-1}, 2^n]$ if $n > 0$ and $I_0 = [0, 1]$. Clearly, the sum $\sum \sqrt{a_n}$ here resembles $\sum \sqrt{A_n}$ in (1.7), which is not a coincidence. In fact, our method allows to improve (1.13) by restricting the sum to $\{n : a_n > c\}$.

Let us state two consequences of Theorem 1.1.

Corollary 1.2 *If*

$$\int_{\mathbb{R}^2} V\,(x)\,(1 + |\ln|x||)\,dx + \sum_{n \in \mathbb{Z}} B_n\,(V) < \infty \tag{1.14}$$

then

$$\mathrm{Neg}\left(\alpha V, \mathbb{R}^2\right) \le C\alpha \sum_{n \in \mathbb{Z}} B_n\,(V) + o\,(\alpha) \quad \textit{as } \alpha \to \infty. \tag{1.15}$$

Corollary 1.3 *Assume that $\mathcal{W}(r)$ is a positive monotone increasing function on $(0,+\infty)$ that satisfies the following Dini type condition both at 0 and at ∞:*

$$\int_0^\infty \frac{r\,|\ln r|^{\frac{p}{p-1}}\,dr}{\mathcal{W}(r)^{\frac{1}{p-1}}} < \infty. \tag{1.16}$$

Then

$$\operatorname{Neg}\left(V,\mathbb{R}^2\right) \le 1 + C\left(\int_{\mathbb{R}^2} V^p(x)\,\mathcal{W}(|x|)\,dx\right)^{1/p}, \tag{1.17}$$

where the constant C depends on p and $\mathcal{W}$.

Here is an example of a weight function $\mathcal{W}(r)$ that satisfies (1.16):

$$\mathcal{W}(r) = r^{2(p-1)}\langle \ln r\rangle^{2p-1}\ln^{p-1+\varepsilon}\langle \ln r\rangle, \tag{1.18}$$

where $\varepsilon > 0$. In particular, for $p = 2$, (1.17) becomes

$$\operatorname{Neg}\left(V,\mathbb{R}^2\right) \le 1 + C\left(\int_{\mathbb{R}^2} V^2(x)\,|x|^2\,\langle \ln |x|\rangle^3 \ln^{1+\varepsilon}\langle \ln |x|\rangle dx\right)^{1/2}. \tag{1.19}$$

Let us emphasize once again that none of the above mentioned estimates (1.8), (1.10), (1.11), (1.12), (1.17) matches the full strength of our main estimate (1.7) even for radial potentials as will be seen on examples below.

1.3 Outline of the paper

As we have already mentioned above, the setting of $\mathbb{R}^2$ versus $\mathbb{R}^n$ with $n > 2$ presents significant difficulties. We try and turn the disadvantages of this setting into an advantage by exploiting specific properties of $\mathbb{R}^2$ such as the presence of a large class of conformal mappings preserving the Dirichlet integral. We use widely the classical idea of Weyl of splitting domains into small enough subdomains with the Neumann boundary condition. A critical issue in this method is estimating the number N of subdomains, which eventually leads to required estimates of $\operatorname{Neg}(V)$. We apply this approach a few times, using at each occurrence different ways of estimating N.

Let us briefly describe the structure of paper that matches the flowchart of the proof. In Section 2 we give examples of application of Theorem 1.1. In Section 3 we define for any open set $\Omega \subset \mathbb{R}^2$ the quantity $\operatorname{Neg}(V,\Omega)$ as the Morse index of the quadratic form

$$\mathcal{E}_{V,\Omega}(u) = \int_\Omega |\nabla u|^2\,dx - \int_\Omega V u^2 dx,$$

and prove various properties of the former including subadditivity with respect to partitioning and the behavior under conformal and bilipschitz mappings. For bounded domains Ω with smooth boundary, $\operatorname{Neg}(V,\Omega)$ coincides with the number of non-positive eigenvalues of the Neumann problem for $-\Delta - V$ in Ω.

In Section 4 we prove Lemma 4.8 that provides an upper bound for $\operatorname{Neg}(V,Q)$ in a unit square Q in terms of $\|V\|_{L^p(Q)}$. The proof involves a careful partitioning of Q into tiles $\Omega_1, ..., \Omega_N$ with small enough $\|V\|_{L^p(\Omega_n)}$ while controlling the number of tiles N via $\|V\|_{L^p(Q)}$. This argument is reminiscent of a Calderon-Zygmund type partition of the cube that was used by Birman and Solomyak [5] for the eigenvalues estimates (cf. also [6], [10], [22], [28]). In contrast, we do not restrict the shape of the tiles to squares[3]. The estimate of Lemma 4.8 leads in the end to the terms B_n in (1.7) reflecting the local properties of the potential.

In Section 5 we make the first step towards the global properties of V. Our starting point is the Green function $g(x,y)$ of the operator $H_0 = -\Delta + V_0$ where $V_0 \in C_0^\infty(\mathbb{R}^2)$ is a fixed potential for which $\operatorname{Neg}(V_0, \mathbb{R}^2) = 1$. We use the following estimate of $g(x,y)$ that was proved in [11]:

$$g(x,y) \simeq \ln\langle x\rangle \wedge \ln\langle y\rangle + \ln_+ \frac{1}{|x-y|}.$$

Considering the integral operator

$$G_V f(x) = \int_{\mathbb{R}^2} g(x,y) f(y) V(y)\, dy$$

acting in $L^2(V dx)$, we show first that

$$\|G_V\| \leq \frac{1}{2} \Rightarrow \operatorname{Neg}\left(V, \mathbb{R}^2\right) = 1$$

(Corollary 5.4). Hence, to characterize the potentials V with $\operatorname{Neg}(V, \mathbb{R}^2) = 1$ it suffices to estimate the norm of G_V. Using the conformal mapping $z \mapsto \ln z$, we translate the problem to a simpler integral operator Γ_V acting in a strip

$$S = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R},\ 0 < x_2 < \pi\right\}.$$

In Section 6 we estimate the norm of a certain integral operator in S using a weighted Hardy inequality (Lemma 6.2).

In Section 7 we obtain an estimate of $\|\Gamma_V\|$ (Lemma 7.1) that leads to conditions for $\operatorname{Neg}(V, S) = 1$ (Proposition 7.3). Then a number of further steps, involving a careful partitioning of the strip into rectangles, is needed to obtain an upper bound for $\operatorname{Neg}(V, S)$ that is stated in Theorem 7.9 and that is interesting on its own right.

In the final Section 8 we translate the estimate for $\operatorname{Neg}(V, S)$ into that for $\operatorname{Neg}(V, \mathbb{R}^2)$ thus finishing the proof of Theorem 1.1.

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[3] Hopefully, this new type of decomposition will find applications elsewhere.

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2 Examples

Let V be a potential in $\mathbb{R}^2$, and let us use the abbreviation $\operatorname{Neg}(V) \equiv \operatorname{Neg}(V, \mathbb{R}^2)$. We write $f \simeq g$ if the ratio $\frac{f}{g}$ is bounded between two positive constants.

1. Assume that, for all $x \in \mathbb{R}^2$,

$$V(x) \leq \frac{\alpha}{|x|^2}$$

for a small enough positive constant α. Then, for all $n \in \mathbb{Z}$,

$$B_n \leq \alpha \left(\int_{e^n}^{e^{n+1}} \frac{1}{r^{2p}} r^{2(p-1)} 2\pi r dr \right)^{1/p} \simeq \alpha$$

so that $B_n < c$ and the last sum in (1.7) is void, whence we obtain

$$\begin{aligned} \operatorname{Neg}(V) &\leq 1 + C \sum_{\{n: A_n > c\}} \sqrt{A_n} && (2.1) \\ &\leq 1 + C \int_{\mathbb{R}^2} V(x) (1 + |\ln |x||) \, dx. && (2.2) \end{aligned}$$

The estimate (2.2) in this case follows also from (1.12).

2. Consider a potential

$$V(x) = \frac{1}{|x|^2 \left(1 + \ln^2 |x|\right)},$$

As in the first example, $B_n \simeq 1$, while A_n can be computed as follows: for $n \geq 1$

$$A_n = \int_{e^{2^{n-1}}}^{e^{2^n}} \frac{1}{r^2 \left(1 + \ln^2 r\right)} (1 + \ln r) \, 2\pi r dr \simeq 1, \tag{2.3}$$

and the same estimate holds for $n \leq 0$. Hence, if $\alpha > 0$ is small enough then $A_n(\alpha V)$ and $B_n(\alpha V)$ are smaller than c for all n, and the both sums in (1.7) are void. It follows that

$$\operatorname{Neg}(\alpha V) = 1.$$

This result cannot be obtained by any of the previously known estimates. Indeed, in the estimates (1.11) and (1.12) the integral $\int_{\mathbb{R}^2} V(x) (1 + |\ln |x||) \, dx$ diverges, and in the estimate (1.8) of Solomyak one has $\|A\|_{1,\infty} = \infty$. As will be shown below, if $\alpha > 1/4$ then $\operatorname{Neg}(\alpha V) = \infty$. Hence, $\operatorname{Neg}(\alpha V)$ exhibits a non-linear behavior with respect to the parameter α, which cannot be captured by linear estimates.

3. Assume that $V(x)$ is locally bounded and

$$V(x) = o\left(\frac{1}{|x|^2 \ln^2 |x|}\right) \quad \text{as } x \to \infty. \tag{2.4}$$

Similarly to the previous example, we see that $A_n(V) \to 0$ and $B_n(V) \to 0$ as $n \to \infty$, which implies that the both sums in (1.7) are finite and, hence,

$$\operatorname{Neg}(V) < \infty.$$

This result is also new.

4. Choose $q > 0$ and consider the potential

$$V(x) = \frac{1}{|x|^2 \ln^2 |x| (\ln \ln |x|)^q} \quad \text{for } |x| > e^2 \tag{2.5}$$

and $V(x) = 0$ for $|x| \le e^2$. A sharp asymptotic for $\operatorname{Neg}(\alpha V)$ as $\alpha \to \infty$ was obtain by Birman and Laptev [2]:

$$\operatorname{Neg}(\alpha V) \sim \text{const} \begin{cases} \alpha, & q \ge 1, \\ \alpha^{1/q}, & q < 1. \end{cases}$$

Let us show how our main estimate (1.7) yields *uniform* upper bounds for $\operatorname{Neg}(\alpha V)$. We have $A_n(V) = 0$ for $n \le 1$, while for $n \ge 2$ we obtain

$$A_n(V) = \int_{e^{2^{n-1}}}^{e^{2^n}} \frac{(1 + \ln r) 2\pi r dr}{r^2 \ln^2 r (\ln \ln r)^q} \simeq \frac{1}{n^q}.$$

Similarly, we have for $n \ge 2$

$$B_n(V) = \left(\int_{e^n}^{e^{n+1}} \frac{r^{2(p-1)} 2\pi r dr}{\left[r^2 \ln^2 r (\ln \ln r)^q\right]^p}\right)^{1/p} \simeq \frac{1}{n^2 \ln^q n}.$$

For a large α we obtain

$$A_n(\alpha V) \simeq \frac{\alpha}{n^q}, \tag{2.6}$$

so that the condition $A_n(\alpha V) > c$ is satisfied for $n \le C\alpha^{1/q}$. It follows that

$$\sum_{\{A_n(\alpha V) > c\}} \sqrt{A_n(\alpha V)} \le C \sum_{n=1}^{\lceil C\alpha^{1/q} \rceil} \sqrt{\frac{\alpha}{n^q}} \simeq C\sqrt{\alpha} \left(\alpha^{1/q}\right)^{1-q/2} = C\alpha^{1/q}.$$

It is clear that $\sum_n B_n(\alpha V) \simeq \alpha$. Hence, we obtain from (1.7)

$$\operatorname{Neg}(\alpha V) \le C\left(\alpha^{1/q} + \alpha\right).$$

If $q \ge 1$ then the leading term here is α, which yields together with (1.2)

$$\operatorname{Neg}(\alpha V) \simeq \alpha.$$

If $q < 1$ then the leading term is $\alpha^{1/q}$, and we obtain

$$\operatorname{Neg}(\alpha V) \leq C\alpha^{1/q}.$$

In the case $q < 1$ we have $\|A\|_{1,\infty} = \infty$, so that neither of the estimates (1.10), (1.11), (1.8), (1.12), (1.17) yields even the finiteness of $\operatorname{Neg}(\alpha V)$, leaving alone the correct rate of growth in α.

5. Let us study the behavior of $\operatorname{Neg}(\alpha V)$ as $\alpha \to \infty$ for a potential V such that

$$\int_{\mathbb{R}^2} V(x)(1 + |\ln|x||)\, dx + \sum_{n \in \mathbb{Z}} B_n(V) < \infty. \tag{2.7}$$

By Corollary 1.2 and (1.2), we obtain

$$c\alpha \int_{\mathbb{R}^2} V dx \leq \operatorname{Neg}(\alpha V) \leq C\alpha \sum_{n \in \mathbb{Z}} B_n(V) + o(\alpha), \quad \alpha \to \infty, \tag{2.8}$$

in particular, $\operatorname{Neg}(\alpha V) \simeq \alpha$. If V satisfies in addition the following condition:

$$\sup_{W_n} V \simeq \inf_{W_n} V, \tag{2.9}$$

for all $n \in \mathbb{Z}$, then

$$B_n(V) \simeq \int_{W_n} V dx,$$

and (2.8) implies that

$$\operatorname{Neg}(\alpha V) \simeq \alpha \int_{\mathbb{R}^2} V(x)\, dx \quad \text{as } \alpha \to \infty. \tag{2.10}$$

For example, (2.7) and, hence, (2.10) are satisfied for the potential (2.5) with $q > 1$. The exact asymptotic for $\operatorname{Neg}(\alpha V)$ as $\alpha \to \infty$ was obtained by Birman and Laptev [2].

6. Set $R = e^{2^m}$ where m is a large integer and consider the following potential on $\mathbb{R}^2$

$$V(x) = \begin{cases} \frac{\alpha}{|x|^2 \ln^2 |x|}, & \text{if } e < |x| < R, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > \frac{1}{4}$. Computing $A_n(V)$ as in (2.3) we obtain $A_n(V) \simeq \alpha$ for any $1 \leq n \leq m$, and $A_n = 0$ otherwise, whence it follows that

$$\sum_{n \in \mathbb{Z}} \sqrt{A_n(V)} \simeq \sqrt{\alpha} m \simeq \sqrt{\alpha} \ln \ln R.$$

Similarly, we have, for $1 \leq n < 2^m$,

$$B_n(V) = \left(\int_{e^n}^{e^{n+1}} \left[\frac{\alpha}{r^2 \ln^2 r} \right]^p r^{2(p-1)} 2\pi r dr \right)^{1/p} \simeq \frac{\alpha}{n^2},$$

and $B_n(V) = 0$ otherwise, whence

$$\sum_{n\in\mathbb{Z}} B_n(V) \simeq \sum_{n=1}^{2^m-1} \frac{\alpha}{n^2} \simeq \alpha.$$

By (1.7) we obtain

$$\operatorname{Neg}(V) \leq C\sqrt{\alpha}\ln\ln R + C\alpha. \tag{2.11}$$

Let us remark that none of the previously known general estimates for $\operatorname{Neg}(V, \mathbb{R}^2)$ yields (2.11). For example, both (1.8) and (1.11) give in this case a weaker estimate

$$\operatorname{Neg}(V) \leq C\alpha\ln\ln R.$$

Obviously, (2.11) requires a full strength of (1.7).

Let us estimate $\operatorname{Neg}(V)$ from below to show the sharpness of (2.11) with respect to the parameters α, R. Consider the function

$$f(x) = \sqrt{\ln|x|}\sin\left(\sqrt{\alpha - \frac{1}{4}}\ln\ln|x|\right)$$

that satisfies in the region $\Omega = \{e < |x| < R\}$ the differential equation $\Delta f + V(x) f = 0$. For any positive integer k, function f does not change sign in the rings

$$\Omega_k := \left\{x \in \mathbb{R}^2 : \pi k < \sqrt{\alpha - \frac{1}{4}}\ln\ln|x| < \pi(k+1)\right\}$$

and vanishes on $\partial\Omega_k$ as long as $\Omega_k \subset \Omega$. Since $\mathcal{E}_{V,\Omega_k}(f) = 0$, using $f|_{\Omega_k}$ as test functions for the energy functional, we obtain $\operatorname{Neg}(V) \geq N$ where N is the number of the rings Ω_k inside Ω. Assuming that $\alpha >> \frac{1}{4}$, we see that $N \simeq \sqrt{\alpha}\ln\ln R$, whence it follows that

$$\operatorname{Neg}(V) \geq c\sqrt{\alpha}\ln\ln R.$$

On the other hand, (1.2) yields $\operatorname{Neg}(V) \geq c\alpha$. Combining these two estimates, we obtain the lower bound

$$\operatorname{Neg}(V) \geq c\left(\sqrt{\alpha}\ln\ln R + \alpha\right),$$

that matches the upper bound (2.11).

Our last example is of a different nature, and we will state it inside the proof of the next proposition.

Proposition 2.1 *No estimate of the type*

$$\operatorname{Neg}(V, \mathbb{R}^2) \leq F\left(\int_{\mathbb{R}^2} V(x)\,\mathcal{W}(x)\,dx\right)$$

can be true for all potentials V on $\mathbb{R}^2$, where $\mathcal{W}$ is non-negative function on $\mathbb{R}^2$ that is bounded in a neighborhood of at least one point and $F : \mathbb{R}_+ \to \mathbb{R}_+$ is any function.

Proof. Assume without loss of generality that $\mathcal{W}(x) \leq C$ for $|x| < \varepsilon$. We will construct a potential V supported in $\{|x| < \varepsilon\}$ such that $\int_{\mathbb{R}^2} V dx < \infty$ while $\operatorname{Neg}(V) = \infty$, which will settle the claim.

