Rend. Sem. Mat. Univ. Pol. Torino
Vol. xx, x (xxxx), 1 – **??**

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STOCHASTIC COMPLETENESS OF SYMMETRIC MARKOV PROCESSES AND VOLUME GROWTH

Abstract. We discuss sufficient conditions for stochastic completeness of various types of Markov processes (diffusions on Riemannian manifolds, jump processes, random walks on graphs) in terms of the volume growth function of the underlying metric measure space.

1. Brownian motion on Riemannian manifolds

Let (M,g) be a Riemannian manifold and μ be the Riemannian measure on M. The Laplace operator (or Laplace-Beltrami operator) Δ is defined to satisfy the Green formula: for all $u,v\in C_0^\infty(M)$

$$\int_M \Delta u\, v d\mu = -\int_M \langle \nabla u, \nabla v\rangle d\mu, \tag{1}$$

where ∇ is the Riemannian gradient and $\langle\cdot,\cdot\rangle$ is the Riemannian inner product (see [2], [6], [10]).

The symmetry of the operator Δ with respect to μ (that follows from (1)) allows to extend it to a self-adjoint operator in $L^2(M,\mu)$. In general, this extension may not be unique, but if M is geodesically complete (which will be assumed throughout) then this extension is unique, that is, Δ is essentially self-adjoint. With some abuse of notation, the self-adjoint extension of Δ will be denoted by the same letter.

As one can see from (1), the operator Δ is non-positive definite, which implies that the operator $P_t := e^{t\Delta}$ is a bounded self-adjoint operator for any $t\geq 0$. The family $\{P_t\}_{t\geq 0}$ is called the *heat semigroup* of Δ for the reason that it resolves the heat equation. More precisely, the following is true:

- for any $f\in L^2$, the function $u(t,x)=P_tf(x)$ is C^∞ smooth in $(t,x)\in(0,+\infty)\times M$, satisfies the heat equation $\frac{\partial u}{\partial t}=\Delta u$ and the initial condition $u(t,\cdot)\overset{L^2}{\to} f$ as $t\to 0+$.
- If $f\geq 0$ then $P_tf\geq 0$; if $f\leq 1$ then $P_tf\leq 1$.
- The semigroup property: $P_tP_s=P_{t+s}$.

Furthermore, the operator P_t is in fact an integral operator with a kernel $p_t(x,y)$ that is a smooth positive function of $t>0$ and $x,y\in M$ such that

$$P_tf(x)=\int_M p_t(x,y)\, f(y)\, d\mu(y) \tag{2}$$

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for all $f \in L^2$. The function $p_t(x,y)$ is called the *heat kernel* of Δ (or of M). It is also the minimal positive fundamental solution of the heat equation and the transition density of Brownian motion on M. For example, if $M = \mathbb{R}^n$ then

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

For general manifolds there is no explicit formula for the heat kernel.

The existence of the heat kernel allows to extend the domain of the operator P_t from L^2 to other spaces. For that, let us use now the identity (2) as the definition of P_t where f is any function such that the integral converges. In particular, P_t extends to a bounded operator on L^∞.

DEFINITION 1. A manifold (M,g) is called *stochastically complete* if $P_t 1 \equiv 1$.

Note that in general we have $0 \leq P_t 1 \leq 1$. If $P_t 1 \not\equiv 1$ then the manifold M is called *stochastically incomplete.*

Easy examples of stochastically incomplete processes are given by diffusions in bounded domains with the Dirichlet boundary condition. A by far less trivial example was discovered by R.Azencott [1] in 1974: he showed that Brownian motion on a geodesically complete non-compact manifold can be stochastically incomplete. In his example, the manifold has negative sectional curvature that grows to $-\infty$ very fast with the distance to an origin. The stochastic incompleteness occurs because negative curvature plays the role of a drift towards infinity, and a very high negative curvature produces an extremely fast drift that sweeps the Brownian particle to infinity in a finite time.

The first sufficient condition for stochastic completeness of geodesically complete manifolds in terms of lower bound of Ricci curvature was proved by S.-T. Yau [15]. Below we present a condition in terms of the volume growth function.

