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ON THE EXISTENCE OF POSITIVE FUNDAMENTAL SOLUTIONS OF THE LAPLACE EQUATION ON RIEMANNIAN MANIFOLDS

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ABSTRACT. A Riemannian manifold is said to be *parabolic* if there does no exist a positive fundamental solution of the Laplace equation on it. The purpose of this article is to obtain geometric conditions, both necessary and sufficient, for a manifold to be parabolic.

Bibliography: 11 titles.

Introduction

A noncompact Riemannian manifold is said to be *parabolic* if there is no positive fundamental solution of the Laplace equation on it. For example, $\mathbf{R}^2$ is parabolic, while $\mathbf{R}^3$ is not. The question of whether a particular manifold is parabolic is of interest from various points of view. One reason is that fundamental solutions bounded from below are natural, physical. For example, temperature is always bounded from below, as is the transition probability density of a Brownian motion. In the language of the theory of random processes, parabolicity means recurrence of a Brownian motion on the manifold. Parabolicity is also connected with Liouville theorems: A manifold is parabolic if and only if every positive superharmonic (all the more so, harmonic) function is equal to a constant (see [1]). This connection is the basis for one of the approaches to proving theorems of Bernstein type for minimal surfaces, namely, the establishment of parabolicity for the surface under consideration (see [2]). In the monograph [1] parabolicity, along with other properties of harmonic functions, serves for classifying Riemannian manifolds.

Our purpose is to find geometric conditions for parabolicity, both necessary conditions and sufficient conditions. In this direction there is the remarkable paper [3] of Cheng and Yau, where it is proved that if the volume of a geodesic ball of radius r on a complete Riemannian manifold does not grow more rapidly than r^2, then this manifold is parabolic. In particular, this result explains why $\mathbf{R}^2$ is parabolic, in contrast to $\mathbf{R}^3$. Some refinements of the Cheng-Yau theorem are given in [4]. We remark that these authors proved parabolicity in the following formulation: every negative subharmonic function is equal to a constant. The proof used the standard device of multiplying by a suitable compactly supported function, with subsequent estimates.

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In the present article we present another, more geometric approach. It is based on two points: 1) the well-known connection between the Green's function and the Wiener capacity (this connection is used, for example, in [5]; see §1 below), and 2) estimates of the capacity in terms of geometric characteristics of the manifold. These estimates are proved in §§2 and 3. They are used to derive estimates of positive fundamental solutions, along with a necessary condition and (separately) a sufficient condition for parabolicity, formulated as follows.

SUFFICIENT CONDITION. *Suppose that M is a complete manifold, σ_r is the measure of codimension* 1 *of a geodesic sphere of radius r* (it is assumed that the sphere is a smooth surface; there is a more general formulation in §2), *and*

$$\int^{\infty} \frac{dr}{\sigma_r} = \infty. \tag{1}$$

Then the manifold M is parabolic.

This gives us results in [3] and [4] as corollaries.

NECESSARY CONDITION. *If the isoperimetric inequality with a function f is valid on a parabolic manifold of infinite volume* (*i.e., if for every compact set with smooth boundary the measure of the boundary is at least $f(v)$, where v is the volume of the compact set*), *then*

$$\int^{\infty} \frac{dv}{f(v)^2} = \infty. \tag{2}$$

For example, $\sigma_r \sim r^{n-1}$ and $f(v)^2 \sim v^{2-2/n}$ in $\mathbf{R}^n$, and conditions (1) and (2) are equivalent to the condition that $n \leqslant 2$.

There are examples in §4 showing that conditions (1) and (2) are sharp.

We mention finally that the manifold can have a boundary in all the questions under consideration. Here a fundamental solution is assumed to satisfy the homogeneous Neumann condition on the boundary.

The main results in this paper were published in the brief communication [6]. They were reported in part at the Ukrainian Republic conference on nonlinear equations in mathematical physics in 1983 (see [7]) and at the joint sessions of the Moscow Mathematical Society and the Petrovskiĭ seminar in 1984.