It will be easier to construct V as a measure but then it can be routinely approximated by a L^1_{loc}-function. For any $r > 0$, let S_r be the circle $\{|x| = r\}$. We will use the arc length measure δ_{S_r} on S_r. Given two sequences $\{a_n\}$ and $\{b_n\}$ of reals such that $0 < a_n < b_n$, consider the measures

$$V_n = \frac{1}{a_n \ln \frac{b_n}{a_n}} \delta_{S_{a_n}}$$

and test functions

$$\varphi_n(x) = \begin{cases} 1, & |x| < a_n, \\ \frac{\ln \frac{b_n}{|x|}}{\ln \frac{b_n}{a_n}}, & a_n \leq |x| \leq b_n, \\ 0, & |x| > b_n. \end{cases} \tag{2.12}$$

An easy computation shows that

$$\int_{\mathbb{R}^2} |\nabla \varphi_n|^2 dx = \frac{2\pi}{\ln \frac{b_n}{a_n}} \tag{2.13}$$

and

$$\int_{\mathbb{R}^2} \varphi_n^2 V_n dx = \int_{\mathbb{R}^2} V_n dx = \frac{2\pi}{\ln \frac{b_n}{a_n}},$$

whence it follows that $\mathcal{E}_{Vn}(\varphi_n) = 0$.

Let us now specify $a_n = 4^{-n^3}$ and $b_n = 2^{-n^3}$. Consider also the following sequence of points in $\mathbb{R}^2$: $y_n = (4^{-n}, 0)$. Then all the disks $D_{b_n}(y_n)$ with large enough n are disjoint and

$$\sum_{n=1}^{\infty} \frac{2\pi}{\ln \frac{b_n}{a_n}} < \infty. \tag{2.14}$$

Consider the generalized function

$$V = \sum_{n=N}^{\infty} V(\cdot - y_n). \tag{2.15}$$

The functions $\psi_n = \varphi_n(\cdot - y_n)$ have disjoint supports and satisfy $\mathcal{E}_V(\psi_n) = 0$ for all $n \geq N$, whence it follows that $\operatorname{Neg}(V) = \infty$. On the other hand, by (2.14) we have

$$\int_{\mathbb{R}^2} V dx < \infty.$$

By taking N large enough, one can make $\int_{\mathbb{R}^2} V dx$ arbitrarily small and $\operatorname{supp} V$ to be located in an arbitrarily small neighborhood of the origin, while still having $\operatorname{Neg}(V) = \infty$. $\square$

3 Generalities of counting functions

3.1 Index of quadratic forms

Let $\Omega \subset \mathbb{R}^2$ be an arbitrary open set. By a *potential* in $\Omega \subset \mathbb{R}^n$ we mean always a non-negative function from $L^1_{loc}(\Omega)$. Given a potential V in Ω, define the energy form

$$\mathcal{E}_{V,\Omega}(f) = \int_\Omega |\nabla f|^2 dx - \int_\Omega V f^2 dx \tag{3.1}$$

in the domain

$$\mathcal{F}_{V,\Omega} = \left\{ f \in L^2_{loc}(\Omega) : \int_\Omega |\nabla f|^2 dx < \infty, \quad \int_\Omega V f^2 dx < \infty \right\}. \tag{3.2}$$

Clearly, $\mathcal{F}_{V,\Omega}$ is a linear space. Note that a more conventional choice for the ambient space for $\mathcal{F}_{V,\Omega}$ would be $L^2(\Omega)$, but for us a larger space $L^2_{loc}(\Omega)$ will be more convenient.

Set

$$\operatorname{Neg}(V,\Omega) := \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{F}_{V,\Omega} : \mathcal{E}_{V,\Omega}(f) \le 0 \text{ for all } f \in \mathcal{V}\right\}, \tag{3.3}$$

where $\mathcal{V} \prec \mathcal{F}_{V,\Omega}$ means that $\mathcal{V}$ is a linear subspace of $\mathcal{F}_{V,\Omega}$, and the supremum of $\dim \mathcal{V}$ is taken over all subspaces $\mathcal{V}$ such that $\mathcal{E}_{V,\Omega} \le 0$ on $\mathcal{V}$. In other words, $\operatorname{Neg}(V,\Omega)$ is the Morse index of the quadratic form $\mathcal{E}_{V,\Omega}$ in $\mathcal{F}_{V,\Omega}$. Observe that one can restrict in (3.3) the class of subspaces $\mathcal{V}$ to those of finite dimension without changing the value of the right hand side.

Note that $\operatorname{Neg}(V,\Omega) \ge 1$ for any potential V. Indeed, if $V \in L^1(\Omega)$ then $1 \in \mathcal{F}_\Omega$ and $\mathcal{E}_{V,\Omega}(1) \le 0$, which implies that $\operatorname{Neg}(V,\Omega) \ge 1$. If $V \notin L^1(\Omega)$, then consider for any positive integer n a function $f_n(x) = \frac{1}{n}(n - |x|)_+$. This function belongs to $\mathcal{F}_{V,\Omega}$ as it has a compact support, $0 \le f_n \le 1$, and $\int_\Omega |\nabla f_n|^2 dx \le \pi$. Since $f_n \uparrow 1$ as $n \to \infty$, it follows that

$$\int_\Omega V f_n^2 dx \to \int_\Omega V dx = \infty.$$

Hence, for large enough n, we obtain $\mathcal{E}_{V,\Omega}(f_n) < 0$ and, hence, $\operatorname{Neg}(V,\Omega) \ge 1$.

If $\Omega = \mathbb{R}^n$ then we use the abbreviations

$$\mathcal{E}_V \equiv \mathcal{E}_{V,\mathbb{R}^n}, \quad \mathcal{F}_V \equiv \mathcal{F}_{V,\mathbb{R}^n}, \quad \operatorname{Neg}(V) \equiv \operatorname{Neg}(V,\mathbb{R}^n).$$

The operator

$$H_V = -\Delta - V$$

is defined as a self-adjoint operator in $L^2(\mathbb{R}^n)$ using the following standard procedure. Firstly, observe that the classical Dirichlet integral

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

with the domain $W^{1,2}(\mathbb{R}^2)$ is a closed form in $L^2(\mathbb{R}^2)$, and the quadratic form

$$u \mapsto \int_{\mathbb{R}^n} V u^2 dx$$

associated with the multiplication operator $u \mapsto Vu$, is closed with the domain $L^2(dx) \cap L^2(Vdx)$. Clearly, the form $\mathcal{E}_V$ is well-defined in the domain

$$\mathcal{D}_V = W^{1,2} \cap L^2(Vdx)$$

that is a subspace of $\mathcal{F}_V$. Under certain assumptions about V, the form $(\mathcal{E}_V, \mathcal{D}_V)$ is closed in L^2 (and, in fact, $\mathcal{D}_V = W^{1,2}$). Consequently, its generator, denoted by H_V, is a self-adjoint, semi-bounded below operator in L^2, whose domain is a subspace of $\mathcal{D}_V$.

For any self-adjoint operator A, denote by $\operatorname{Neg}(A)$ the rank of the operator $\mathbf{1}_{(-\infty,0]}(A)$, that is,

$$\operatorname{Neg}(A) = \dim \operatorname{Im} \mathbf{1}_{(-\infty,0]}(A).$$

If the spectrum of A below 0 is discrete then $\operatorname{Neg}(A)$ coincides with the number of non-positive eigenvalues of A counted with multiplicities.

Lemma 3.1 *If the form $(\mathcal{E}_V, \mathcal{D}_V)$ is closed and, hence, H_V is well-defined, then*

$$\operatorname{Neg}(H_V) \leq \operatorname{Neg}(V). \tag{3.4}$$

Proof. It is well-known that

$$\operatorname{Neg}(H_V) = \sup\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_V \text{ and } \mathcal{E}_V(f) \leq 0\ \forall f \in \mathcal{V}\}$$

(cf. [12, Lemma 2.7]). Since $\mathcal{D}_V \subset \mathcal{F}_V$, (3.4) holds by monotonicity argument. □

Theorem 1.1 states the upper bound for $\operatorname{Neg}(V)$, which implies then by Lemma 3.1 the same bound for $\operatorname{Neg}(H_V)$ whenever H_V is well-defined. If this method were applied in $\mathbb{R}^n$ with $n \geq 3$ then the resulting estimate would not have been satisfactory, because $\operatorname{Neg}(H_V)$ can be 0 (as follows, for example, from (1.1)), whereas $\operatorname{Neg}(V) \geq 1$ for all potentials V as it was remarked above. However, our aim is $\mathbb{R}^2$, where $\operatorname{Neg}(H_V) \geq 1$ for any non-zero potential V, so that we do not lose 1 in the estimate.

In the rest of this section we prove some general properties of $\operatorname{Neg}(V, \Omega)$ that will be used in the next sections. For a bounded domain Ω with smooth boundary, the form $\mathcal{E}_{V,\Omega}$ can be associated with the operator $\Delta + V$ in Ω with the Neumann boundary condition on $\partial\Omega$. In this case $\operatorname{Neg}(V, \Omega)$ is equal to the number of non-positive eigenvalues of the Neumann problem in Ω for the operator $\Delta + V$. This understanding helps the intuition, but technically we never need to use the operator $\Delta + V$. Nor the closability of the form $\mathcal{E}_{V,\Omega}$ is needed, except for Lemma 3.1.

Lemma 3.2 *Let $\Omega, \widetilde{\Omega}$ be open subsets of $\mathbb{R}^2$ and V and $\widetilde{V}$ be potentials in Ω and $\widetilde{\Omega}$, respectively. Let $\mathcal{L} : \mathcal{F}_{V,\Omega} \to \mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$ be a linear injective mapping.*

(a) *If $\mathcal{E}_{V,\Omega}(u) \leq 0$ implies $\mathcal{E}_{\widetilde{V},\widetilde{\Omega}}(\widetilde{u}) \leq 0$ for $\widetilde{u} = \mathcal{L}(u)$ then*

$$\operatorname{Neg}(V, \Omega) \leq \operatorname{Neg}(\widetilde{V}, \widetilde{\Omega}). \tag{3.5}$$

(*b*) *Assume that there are positive constants* c_1, c_2, *such that, for any* $u \in \mathcal{F}_{V,\Omega}$, *the function* $\widetilde{u} = \mathcal{L}(u)$ *satisfies*

$$\int_{\widetilde{\Omega}} |\nabla \widetilde{u}|^2 \, dx \leq c_1 \int_{\Omega} |\nabla u|^2 \, dx \tag{3.6}$$

and

$$\int_{\widetilde{\Omega}} \widetilde{V} \widetilde{u}^2 dx \geq c_2 \int_{\Omega} V u^2 dx. \tag{3.7}$$

Then

$$\operatorname{Neg}(V, \Omega) \leq \operatorname{Neg}(\frac{c_1}{c_2} \widetilde{V}, \widetilde{\Omega}). \tag{3.8}$$

Proof. (*a*) Let $\mathcal{V}$ be a finitely dimensional linear subspace of $\mathcal{F}_\Omega$ where $\mathcal{E}_{V,\Omega} \leq 0$. Then $\widetilde{\mathcal{V}} := \mathcal{L}(\mathcal{V})$ is a linear subspace of $\mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$ of the same dimension. For any $\widetilde{u} \in \widetilde{\mathcal{V}}$ we have $\mathcal{E}_{\widetilde{V},\widetilde{\Omega}}(\widetilde{u}) \leq 0$, which implies $\dim \widetilde{\mathcal{V}} \leq \operatorname{Neg}(\widetilde{V}, \widetilde{\Omega})$. Since $\dim \mathcal{V} = \dim \widetilde{\mathcal{V}}$, we have also $\dim \mathcal{V} \leq \operatorname{Neg}(\widetilde{V}, \widetilde{\Omega})$, whence (3.5) follows.

(*b*) If $\mathcal{E}_{V,\Omega}(u) \leq 0$ then

$$\begin{aligned} \mathcal{E}_{\frac{c_1}{c_2}\widetilde{V},\widetilde{\Omega}}(\widetilde{u}) &= \int_{\widetilde{\Omega}} |\nabla \widetilde{u}|^2 \, dx - \frac{c_1}{c_2} \int_{\widetilde{\Omega}} \widetilde{V} \widetilde{u}^2 dx \\ &\leq c_1 \int_{\Omega} |\nabla u|^2 \, dx - c_1 \int V u^2 dx = c_1 \mathcal{E}_{V,\Omega}(u) \leq 0. \end{aligned}$$

Applying part (*a*) with $\frac{c_1}{c_2}\widetilde{V}$ instead of $\widetilde{V}$, we obtain (3.8). □

Lemma 3.3 *Let* Ω *be any open subset of* $\mathbb{R}^2$, *and* K *be a closed subset of* $\mathbb{R}^n$ *of measure* 0. *Set* $\Omega' = \Omega \setminus K$. *Then we have*

$$\operatorname{Neg}(V, \Omega) \leq \operatorname{Neg}(V, \Omega'). \tag{3.9}$$

Proof. Every function $u \in \mathcal{F}_{V,\Omega}$ can be considered as an element of $\mathcal{F}_{V,\Omega'}$ simply by restricting u to Ω'. Since the difference $\Omega \setminus \Omega'$ has measure 0, we have $\mathcal{E}_{V,\Omega}(u) = \mathcal{E}_{V,\Omega'}(u)$. Then Lemma 3.2(*a*) implies (3.9). □

Definition 3.4 We say that a (finite or infinite) sequence $\{\Omega_k\}$ of non-empty open sets $\Omega_k \subset \mathbb{R}^2$ is a *partition* of an open set $\Omega \subset \mathbb{R}^n$ if all the sets Ω_k are disjoint, $\Omega_k \subset \Omega$, and $\overline{\Omega} \setminus \bigcup_k \Omega_k$ has measure 0 (cf. Fig. 1).

Lemma 3.5 *If* $\{\Omega_k\}$ *is a partition of* Ω, *then*

$$\operatorname{Neg}(V, \Omega) \leq \sum_k \operatorname{Neg}(V, \Omega_k). \tag{3.10}$$

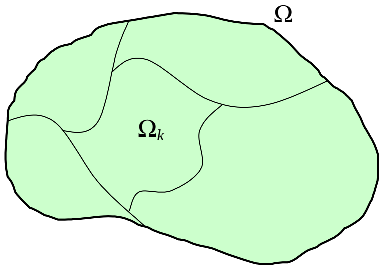

Figure 1: A partition of Ω

Proof. Set $\Omega' = \bigcup_k \Omega_k$ and $K = \overline{\Omega} \setminus \Omega'$. Since K is closed, K has measure 0, and $\Omega' = \Omega \setminus K$, we obtain by Lemma 3.3 that

$$\operatorname{Neg}(V, \Omega) \leq \operatorname{Neg}(V, \Omega').$$

Next, we claim that

$$\operatorname{Neg}(V, \Omega') \leq \sum_k \operatorname{Neg}(V, \Omega_k). \tag{3.11}$$

If the sum in (3.11) is infinite then there is nothing to prove. Assume that this sum is finite. Since $\operatorname{Neg}(V, \Omega_k) \geq 1$, the number of elements in the partition $\{\Omega_k\}$ must be finite, which will be assumed in the sequel. Denote for simplicity $\mathcal{F}' = \mathcal{F}_{V,\Omega'}$, $\mathcal{E}' = \mathcal{E}_{V,\Omega'}$, $\mathcal{F}_k = \mathcal{F}_{V,\Omega_k}$ and $\mathcal{E}_k = \mathcal{E}_{V,\Omega_k}$.

For any $f \in \mathcal{F}'$ and index k, set $f_k = f|_{\Omega_k}$ so that $f_k \in \mathcal{F}_k$. Clearly, we have $f = \sum_k f_k$ and

$$\mathcal{E}'(f) = \sum_k \mathcal{E}_k(f_k). \tag{3.12}$$

Hence, $\mathcal{F}'$ can be identified as a subspace of the direct sum $\mathcal{F} = \bigoplus \mathcal{F}_k$, and $\mathcal{E}'$ can be extended from $\mathcal{F}'$ to $\mathcal{F}$ by (3.12), as the direct sum of all $\mathcal{E}_k$.