Let us first state various equivalent conditions for the stochastic completeness. Fix $0 < T \leq \infty$, set $I = (0,T)$ and consider the Cauchy problem in $I \times M$

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } I \times M, \\ u|_{t=0} = 0. \end{cases} \tag{3}$$

The problem (3) is understood in the classical sense, that is, $u \in C^\infty(I \times M)$ and $u(t,x) \to 0$ locally uniformly in $x \in M$ as $t \to 0$. We are interested in the uniqueness of the trivial solution $u \equiv 0$ of (3).

THEOREM 1. (Khas'minskii [9]) *For any* $\alpha > 0$ *and* $T \in (0,\infty]$, *the following conditions are equivalent.*

(a) *M is stochastically complete.*

(b) *The equation $\Delta v = \alpha v$ in M has the only bounded non-negative solution $v \equiv 0$.*

(c) *The Cauchy problem in* $(0,T)\times M$ *has a unique bounded solution* $u\equiv 0$.

DEFINITION 2. Define the *volume function* $V(x,r)$ of a manifold (M,g) by $V(x,r):=\mu(B(x,r))$, where $B(x,r)$ is the geodesic ball of radius r centered at x.

Note that $0<V(x,r)<\infty$ for all $x\in M$ and $r>0$ provided M is geodesically complete.

THEOREM 2. *Let* (M,g) *be a geodesically complete connected Riemannian manifold. If, for some point* $x_0\in M$,

$$\int^{\infty}\frac{rdr}{\log V(x_0,r)}=\infty, \tag{4}$$

then M is stochastically complete.

Condition (4) holds, in particular, if

$$V(x_0,r)\le\exp\left(Cr^2\right) \tag{5}$$

for all r large enough or even if (5) holds for a sequence $\{r_k\}$ of values r that goes to ∞ as $k\longrightarrow\infty$.

Theorem 2 follows from the equivalence $(a)\Leftrightarrow(c)$ of Theorem 1 and the following more general result.

THEOREM 3. *Let* (M,g) *be a complete connected Riemannian manifold, and let* $u(x,t)$ *be a solution to the Cauchy problem* (3). *Assume that, for some* $x_0\in M$ *and for all* $R>0$,

$$\int_0^T\int_{B(x_0,R)}u^2(x,t)\,d\mu(x)dt\le\exp\left(f(R)\right), \tag{6}$$

where $f(r)$ *is a positive increasing function on* $(0,+\infty)$ *such that*

$$\int^{\infty}\frac{rdr}{f(r)}=\infty. \tag{7}$$

Then $u\equiv 0$ *in* $I\times M$.

Condition (6) determines hence a uniqueness class for the Cauchy problem. Clearly, (7) holds for $f(r)=Cr^2$, but fails for $f(r)=Cr^{2+\varepsilon}$ with $\varepsilon>0$.

Theorems 2 and 3 were proved in [4] (see also [5] and [6]). Without going into details, let us emphasize, that the argument repeatedly uses the following property of the geodesic distance function d on the manifold: $|\nabla d|\le 1$.

Let us mention the following consequence for $\mathbb{R}^n$.

COROLLARY 1. *If* $M=\mathbb{R}^n$ *and* $u(t,x)$ *be a solution to* (3) *satisfying the condition*

$$|u(t,x)|\le C\exp\left(C|x|^2\right)\quad \text{for all } t\in I,\ x\in\mathbb{R}^n, \tag{8}$$

then $u \equiv 0$. *Moreover, the same is true if u satisfies instead of* (8) *the condition*

$$|u(t,x)| \le C \exp\left(f\left(|x|\right)\right) \quad \textit{for all } t \in I,\ x \in \mathbb{R}^n, \tag{9}$$

where $f(r)$ *is a convex increasing function on* $(0,+\infty)$ *satisfying* (7).

The class of functions u satisfying (8) is called the *Tikhonov class*, and the conditions (9) and (7) define the *Täcklind class*. The uniqueness of the Cauchy problem in $\mathbb{R}^n$ in each of these classes is a classical result of Tikhonov [13] and Täcklind [12], respectively.

The hypothesis (4) of Theorem 2 is sufficient for the stochastic completeness of M but not necessary. Moreover, there are examples of stochastically complete manifolds with arbitrarily large volume function.

Nevertheless, the condition (4) is sharp in the following sense: if $f(r)$ is a smooth positive convex function on $(0,+\infty)$ with $f'(r) > 0$ and such that

$$\int^{\infty} \frac{rdr}{f(r)} < \infty,$$

then there exists a geodesically complete but stochastically incomplete manifold M such that $\log V(x_0,r) = f(r)$, for some $x_0 \in M$ and large enough r (see [5]).