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Notation and terminology. Everywhere below, M is a smooth connected noncompact Riemannian manifold, and ∂M is its boundary (possibly empty). If A is a proper subset of M, then $\partial_0 A$ denotes the part $\partial_0 A = \overline{\partial A \cap (M \setminus \partial M)}$ of its boundary. A smooth hypersurface is defined to be a submanifold of codimension 1 that is transversal to ∂M. The words "the set $A \subset M$ has smooth boundary" usually mean that $\partial_0 A$ is a smooth hypersurface. The letter μ stands for the measure on M induced by the Riemannian metric; μ_1 is the measure on submanifolds of codimension 1; $|A|$ stands for μA or $\mu_1 A$, depending on the context. Finally, Δ, ∇, and $\partial/\partial\nu$ are the standard notation for the Laplace operator, the gradient, and the normal derivative in the Riemannian metric (see [8]).

§1. Capacity and fundamental solutions

Suppose that A and B are precompact subsets of M, B is open, and $\bar{A} \subset B$.

DEFINITION. *The capacity of A with respect to B* is defined to be the number

$$\operatorname{cap}(A, B) = \inf_{\varphi} \int_M |\nabla\varphi|^2 \, d\mu,$$

where the infimum is over all Lipschitz functions φ such that $\varphi|_A \geqslant 1$ and $\varphi|_{M\setminus B} \leqslant 0$.

The capacity decreases as B increases in size; therefore the limit

$$\lim_{B\to M} \operatorname{cap}(A, B) = \operatorname{cap} A$$

exists, and is called the *capacity of A* (here $B \to M$ is an exhaustion of M by precompact open sets).

PROPOSITION 1. *Let u be a weak solution in $H_1(B\setminus \bar{A})$ of the problem*

$$\Delta u = 0, \quad u|_{\partial_0 A} = 1, \quad u|_{\partial_0 B} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial M\cap \overline{B\setminus\bar{A}}} = 0. \tag{3}$$

Then

$$\operatorname{cap}(A, B) = \int_{B\setminus\bar{A}} |\nabla u|^2\, d\mu; \tag{4}$$

if D is an open set with smooth boundary $\partial_0 D$, and if $A \subset D \subset B$, then

$$\operatorname{cap}(A, B) = \int_{\partial_0 D} \frac{\partial u}{\partial \nu}\, d\mu_1 \tag{5}$$

(*where the normal ν is directed into D*).

The proof is completely analogous to the case of $\mathbf{R}^n$ (see [5]).

The function u determined in (3) is called *the capacity potential of A with respect to B*.

DEFINITION. Every function $E(x) \in C^1(M\setminus\{O\})$ such that

$$-\Delta E(x) = \delta(x), \qquad \left.\frac{\partial E}{\partial \nu}\right|_{\partial M} = 0,$$

where $\delta(x)$ is the Dirac delta-function with pole at O, is called a *fundamental solution* of the Laplace equation with pole at the point $O \in \mathring{M}$.

Among the positive fundamental solutions with a given pole there is always a smallest one, constructed as follows. For every precompact open set B with smooth boundary we construct a Green's function, i.e., a solution of the problem

$$-\Delta G_B = \delta(x), \quad G_B|_{\partial_0 B} = 0, \quad \left.\frac{\partial G_B}{\partial \nu}\right|_{\partial M\cap\bar{B}} = 0. \tag{6}$$

It follows from the maximum principle that $G_B \leqslant E$ for an arbitrary positive fundamental solution E. As B increases in size, the sequence G_B increases and converges to a function $G \leqslant E$ that is the smallest positive fundamental solution and is called the *Green's function of the manifold M*.