Let $\mathcal{V}$ be a finite dimensional subspace of $\mathcal{F}'$ (or even of $\mathcal{F}$) where $\mathcal{E}' \leq 0$. Restricting as above the functions from $\mathcal{V}$ to Ω_k, we obtain a finite dimensional subspace $\mathcal{V}_k$ of $\mathcal{F}_k$. Set $\mathcal{U} = \bigoplus \mathcal{V}_k$, so that $\mathcal{V} \prec \mathcal{U} \prec \mathcal{F}$. The quadratic form $\mathcal{E}_k$ is diagonalizable on the finite dimensional space $\mathcal{V}_k$, and the number N_k of the non-positive terms in the signature of $\mathcal{E}_k$ on $\mathcal{V}_k$ is clearly bounded by $\operatorname{Neg}(V, \Omega_k)$. Hence, denoting by N the number of the non-positive terms in the signature of $\mathcal{E}'$ on $\mathcal{U}$, we obtain

$$N = \sum_k N_k \leq \sum_k \operatorname{Neg}(V, \Omega_k).$$

If $\dim \mathcal{V} > N$ then $\mathcal{V}$ intersects the subspace of $\mathcal{U}$ where $\mathcal{E}'$ is positive definite, which contradicts the assumption that $\mathcal{E}' \leq 0$ on $\mathcal{V}$. Therefore, $\dim \mathcal{V} \leq N$, whence (3.11) follows. $\square$

Lemma 3.6 *If V_1, V_2 are two potentials in Ω then*

$$\operatorname{Neg}(V_1 + V_2, \Omega) \leq \operatorname{Neg}(2V_1, \Omega) + \operatorname{Neg}(2V_2, \Omega). \tag{3.13}$$

Proof. Let us write for simplicity $\mathcal{E}_{V,\Omega} \equiv \mathcal{E}_V$ and $\mathcal{F}_{V,\Omega} \equiv \mathcal{F}_V$. Set $V = V_1 + V_2$ and observe that by (3.2)

$$\mathcal{F}_V = \mathcal{F}_{V_1} \cap \mathcal{F}_{V_2}$$

and by (3.1)

$$2\mathcal{E}_V = \mathcal{E}_{2V_1} + \mathcal{E}_{2V_2} \text{ on } \mathcal{F}_V. \tag{3.14}$$

Assume that (3.13) is not true. Then there exists a finite-dimensional subspace $\mathcal{V}$ of $\mathcal{F}_V$ where $\mathcal{E}_V \leq 0$ and such that

$$\dim \mathcal{V} > \operatorname{Neg}(2V_1) + \operatorname{Neg}(2V_2). \tag{3.15}$$

Set $N = \dim \mathcal{V}$ and denote by N_i, $i = 1, 2$, the maximal dimension of a subspace of $\mathcal{V}$ where $\mathcal{E}_{2V_i} \leq 0$. Then there exists a subspace $\mathcal{P}_i$ of $\mathcal{V}$ of dimension $N - N_i$ where $\mathcal{E}_{2V_i} \geq 0$. The intersection $\mathcal{P}_1 \cap \mathcal{P}_2$ has dimension at least

$$(N - N_1 + N - N_2) - N = N - (N_1 + N_2) > 0,$$

where the positivity holds by (3.15). By (3.14) the form $\mathcal{E}_V$ is non-negative on $\mathcal{P}_1 \cap \mathcal{P}_2$, which contradicts the assumption that $\mathcal{E}_V \leq 0$ on $\mathcal{V}$. $\square$

3.2 Transformation of potentials and weights

Given a 2×2 matrix $A = (a_{ij})$, denote by $\|A\|$ the norm of A as an linear operator in $\mathbb{R}^2$ with the Euclidean norm. Denote also

$$\|A\|_2 := \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}.$$

It is easy to see that

$$\frac{1}{\sqrt{2}} \|A\|_2 \leq \|A\| \leq \|A\|_2$$

Assuming further that A is non-singular, define the quantities

$$M(A) := \frac{\|A\|^2}{\det A} \quad \text{and} \quad M_2(A) := \frac{\|A\|_2^2}{\det A}$$

For example, if A is a conformal matrix, that is, $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ or $\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$, then

$$\det A = \alpha^2 + \beta^2 = \|A\|^2,$$

whence $M(A) = 1$.

For a general non-singular matrix A, the following identity holds:

$$M_2(A) = M_2\left(A^{-1}\right). \tag{3.16}$$

Indeed, denoting $a = \det A$, we obtain

$$A^{-1} = \frac{1}{a} \begin{pmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{pmatrix},$$

whence $\|A^{-1}\|_2^2 = \frac{1}{a^2}\|A\|_2^2$, which implies (3.16). Consequently, we obtain that, for any non-singular matrix A,

$$\frac{1}{2}M(A) \le M\left(A^{-1}\right) \le 2M(A). \tag{3.17}$$

Let Ω and $\widetilde{\Omega}$ be two open subsets of $\mathbb{R}^2$ and $\Phi : \widetilde{\Omega} \to \Omega$ be a C^1-diffeomorphism. Denote by Φ' its Jacobi matrix and by J_Φ - its Jacobian, that it $J_\Phi = \det \Phi'$. Set

$$M_\Phi := \sup_{x\in\widetilde{\Omega}} M(\Phi'(x)) = \sup_{x\in\widetilde{\Omega}} \frac{\|\Phi'(x)\|^2}{|J_\Phi(x)|}.$$

We will use two types of mappings Φ : bilipschitz and conformal. If Φ is conformal then we have $M_\Phi = 1$. Moreover, if Φ is holomorphic then

$$J_\Phi(z) = |\Phi'(z)|^2, \tag{3.18}$$

where now $\Phi' = \frac{d\Phi}{dz}$ is a complex derivative in $z \in \mathbb{C}$.

If Φ is bilipschitz and with bilipschitz constant L then an easy calculation shows that $\|\Phi'(x)\|^2 \le 4L^2$ and that both $|J_\Phi|$ and $|J_{\Phi^{-1}}|$ are bounded by $2L^2$ whence $M_\Phi \le 8L^4$.

By (3.17), we always have

$$\frac{1}{2}M_\Phi \le M_{\Phi^{-1}} \le 2M_\Phi \tag{3.19}$$

The next lemma establishes the behavior of $\operatorname{Neg}(V,\Omega)$ and certain integrals over Ω under transformations of Ω. By a weight function on Ω we mean any non-negative function from $L^1_{loc}(\Omega)$.

Lemma 3.7 *Let $\Omega, \widetilde{\Omega}$ be two open subsets of $\mathbb{R}^2$ and*

$$\Psi : \Omega \to \widetilde{\Omega}$$

be a C^1 diffeomorphism with a finite M_Ψ. Set $\Phi = \Psi^{-1}$.

(a) For any potential V on Ω, define a Ψ-push-forward potential $\widetilde{V}$ on $\widetilde{\Omega}$ by

$$\widetilde{V}(y) = M_\Phi |J_\Phi(y)| V(\Phi(y)). \tag{3.20}$$

Then

$$\operatorname{Neg}(V,\Omega) \le \operatorname{Neg}(\widetilde{V},\widetilde{\Omega}). \tag{3.21}$$

(b) For any $p \ge 1$ and any weight function W on Ω, define a Ψ-push-forward weight function $\widetilde{W}$ on $\widetilde{\Omega}$ by

$$\widetilde{W}(y) = M_\Phi^{-p} |J_\Phi(y)|^{1-p} W(\Phi(y)). \tag{3.22}$$

Then we the following identity holds

$$\int_\Omega V(x)^p W(x)\,dx = \int_{\widetilde{\Omega}} \widetilde{V}(y)^p \widetilde{W}(y)\,dy \tag{3.23}$$

As one sees from (3.20) and (3.22), the rules of change of a potential and a weight function under a mapping Ψ are different.

Proof. (a) Let $\mathcal{V}$ be a subspace of $\mathcal{F}_{V,\Omega}$ as in (3.3). Define $\widetilde{\mathcal{V}}$ as the pullback of $\mathcal{V}$ under the mapping Φ, that is, any function $\widetilde{f} \in \widetilde{\mathcal{V}}$ has the form

$$\widetilde{f}\left(y\right) = f\left(\Phi\left(y\right)\right)$$

for some $f \in \mathcal{V}$. Let us show that $\widetilde{f} \in \mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$. That $\widetilde{f} \in L^2_{loc}(\widetilde{\Omega})$ is obvious. Using the change $y = \Psi\left(x\right)$ (or $x = \Phi\left(y\right)$), we obtain

$$\begin{aligned}
\int_{\widetilde{\Omega}} \left|\widetilde{f}\left(y\right)\right|^2 \widetilde{V}\left(y\right) dy &= \int_{\Omega} \left|\widetilde{f}\left(y\right)\right|^2 \widetilde{V}\left(y\right) \left|J_\Psi\left(x\right)\right| dx \\
&= \int_{\Omega} \left|f\left(x\right)\right|^2 M_\Phi V\left(x\right) \left|J_\Phi\left(y\right)\right| \left|J_\Phi\left(y\right)\right|^{-1} dx \\
&= M_\Phi \int_{\Omega} \left|f\left(x\right)\right|^2 V\left(x\right) dx \qquad (3.24)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\widetilde{\Omega}} \left|\nabla \widetilde{f}\left(y\right)\right|^2 dy &= \int_{\widetilde{\Omega}} \left|\left(\nabla f\right)\left(\Phi\left(y\right)\right) \cdot \Phi'\left(y\right)\right|^2 dy \\
&\leq \int_{\widetilde{\Omega}} \left\|\Phi'\left(y\right)\right\|^2 \left|\nabla f\right|^2 \left(\Phi\left(y\right)\right) dy \\
&\leq M_\Phi \int_{\widetilde{\Omega}} \left|J_\Phi\left(y\right)\right| \left|\nabla f\right|^2 \left(\Phi\left(y\right)\right) dy \\
&= M_\Phi \int_{\Omega} \left|\nabla f\right|^2 \left(x\right) dx. \qquad (3.25)
\end{aligned}$$

It follows from (3.24) and (3.25) that $\widetilde{f} \in \mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$ and $\mathcal{E}_{\widetilde{V},\widetilde{\Omega}}(\widetilde{f}) \leq M_\Phi \mathcal{E}_{V,\Omega}\left(f\right)$. Applying Lemma 3.2 to the mapping $f \mapsto \widetilde{f}$, we obtain (3.21).

(b) Using the same change in integral, we obtain

$$\begin{aligned}
\int_{\widetilde{\Omega}} \widetilde{V}\left(y\right)^p \widetilde{W}\left(y\right) dy &= \int_{\Omega} \widetilde{V}\left(\Psi\left(x\right)\right)^p \widetilde{W}\left(\Psi\left(x\right)\right) \left|J_\Psi\left(x\right)\right| dx \\
&= \int_{\Omega} \left(M_\Phi V\left(x\right) \left|J_\Phi\left(y\right)\right|\right)^p M_\Phi^{-p} \left|J_\Phi\left(y\right)\right|^{1-p} W\left(x\right) \left|J_\Phi\left(y\right)\right|^{-1} dx \\
&= \int_{\Omega} V\left(x\right)^p W\left(x\right) dx.
\end{aligned}$$

$\square$

Remark 3.8 If Ψ is conformal then it follows from Lemma 3.7 that

$$\operatorname{Neg}\left(V, \Omega\right) = \operatorname{Neg}(\widetilde{V}, \widetilde{\Omega}),$$

where

$$\widetilde{V}\left(y\right) = \left|J_\Phi\left(y\right)\right| V\left(\Phi\left(y\right)\right).$$

Furthermore, if Ψ is holomorphic then the formulas (3.20) and (3.22) can be simplified as follows:

$$\widetilde{V}(z) = |\Phi'(z)|^2 V(\Phi(z))$$

and

$$\widetilde{W}(z) = \frac{W(\Phi(z))}{|\Phi'(z)|^{2(p-1)}},$$

where Φ' is a $\mathbb{C}$-derivative.

3.3 Bounded test functions

Consider the following modification of the space $\mathcal{F}_{V,\Omega}$:

$$\mathcal{F}^b_{V,\Omega} = \left\{ f \in L^\infty(\Omega) : \int_\Omega |\nabla f|^2 dx < \infty, \quad \int_\Omega V f^2 dx < \infty \right\} \tag{3.26}$$

and of the counting function:

$$\mathrm{Neg}^b(V,\Omega) := \sup \left\{ \dim \mathcal{V} : \mathcal{V} \prec \mathcal{F}^b_{V,\Omega} : \mathcal{E}_{V,\Omega}(f) \le 0 \text{ for all } f \in \mathcal{V} \right\}. \tag{3.27}$$

In short, we restrict consideration to the class of bounded test functions. By monotonicity we have

$$\mathrm{Neg}^b(V,\Omega) \le \mathrm{Neg}(V,\Omega).$$

The following claim will be used in Section 7.1.

Lemma 3.9 *Let Ω be a connected domain in $\mathbb{R}^2$ such that $\mathrm{Neg}^b(2V,\Omega) = 1$. Then $\mathrm{Neg}(V,\Omega) = 1$.*

Proof. Assume that $\mathrm{Neg}(V,\Omega) > 1$. Then there exists a two-dimensional subspace $\mathcal{V}$ of $\mathcal{F}_{V,\Omega}$ such that $\mathcal{E}_{V,\Omega} \le 0$ on $\mathcal{V}$. Consider the following two functions on $\mathcal{V}$:

$$X(f) = \int_\Omega |\nabla f_+|^2 dx - 2\int_\Omega V f_+^2 dx \tag{3.28}$$

and

$$Y(f) = \int_\Omega |\nabla f_-|^2 dx - 2\int_\Omega V f_-^2 dx, \tag{3.29}$$

where $f_\pm = \frac{1}{2}(|f| \pm f)$ are the positive and negative parts of f. Clearly, we have

$$X(f) + Y(f) = \int_\Omega |\nabla f|^2 dx - 2\int_\Omega V f^2 dx = \mathcal{E}_{2V,\Omega}(f) \le 0.$$

Let us show that in fact a strict inequality holds for all $f \in \mathcal{V} \setminus \{0\}$:

$$X(f) + Y(f) < 0. \tag{3.30}$$

Indeed, if this is not true, that is,

$$\int_\Omega |\nabla f|^2 dx \ge 2\int_\Omega V f^2 dx, \tag{3.31}$$

then combining with

$$2\int_{\Omega}|\nabla f|^2\,dx\leq 2\int_{\Omega}Vf^2dx,$$

we obtain $\int_{\Omega}|\nabla f|^2\,dx=0$ and, hence, $f=\text{const}$ in Ω. Then (3.31) implies $V=0$ in Ω, which is not possible by the assumption $\operatorname{Neg}(V,\Omega)>1$. This proves (3.30).

A second observation that we need is the identities

$$X(-f)=Y(f)\quad\text{and}\quad Y(-f)=X(f)\,, \tag{3.32}$$

that follow immediately from the definitions (3.28), (3.29).

Now consider a mapping $F:\mathcal{V}\to\mathbb{R}^2$ given by

$$F(f)=(X(f)\,,Y(f))\,.$$

Let T be the unit circle in $\mathcal{V}$ (with respect to some arbitrary norm in $\mathcal{V}$). Then the image $F(T)$ is a compact connected subset of $\mathbb{R}^2$ that by (3.30) lies in the half-plane $\{x+y<0\}$, and by (3.32) is symmetric in the diagonal $x=y$. It follows that there is a point in $F(T)$ that lies on the diagonal $x=y$, that is, there is a function $f\in\mathcal{V}\setminus\{0\}$ such that

$$X(f)=Y(f)<0.$$

This can be rewritten in the form

$$\mathcal{E}_{2V,\Omega}(f_+)=\mathcal{E}_{2V,\Omega}(f_-)<0.$$

Since

$$\mathcal{E}_{2V,\Omega}(f\wedge n)\to\mathcal{E}_{2V,\Omega}(f)\quad\text{ as } n\to+\infty,$$

it follows that there is large enough n such that

$$\mathcal{E}_{2V,\Omega}(f_+\wedge n)<0,\quad\mathcal{E}_{2V,\Omega}(f_-\wedge n)<0.$$

The functions $f_+\wedge n$ and $f_-\wedge n$ are bounded and have "almost" disjoint supports. It follows that $\mathcal{E}_{2V,\Omega}(f)\leq 0$ holds for all linear combinations f of these two functions. Hence, we obtain a two dimensional subspace of $\mathcal{F}^b_{2V,\Omega}$ where $\mathcal{E}_{2V,\Omega}\leq 0$, which implies $\operatorname{Neg}^b(2V,\Omega)\geq 2$. This contradiction finishes the proof. $\square$

4 L^p-estimate in bounded domains

In this section we obtain upper bound for $\operatorname{Neg}(V,\Omega)$ for certain bounded domains $\Omega\subset\mathbb{R}^2$.

4.1 Extension of functions from $\mathcal{F}_{V,\Omega}$

Here we consider auxiliary techniques for extending functions from $\mathcal{F}_{V,\Omega}$ to larger domains. Denote by $D_r(x)$ an open disk in $\mathbb{R}^2$ of radius r centered at x.

Lemma 4.1 *Let Ω be a domain in $\mathbb{R}^2$ with piecewise smooth boundary. Then $\mathcal{F}_{V,\Omega} \subset L^2_{loc}\left(\overline{\Omega}\right)$, where $\overline{\Omega}$ is the closure of Ω. If in addition Ω is bounded then $\mathcal{F}_{V,\Omega} \subset L^2(\Omega)$.*

Proof. Fix a point $x \in \partial\Omega$ and consider the domain $U = \Omega \cap D_r(x)$ where $r > 0$ is sufficiently small. It suffices to verify that

$$f \in L^2_{loc}(\Omega), \quad \nabla f \in L^2(\Omega) \ \Rightarrow \ f \in L^2(U). \tag{4.1}$$

Choose a little disk K inside U. For any function $f \in W^{1,2}_{loc}(U)$ we have the following Poincaré type inequality:

$$\int_U f^2 dx \le C \int_U |\nabla f|^2 dx + C \int_K f^2 dx \tag{4.2}$$

where $C = C(K,U)$. Since the right hand side of (4.2) is finite by hypotheses, it follows that $f \in L^2(U)$, which was to be proved. $\square$

Lemma 4.1 can be used to extend functions from $\mathcal{F}_{V,\Omega}$ to $\mathcal{F}_{V,\Omega'}$ where Ω' is a larger domain. Any potential V in a domain Ω can be extended to a larger domain Ω' by setting $V = 0$ outside Ω. We will refer to such an extension as a trivial one.

Let us give two examples, which will be frequently used in the next sections. In all cases we assume that V is trivially extended from Ω to Ω'.

Example 4.2 Let Ω be a rectangle and let L be one of its sides. Merging Ω with its image under the axial symmetry around L, we obtain a larger rectangle Ω'. Any function f on Ω can be extended to Ω' using push-forward under the axial symmetry. We claim that if $f \in \mathcal{F}_{V,\Omega}$ then the extended function f belongs to $\mathcal{F}_{V,\Omega'}$. By Lemma 4.1 we have $f \in L^2(\Omega)$ and, hence, $f \in W^{1,2}(\Omega)$. It is well-known that if a $W^{1,2}$ function extends by axial symmetry then the resulting function is again from $W^{1,2}$, which implies that $f \in \mathcal{F}_{V,\Omega'}$.