2. Jump processes

Let (M,d) be a metric space such that all closed metric balls

$$B(x,r) = \{y \in M : d(x,y) \le r\}$$

are compact. In particular, (M,d) is locally compact and separable. Let μ be a Radon measure on M with a full support.

Recall that a *Dirichlet form* $(\mathcal{E},\mathcal{F})$ in $L^2(M,\mu)$ is a symmetric, non-negative definite, bilinear form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ defined on a dense subspace $\mathcal{F}$ of $L^2(M,\mu)$, which satisfies in addition the following properties:

- Closedness: $\mathcal{F}$ is a Hilbert space with respect to the following inner product:

$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + (f,g). \tag{10}$$

- The Markov property: if $f \in \mathcal{F}$ then also $\widetilde{f} := (f \wedge 1)_+$ belongs to $\mathcal{F}$ and $\mathcal{E}(\widetilde{f}) \le \mathcal{E}(f)$, where $\mathcal{E}(f) := \mathcal{E}(f,f)$.

Then $(\mathcal{E},\mathcal{F})$ has the *generator* $\mathcal{L}$ that is a non-positive definite, self-adjoint operator on $L^2(M,\mu)$ with domain $\mathcal{D} \subset \mathcal{F}$ such that $\mathcal{E}(f,g) = (-\mathcal{L}f,g)$ for all $f \in \mathcal{D}$ and $g \in \mathcal{F}$. The generator $\mathcal{L}$ determines the *heat semigroup* $\{P_t\}_{t\ge0}$ by $P_t = e^{t\mathcal{L}}$ in the sense of functional calculus of self-adjoint operators. It is known that $\{P_t\}_{t\ge0}$ is strongly continuous, contractive, symmetric semigroup in L^2, and is *Markovian*, that is, $0 \leq P_t f \leq 1$ for any $t > 0$ if $0 \leq f \leq 1$.

The Markovian property of the heat semigroup implies that the operator P_t preserves the inequalities between functions, which allows to use monotone limits to extend P_t from L^2 to L^∞ (in fact, P_t extends to any L^q, $1 \leq q \leq \infty$ as a contraction). In particular, $P_t 1$ is defined.

DEFINITION 3. The form $(\mathcal{E}, \mathcal{F})$ is called *conservative* or *stochastically complete* if $P_t 1 = 1$ for every $t > 0$.

Assume in addition that $(\mathcal{E}, \mathcal{F})$ is *regular*, that is, the set $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ with respect to the norm (10) and in $C_0(M)$ with respect to the sup-norm. By a theory of Fukushima [3], for any regular Dirichlet form there exists a Hunt process $\{X_t\}_{t \geq 0}$ such that, for all bounded Borel functions f on M,

$$\mathbb{E}_x f(X_t) = P_t f(x) \tag{11}$$

for all $t > 0$ and almost all $x \in M$, where $\mathbb{E}_x$ is expectation associated with the law of $\{X_t\}$ started at x. Using the identity (11), one can show that the lifetime of X_t is almost surely ∞ if and only if $P_t 1 = 1$ for all $t > 0$, which motivates the term “stochastic completeness” in the above definition.

One distinguishes local and non-local Dirichlet forms. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *local* if $\mathcal{E}(f, g) = 0$ for all functions $f, g \in \mathcal{F}$ with disjoint compact support. It is called *strongly local* if the same is true under a milder assumption that $f = \mathrm{const}$ on a neighborhood of $\operatorname{supp} g$.

For example, the classical Dirichlet form on a Riemannian manifold

$$\mathcal{E}(f, g) = \int_M \nabla f \cdot \nabla g d\mu$$

is strongly local. The domain of this form is the Sobolev space H^1, the generator is the self-adjoint Laplace-Beltrami operator Δ, and the Hunt process is Brownian motion on M.

A well-studied non-local Dirichlet form in $\mathbb{R}^n$ is given by

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{n+\alpha}} dx dy \tag{12}$$

where $0 < \alpha < 2$. The domain of this form is the Besov space $B_{2,2}^{\alpha/2}$, the generator is (up to a constant multiple) the operator $-(-\Delta)^{\alpha/2}$, where Δ is the Laplace operator in $\mathbb{R}^n$, and the Hunt process is the the symmetric stable process of index α.