PROPOSITION 2. a) *Suppose that G_B is a solution of problem* (6), *A is an open set containing the point O, and $\bar{A} \subset B$. Then*

$$\max_{\partial_0 A} G_B \geqslant \operatorname{cap}(A, B)^{-1} \geqslant \min_{\partial_0 A} G_B.$$

b) *If M has a Green's function $G(x)$, then*

$$\max_{\partial_0 A} G \geqslant (\operatorname{cap} A)^{-1} \geqslant \min_{\partial_0 A} G.$$

The proof repeats an argument in [5]. Let $B_t = \{x \in B \mid G_B > t\}$. Then

$$\operatorname{cap}(B_t, B) = \int_{\partial_0 B_t} \frac{\partial}{\partial \nu}\left(\frac{G_B}{t}\right) d\mu_1 = \frac{1}{t},$$

since, by (6),

$$\int_{\partial_0 B} \frac{\partial G_B}{\partial \nu} d\mu_1 = 1.$$

Let $a = \max_{\partial_0 A} G$ and $b = \min_{\partial_0 A} G$. It follows from the monotonicity of the capacity and the obvious inclusions $B_a \subset A \subset B_b$ that $1/a = \operatorname{cap}(B_a, B) \leqslant \operatorname{cap}(A, B) \leqslant \operatorname{cap}(B_b, B) = 1/b$.

Assertion b) is obtained by an obvious passage to the limit.

PROPOSITION 3. *The manifold M is parabolic if and only if the capacity of any compact set is equal to zero.*

PROOF. If the capacity of any compact set is equal to zero, then the Green's function cannot exist because of the estimate

$$\max_{\partial_0 A} G \geqslant (\operatorname{cap} A)^{-1} = \infty.$$

If the capacity of some compact set is positive, then there is a precompact open set A of positive capacity containing the pole O. For any $B \supset \bar{A}$

$$\min_{\partial_0 A} G_B \leqslant \operatorname{cap}(A, B)^{-1} \leqslant (\operatorname{cap} A)^{-1};$$

consequently, $\min_{\partial_0 A} G_B$ has a finite limit as $B \to M$. Therefore, the limit $\lim_{B \to M} G_B(x)$ exists at each point $x \neq O$ (by the local Harnacks' inequality [9], for example) and is the Green's function.

COROLLARIES. 1. *Positive fundamental solutions with poles at points $O_1, O_2 \in M$ exist or fail to exist simultaneously.*

2. *If g_1 and g_2 are two Riemannian metrics on M that have finite ratios, then the Riemannian manifolds (M, g_1) and (M, g_2) are or are not parabolic simultaneously.*

Indeed, the g_1- and g_2-capacities have finite ratios by the definition of capacity.

3. *If M is covered by a compact set K and finitely many disjoint open subsets $M_1, \dots, M_s$ with smooth boundaries ∂M_i, then M is parabolic if and only if all the M_i, regarded as manifolds with boundary, are parabolic.*

The proof follows from the relation $\operatorname{cap} A = \sum_{i=1}^{s} \operatorname{cap}_{M_i}(A \cap M_i)$, which is valid for any compact set $A \supset K$.

§2. Sufficient conditions for parabolicity

A function $\rho(x) \in C^\infty(M)$ is called an *exhaustion function* if for any $t \in (-\infty, +\infty)$ the set $\Omega_t = \{x \in M \mid \rho(x) < t\}$ is precompact. Let $S_t = \{x \in M \mid \rho(x) = t\}$. It follows from Sard's theorem that S_t is a smooth hypersurface and $\partial_0 \Omega_t = S_t$ for almost all t. Let

$$P_t = \int_{S_t} |\nabla \rho| \, d\mu_1$$

(the integral is taken to be zero for empty S_t).

THEOREM 1. *If M has an exhaustion function $\rho(x)$ such that*

$$\int^{\infty} \frac{dt}{P_t} = \infty, \tag{7}$$

then M is a parabolic manifold.