Example 4.3 Let Ω be a sector of a disk $D_r(x_0)$ and let C be a circular part of ∂U. Let us merge Ω with its image under the inversion in C and denote the resulting wedge by Ω'. Extend any function f from Ω to Ω' using push-forward under the inversion. Let us show that if $f \in \mathcal{F}_{V,\Omega}$ then the extended function f belongs to $\mathcal{F}_{V,\Omega'}$. Set $U = \Omega \setminus \overline{D_\varepsilon(x_0)}$ with some $\varepsilon > 0$ so that U is away from the center of inversion. Let U' be obtained by merging U with its image under inversion. By Lemma 4.1, any function $f \in \mathcal{F}_{V,\Omega}$ belongs to $L^2(U)$ and, hence, to $W^{1,2}(U)$. Since U' is bounded, the extended function f belongs also to $W^{1,2}(U')$, which implies that $f \in W^{1,2}_{loc}(\Omega')$. By the conformal invariance of the Dirichlet integral we have

$$\int_{\Omega'\setminus\Omega} |\nabla f|^2 dx = \int_\Omega |\nabla f|^2 dx,$$

which implies that $\int_{\Omega'} |\nabla f|^2 dx < \infty$ and, hence, $f \in \mathcal{F}_{V,\Omega'}$.

Let $H_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ be an upper half-plane.

Lemma 4.4 *For any potential V in H_+, we have*

$$\operatorname{Neg}(V, H_+) \le \operatorname{Neg}\left(2V, \mathbb{R}^2\right), \tag{4.3}$$

assuming that V is trivially extended from H_+ to $\mathbb{R}^2$.

Proof. Any function $f \in \mathcal{F}_{V,H_+}$ can be extended to a function f on $\mathbb{R}^2$ by the axial symmetry around the axis x_1. Since by Lemma 4.1 $f \in L^2(U)$ for any bounded open subset U of H_+, in particular, for any rectangle U attached to ∂H_+, we obtain as in Example 4.2 that $f \in W^{1,2}_{loc}(\mathbb{R}^2)$. Since also

$$\int_{\mathbb{R}^2} |\nabla f|^2 \, dx = 2 \int_{H_+} |\nabla f|^2 \, dx$$

and

$$\int_{\mathbb{R}^2} V f^2 dx = \int_{H_+} V f^2 dx,$$

we see that $f \in \mathcal{F}_{V,\mathbb{R}^2}$, and the estimate (4.3) follows by Lemma 3.2. $\square$

Let $D_r = D_r(0)$ be an open disk of radius r centered at the origin.

Lemma 4.5 *For any potential V in a disk D_r,*

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \le \operatorname{Neg}\left(2V, D_r\right), \tag{4.4}$$

assuming that V is trivially extended from D to $\mathbb{R}^2$.

Proof. Any function $f \in \mathcal{F}_{V,D_r}$ can be extended to a function $f \in \mathcal{F}_{V,\mathbb{R}^2}$ using inversion in the circle $\{|x| = r\}$ as in Example 4.3. Then we have

$$\int_{\mathbb{R}^2} |\nabla f|^2 \, dx = 2 \int_{D_r} |\nabla f|^2 \, dx$$

and

$$\int_{\mathbb{R}^2} V f^2 dx = \int_{D_r} V f^2 dx,$$

which implies (4.4) by Lemma 3.2. $\square$

A more complicated result analogous to Lemmas 4.4 and 4.5 will be considered in Section 7.2.

4.2 One negative eigenvalue in a disc

Let $D = \{|x| < 1\}$ be the open unit disk in $\mathbb{R}^2$.

Lemma 4.6 *For any $p > 1$ there is $\varepsilon > 0$ such that, for any potential V in D,*

$$\|V\|_{L^p(D)} \le \varepsilon \ \Rightarrow \ \operatorname{Neg}(V, D) = 1. \tag{4.5}$$

Proof. Extend V to entire $\mathbb{R}^2$ by setting $V(x) = 0$ for all $|x| \ge 1$. Given a function $u \in \mathcal{F}_{V,D}$, extend u to the entire $\mathbb{R}^2$ using the inversion $\Phi(x) = \frac{x}{|x|^2}$: for any $|x| > 1$, set $u(x) = u(\Phi(x))$. As in Example 4.3, we have $u \in \mathcal{F}_{V,\mathbb{R}^2}$. By the conformal invariance of the Dirichlet integral, we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx = 2 \int_D |\nabla u|^2 \, dx. \tag{4.6}$$

Choose a cutoff function φ such that $\varphi|_{D_2} \equiv 1$, $\varphi|_{\mathbb{R}^2\setminus D_3} = 0$ and $\varphi = \varphi(|x|)$ is linear in $|x|$ in $D_3 \setminus D_2$, and define a function u^* by

$$u^* = u\varphi.$$

Then $u^* \in W^{1,2}(\mathbb{R}^2)$ and u^* vanishes outside D_3. Next, we prove some estimates for the function u^*.

Claim 1. *We have*

$$\int_{D_3} |\nabla u^*|^2 dx \leq 4 \int_D |\nabla u|^2 dx + 162 \int_D u^2 dx. \tag{4.7}$$

Indeed, since $\nabla u^* = \varphi \nabla u + u \nabla \varphi$, we have

$$\begin{aligned} \int_{D_3} |\nabla u^*|^2 dx &\leq 2 \int_{D_3} \varphi^2 |\nabla u|^2 dx + 2 \int_{D_3} u^2 |\nabla \varphi|^2 dx \\ &\leq 2 \int_{\mathbb{R}^2} |\nabla u|^2 dx + 2 \int_{D_3\setminus D_2} u^2 dx, \end{aligned}$$

where we have used that $|\nabla \varphi| = 1$ in $D_3 \setminus D_2$ and $\nabla \varphi = 0$ otherwise. Next, use the change $y = \Phi(x)$ to map $D_3 \setminus D_2$ to $D_{1/2} \setminus D_{1/3}$. Since $|J_{\Phi^{-1}}(y)| = \frac{1}{|y|^4}$, we obtain

$$\int_{D_3\setminus D_2} u^2(x)\, dx = \int_{D_{1/2}\setminus D_{1/3}} u^2(y) \frac{1}{|y|^4} dy \leq 3^4 \int_D u^2 dy.$$

Combining the above estimates and using also (4.6), we obtain (4.7).

Claim 2. *If $u \perp 1$ in $L^2(D)$ and $\mathcal{E}_{V,D}(u) \leq 0$ then*

$$\int_{D_4} |\nabla u^*|^2 dx \leq C \int_D V u^2 dx, \tag{4.8}$$

with some absolute constant C.

Indeed, the assumption $u \perp 1$ implies by the Poincaré inequality

$$\int_D u^2 dx \leq c \int_D |\nabla u|^2 dx,$$

which together with (4.7) yields

$$\int_{D_4} |\nabla u^*|^2 dx \leq (4 + 162c) \int_D |\nabla u|^2 dx.$$

Combining this with the hypothesis $\mathcal{E}_V(u) \leq 0$, that is,

$$\int_D |\nabla u|^2 dx \leq \int_D V u^2 dx, \tag{4.9}$$

we obtain (4.8).

Now we prove the implication (4.5). Applying the Hölder inequality to the right hand side of (4.8), we obtain

$$\begin{aligned}
\int_D Vu^2 dx &\le \left(\int_D V^p dx\right)^{1/p} \left(\int_D |u|^{\frac{2p}{p-1}}\, dx\right)^{1-1/p} \\
&\le \left(\int_D V^p dx\right)^{1/p} \left(\int_{D_4} |u^*|^{\frac{2p}{p-1}}\, dx\right)^{1-1/p}. \qquad (4.10)
\end{aligned}$$

Next, let us use Sobolev inequality for Lipschitz functions f supported in $\overline{D_3}$:

$$\left(\int_{D_3} |f|^\alpha\, dx\right)^{1/\alpha} \le C \int_{D_3} |\nabla f|\, dx$$

where $\alpha \in (1,2)$ is arbitrary and $C = C(\alpha)$. Replacing f by f^β (where $\beta > 1$), we obtain

$$\begin{aligned}
\left(\int_{D_3} |f|^{\alpha\beta}\, dx\right)^{1/\alpha} &\le C \int_{D_3} |\nabla f|\, |f|^{\beta-1}\, dx \\
&\le C \left(\int_{D_3} |\nabla f|^2\, dx\right)^{1/2} \left(\int_{D_3} |f|^{2(\beta-1)}\, dx\right)^{1/2}.
\end{aligned}$$

Choosing β to satisfy the identity $\alpha\beta = 2(\beta-1)$, that is, $\beta = \frac{2}{2-\alpha}$, we obtain

$$\left(\int_{D_3} |f|^{\frac{2\alpha}{2-\alpha}}\, dx\right)^{\frac{2-\alpha}{\alpha}} \le C \int_{D_3} |\nabla f|^2\, dx. \qquad (4.11)$$

This inequality extends routinely to $W^{1,2}$ functions f supported in $\overline{D_3}$. Applying (4.11) for $f = u^*$ with $\alpha = \frac{2p}{2p-1}$ we obtain

$$\left(\int_{D_3} |u^*|^{\frac{2p}{p-1}}\, dx\right)^{1-1/p} \le C \int_{D_3} |\nabla u^*|^2\, dx,$$

which together with (4.8), (4.10) yields

$$\int_{D_3} |\nabla u^*|^2\, dx \le C \left(\int_D V^p dx\right)^{1/p} \int_{D_3} |\nabla u^*|^2\, dx. \qquad (4.12)$$

Assuming that

$$\|V\|_{L^p(D)} \le \varepsilon := \frac{1}{2C}, \qquad (4.13)$$

we see that (4.12) is only possible if $u^* = \text{const}$. Since $u \perp 1$ in $L^2(D)$, it follows that $u \equiv 0$.

Hence, $\mathcal{E}_{V,D}(u) \le 0$ and $u \perp 1$ imply $u \equiv 0$, whence $\operatorname{Neg}(V,D) \le 1$ follows. □

Corollary 4.7 *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\Phi : D \to \Omega$ be a C^1-diffeomorphism with finite M_Φ and $\sup |J_\Phi|$. Then there is $\varepsilon_\Omega > 0$ such that*

$$\|V\|_{L^p(\Omega)} \leq \varepsilon_\Omega \Rightarrow \mathrm{Neg}\,(V, \Omega) = 1,$$

where ε_Ω depends on p, M_Φ and $\sup |J_\Phi|$.

Consequently, if Ω is bilipschitz equivalent to D_r then

$$\|V\|_{L^p(\Omega)} \leq c r^{2/p-2} \Rightarrow \mathrm{Neg}\,(V, \Omega) = 1, \tag{4.14}$$

where $c > 0$ depends on p and on the bilipschitz constant of the mapping between D_r and Ω.

Proof. By Lemma 3.7, we have

$$\mathrm{Neg}\,(V, \Omega) \leq \mathrm{Neg}(\widetilde{V}, D),$$

where $\widetilde{V}$ is given by (3.20). By Lemma 4.6,

$$\|\widetilde{V}\|_{L^p(D)} \leq \varepsilon \Rightarrow \mathrm{Neg}(\widetilde{V}, D) = 1.$$

Using the notation of Lemma 3.7, set $\Psi = \Phi^{-1}$, $\widetilde{W} \equiv 1$ and define a function $W(x)$ on Ω by (3.22), that is,

$$W(x) = M_\Phi^p \left|J_\Psi(x)\right|^{1-p} \leq M_\Phi^p \sup \left|J_\Phi\right|^{p-1}.$$

Then by (3.23) we have

$$\int_D \widetilde{V}(y)^p \, dy = \int_\Omega V(x)^p W(x) \, dx \leq M_\Phi^p \sup |J_\Phi|^{p-1} \int_\Omega V(x)^p \, dx,$$

whence

$$\|\widetilde{V}\|_{L^p(D)} \leq M_\Phi \sup |J_\Phi|^{\frac{p-1}{p}} \|V\|_{L^p(\Omega)}.$$

Therefore, if

$$\|V\|_{L^p(\Omega)} \leq \varepsilon_\Omega := \frac{\varepsilon}{M_\Phi \sup |J_\Phi|^{\frac{p-1}{p}}}, \tag{4.15}$$

then $\|\widetilde{V}\|_{L^p(D)} \leq \varepsilon$, which implies by the above argument $\mathrm{Neg}\,(V, \Omega) = 1$.

Let $\Omega = D_r$. Then, for the mapping $\Phi(x) = rx$, we have $M_\Phi = 1$ and $|J_\Phi| = r^2$ whence we obtain

$$\varepsilon_{D_r} = \varepsilon r^{2/p-2}. \tag{4.16}$$

More generally, assume that there is a bilipschitz mapping $\Phi : D_r \to \Omega$ with a bilipschitz constant L. Arguing as in the first part of the proof but using D_r instead of D, we obtain similarly to (4.15) that ε_Ω can be determined by

$$\varepsilon_\Omega = \frac{\varepsilon_{D_r}}{M_\Phi \sup |J_\Phi|^{\frac{p-1}{p}}} \geq c r^{2/p-2},$$

where $c > 0$ depends on p and L, which was to be proved. $\square$

4.3 Negative eigenvalues in a square

Denote by Q the unit square in $\mathbb{R}^2$, that is,

$$Q = \left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < 1 \right\}.$$

Lemma 4.8 *For any $p > 1$ and for any potential V in Q,*

$$\operatorname{Neg}(V, Q) \le 1 + C \|V\|_{L^p(Q)}, \tag{4.17}$$

where C depends only on p.

This lemma follows also from the result of Birman and Solomyak [4]. However, we give here a different proof using specific properties of $\mathbb{R}^2$, although it is also based on some tiling of a square. Besides we use some counting argument that will appear later (in some disguise) also in Section 7.3.

Remark 4.9 Combining Lemma 4.8 with Lemma 3.7 we obtain that if an open set $\Omega \subset \mathbb{R}^2$ is bilipschitz equivalent to Q, then

$$\operatorname{Neg}(V, \Omega) \le 1 + C \|V\|_{L^p(\Omega)},$$

where the constant C depends on p and on the Lipschitz constant.

Proof. It suffices to construct a partition $\mathcal{P}$ of Q into a family of N disjoint subsets such that

1. $\operatorname{Neg}(V, \Omega) = 1$ for any $\Omega \in \mathcal{P}$;
2. $N \le 1 + C \|V\|_{L^p(Q)}$.

Indeed, if such a partition exists then we obtain by Lemma 3.5 that

$$\operatorname{Neg}(V, Q) \le \sum_{\Omega \in \mathcal{P}} \operatorname{Neg}(V, \Omega) = N, \tag{4.18}$$

and (4.17) follows from the above bound of N.

The elements of a partition – tiles, will be of two shapes: any tile is either a square of the side length $l \in (0, 1]$ or a *step*, that is, a set of the form $\Omega = A \setminus B$ where A is a square of the side length l, and B is a square of the side length $\le l/2$ that is attached to one of corners of A (see Fig. 2).

In the both cases we refer to l as the size of Ω. By Corollary 4.7, the condition $\operatorname{Neg}(V, \Omega) = 1$ for a tile Ω will follow from

$$\int_\Omega V^p dx \le c l^{2-2p}, \tag{4.19}$$

with some constant $c > 0$ depending only on p.

Apart from the shape, we will distinguish also the *type* of a tile $\Omega \in \mathcal{P}$ of size l as follows: we say that

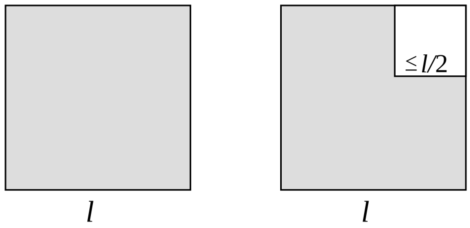

Figure 2: A square and a step of size l

- Ω is of a large type, if
$$\int_\Omega V^p dx > cl^{2-2p};$$
- Ω is of a medium type if
$$c'l^{2-2p} < \int_\Omega V^p dx \le cl^{2-2p}; \tag{4.20}$$
- Ω is of small type if
$$\int_\Omega V^p dx \le c'l^{2-2p}. \tag{4.21}$$
Here c is the constant from (4.19) and $c' > 0$ is another constant that satisfies
$$4c'2^{2p-2} < c. \tag{4.22}$$

The construction of the partition $\mathcal{P}$ will be done by induction. At each step $i \ge 1$ of induction we will have a partition $\mathcal{P}^{(i)}$ of Q such that

1. each tile $\Omega \in \mathcal{P}^{(i)}$ is either a square or a step;
2. If $\Omega \in \mathcal{P}^{(i)}$ is a step then Ω is of a medium type.

At step 1 we have just one set: $\mathcal{P}^{(1)} = \{Q\}$. At any step $i \ge 1$, partition $\mathcal{P}^{(i+1)}$ is obtained from $\mathcal{P}^{(i)}$ as follows. If $\Omega \in \mathcal{P}^{(i)}$ is small or medium then Ω becomes one of the elements of the partition $\mathcal{P}^{(i+1)}$. If $\Omega \in \mathcal{P}^{(i)}$ is large, then it is a square, and it will be further partitioned into a few smaller tiles that will become elements of $\mathcal{P}^{(i+1)}$. Denoting by l the side length of the square Ω, let us first split Ω into four equal squares $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ of side length $l/2$ and consider the following cases (see Fig. 3).

Case 1. If among $\Omega_1, ..., \Omega_4$ the number of small type squares is at most 2, then all the sets $\Omega_1, ..., \Omega_4$ become elements of $\mathcal{P}^{(i+1)}$.