By a theorem of Beurling and Deny (cf. [3]), any regular Dirichlet form can be represented in the form

$$\mathcal{E} = \mathcal{E}^{(c)} + \mathcal{E}^{(j)} + \mathcal{E}^{(k)},$$

where $\mathcal{E}^{(c)}$ is a strongly local part that has the form

$$\mathcal{E}^{(c)}(f, g) = \int_M \Gamma(f, g) d\mu,$$

where $\Gamma(f,g)$ is a so called *energy density* (generalizing $\nabla f \cdot \nabla g$ on manifolds); $\mathcal{E}^{(j)}$ is a jump part that has the form

$$\mathcal{E}^{(j)}(f,g) = \frac{1}{2}\int\int_{M\times M}(f(x)-f(y))(g(x)-g(y))\,dJ(x,y)$$

with some measure J on $M\times M$ that is called a *jump measure*; and $\mathcal{E}^{(k)}$ is a killing part that has the form

$$\mathcal{E}^{(k)}(f,g) = \int_M fg\,dk$$

where k is a measure on M that is called a *killing measure*.

In terms of the associated process this means that X_t is in some sense a mixture of a diffusion process, jump process and a killing condition.

The log-volume test of Theorem 2 can be extended to strongly local Dirichlet forms, provided the distance function satisfies the condition

$$\Gamma(d(\cdot,x_0), d(\cdot,x_0)) \le C, \tag{13}$$

for some point $x_0 \in M$ and constant C, and the volume function $V(x,r) := \mu(B(x,r))$ satisfies (4). The method of the proof is basically the same as in Theorem 2 because for strongly local forms the same chain rule and product rules are available, and the condition (13) replaces the condition $|\nabla d| \le 1$ (see [11]).

Now let us turn to jump processes. For simplicity let us assume that the jump measure J has a density $j(x,y)$. Namely, let $j(x,y)$ be is a non-negative Borel function on $M\times M$ that satisfies the following two conditions:

(a) $j(x,y)$ is symmetric: $j(x,y) = j(y,x)$;

(b) there is a positive constant C such that

$$\int_M (1\wedge d(x,y)^2)\,j(x,y)\,d\mu(y) \le C \text{ for all } x\in M. \tag{14}$$

DEFINITION 4. We say that a distance function d is *adapted* to a kernel $j(x,y)$ (or j is adapted to d) if (b) is satisfied.

For the purpose of investigation of stochastic completeness the condition (b) plays the same role as (13) does for diffusion.

Consider the following bilinear functional

$$\mathcal{E}(f,g) = \frac{1}{2}\int\int_{M\times M}(f(x)-f(y))(g(x)-g(y))\,j(x,y)\,d\mu(x)\,d\mu(y) \tag{15}$$

that is defined on Borel functions f and g whenever the integral makes sense. Define the maximal domain of $\mathcal{E}$ by

$$\mathcal{F}_{\max} = \left\{ f\in L^2 : \mathcal{E}(f,f) < \infty \right\},$$

where $L^2 = L^2(M,\mu)$. By the polarization identity, $\mathcal{E}(f,g)$ is finite for all $f,g \in \mathcal{F}_{\max}$. Moreover, $\mathcal{F}_{\max}$ is a Hilbert space with the following norm:

$$\|f\|^2_{\mathcal{F}_{\max}} = \mathcal{E}_1(f,f) := \|f\|^2_{L^2} + \mathcal{E}(f,f).$$

Denote by $\mathrm{Lip}_0(M)$ the class of Lipschitz functions on M with compact support. It follows from (14) that $\mathrm{Lip}_0(M) \subset \mathcal{F}_{\max}$. Define the space $\mathcal{F}$ as the closure of $\mathrm{Lip}_0(M)$ in $(\mathcal{F}_{\max}, \|\cdot\|_{\mathcal{F}_{\max}})$. Under the above hypothesis, $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2(M,\mu)$. The associated Hunt process $\{X_t\}$ is a pure jump process with the jump density $j(x,y)$.

