A correction

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This is a correction to [1, Lemma3.2].

Let M be a connected Riemannian manifold, μ be the Riemannian measure and V be a non-negative L^1_{loc} function (potential) on M. For any non-empty open set $\Omega \subset M$, define the linear space

$$\mathcal{F}_{V,\Omega} = \left\{ f \in W^{1,2}_{loc}(\Omega) : \int_\Omega |\nabla f|^2 \, d\mu < \infty, \quad \int_\Omega V f^2 d\mu < \infty \right\} \tag{1}$$

that is a natural domain for the quadratic form

$$\mathcal{E}_{V,\Omega}(f) = \int_\Omega |\nabla f|^2 \, d\mu - \int_\Omega V f^2 d\mu. \tag{2}$$

Define the Morse index of $\mathcal{E}_{V,\Omega}$ by

$$\operatorname{Neg}(V,\Omega) = \sup \{ \dim \mathcal{V} : \mathcal{V} \prec \mathcal{F}_{V,\Omega} \ \text{ s.t. } \mathcal{E}_{V,\Omega}(f) \le 0 \ \forall f \in \mathcal{V} \},$$

where $\mathcal{V} \prec \mathcal{F}_{V,\Omega}$ means that $\mathcal{V}$ is a linear subspace of $\mathcal{F}_{V,\Omega}$ of finite dimension. Clearly, we have also

$$\operatorname{Neg}(V,\Omega) = \sup \left\{ \dim \mathcal{V} : \mathcal{V} \prec W^{1,2}_{loc}(\Omega) \ \text{ s.t. } \int_M |\nabla f|^2 \, d\mu \le \int_M V f^2 d\mu < \infty \ \forall f \in \mathcal{V} \right\}.$$

Lemma 1. *If Ω is a non-empty open subset of M and V vanishes μ-a.e. outside Ω then*

$$\operatorname{Neg}(V,M) \le \operatorname{Neg}(V,\Omega). \tag{3}$$

Proof. By definition, we have

$$\operatorname{Neg}(V,M) = \sup \left\{ \dim \mathcal{V} : \mathcal{V} \prec W^{1,2}_{loc}(M) \ \text{ s.t. } \int_M |\nabla f|^2 \, d\mu \le \int_M V f^2 d\mu < \infty \ \forall f \in \mathcal{V} \right\}. \tag{4}$$

Let $\mathcal{V}$ be a subspace as in (4). For any $f \in \mathcal{V}$ we have

$$\int_\Omega |\nabla f|^2 \, d\mu \le \int_M |\nabla f|^2 \, d\mu \le \int_M V f^2 d\mu = \int_\Omega V f^2 d\mu,$$

since V vanishes outside Ω. Denote by $\mathcal{V}_\Omega$ the set of functions on Ω that are obtained by restricting functions from M to Ω. The above computation shows that

$$\int_\Omega |\nabla f|^2 \, d\mu \le \int_\Omega V f^2 d\mu < \infty \ \ \forall f \in \mathcal{V}_\Omega.$$

Let us verify that

$$\dim \mathcal{V}_\Omega = \dim \mathcal{V}. \tag{5}$$

Indeed, by construction we have a surjective linear mapping

$$\begin{aligned} A \ &: \ \mathcal{V} \to \mathcal{V}_\Omega \\ Af &= f|_\Omega. \end{aligned}$$

It suffices to prove that A is also injective. Indeed, if $Af = 0$ for some $f \in \mathcal{V}$ then $f = 0$ in Ω and, hence,

$$\int_M V f^2 d\mu = \int_\Omega V f^2 d\mu + \int_{\Omega^c} V f^2 d\mu = 0,$$

which implies by (4) that

$$\int_M |\nabla f|^2 \, d\mu = 0.$$

By connectedness of M, we conclude that $f = \text{const}$. Since f vanishes in Ω, it follows that $f \equiv 0$, which proves (5).

Since

$$\operatorname{Neg}(V, \Omega) \geq \dim \mathcal{V}_\Omega,$$

it follows from (5) that

$$\operatorname{Neg}(V, \Omega) \geq \dim \mathcal{V}.$$

Since this is true for any subspace $\mathcal{V}$ from (4), we conclude that

$$\operatorname{Neg}(V, \Omega) \geq \operatorname{Neg}(V, M),$$

which was to be proved. ■

Remark 2. [1, Lemma 4.5] states that if D is a disk in $\mathbb{R}^2$ and V is supported in D then

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \leq \operatorname{Neg}(2V, D). \tag{6}$$

However, the proof of this statement in [1] is wrong. It is based on extension of functions from D to $\mathbb{R}^2$ and on [1, Lemma 3.2], and yields, in fact, another inequality:

$$\operatorname{Neg}(V, D) \leq \operatorname{Neg}\left(2V, \mathbb{R}^2\right).$$

This argument does not require V to vanish outside D (compare also with correct [1, Lemma 7.4]).

Nevertheless, (6) is true because by (3) we have

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \leq \operatorname{Neg}(V, D) \leq \operatorname{Neg}(2V, D).$$

As we see, (6) is much simpler than it is meant to be in [1], as it is based not on extension of functions but on restriction.

References

[1] Grigor'yan A., Nadirashvili N., *Negative eigenvalues of two-dimensional Schrödinger equations*, Archive Rat. Mech. Anal. **217** (2015) 975–1028.