Case 2. If among $\Omega_1, ..., \Omega_4$ there are exactly 3 small type squares, say, $\Omega_2, \Omega_3, \Omega_4$, then we have
$$\int_{\Omega\setminus\Omega_1} V^p dx = \int_{\Omega_2\cup\Omega_3\cup\Omega_4} V^p dx \le 3c'\left(\frac{l}{2}\right)^{2-2p} = 3c'2^{2p-2}l^{2-2p} < cl^{2-2p},$$

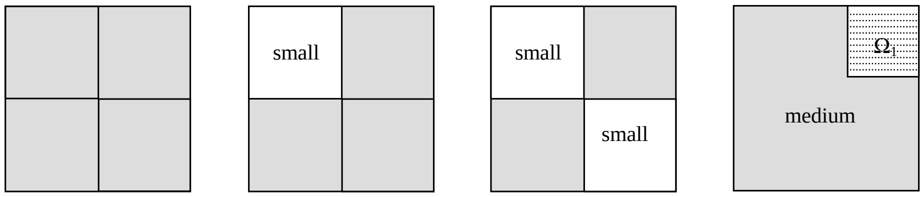

Figure 3: Various possibilities of partitioning of a square Ω (the shaded tiles are of medium or large type, the hatched tile Ω_1 can be of any type)

where we have used (4.22). On the other hand, we have

$$\int_\Omega V^p dx > cl^{2-2p}.$$

Therefore, by reducing the size of Ω_1 (but keeping Ω_1 attached to the corner of Ω) one can achieve the equality

$$\int_{\Omega\setminus\Omega_1} V^p dx = cl^{2-2p}.$$

Hence, we obtain a partition of Ω into two sets Ω_1 and $\Omega\setminus\overline{\Omega_1}$, where the step $\Omega\setminus\overline{\Omega_1}$ is of medium type, while the square Ω_1 can be of any type. The both sets Ω_1 and $\Omega\setminus\overline{\Omega_1}$ become elements of $\mathcal{P}^{(i+1)}$.

Case 3. Let us show that all 4 squares $\Omega_1, ..., \Omega_4$ cannot be small. Indeed, in this case we would have by (4.22)

$$\int_\Omega V^p dx = \sum_{k=1}^4 \int_{\Omega_k} V^p dx \le 4c' \left(\frac{l}{2}\right)^{2-2p} = \left(4c' 2^{2p-2}\right) l^{2-2p} < cl^{2-2p},$$

which contradicts to the assumption that Ω is of large type.

As we see from the construction, at each step i only large type squares get partitioned further, and the size of the large type squares in $\mathcal{P}^{(i+1)}$ reduces at least by a factor 2. If the size of a square is small enough then it is necessarily of small type, because the right hand side of (4.21) goes to ∞ as $l \to 0$. Hence, the process stops after finitely many steps, and we obtain a partition $\mathcal{P}$ where all the tiles are either of small or medium types (see Fig. 4). In particular, we have $\operatorname{Neg}(V,\Omega) = 1$ for any $\Omega \in \mathcal{P}$.

Let N be the number of tiles in $\mathcal{P}$. We need to show that

$$N \le 1 + C \|V\|_{L^p(Q)}. \tag{4.23}$$

At each step of construction, denote by L the number of large tiles, by M the number of medium tiles, and by S the number of small tiles. Let us show that the quantity $2L+3M-S$ is non-decreasing during the construction. Indeed, at each step we split one large square Ω, so that by removing this square, L decreases by 1. However, we add new tiles that contribute to the quantity $2L + 3M - S$ as follows.

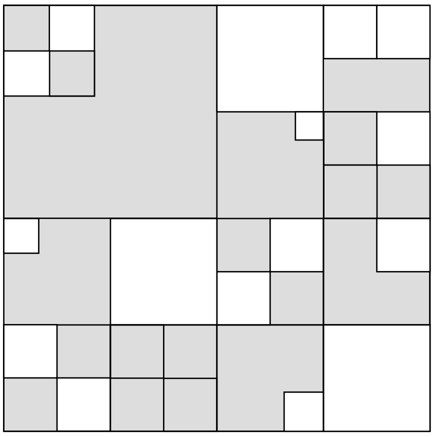

Figure 4: An example of a final partition $\mathcal{P}$. The shaded tiles are of medium type, the white squares are of small type.

1. If Ω is split into $s \leq 2$ small and $4-s$ medium/large squares as in Case 1, then the value of $2L+3M-S$ has the increment at least
$$-2+2(4-s)-s=6-3s\geq 0.$$
2. If Ω is split into 1 square and 1 step as in Case 2, then one obtains at least 1 medium tile and at most 1 small tile, so that $2L+3M-S$ has the increment at least
$$-2+3-1=0.$$

(Luckily, Case 3 cannot occur. In that case, we would have 4 new small squares so that L and M would not have increased, whereas S would have increased at least by 3, so that no quantity of the type C_1L+C_2M-S would have been monotone increasing.)

Since for the partition $\mathcal{P}^{(1)}$ we have $2L+3M-S\geq -1$, this inequality remains true at all steps of construction and, in particular, it is satisfied for the final partition $\mathcal{P}$. For the final partition we have $L=0$, whence it follows that $S\leq 1+3M$ and, hence,
$$N=S+M\leq 1+4M. \tag{4.24}$$

Let us estimate M. Let $\Omega_1,...,\Omega_M$ be the medium type tiles of $\mathcal{P}$ and let l_k be the size of Ω_k. Each Ω_k contains a square $\Omega_k'\subset\Omega_k$ of the size $l_k/2$, and all the squares $\{\Omega_k'\}_{k=1}^M$ are disjoint, which implies that
$$\sum_{k=1}^M l_k^2\leq 4. \tag{4.25}$$

Using the Hölder inequality and (4.25), we obtain
$$M=\sum_{k=1}^M l_k^{\frac{2}{p'}}l_k^{-\frac{2}{p'}}\leq\left(\sum_{k=1}^M l_k^2\right)^{1/p'}\left(\sum_{k=1}^M l_k^{-\frac{2p}{p'}}\right)^{1/p}\leq 4^{1/p'}\left(\sum_{k=1}^M l_k^{2-2p}\right)^{1/p}.$$

Since by (4.20) $c' l_k^{2-2p} < \int_{\Omega_k} V^p dx$, it follows that

$$M \leq C \left(\sum_{k=1}^{M} \int_{\Omega_k} V^p dx \right)^{1/p} \leq C \left(\int_Q V^p dx \right)^{1/p}.$$

Combining this with $N \leq 1 + 4M$, we obtain $N \leq 1 + C \|V\|_{L^p(Q)}$, thus finishing the proof. $\square$

5 Negative eigenvalues and Green operator

5.1 Green operator in $\mathbb{R}^2$

We start with the following statement.

Lemma 5.1 *There exists non-negative non-zero function* $V_0 \in C_0^\infty (\mathbb{R}^2)$ *such that* $\operatorname{Neg}(V_0) = 1$.

Proof. Choose V_0 to be supported in the unit disk D and such that $\|2V_0\|_{L^p(D)}$ is small enough as in Lemma 4.6, so that $\operatorname{Neg}(2V_0, D) = 1$. By Lemma 4.5 we have $\operatorname{Neg}(V_0, \mathbb{R}^2) \leq \operatorname{Neg}(2V_0, D)$, whence the claim follows. $\square$

From now on let us fix a potential V_0 as in Lemma 5.1. We can always assume that V_0 is spherically symmetric. Consider the quadratic form $\mathcal{E}_0$

$$\mathcal{E}_0(u) := \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} V_0 u^2 dx = \mathcal{E}_{-V_0}(u),$$

defined on the space

$$\mathcal{F}_0 = \left\{ u \in L^2_{loc}\left(\mathbb{R}^2\right) : \int_{\mathbb{R}^2} |\nabla u|^2 dx < \infty \right\} = \mathcal{F}_{V_0}.$$

Since V_0 is bounded and has compact support, the condition $\int_{\mathbb{R}^2} V_0 u^2 dx < \infty$ is satisfied for any $u \in L^2_{loc}$. Note also that $\mathcal{F}_V \subset \mathcal{F}_0$ for any potential V.

Lemma 5.2 *If, for all* $u \in \mathcal{F}_0$,

$$\mathcal{E}_0(u) \geq 2 \int_{\mathbb{R}^2} V u^2 dx, \tag{5.1}$$

then $\operatorname{Neg}(V) = 1$.

Proof. If $\mathcal{E}_V(u) \leq 0$ that is, if

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq \int_{\mathbb{R}^2} V u^2 dx,$$

then, substituting this into the right hand side of (5.1), we obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} V_0 u^2 dx \geq 2 \int_{\mathbb{R}^2} |\nabla u|^2 dx,$$

whence

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^2} V_0 u^2 dx,$$

that is, $\mathcal{E}_{V_0}(u) \leq 0$. By Lemma 3.2 this implies $\operatorname{Neg}(V) \leq \operatorname{Neg}(V_0)$, whence the claim follows. □

Lemma 5.2 provides the following method of proving that $\operatorname{Neg}(V) = 1$: it suffices to prove the inequality (5.1) for all $u \in \mathcal{F}_0$. For the latter, we will use the Green function of the operator

$$H_0 = -\Delta + V_0.$$

It was shown in [11, Example 10.14] that the operator H_0 has a symmetric positive Green function $g(x, y)$ that satisfies the following estimate

$$g(x, y) \simeq \ln\langle y\rangle + \frac{\ln\langle y\rangle}{\ln\langle x\rangle} \ln_+ \frac{1}{|x-y|} \quad \text{if} \quad |y| \leq |x|, \tag{5.2}$$

and a symmetric estimate if $|y| \geq |x|$, where we use the notation

$$\langle x\rangle = e + |x|.$$

It follows from (5.2) that, for all $x, y \in \mathbb{R}^2$,

$$g(x, y) \simeq \ln\langle x\rangle \wedge \ln\langle y\rangle + \ln_+ \frac{1}{|x-y|}, \tag{5.3}$$

where $a \wedge b := \min(a, b)$. Here we have used the fact that $\langle x\rangle \simeq \langle y\rangle$ provided $|x - y| < 1$; note that the latter is equivalent to $\ln_+ \frac{1}{|x-y|} > 0$.

For comparison, let us recall that the operator $-\Delta$ in $\mathbb{R}^2$ has no positive Green function, so that adding a small perturbation V_0 changes this property.

Fix a potential V on $\mathbb{R}^2$, consider a measure ν on $\mathbb{R}^2$ given by

$$d\nu = V(x)\, dx,$$

and the integral operator G_V in $L^2(\nu) = L^2(\mathbb{R}^2, \nu)$ that acts by the rule

$$G_V f(x) = \int_{\mathbb{R}^2} g(x, y) f(y) \, d\nu(y).$$

Denote by $\|G_V\|$ the norm of the operator G from $L^2(\nu)$ to $L^2(\nu)$ (if G_V does not map $L^2(\nu)$ into itself then set $\|G_V\| = \infty$).

Lemma 5.3 *Assume that*

$$\frac{1}{V} \in L^1_{loc}. \tag{5.4}$$

Then following inequality holds for all $u \in \mathcal{F}_0$:

$$\mathcal{E}_0(u) \geq \frac{1}{\|G_V\|} \int_{\mathbb{R}^2} V u^2 dx. \tag{5.5}$$

Proof. If $\|G_V\| = \infty$ then (5.5) is trivially satisfied, so assume that $\|G_V\| < \infty$. Consider first the case when $u \in C_0^\infty(\mathbb{R}^2)$. Set $f = \frac{1}{V} H_0 u$ so that $H_0 u = fV$. Then function u can be recovered from f using the Green operator $G = G_V$ as follows:

$$u(x) = \int_{\mathbb{R}^2} g(x,y)(fV)(y)\,dy = Gf(x).$$

Observe that $f \in L^2(\nu)$ because by (5.4)

$$\int_{\mathbb{R}^2} f^2 d\nu = \int_{\mathbb{R}^2} \frac{(H_0 u)^2}{V} dx \le \sup |H_0 u|^2 \int_{\operatorname{supp} u} \frac{1}{V} dx < \infty.$$

It follows that

$$\mathcal{E}_0(u) = (H_0 u, u)_{L^2(dx)} = (fV, Gf)_{L^2(dx)} = (f, Gf)_{L^2(\nu)} \tag{5.6}$$

and

$$\int_{\mathbb{R}^2} V u^2 dx = (u,u)_{L^2(\nu)} = (Gf, Gf)_{L^2(\nu)}.$$

Inequality (5.5) will follows if we prove that, for all $f \in L^2(\nu)$,

$$(f, Gf) \ge \frac{1}{\|G\|}(Gf, Gf), \tag{5.7}$$

where the both inner products are in $L^2(\nu)$.

Recall that G is a bounded symmetric (hence, self-adjoint) operator in $L^2(\nu)$. Observe that G is non-negative definite. Indeed, if $f \in C_0^\infty(\mathbb{R}^2)$ then, setting $u = Gf$, we obtain the identities (5.6) so that

$$(f, Gf) = \mathcal{E}_0(u) \ge 0.$$

Then $(f, Gf) \ge 0$ follows from the fact that $C_0^\infty(\mathbb{R}^2)$ is dense in $L^2(\nu)$.

Now, let us prove (5.7). For non-negative definite self-adjoint operators the following inequality holds, for all $f, h \in L^2(\nu)$:

$$(Gf, h)^2 \le (Gf, f)(Gh, h).$$

Setting $h = Gf$, we obtain

$$(Gf, Gf)^2 \le (Gf, f)\|G\|\|h\|^2 = \|G\|(Gf, f)(Gf, Gf).$$

Dividing by (Gf, Gf), we obtain (5.7).

Hence, we have proved (5.5) for $u \in C_0^\infty(\mathbb{R}^2)$. Let us extend this inequality to all $u \in \mathcal{F}_0$. Assume first that $u \in \mathcal{F}_0$ has a compact support. Then it follows that $u \in W^{1,2}(\mathbb{R}^2)$. Approximating u in $W^{1,2}$ by a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^2)$, applying (5.5) for each u_n and passing to the limit using Fatou's lemma, we obtain (5.5) for u.

Let us now prove (5.5) for the case when the function $u \in \mathcal{F}_0$ is essentially bounded. There is a sequence of non-negative Lipschitz functions φ_n on $\mathbb{R}^2$ with compact supports such that $\varphi_n \uparrow 1$ as $n \to \infty$ and

$$\int_{\mathbb{R}^2} |\nabla \varphi_n|^2 \, dx \to 0. \tag{5.8}$$

For example, one can take φ_n as in (2.12) with $a_n = n$ and $b_n = n^2$, that is,

$$\varphi_n(x) = \min\left(1, \frac{1}{\ln n} \ln_+ \frac{n^2}{|x|}\right). \tag{5.9}$$

Clearly, $\varphi_n \in \mathcal{F}_0$. By (2.13) we have

$$\int_{\mathbb{R}^2} |\nabla \varphi_n|^2 \, dx = \frac{2\pi}{\ln n} \to 0 \text{ as } n \to \infty.$$

Since (5.5) holds for the functions $u_n = u\varphi_n$ with compact support, it suffices to show that passing to the limit as $n \to \infty$, we obtain (5.5) for the function u. The terms $\int V_0 u_n^2 dx$ and $\int V u_n^2 dx$ are obviously survive under the monotone limit. We are left to verify that

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx \to \int_{\mathbb{R}^2} |\nabla u|^2 \, dx. \tag{5.10}$$

We have

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla (u\varphi_n)|^2 \, dx \\ =\ &\int_{\mathbb{R}^2} |\nabla u|^2 \varphi_n^2 dx + 2\int_{\mathbb{R}^2} \langle \nabla u, \nabla \varphi_n \rangle u\varphi_n dx + \int_{\mathbb{R}^2} u^2 |\nabla \varphi_n|^2 \, dx. \end{aligned} \tag{5.11}$$

The first term in the right hand side of (5.11) converges to $\int_{\mathbb{R}^2} |\nabla u|^2 \, dx$. For the third term we have by (5.8)

$$\int_{\mathbb{R}^2} u^2 |\nabla \varphi_n|^2 \, dx \le \|u\|_{L^\infty}^2 \int_{\mathbb{R}^2} |\nabla \varphi_n|^2 \, dx \to 0 \text{ as } n \to \infty.$$

Similarly, the middle term converges to 0 as $n \to \infty$ by

$$\left| \int_{\mathbb{R}^2} \langle \nabla u, \nabla \varphi_n \rangle u\varphi_n dx \right| \le \left(\int_{\mathbb{R}^2} |\nabla u|^2 \varphi_n^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^2} u^2 |\nabla \varphi_n|^2 \, dx \right) \to 0,$$

which proves (5.10).

Finally, for a general function $u \in \mathcal{F}_0$, consider an approximating sequence

$$u_n = \max(\min(u, n), -n).$$

The function u_n is bounded so that (5.5) holds for u_n. Letting $n \to \infty$, we obtain (5.5) for the function u. $\square$

Corollary 5.4 *Under the hypothesis* (5.4),

$$\|G_V\| \le \frac{1}{2} \Rightarrow \operatorname{Neg}(V) = 1. \tag{5.12}$$

Proof. Indeed, (5.12) implies (5.1), whence $\operatorname{Neg}(V) = 1$ holds by Lemma 5.2. $\square$

5.2 Green operator in a strip

Consider a strip

$$S = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R},\ 0 < x_2 < \pi \right\}$$

and a potential V on S. The analytic function $\Psi(z) = e^z$ provides a biholomorphic mapping from S onto the upper half-plane H_+. Set $\Phi = \Psi^{-1}$ so that $\Phi(z) = \ln z$. Consider the function

$$\gamma(z, w) = g(\Psi(z), \Psi(w)), \tag{5.13}$$

where $z, w \in S$ and $g(x, y)$ is the Green function from Section 5. Consider also the corresponding integral operator

$$\Gamma_V f(z) = \int_S \gamma(z, \cdot) f(\cdot)\, d\nu, \tag{5.14}$$

where measure ν is defined as above by $d\nu = V(x)\, dx$. Denote by $\|\Gamma_V\|$ the norm of Γ_V in $L^2(S, \nu)$.