THEOREM 2. *If $\rho(x)$ is an exhaustion function and $G(x)$ is a Green's function with pole at the point $O \in \Omega_a$, then*

$$\max_{S_a} G(x) \geqslant \int_a^{\infty} \frac{dt}{P_t}. \tag{8}$$

Theorems 1 and 2 are easy to get from the following lemma and Propositions 2 and 3.

LEMMA 1. *If the set Ω_a is nonempty, then for $b > a$*

$$\operatorname{cap}(\Omega_a, \Omega_b)^{-1} \geqslant \int_a^b \frac{dt}{P_t}$$

and, consequently,

$$(\operatorname{cap} \Omega_a)^{-1} \geqslant \int_0^{\infty} \frac{dt}{P_t}.$$

In particular, if the exhaustion function is such that $|\nabla \rho| \leqslant 1$, then

$$(\operatorname{cap} \Omega_a)^{-1} \geqslant \int_a^{\infty} \frac{dt}{|S_t|}.$$

The following formula for integrating over level surfaces is used in proving the lemma.

PROPOSITION 4. *If $h(x) \in L^1(M, \mu)$, $h \geqslant 0$, then*

$$\int_M h\, d\mu \geqslant \int_{-\infty}^{\infty} dt \left(\int_{S_t} h |\nabla \rho|^{-1}\, d\mu_1 \right) \tag{9}$$

and for almost all t

$$\frac{d}{dt} \int_{\Omega_t} h\, d\mu \geqslant \int_{S_t} h |\nabla \rho|^{-1}\, d\mu_1. \tag{10}$$

See [10] for a proof.

PROOF OF LEMMA 1. Let u be the capacity potential of Ω_a with respect to Ω_b. Then (4) and (9) give us that

$$\operatorname{cap}(\Omega_a, \Omega_b) = \int_{\Omega_b \setminus \Omega_a} |\nabla u|^2\, d\mu \geqslant \int_a^b dt \int_{S_t} |\nabla u|^2 |\nabla \rho|^{-1}\, d\mu_1.$$

By the Cauchy-Schwarz-Bunyakovskiĭ inequality,

$$\left(\int_{S_t} |\nabla u|\, d\mu_1 \right)^2 \leqslant \int_{S_t} |\nabla u|^2 |\nabla \rho|^{-1}\, d\mu_1 \int_{S_t} |\nabla \rho|\, d\mu_1,$$

i.e.,

$$\int_{S_t} |\nabla u|^2 |\nabla \rho|^{-1}\, d\mu_1 \geqslant P_t^{-1} \left(\int_{S_t} |\nabla u|\, d\mu_1 \right)^2.$$

Using the fact

$$\left(\int_{S_t} |\nabla u|\, d\mu_1 \right)^2 \geqslant \left(\int_{S_t} \frac{\partial u}{\partial \nu}\, d\mu_1 \right)^2 = \operatorname{cap}(\Omega_a, \Omega_b)^2,$$

we get that

$$\operatorname{cap}(\Omega_a, \Omega_b) \geqslant \int_a^b P_t^{-1} \operatorname{cap}(\Omega_a, \Omega_b)^2 \, dt,$$

which implies the desired result.

We give some sufficient conditions for parabolicity that follow from Theorem 1.

COROLLARY 1. *If M has a Lipschitz exhaustion function such that*

$$\int^{\infty} \frac{dt}{|S_t|} = \infty, \tag{11}$$

then M is parabolic.

Indeed, it follows from the Lipschitz condition $|\nabla\rho| \leqslant \text{const}$ that

$$P_t = \int_{S_t} |\nabla\rho| \, d\mu_1 \leqslant \text{const}|S_t|. \tag{12}$$

We remark that a complete manifold always has a Lipschitz exhaustion function—for example, it can be obtained by smoothing the distance function (see [11] for corresponding approximation methods).

COROLLARY 2. *Suppose that M is a complete manifold and that for some $O \in M$ the volume V_r of a geodesic ball of radius r about O satisfies the relation*

$$\int^{\infty} \frac{r dr}{V_r} = \infty. \tag{13}$$

Then M is a parabolic manifold.