Many examples of jump processes are provided by Lévy-Khintchine theorem where the Lévy measure corresponds to $j(x,y)\,d\mu(y)$. The condition (14) appears also in Lévy-Khintchine theorem, so that the Euclidean distance in $\mathbb{R}^n$ is adapted to any Lévy measure. An explicit example of a jump density in $\mathbb{R}^n$ is

$$j(x,y) = \frac{\mathrm{const}}{|x-y|^{n+\alpha}},$$

where $\alpha \in (0,2)$, which defines the Dirichlet form (12).

Sufficient condition for stochastic completeness of the Dirichlet form of jump type is given in the following theorem that was proved in [7].

THEOREM 4. *Assume that j satisfies (a) and (b) and let $(\mathcal{E},\mathcal{F})$ be the jump form defined as above. Fix a constant $b < \frac{1}{2}$. If, for some $x_0 \in M$ and for all large enough[*] r,*

$$V(x_0,r) \le \exp(br\log r), \tag{16}$$

then the Dirichlet form $(\mathcal{E},\mathcal{F})$ is stochastically complete.

It is not known if the borderline value $\frac{1}{2}$ for b is sharp.

For example, (16) is satisfied if, for some constant C and all large r,

$$V(x_0,r) \le \exp(Cr)$$

For the proof of Theorem 4 we split the jump kernel $j(x,y)$ into the sum of two parts:

$$j'(x,y) = j(x,y)\mathbf{1}_{\{d(x,y)\le 1\}} \text{ and } j''(x,y) = j(x,y)\mathbf{1}_{\{d(x,y)>1\}}$$

and show first the stochastic completeness of the Dirichlet form $(\mathcal{E}',\mathcal{F})$ associated with j'. For that we adapt the methods used for stochastic completeness for the local form. The bounded range of j' allows to treat the Dirichlet form $(\mathcal{E}',\mathcal{F})$ as "almost" local: if f,g are two functions from $\mathcal{F}$ such that $d(\mathrm{supp}\, f, \mathrm{supp}\, g) > 1$ then $\mathcal{E}(f,g) = 0$. The condition (14) plays in the proof the same role as the condition (13) in the local case. However, the lack of locality brings up in the estimates various additional terms that

[*]In fact it suffices to have (16) for $r = r_k$ where $\{r_k\}$ is any sequence such that $r_k \to \infty$ as $k \to \infty$.

have to be compensated by a stronger hypothesis of the volume growth (16), instead of the quadratic exponential growth in Theorem 2.

The tail j'' can regarded as a small perturbation of j' in the following sense: $(\mathcal{E},\mathcal{F})$ is stochastically complete if and only if $(\mathcal{E}',\mathcal{F})$ is so. The proof is based on the fact that the integral operator with the kernel j'' is a bounded operator in $L^2(M,\mu)$, because by (14)

$$\int_M j''(x,y)\,d\mu(y) \le C.$$

It is not clear if the volume growth condition (16) in Theorem 4 is sharp.

Let us briefly mention a recent result of Xueping Huang [8], that is analogous of Theorem 3 about the uniqueness class for the Cauchy problem on a geodesically complete manifold. X.Huang proved a similar theorem for the heat equation associated with the jump Dirichlet form on graphs satisfying (a) and (b): namely, the associated heat equation has the following uniqueness class

$$\int_0^T \int_{B(x,R)} u^2(t,x)\,d\mu(x)\,dt \le \exp(br\log r)$$

where b is as above any constant smaller than $\frac{1}{2}$. Moreover, he has shown that for $b > 2\sqrt{2}$ this statement fails. The optimal value of b remains unknown. Note that the function u in that example is unbounded, so that it cannot serve to show the sharpness of the condition (16) in Theorem 4.

3. Random walks on graphs

Let us now turn to random walks on graphs. Let (X,E) be a locally finite, infinite, connected graph, where X is the set of vertices and E is the set of edges. We assume that the graph is undirected, simple, without loops. Let μ be the counting measure on X. Define the jump kernel by $j(x,y) = 1_{\{x\sim y\}}$, where $x\sim y$ means that x,y are neighbors, that is, $(x,y)\in E$. The corresponding Dirichlet form is

$$\mathcal{E}(f) = \frac{1}{2}\sum_{\{x,y:x\sim y\}}(f(x)-f(y))^2,$$

and its generator is

$$\Delta f(x) = \sum_{y\sim x}(f(y)-f(x)).$$

The operator Δ is called *unnormalized* (or *physical)* Laplace operator on (X,E). This is to distinguish from the *normalized* or *combinatorial* Laplace operator

$$\hat{\Delta} f(x) = \frac{1}{\deg(x)}\sum_{y\sim x}(f(y)-f(x)),$$

where $\deg(x)$ is the number of neighbors of x. The normalized Laplacian $\hat{\Delta}$ is the generator of the same Dirichlet form but with respect to the degree measure $\deg(x)$.