Lemma 5.5 *Let*

$$\frac{1}{V} \in L^1_{loc}(S). \tag{5.15}$$

Then

$$\|\Gamma_V\| \le \frac{1}{8} \Rightarrow \operatorname{Neg}(V, S) = 1.$$

Proof. Consider the potential $\widetilde{V}$ on the half-plane $H_+ = \{x_2 > 0\}$ given by

$$\widetilde{V}(x) = V(\Phi(x)) |\Phi'(x)|^2,$$

for which we have by Lemma 3.7 that

$$\operatorname{Neg}(V, S) \le \operatorname{Neg}(\widetilde{V}, H_+). \tag{5.16}$$

Let us extend $\widetilde{V}$ from H_+ to $\mathbb{R}^2$ by symmetry in the axis x_1. By Lemma 4.4 we have

$$\operatorname{Neg}(\widetilde{V}, H_+) \le \operatorname{Neg}(2\widetilde{V}, \mathbb{R}^2).$$

Consider the operator $G_{\widetilde{V}}$ that acts in $L^2(\mathbb{R}^2, \widetilde{\nu})$ where $d\widetilde{\nu} = \widetilde{V} dx$, and $G_{2\widetilde{V}}$ that acts in $L^2(\mathbb{R}^2, 2\widetilde{\nu})$. It is easy to see that

$$\left\| G_{2\widetilde{V}} \right\| = 2 \left\| G_{\widetilde{V}} \right\|, \tag{5.17}$$

Denote by $\left\| G_{\widetilde{V}} \right\|_+$ the norm of the operator $G_{\widetilde{V}}$ acting in $L^2(H_+, \widetilde{\nu})$. Using the symmetry of the potential $\widetilde{V}$ in the axis x_1 and that of the Green function $g(x, y)$, one can easily show that

$$\left\| G_{\widetilde{V}} \right\| \le 2 \left\| G_{\widetilde{V}} \right\|_+. \tag{5.18}$$

Let us verify that

$$\left\| G_{\widetilde{V}} \right\|_+ = \|\Gamma_V\|. \tag{5.19}$$

In fact, the operators Γ_V in $L^2(S,\nu)$ and $G_{\widetilde{V}}$ in $L^2(H_+,\widetilde{\nu})$ are unitary equivalent. Indeed, consider a mapping $f \mapsto \widetilde{f}$ from $L^2(S,\nu)$ to $L^2(H_+,\widetilde{\nu})$ defined by

$$\widetilde{f}(x) = f(\Phi(x)).$$

Then we have

$$\begin{aligned}\|\widetilde{f}\|^2_{L^2(H_+,\widetilde{\nu})} &= \int_{H_+} \widetilde{f}^2(x)\,\widetilde{V}(x)\,dx = \int_{H_+} f(\Phi(x))^2\,V(\Phi(x))\,|\Phi'(x)|^2\,dx \\ &= \int_S f(z)^2\,V(z)\,dz = \|f\|^2_{L^2(S,\nu)},\end{aligned}$$

so that this mapping is unitary. Next, we have, for any $x \in H_+$,

$$\begin{aligned}\widetilde{\Gamma_V f}(x) &= \Gamma_V f(\Phi(x)) = \int_S \gamma(\Phi(x),w)\,f(w)\,V(w)\,dw \\ &= \int_{H_+} \gamma(\Phi(x),\Phi(y))\,f(\Phi(y))\,V(\Phi(y))\,|\Phi'(y)|^2\,dy \\ &= \int_{H_+} g(x,y)\,\widetilde{f}(y)\,\widetilde{V}(y)\,dy \\ &= G_{\widetilde{V}}\widetilde{f}(x),\end{aligned}$$

that is, $\widetilde{\Gamma_V f} = G_{\widetilde{V}}\widetilde{f}$ which implies the unitary equivalence of Γ_V and $G_{\widetilde{V}}$.

Combining (5.17)-(5.19), we conclude that

$$\left\|G_{2\widetilde{V}}\right\| \le 4\,\|\Gamma_V\| \le \frac{1}{2}.$$

Since $\frac{1}{\widetilde{V}} \in L^1_{loc}$, we have by Corollary 5.4 that $\mathrm{Neg}(2\widetilde{V},\mathbb{R}^2) = 1$. $\square$

In the next lemma, we prove an upper bound for the Green kernel γ.

Lemma 5.6 *For all $x,y \in S$, we have*

$$\gamma(x,y) \le C(1+|x_1| \wedge |y_1|) + C\ln_+ \frac{1}{|x-y|} \tag{5.20}$$

with an absolute constant C.

Proof. By (5.3) and (5.13) we have

$$\gamma(x,y) \le C\ln\langle e^x\rangle \wedge \ln\langle e^y\rangle + C\ln_+ \frac{1}{|e^x-e^y|},$$

where in the expressions e^x, e^y we regards x,y are complex numbers. Observe that

$$\ln\langle e^x\rangle = \ln(e+|e^x|) = \ln(e+e^{x_1}) \le e+|x_1|. \tag{5.21}$$

Let us show that

$$\ln_+ \frac{1}{|e^x-e^y|} \le C + |x| \wedge |y| + \ln_+ \frac{1}{|x-y|}, \tag{5.22}$$

with some absolute constant C. Indeed, by symmetry between x, y, it suffices to prove that

$$\left|e^{x}-e^{y}\right| \geq c e^{-|x|} \min (1,|x-y|) \tag{5.23}$$

for all $x, y \in S$ and for some positive constant c. Indeed, setting $z=y-x$ we see that (5.23) is equivalent to

$$\left|1-e^{z}\right| \geq c \min (1,|z|),$$

and the latter is true for $|z| \leq 1$ because

$$\left|1-e^{z}\right|=\left|z+\sum_{k=2}^{\infty} \frac{z^{k}}{k!}\right| \geq|z|-|z| \sum_{k=2}^{\infty} \frac{1}{k!}=(3-e)|z|,$$

and for $|z|>1$ because the set $\{z \in S:|z|>1\}$ is separated from the only point $z=0$ in $\overline{S}$ where $e^{z}=1$.

Combining (5.21), (5.22) and noticing that $|x| \leq \pi+\left|x_{1}\right|$, we obtain (5.20). $\square$

6 Estimates of the norms of some integral operators

In this section we introduce tools for estimating the norm of the operator Γ_{V} from the previous section. We start with an one-dimensional case that contains already all difficulties.

Lemma 6.1 *Let μ be a Radon measure on $\mathbb{R}$ and consider the following operator acting on $L^{2}(\mathbb{R}, \mu)$:*

$$T f(x)=\int_{\mathbb{R}}(1+|x| \wedge|y|) f(y) d \mu(y).$$

For any $n \in \mathbb{Z}$, set

$$I_{n}=\left[2^{n-1}, 2^{n}\right] \text { for } n>0, \quad I_{0}=[-1,1], \quad I_{n}=\left[-2^{|n|},-2^{|n|-1}\right] \text { for } n<0$$

and

$$\alpha_{n}=2^{|n|} \mu\left(I_{n}\right). \tag{6.1}$$

Then the following estimate holds:

$$\|T\| \leq 64 \sup _{n \in \mathbb{Z}} \alpha_{n}.$$

Proof. Let us represent the operator T as the sum $T=T_{1}+T_{2}$ where

$$\begin{aligned}
T_{1} f(x) & =\int_{\{|y| \leq|x|\}}(1+|y|) f(y) d \mu(y) \\
T_{2} f(x) & =\int_{\{|y| \geq|x|\}}(1+|x|) f(y) d \mu(y).
\end{aligned}$$

These operators are clearly adjoint in $L^2(\mathbb{R},\mu)$ which implies that $\|T_1\| = \|T_2\|$. Hence, $\|T\| \leq 2\|T_1\|$. The operator T_1 can be further split into the sum $T_1 = T_3 + T_4$ where

$$\begin{aligned} T_3 f(x) &= \int_0^{|x|} (1+|y|) f(y)\, d\mu(y) \\ T_4 f(x) &= \int_{-|x|}^{0} (1+|y|) f(y)\, d\mu(y). \end{aligned}$$

We will estimate $\|T_3\|$ via $\alpha = \sup \alpha_n$, and by symmetry $\|T_4\|$ could be estimated in the same way. The operator T_3 splits further into the sum $T_3 = T_5 + T_6$ where

$$\begin{aligned} T_5 f(x) &= \int_0^{x_+} (1+y) f(y)\, d\mu(y) \\ T_6 f(x) &= \int_0^{x_-} (1+y) f(y)\, d\mu(y). \end{aligned}$$

Clearly, we have $\|T_5\| = \|T_6\|$ and, hence, $\|T_3\| \leq 2\|T_5\|$. Since $T_5 f(x)$ vanishes for $x \leq 0$, it suffices to estimate $\|T_5\|$ in the space $L^2(\mathbb{R}_+,\mu)$.

In what follows we redefine I_0 to be $I_0 = [0,1]$, which only reduces α_0 and improves the estimates. Fix a non-negative function $f \in L^2(\mathbb{R}_+,\mu)$ and set for any non-negative integer n

$$w_n = \frac{1}{\sqrt{\alpha_n}} \int_{I_n} f d\mu.$$

For any $x \in I_n$ we have

$$T_5 f(x) \leq \int_0^{2^n} (1+y) f(y)\, d\mu(y) \leq \sum_{k=0}^{n} \left(1+2^k\right) \int_{I_k} f d\mu \leq \sum_{k=0}^{n} 2^{k+1} w_k \sqrt{\alpha_k}.$$

It follows that

$$\begin{aligned} \|T_5 f\|^2_{L^2(\mathbb{R}_+,\mu)} &= \sum_{n=0}^{\infty} \int_{I_n} (T_5 f(x))^2\, d\mu(x) \\ &\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} 2^{k+1} w_k \sqrt{\alpha_k} \right)^2 \mu(I_n) \\ &= 4 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} 2^k w_k \sqrt{\alpha_k} \right)^2 \frac{\alpha_n}{2^n}. \end{aligned}$$

Using $\alpha_n \leq \alpha$, we obtain

$$\|T_5 f\|^2_{L^2(\mathbb{R}_+,\mu)} \leq 4\alpha^2 \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\sum_{k=0}^{n} 2^k w_k \right)^2. \tag{6.2}$$

On the other hand, we have

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}_+,\mu)} &= \sum_{n=0}^{\infty}\int_{I_n} f^2 d\mu \geq \sum_{n=0}^{\infty}\frac{1}{\mu(I_n)}\left(\int_{I_n} f d\mu\right)^2 \\ &= \sum_{n=0}^{\infty}\frac{2^n}{\alpha_n} w_n^2 \alpha_n = \sum_{n=0}^{\infty} 2^n w_n^2. \end{aligned} \tag{6.3}$$

Let us prove that

$$\sum_{n=0}^{\infty}\frac{1}{2^n}\left(\sum_{k=0}^{n} 2^k w_k\right)^2 \leq 16\sum_{n=0}^{\infty} 2^n w_n^2. \tag{6.4}$$

This is nothing other than a discrete weighted Hardy inequality. By [1], if, for some $r > s \geq 1$ and for non-negative sequences $\{u_k\}_{k=0}^N, \{v_k\}_{k=0}^N$, the following inequality is satisfied

$$\sum_{n=0}^{m} u_n\left(\sum_{k=0}^{n} v_k\right)^r \leq \left(\sum_{k=0}^{m} v_k\right)^s \quad \text{for } m = 0, ..., N, \tag{6.5}$$

then, for all non-negative sequences $\{w_k\}_{k=0}^N$,

$$\sum_{n=0}^{N} u_n\left(\sum_{k=0}^{n} v_k w_k\right)^r \leq \left(\frac{r}{r-s}\right)^r\left(\sum_{k=0}^{N} v_k w_k^{r/s}\right)^s. \tag{6.6}$$

We apply this result with $r = 2$, $s = 1$, $v_n = 2^k$ and $u_n = 2^{-n-2}$. Then (6.5) holds because

$$\sum_{n=0}^{m} 2^{-n-2}\left(\sum_{k=0}^{n} 2^k\right)^2 = \sum_{n=0}^{m} 2^{-n-2}\left(2^{n+1}-1\right)^2 \leq \sum_{n=0}^{m} 2^n,$$

and (6.6) yields

$$\sum_{n=0}^{N}\frac{1}{2^{n+2}}\left(\sum_{k=0}^{n} 2^k w_k\right)^2 \leq 4\sum_{k=0}^{N} 2^k w_k^2,$$

which is equivalent to (6.4). The latter together with (6.2) and (6.3) implies that $\|T_5\| \leq 8\alpha$. It follows that $\|T_3\| \leq 16\alpha$. As T_4 admits the same estimate, we obtain $\|T_1\| \leq 32\alpha$, whence $\|T\| \leq 64\alpha$, which was to be proved. □

Consider the strip

$$S = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < \pi\right\}$$

and its partition into rectangles S_n, $n \in \mathbb{Z}$, defined by

$$S_n = \begin{cases} \{x \in S : 2^{n-1} < x_1 < 2^n\}, & n > 0, \\ \{x \in S : -1 < x_1 < 1\}, & n = 0, \\ \left\{x \in S : -2^{|n|} < x_1 < -2^{|n|-1}\right\}, & n < 0. \end{cases} \tag{6.7}$$

Lemma 6.2 *Let ν be a Radon measure on the strip S, absolutely continuous with respect to the Lebesgue measure. For any $n \in \mathbb{Z}$, set*

$$a_n = 2^{|n|}\nu(S_n).$$

Then the following integral operator

$$Tf(x) = \int_S (1 + |x_1| \wedge |y_1|) f(y) \, d\nu(y)$$

admits the following norm estimate in $L^2(S, \nu)$:

$$\|T\| \leq 64 \sup_{n\in\mathbb{Z}} a_n. \tag{6.8}$$

Proof. Introduce a Radon measure μ on $\mathbb{R}$ by

$$\mu(A) = \nu(A \times (0, \pi)).$$

For the quantities α_n, defined for measure μ by (6.1), we obviously have the identity $\alpha_n = a_n$. The estimate (6.8) will follow from Lemma 6.1 if we prove that $\|T\| \leq \|T'\|$ where T' is the following operator in $L^2(\mathbb{R}, \mu)$:

$$T'f(t) = \int_{\mathbb{R}} (1 + t \wedge s) f(s) \, d\mu(s).$$

It suffices to prove that, for any non-zero bounded function $f \in L^2(S, \nu)$, there is a function $f' \in L^2(\mathbb{R}, \mu)$ such that

$$\frac{\|Tf\|}{\|f\|} \leq \frac{\|T'f'\|}{\|f'\|},$$

where the norms are taken in the appropriate spaces. For any measurable set $A \subset \mathbb{R}$, set

$$\mu_f(A) = \int_{A\times(0,\pi)} f d\nu.$$

Since f is bounded, measure μ_f is absolutely continuous with respect to μ, so that there exists a function f' such that $d\mu_f = f'd\mu$. In the same way, there is a function f'' such that $d\mu_{f^2} = f''d\mu$. Since

$$\left(\int_{A\times(0,\pi)} f d\nu\right)^2 \leq \left(\int_{A\times(0,\pi)} f^2 d\nu\right) \mu(A),$$

it follows that

$$\mu_f(A)^2 \leq \mu_{f^2}(A) \mu(A)$$

whence

$$(f')^2 \leq f''.$$

It follows that

$$\|f'\|^2_{L^2(\mathbb{R},\mu)} = \int_{\mathbb{R}} (f')^2\, d\mu \le \int_{\mathbb{R}} f'' d\mu = \int_{\mathbb{R}} \mu_{f^2} = \int_S f^2 d\nu = \left\|f^2\right\|^2_{L^2(S,\nu)}.$$

It remains to show that $\|Tf\| = \|T'f'\|$. Using the notation $\tau(t,s) = 1 + t\wedge s$, we have

$$\begin{aligned} Tf(x) &= \int_S \tau(x_1,y_1)\, f(y)\, d\nu(y) = \int_{\mathbb{R}} \tau(x_1,y_1)\, d\mu_f(y_1) \\ &= \int_{\mathbb{R}} \tau(x_1,y_1)\, f'(y_1)\, d\mu(y_1) = T'f'(x_1). \end{aligned}$$

It follows that

$$\|Tf\|^2_{L^2(S,\nu)} = \int_S (Tf(x))^2\, d\nu = \int_{\mathbb{R}} (T'f'(x_1))^2\, d\mu = \|T'f'\|^2_{L^2(\mathbb{R},\mu)},$$

which finishes the proof. □

7 Estimating the number of negative eigenvalues in a strip

Let V be a potential in the strip

$$S = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R},\ 0 < x_2 < \pi\right\}.$$

For any $n \in \mathbb{Z}$ set

$$a_n(V) = \int_{S_n} (1 + |x_1|)\, V(x)\, dx, \tag{7.1}$$

where S_n is defined by (6.7). It is easy to see that

$$2^{|n|-1} \int_{S_n} V(x)\, dx \le a_n(V) \le 2^{|n|+1} \int_{S_n} V(x)\, dx. \tag{7.2}$$

Fix $p > 1$ and set also

$$b_n(V) = \left(\int_{S\cap\{n<x_1<n+1\}} V^p(x)\, dx\right)^{1/p}. \tag{7.3}$$

7.1 Condition for one negative eigenvalue

Lemma 7.1 *The operator Γ_V defined by (5.13)-(5.14) admits the following norm estimate in $L^2(S, Vdx)$:*

$$\|\Gamma_V\| \le C \sup_{n\in\mathbb{Z}} a_n(V) + C \sup_{x\in S} \int_S \ln_+ \frac{1}{|x-y|} V(y)\, dy. \tag{7.4}$$

where C is an absolute constant. Consequently,

$$\|\Gamma_V\| \le C \sup_{n\in\mathbb{Z}} a_n(V) + C_p \sup_{n\in Z} b_n(V), \tag{7.5}$$

where C_p depends on p.

Proof. Consider measure ν in S given by $d\nu = V dx$. By (5.20) we have, for any non-negative function f on S,

$$\Gamma_V f(x) \le C \int_S (1 + |x_1| \wedge |y_1|) f(y) \, d\nu(y) + C \int_S \ln_+ \frac{1}{|x-y|} f(y) \, d\nu(y) . \quad (7.6)$$

By Lemma 6.2, the norm of the first integral operator in (7.6) is bounded by

$$64 \sup_n 2^{|n|} \nu(S_n) \le 128 \sup_n a_n(V),$$

where we have used (7.2). The norm of the second integral operator in (7.6) is trivially bounded by

$$\sup_{x \in S} \int_S \ln_+ \frac{1}{|x-y|} d\nu(y),$$

whence (7.4) follows.

For the second part, we have by the Hölder inequality

$$\begin{aligned} \int_S \ln_+ \frac{1}{|x-y|} V(y) \, dy \ &\le \ \left(\int_{D_1(x)} \left(\ln_+ \frac{1}{|x-y|} \right)^{p'} dy \right)^{1/p'} \\ &\quad \times \left(\int_{D_1(x) \cap S} V^p(y) \, dy \right)^{1/p}, \end{aligned}$$

where p' is the Hölder conjugate to p and $D_1(x)$ is the disk of radius 1 centered at x. The first integral is equal to a finite constant depending only on p, but independent of x. Since $D_1(x) \cap S$ is covered by at most 3 rectangles Q_n, the second integral is bounded by $3 \sup_n b_n(V)$. Substituting into (7.4), we obtain (7.5). $\square$

Remark 7.2 Although the norm of the first integral operator in (7.6) can be trivially bounded by

$$\sup_{x \in S} \int_S (1 + |x_1| \wedge |y_1|) \, d\nu(y),$$

this estimate is weaker than the one by Lemma 6.2 and is certainly not good enough for our purposes.

Proposition 7.3 *There is a constant $c > 0$ such that*

$$\sup_n a_n(V) \le c \quad \textit{and} \quad \sup_n b_n(V) \le c \quad \Rightarrow \ \operatorname{Neg}(V, S) = 1. \quad (7.7)$$

Proof. Assume first that $\frac{1}{V} \in L^1_{loc}(S)$. By Lemma 5.5 it suffices to show that $\|\Gamma_V\| \le \frac{1}{8}$. Assuming that the constant c in (7.7) is small enough, we obtain from (7.5) that indeed $\|\Gamma_V\| \le \frac{1}{8}$ and, hence, $\operatorname{Neg}(V, S) = 1$.

Consider now a general potential V. In this case consider bit larger potential

$$V' = V + \varepsilon e^{-|x|},$$

where $\varepsilon > 0$. Clearly, $\frac{1}{V'} \in L^1_{loc}$ while $a_n(V')$ and $b_n(V')$ are still small enough provided ε is chosen sufficiently small. Assuming that the constant c in (7.7) is small enough, we obtain by the first part of the proof that

$$\operatorname{Neg}(2V', S) = 1. \tag{7.8}$$

We would like to deduce from (7.8) that $\operatorname{Neg}(V, S) = 1$. Since in general $\operatorname{Neg}(V, S)$ is not monotone with respect to V, we have to use an additional argument. We use the counting function $\operatorname{Neg}^b(V, S)$ based on bounded test functions (cf. Section 3.3).

Observe first that

$$\operatorname{Neg}^b(2V, S) \le \operatorname{Neg}^b(2V', S). \tag{7.9}$$

Since $\mathcal{E}_{2V,S} \le \mathcal{E}_{2V',S}$, (7.9) will follow from the identity of the spaces $\mathcal{F}^b_{2V,S}$ and $\mathcal{F}^b_{2V',S}$, where the latter amounts to

$$\int_\Omega V f^2 dx < \infty \;\Leftrightarrow\; \int_\Omega V' f^2 dx < \infty.$$

The implication $\Leftarrow$ here is trivial, while the opposite direction $\Rightarrow$ follows from

$$\int_\Omega V' f^2 dx = \int_\Omega V f^2 dx + \varepsilon \int_\Omega f^2 e^{-|x|} dx$$

and the finiteness of the last integral, which is true by the boundedness of the test function f.

Since

$$\operatorname{Neg}^b(2V', S) \le \operatorname{Neg}(2V', S),$$

combining this with (7.8) and (7.9) we obtain

$$\operatorname{Neg}^b(2V, S) = 1.$$

Finally, we conclude by Lemma 3.9 that $\operatorname{Neg}(V, S) = 1$. $\square$

7.2 Extension of functions from a rectangle to a strip

For all $\alpha \in [-\infty, +\infty)$, $\beta \in (-\infty, +\infty]$ such that $\alpha < \beta$, denote by $P_{\alpha,\beta}$ the rectangle

$$P_{\alpha,\beta} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \alpha < x_1 < \beta, \;\; 0 < x_2 < \pi \right\}.$$

Lemma 7.4 *For any potential V in a rectangle $P_{\alpha,\beta}$ with $\beta - \alpha \ge 1$, we have*

$$\operatorname{Neg}(V, P_{\alpha,\beta}) \le \operatorname{Neg}(17V, S), \tag{7.10}$$

assuming that V is extended to S by setting $V = 0$ outside $P_{\alpha,\beta}$.

Proof. By Lemma 3.2 it suffices to show that any function $u \in \mathcal{F}_{V,P}$ can be extended to a function $u \in \mathcal{F}_{V,S}$ so that

$$\int_S |\nabla u|^2 \, dx \le 17 \int_P |\nabla u|^2 \, dx. \tag{7.11}$$

Assume first that both α, β are finite. Attach to P from each side one rectangle, say P' from the left and P'' from the right, each having the length $4(\beta - \alpha)$ (to ensure that the latter is $> \pi$). Extend function u to P' by applying four times symmetries in the vertical sides (cf. Example 4.2). Then we have

$$\int_{P'} |\nabla u|^2 \, dx = 4 \int_P |\nabla u|^2 \, dx.$$

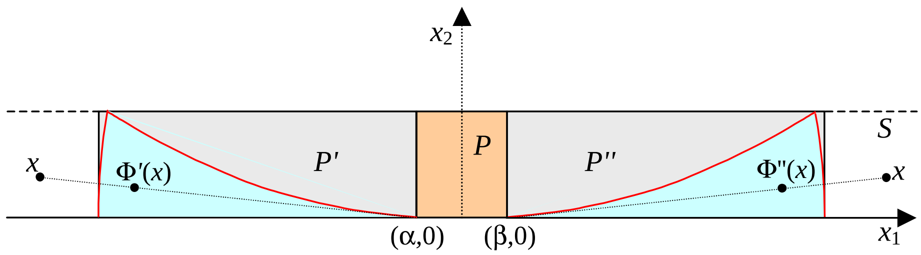

Figure 5: Extension of function u from P to S.

Then slightly reduce P' by taking its intersections with the disk of radius $\beta - \alpha$ centered at $(\alpha, 0)$ (cf. Fig. 5). Now we extend u from P' to the left by using the inversion Φ' at the point $(\alpha, 0)$ in the circle of radius $\beta - \alpha$ centered at $(\alpha, 0)$ (cf. Example 4.3). By the conformal invariance of the Dirichlet integral, we have

$$\int_{S \cap \{x_1 < \alpha\}} |\nabla u|^2 \le 8 \int_P |\nabla u|^2 \, dx.$$

Extending u in the same way to the right of P, we obtain (7.11). The case when one of the endpoints α, β is at infinity is treated similarly. $\square$

7.3 Sparse potentials

Definition 7.5 We say that a potential V in S is *sparse* if

$$\sup_n b_n(V) < c_0, \tag{7.12}$$

where c_0 is a small enough positive constant, depending only on p. We say that a potential V is sparse in a domain $\Omega \subset S$ if its trivial extension to S is sparse.

Let us choose c_0 smaller that the constant c from (7.7). It follows from Proposition 7.3 that, for a sparse potential,

$$\sup_n a_n(V) \leq c \ \Rightarrow \operatorname{Neg}(V, S) = 1.$$

Consider some estimates for $\operatorname{Neg}(V, \Omega)$ for sparse potentials.

Corollary 7.6 *Let V be a sparse potential on a rectangle $P_{\alpha,\beta}$ with $\beta - \alpha \geq 1$. Then*

$$(\beta - \alpha) \int_{P_{\alpha,\beta}} V(x)\, dx \leq c \Rightarrow \operatorname{Neg}(V, P_{\alpha,\beta}) = 1, \tag{7.13}$$

where c is a positive constant depending only on p.

Proof. By shifting $P_{\alpha.\beta}$ and V along the axis x_1, we can assume that $\alpha = 0$, so that $\beta \geq 1$. Let m be a non-negative integer such that $2^{m-1} < \beta \leq 2^m$ (cf. Fig. 6).

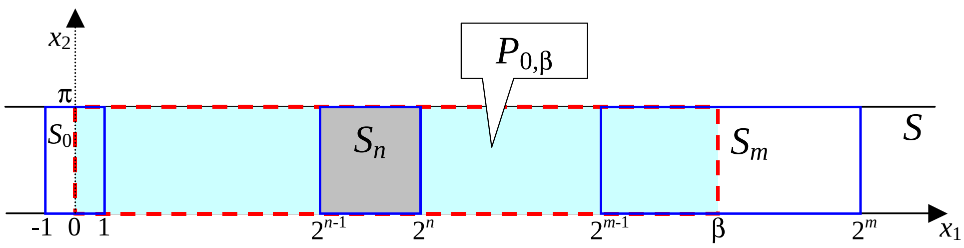

Figure 6: Rectangle $P_{0,\beta}$ is covered by the sequence $S_n, 0 \leq n \leq m$

Then $a_n(V) = 0$ for $n < 0$ and for $n \geq m + 1$. For $0 \leq n \leq m$ we have by (7.2)

$$a_n(V) \leq 2^{n+1} \int_{S_n} V(x)\, dx \leq 2^{m+1} \int_{P_{0,\beta}} V(x)\, dx \leq 4\beta \int_{P_{0,\beta}} V(x)\, dx. \tag{7.14}$$

The hypotheses (7.13) with small enough c and (7.14) imply that $a_n(17V)$ are sufficiently small for all $n \in \mathbb{Z}$. By Proposition 7.3 we obtain $\operatorname{Neg}(17V, S) = 1$, and by Lemma 7.4 $\operatorname{Neg}(V, P_{0,\beta}) = 1$. $\square$

The next statement is the main technical lemma about sparse potentials.

Lemma 7.7 *Let V be a sparse potential in a rectangle $P_{\alpha,\beta}$ with $\beta - \alpha \geq 1$. Then*

$$\operatorname{Neg}(V, P_{\alpha,\beta}) \leq 1 + C \left((\beta - \alpha) \int_{P_{\alpha,\beta}} V(x)\, dx \right)^{1/2}, \tag{7.15}$$

where the constant C depends only on p. In particular, for any $n \in \mathbb{Z}$,

$$\operatorname{Neg}(V, S_n) \leq 1 + C\sqrt{a_n(V)}. \tag{7.16}$$

Proof. Without loss of generality set $\alpha = 0$. Set also

$$J = \int_{P_{0,\beta}} V(x)\,dx$$

and recall that, by Corollary 7.6, if $\beta J \leq c$ for sufficiently small c then $\mathrm{Neg}(V, P_{0,\beta}) = 1$. Hence, in this case (7.15) is trivially satisfied, and we assume in the sequel that $\beta J > c$.

Due to Lemma 7.4, it suffices to prove the estimate

$$\mathrm{Neg}(V, S) \leq C(\beta J)^{1/2},$$

assuming that V vanishes outside $P_{0,\beta}$. Consider a sequence of reals $\{r_k\}_{k=0}^{N}$ such that

$$0 = r_0 < r_1 < ... < r_{N-1} < \beta \leq r_N$$

and the corresponding sequence of rectangles

$$R_k := P_{r_{k-1},r_k} = \{(x_1, x_2) : r_{k-1} < x_1 < r_k, \quad 0 < x_2 < \pi\}$$

where $k = 1, ..., N$, that covers $P_{0,\beta}$ (see Fig. 7).

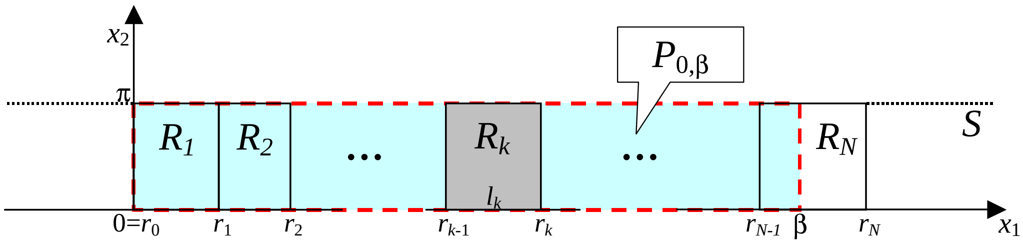

Figure 7: The sequence $\{R_k\}_{k=1}^{N}$ of rectangles covering $P_{0,\beta}$

Denote $l_k = r_k - r_{k-1}$ and

$$J_k = \int_{R_k} V(x)\,dx.$$

By Corollary 7.6, if

$$l_k \geq 1 \text{ and } l_k J_k \leq c \tag{7.17}$$

then

$$\mathrm{Neg}(V, R_k) = 1.$$

It is easy to construct the sequence $\{r_k\}$ inductively so that both conditions in (7.17) are satisfied for all $k = 1, ..., N$ (where N is yet to be determined). If r_{k-1} is already defined and is smaller than β then choose $r_k > r_{k-1}$ to satisfy the identity

$$l_k J_k = c. \tag{7.18}$$

If such r_k does not exist then set $r_k = \beta + 1$; in this case, we have

$$l_k J_k < c.$$

Let us show that in the both cases $l_k = r_k - r_{k-1} \geq 1$. Indeed, if $l_k < 1$ then $r_k < \beta + 1$ so that (7.18) is satisfied. Using the Hölder inequality, (7.18) and $l_k < 1$, we obtain

$$\left(\int_{R_k} V^p dx\right)^{1/p} \geq \frac{1}{(\pi l_k)^{1/p'}} \int_{R_k} V dx = \frac{c}{(\pi l_k)^{1/p'} l_k} \geq \frac{c}{\pi^{1/p'}}. \tag{7.19}$$

However, if the constant c_0 in the definition (7.12) of a sparse potential is small enough, then we obtain that (7.19) and (7.12) contradict each other, which proves that $l_k \geq 1$.

As soon as we reach $r_k \geq \beta$ we stop the process and set $N = k$. Since always $l_k \geq 1$, the process will indeed stop in a finite number of steps.

We obtain a partition of S into N rectangles $R_1, ..., R_N$ and two half-strips: $S \cap \{x_1 < 0\}$ and $S \cap \{x_1 > r_N\}$, and in the both half-strips we have $V \equiv 0$. In each R_k we have $\operatorname{Neg}(V, R_k) = 1$ whence it follows that

$$\operatorname{Neg}(V, S) \leq 2 + \sum_{k=1}^{N} \operatorname{Neg}(V, R_k) = N + 2.$$

Let us estimate N from above. In each R_k with $k \leq N - 1$ we have by (7.18) $\frac{1}{J_k} = \frac{1}{c} l_k$. Therefore, we have

$$N - 1 = \sum_{k=1}^{N-1} \frac{1}{\sqrt{J_k}} \sqrt{J_k} \leq \left(\frac{1}{c} \sum_{k=1}^{N-1} l_k\right)^{1/2} \left(\sum_{k=1}^{N-1} J_k\right)^{1/2} \leq \left(\frac{1}{c}\beta\right)^{1/2} J^{1/2}.$$

Using also $3 \leq 3\left(\frac{1}{c}\beta J\right)^{1/2}$, we obtain $N + 2 \leq 4\left(c^{-1}\beta J\right)^{1/2}$, which finishes the proof of (7.15).

The estimate (7.16) follows trivially from (7.15). Indeed, S_n is a rectangle $P_{\alpha,\beta}$ with the length $1 \leq \beta - \alpha \leq 2^{|n|+1}$. Using (7.15) and (7.2), we obtain

$$\operatorname{Neg}(V, S_n) \leq 1 + C\left(2^{n+1} \int_{S_n} V(x)\, dx\right)^{1/2} \leq 1 + C' \sqrt{a_n(V)}$$

with $C' = 2C$, which proves (7.16). $\square$

Proposition 7.8 *For any sparse potential V in the strip S,*

$$\operatorname{Neg}(V, S) \leq 1 + C \sum_{\{n: a_n(V) > c\}} \sqrt{a_n(V)}, \tag{7.20}$$

for some constant $C, c > 0$ depending only on p.