Both Δ and $\hat{\Delta}$ generate the heat semigroups $e^{t\Delta}$ and $e^{t\hat{\Delta}}$ and, hence, associated continuous time random walks on X. It is easy to prove that $\hat{\Delta}$ is a bounded operator in $L^2(X,\deg)$, which then implies that the associated random walk is always stochastically complete. On the contrary, the random walk associated with the unnormalized Laplace operator can be stochastically incomplete.

We say that the graph (X,E) is stochastically complete if the heat semigroup $e^{t\Delta}$ is stochastically complete.

Denote by $\rho(x,y)$ the graph distance on X, that is the minimal number of edges in an edge chain connecting x and y. Let $B_\rho(x,r)$ be closed metric balls with respect to this distance ρ and set $V_\rho(x,r) = \left|B_\rho(x,r)\right|$ where $|\cdot| := \mu(\cdot)$ denotes the number of vertices in a given set.

The stochastic completeness can be determined in terms of the function V_ρ as follows.

THEOREM 5. *If there is a point $x_0 \in X$ and a constant $c > 0$ such that*

$$V_\rho(x_0,r) \le cr^3 \tag{17}$$

for all large enough r, then the graph (X,E) is stochastically complete.

Note that the cubic rate of the volume growth is sharp here. Indeed, Wojciechowski [14] has shown that, for any $\varepsilon > 0$ there is a stochastically incomplete graph that satisfies $V_\rho(x_0,r) \le cr^{3+\varepsilon}$. For any non-negative integer r, set

$$S_r = \{x \in X : \rho(x_0,x) = r\}.$$

In the example of Wojciechowski every vertex on S_r is connected to all vertices on S_{r-1} and S_r.

For this type of graphs, that are called *anti-trees*, the stochastic incompleteness is equivalent to the following condition ([14]):

$$\sum_{r=1}^{\infty} \frac{V_\rho(x_0,r)}{|S_{r+1}|\,|S_r|} < \infty. \tag{18}$$

Indeed, assuming (18), one constructs a non-trivial bounded solution to the equation $\Delta u - u = 0$, which is enough to ensure the stochastic incompleteness (cf. Theorem 1). For a radial function $u = u(r)$ this equation acquires the form

$$u(r+1) = u(r) + \frac{1}{|S_{r+1}|\,|S_r|} \sum_{i=0}^{r} |S_i|\, u(i).$$

Setting $u(0) = 1$ and solving this equation inductively in r, we obtain a positive solution $u(r)$ that increases in r. It follows that

$$u(r+1) \le \left(1 + \frac{1}{|S_{r+1}|\,|S_r|} \sum_{i=0}^{r} |S_i|\right) u(r)$$

whence by induction

$$u(R) \leq \prod_{r=0}^{R-1} \left(1 + \frac{V_\rho(x_0, r)}{|S_{r+1}|\,|S_r|}\right).$$

The condition (18) implies that the product in the right hand side is bounded so that u is a bounded function.

If $|S_r| \simeq r^{2+\varepsilon}$ then $V_\rho(x_0, r) \simeq r^{3+\varepsilon}$ and the condition (18) is satisfied so that the graph is stochastically incomplete.

The proof of Theorem 5 is based on the following ideas. First observe that the graph distance ρ is in general not adapted. More precisely, ρ is adapted if and only if the graph has uniformly bounded degree, which is not an interesting case.

Let us construct an adapted distance as follows. For any edge $x \sim y$ define first its length $\sigma(x,y)$ by

$$\sigma(x,y) = \frac{1}{\sqrt{\deg(x)}} \wedge \frac{1}{\sqrt{\deg(y)}}.$$

Then, for all $x, y \in X$ define $d(x,y)$ as the smallest total length of all edges in an edge chain connecting x and y. It is easy to verify that d satisfies (14) with $C = 1$.

Next one proves that (17) for ρ-balls implies that the d-balls have at most exponential volume growth, so that the stochastic completeness follows by Theorem 4.

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