Proof. Let us enumerate in the increasing order those values n where $a_n(V) > c$. So, we obtain an increasing sequence $\{n_i\}$, finite or infinite, such that $a_{n_i}(V) > c$ for any index i. The difference $S \setminus \bigcup_i S_{n_i}$ can be partitions into a sequence $\{T_j\}$ of rectangles, where each rectangle T_j either fills the gap in S between successive rectangles $S_{n_i}, S_{n_{i+1}}$ as on Fig. 8 or T_j may be a half-strip that fills the gap between S_{n_i} and $+\infty$ or $-\infty$, when n_i is the maximal, respectively minimal, value in the sequence $\{n_i\}$.

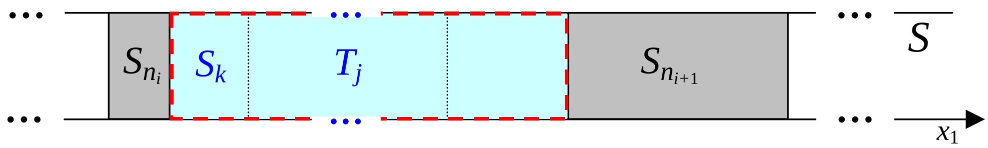

Figure 8: Partitioning of the strip S into rectangles S_{n_i} and T_j

By construction, each T_j is a union of some rectangles S_k with $a_k(V) \le c$. Consider the potential $V_j = V\mathbf{1}_{T_j}$. By Lemma 7.4 we have

$$\operatorname{Neg}(V, T_j) = \operatorname{Neg}(V_j, T_j) \le \operatorname{Neg}(17V_j, S).$$

For those k where $S_k \subset T_j$, we have $a_k(17V_j) \le 17c$, while $a_k(17V_j) = 0$ otherwise. Assuming that c is small enough, we conclude by Proposition 7.3 that $\operatorname{Neg}(17V_j, S) = 1$ and, hence, $\operatorname{Neg}(V, T_j) = 1$.

Since by construction

$$\#\{T_j\} \le 1 + \#\{S_{n_i}\},$$

it follows that

$$\begin{aligned}
\operatorname{Neg}(V, S) &\le \sum_j \operatorname{Neg}(V, T_i) + \sum_i \operatorname{Neg}(V, S_{n_i}) \\
&\le 1 + \#\{S_{n_i}\} + \sum_i \operatorname{Neg}(V, S_{n_i}) \\
&\le 1 + 2\sum_i \operatorname{Neg}(V, S_{n_i}).
\end{aligned}$$

In each S_{n_i} we have by (7.16) and $a_{n_i}(V) > c$ that

$$\operatorname{Neg}(V, S_{n_i}) \le C\sqrt{a_{n_i}(V)}.$$

Substituting into the previous estimate, we obtain (7.20). $\square$

7.4 Arbitrary potentials in a strip

We use notation $a_n(V)$ and $b_n(V)$ defined by (7.1) and (7.3), respectively.

Theorem 7.9 *For any $p > 1$ and for any potential V in the strip S, we have*

$$\operatorname{Neg}(V, S) \leq 1 + C \sum_{\{n \in \mathbb{Z}: a_n(V) > c\}} \sqrt{a_n(V)} + C \sum_{\{n \in \mathbb{Z}: b_n(V) > c\}} b_n(V), \tag{7.21}$$

where the positive constants C, c depend only on p.

Proof. Define $Q_n = S \cap \{n < x_1 < n+1\}$ so that

$$b_n(V) = \left(\int_{Q_n} V^p dx \right)^{1/p}.$$

Let $\{n_i\}$ be a sequence of all those $n \in \mathbb{Z}$ for which

$$b_n(V) > c, \tag{7.22}$$

where c is a positive constant whose value will be determined below. If this sequence is empty then the potential V is sparse, and (7.21) follows from Proposition 7.8.

Assume in the sequel that the sequence $\{n_i\}$ is non-empty. Denote by $\{T_j\}$ a sequence of rectangles that fill the gaps in S between successive rectangles Q_{n_i} or between one of Q_{n_i} and $\pm\infty$ (cf. Fig. 9).

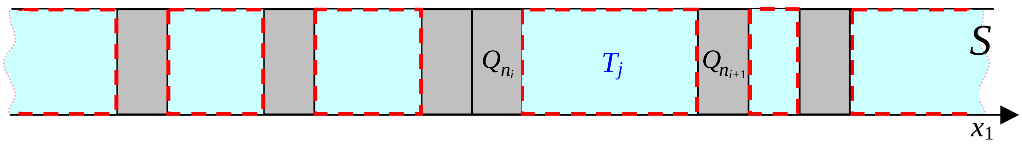

Figure 9: Partitioning of the strip S into rectangles Q_{n_i} and T_j

Consider the potentials $V' = V \mathbf{1}_{\cup T_j}$ and $V'' = V \mathbf{1}_{\cup Q_{n_i}}$. Since $V = V' + V''$, by Lemma 3.6 we obtain

$$\operatorname{Neg}(V, S) \leq \operatorname{Neg}(2V', S) + \operatorname{Neg}(2V'', S).$$

The potential $2V'$ is sparse by construction provided the constant c in (7.22) is small enough. Hence, we obtain by Proposition 7.8

$$\operatorname{Neg}(2V', S) \leq 1 + C \sum_{\{n: a_n(V') > c\}} \sqrt{a_n(V')}. \tag{7.23}$$

By Lemma 3.5 and Lemma 4.8, we obtain

$$\begin{aligned} \operatorname{Neg}(2V'', S) &\leq \sum_j \operatorname{Neg}(2V'', T_j) + \sum_i \operatorname{Neg}(2V'', Q_{n_i}) \\ &= \#\{T_j\} + \sum_i \left(1 + C \|2V''\|_{L^p(Q_{n_i})} \right) \\ &= \#\{T_j\} + \#\{Q_{n_i}\} + 2C \sum_i b_{n_i}(V). \end{aligned}$$

By construction we have $\#\{T_j\} \leq 1 + \#\{Q_{n_i}\}$. By the choice of n_i, we have $1 < c^{-1} b_{n_i}(V)$, whence

$$\begin{aligned}
\#\{T_j\} + \#\{Q_{n_i}\} &\leq 1 + 2\#\{Q_{n_i}\} \\
&\leq 1 + 2c^{-1} \sum_i b_{n_i}(V) \leq 3c^{-1} \sum_i b_{n_i}(V)
\end{aligned}$$

Combining these estimates together, we obtain

$$\operatorname{Neg}(2V'', S) \leq C' \sum_i b_{n_i}(V) = C' \sum_{\{n: b_n(V) > c\}} b_n(V) \tag{7.24}$$

Adding up (7.23) and (7.24) yields

$$\operatorname{Neg}(V, S) \leq 1 + C \sum_{\{n: a_n(V') > c\}} \sqrt{a_n(V')} + C \sum_{\{n: b_n(V) > c\}} b_n(V). \tag{7.25}$$

Since $V' \leq V$, (7.25) implies (7.21), which finishes the proof. $\square$

Remark 7.10 In fact, we have proved a slightly better inequality (7.25) than (7.21).

8 Negative eigenvalues in $\mathbb{R}^2$

Here we prove the main Theorem 1.1. Recall that Theorem 1.1 states the following: for any potential V in $\mathbb{R}^2$,

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \leq 1 + C \sum_{\{n \in \mathbb{Z}: A_n > c\}} \sqrt{A_n} + C \sum_{\{n \in \mathbb{Z}: B_n > c\}} B_n, \tag{8.1}$$

where A_n and B_n are defined in (1.5) and (1.6), and c, C are positive constants that depend only on $p > 1$.

Proof of Theorem 1.1. Consider an open set $\Omega = \mathbb{R}^2 \setminus L$ where $L = \{ x_1 \geq 0, x_2 = 0\}$ is a ray. By Lemma 3.3 we have

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \leq \operatorname{Neg}(V, \Omega). \tag{8.2}$$

The function $\Psi(z) = \ln z$ is holomorphic in Ω and provides a biholomorphic mapping from Ω onto the strip

$$\widetilde{S} = \left\{(y_1, y_2) \in \mathbb{R}^2 : \quad 0 < y_2 < 2\pi\right\}$$

(see Fig. 10).

Let $\widetilde{V}$ be a push-forward of V under Ψ (cf. by (3.20)), so that by Lemma 3.7

$$\operatorname{Neg}(V, \Omega) = \operatorname{Neg}(\widetilde{V}, \widetilde{S}). \tag{8.3}$$

Since $\widetilde{S}$ and the strip S from Section 7 are bilipschitz equivalent, the estimate (7.21) of Theorem 7.9 holds also for $\widetilde{S}$, that is,

$$\operatorname{Neg}(\widetilde{V}, \widetilde{S}) \leq 1 + C \sum_{\{n: a_n > c\}} \sqrt{a_n} + C \sum_{\{n: b_n(V) > c\}} b_n, \tag{8.4}$$

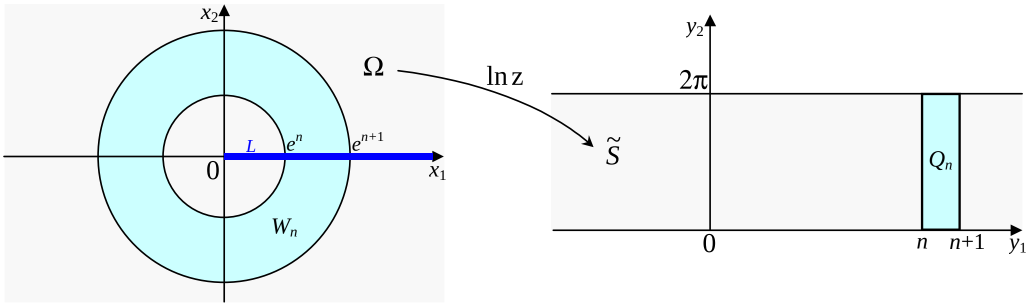

Figure 10: Conformal mapping $\Psi : \Omega \to \widetilde{S}$

where we use the following notation:

$$a_n = \int_{S_n} (1 + |y_1|)\, \widetilde{V}(y)\, dy, \quad b_n = \left(\int_{Q_n} \widetilde{V}^p dy \right)^{1/p},$$

where

$$S_n = \begin{cases} \left\{ y \in \widetilde{S} : 2^{n-1} < y_1 < 2^n \right\}, & n > 0, \\ \left\{ y \in \widetilde{S} : -1 < y_1 < 1 \right\}, & n = 0, \\ \left\{ y \in \widetilde{S} : -2^{|n|} < y_1 < -2^{|n|-1} \right\}, & n < 0. \end{cases}$$

and

$$Q_n = \left\{ y \in \widetilde{S} : n < y_1 < n + 1 \right\}.$$

Consider also the rings U_n and W_n in $\mathbb{R}^2$ defined by (1.3) and (1.4). Obviously, we have

$$\Psi(U_n \setminus L) = S_n \quad \text{and} \quad \Psi(W_n \setminus L) = Q_n.$$

Since $J_\Psi = |\Psi'(x)|^2 = \frac{1}{|x|^2}$, where x is treated as a complex variable, we obtain by (3.23) that

$$\begin{aligned} b_n^p &= \int_{Q_n} \widetilde{V}^p(y)\, dy = \int_{W_n} V^p(x)\, |J_\Psi(x)|^{1-p} dx \\ &= \int_{W_n} V^p(x)\, |x|^{2(p-1)} dx = B_n^p. \end{aligned} \tag{8.5}$$

Since for $y = \Psi(x)$ we have $y_1 = \operatorname{Re} \ln x = \ln |x|$, it follows from (3.23) that

$$a_n = \int_{S_n} \widetilde{V}(y)\, (1 + |y_1|)\, dy = \int_{U_n} V(x)\, (1 + |\ln |x||)\, dx = A_n.$$

Combining together (8.2), (8.3), (8.4), we obtain (8.1). ☐

Proof of Corollary 1.2. If a stronger hypothesis

$$\sum_{n \in \mathbb{Z}} \sqrt{A_n(V)} + \sum_{n \in \mathbb{Z}} B_n(V) < \infty$$

is satisfied then (1.15) is an immediate consequence of (1.7). To prove (1.15) under the hypothesis (1.14), we need an improved version of (1.7). Let us come back to the proof of Theorem 1.1 and use instead of (8.4) the estimate (7.25), that is,

$$\operatorname{Neg}(\widetilde{V},\widetilde{S}) \le 1 + C \sum_{\left\{n: a_n(\widetilde{V}')>c\right\}} \sqrt{a_n(\widetilde{V}')} + C \sum_{\left\{n: b_n(\widetilde{V})>c\right\}} b_n(\widetilde{V}),$$

where $\widetilde{V}'$ is a modification of $\widetilde{V}$ that vanishes on the rectangles Q_n with $b_n(\widetilde{V}) > c$. Then we obtain instead of (8.1) the following estimate:

$$\operatorname{Neg}\left(V,\mathbb{R}^2\right) \le 1 + C \sum_{\{n\in\mathbb{Z}: A_n(V')>c\}} \sqrt{A_n\left(V'\right)} + C \sum_{\{n\in\mathbb{Z}: B_n(V)>c\}} B_n\left(V\right), \tag{8.6}$$

where V' is a modification of V that vanishes on the annuli W_n with $B_n\left(V\right) > c$. As it was explained in Introduction, (8.6) implies

$$\operatorname{Neg}\left(V,\mathbb{R}^2\right) \le 1 + C \int_{\mathbb{R}^2} V'\left(x\right)\left(1+\left|\ln\left|x\right|\right|\right) dx + C \sum_{\{n\in\mathbb{Z}: B_n(V)>c\}} B_n\left(V\right). \tag{8.7}$$

Let us apply (8.7) to the potential αV with $\alpha \to \infty$. Denote by $W\left(\alpha\right)$ the union of all annuli W_n with $B_n\left(\alpha V\right) \le c$, so that $\left(\alpha V\right)' = \alpha V \mathbf{1}_{W(\alpha)}$. Then (8.7) implies

$$\operatorname{Neg}\left(\alpha V,\mathbb{R}^2\right) \le 1 + C\alpha \int_{\mathbb{R}^2} V\left(x\right) \mathbf{1}_{W(\alpha)} \left(1+\left|\ln\left|x\right|\right|\right) dx + C\alpha \sum_{n\in\mathbb{Z}} B_n\left(V\right). \tag{8.8}$$

For any n with $B_n\left(V\right) > 0$, the condition $B_n\left(\alpha V\right) > c$ will be satisfied for large enough α, so that for such α the function $V\mathbf{1}_{W(\alpha)}$ vanishes on W_n. If $B_n\left(V\right) = 0$ then $V = 0$ on W_n and, hence, $V\mathbf{1}_{W(\alpha)} = 0$ on W_n again. We see that $V\mathbf{1}_{W(\alpha)} \to 0$ a.e. as $\alpha \to \infty$, and by the dominated convergence theorem

$$\int_{\mathbb{R}^2} V\left(x\right) \mathbf{1}_{W(\alpha)} \left(1+\left|\ln\left|x\right|\right|\right) dx \to 0.$$

Substituting into (8.8) we obtain (1.15). □

Proof of Corollary 1.3. Let us estimate the both terms in the right hand side of (1.10) using the Hölder inequality. For the first term we have

$$\int_{\mathbb{R}^2} V\left(x\right)\left(1+\left|\ln\left|x\right|\right|\right) dx \le \left(\int_{\mathbb{R}^2} V^p \mathcal{W} dx\right)^{1/p} \left(\int_{\mathbb{R}^2} \frac{\left(1+\left|\ln\left|x\right|\right|\right)^{p'}}{\left(\mathcal{W}\left(\left|x\right|\right)\right)^{\frac{p'}{p}}} dx\right)^{1/p'}.$$

The second integral can be computed in the polar coordinates and it is equal to

$$\int_{\mathbb{R}^2} \frac{\left(1+\left|\ln r\right|\right)^{\frac{p}{p-1}}}{\mathcal{W}\left(r\right)^{\frac{1}{p-1}}} 2\pi r dr,$$

which is finite by (1.16). Hence, we obtain that

$$\int_{\mathbb{R}^2} V\left(x\right)\left(1+\left|\ln\left|x\right|\right|\right) dx \le C \left(\int_{\mathbb{R}^2} V^p \mathcal{W} dx\right)^{1/p} \tag{8.9}$$

To estimate the second term in (1.10), take any sequence $\{l_n\}_{n\in\mathbb{Z}}$ of positive reals and write

$$\begin{aligned}\sum_n B_n &= \sum_n l_n^{-1/p} l_n^{1/p} \left(\int_{W_n} V^p |x|^{2(p-1)}\, dx\right)^{1/p} \\ &\le \left(\sum_n l_n^{-\frac{1}{p-1}}\right)^{1/p'} \left(\sum_n l_n \int_{W_n} V^p |x|^{2(p-1)}\, dx\right)^{1/p}.\end{aligned}$$

Choose here

$$l_n = \frac{\mathcal{W}(e^n)}{(e^{n+1})^{2(p-1)}}$$

so that, for $x \in W_n$,

$$l_n |x|^{2(p-1)} \le \mathcal{W}(e^n) \le \mathcal{W}(|x|)$$

and

$$\sum_n l_n \int_{W_n} V^p(x) |x|^{2(p-1)}\, dx \le \int_{\mathbb{R}^2} V^p(x) \mathcal{W}(|x|)\, dx.$$

On the other hand, we have

$$\sum_n l_n^{-\frac{1}{p-1}} = \sum_n \frac{e^{2(n+1)}}{\mathcal{W}(e^n)^{\frac{1}{p-1}}} \simeq \int_0^\infty \frac{r dr}{\mathcal{W}(r)^{\frac{1}{p-1}}} < \infty$$

by (1.16). Hence,

$$\sum_n B_n \le C\left(\int_{\mathbb{R}^2} V^p \mathcal{W} dx\right)^{1/p}. \tag{8.10}$$

Substituting (8.9) and (8.10) into (1.10), we obtain (1.17). □

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