PROOF. Let $\rho(x)$ be the smoothed distance function to the point O, and let $\varphi(t) = |\Omega_t|$. Obviously,

$$\int^{\infty} \frac{t\, dt}{\varphi(t)} = \infty.$$

It follows from (10) with $h \equiv 1$ that

$$\varphi'(t) = \frac{d}{dt}|\Omega_t| \geqslant \int_{S_t} |\nabla\rho|^{-1} \, d\mu_1 \geqslant \text{const}|S_t|;$$

therefore, by Corollary 1, it suffices for us to prove that

$$\int^{\infty} \frac{dt}{\varphi'(t)} = \infty.$$

By induction on k it is easy to construct a sequence of positive numbers c_k and a sequence of measurable sets $E_k \subset [0, 2k]$ having the following properties: 1) the E_k are disjoint; 2) $\operatorname{meas} E_k = 1$; and 3) $\varphi'(t) \leqslant c_k$ for $t \in E_k$, and $\varphi'(t) \geqslant c_k$ for $t \in F_k = [0, 2k] \setminus (E_1 \cup \cdots \cup E_k)$.

Note that $\operatorname{meas} F_k = k$ by construction. We have that

$$\varphi(2k) \geqslant \int_0^{2k} \varphi'(t)\, dt \geqslant \int_{F_k} c_k \, dt = k c_k.$$

If $t \in [2k, 2k+2]$, then $\varphi(t) \geqslant \varphi(2k) \geqslant k c_k = (k/t) t c_k \geqslant t c_k/4$. Therefore,

$$\int_2^{\infty} \frac{t\, dt}{\varphi(t)} \leqslant 4 \sum_{k=1}^{\infty} \int_{2k}^{2k+2} c_k^{-1} \, dt \leqslant 8 \sum_{k=1}^{\infty} c_k^{-1}; \qquad \sum_{k=1}^{\infty} c_k^{-1} = \infty.$$

From this we get

$$\int_2^\infty \frac{dt}{\varphi'(t)} \geqslant \sum_{k=2}^\infty \int_{E_k} \frac{dt}{\varphi'(t)} \geqslant \sum_{k=2}^\infty c_k^{-1} = \infty.$$

We mention that if we assume a growth of volume such that the integral (13) converges, then it is possible to construct an example of a complete nonparabolic manifold on which this growth is realized (see §4).

COROLLARY 3 (CHENG AND YAU [3]). *If* $\underline{\lim}_{r\to\infty} V_r/r^2 < \infty$ *in the notation of Corollary* 2, *then* M *is parabolic.*

Indeed, if $V_{r_k} \leqslant Cr_k^2$ for some sequence $r_k \uparrow \infty$, then

$$\int_{r_1}^\infty \frac{r\,dr}{V_r} \geqslant \sum_{k=1}^\infty \int_{r_k}^{r_{k+1}} \frac{r\,dr}{Cr_{k+1}^2} = \frac{1}{2C}\sum_{k=1}^\infty \left(1 - \frac{r_k^2}{r_{k+1}^2}\right) = \infty,$$

since the product $\prod_1^\infty r_k^2/r_{k+1}^2$ diverges.

COROLLARY 4. *If a manifold has an exhaustion function that is subharmonic outside some compact set* (*i.e.*, $\Delta\rho \leqslant 0$ *and* $\partial\rho/\partial\nu|_{\partial M} \geqslant 0$, *where* ν *is the outward normal to* ∂M), *then it is parabolic.*

Indeed, using Green's formula, we see that the flow decreases; in particular, (7) holds.

§3. A necessary condition for parabolicity

Suppose that the isoperimetric inequality with function $f(v)$ holds on M, i.e., $|\partial_0 K| \geqslant f(|K|)$ for any compact set K with smooth boundary $\partial_0 K$.

THEOREM 3. *If the isoperimetric function* f *satisfies the condition*

$$\int^{|M|} \frac{dv}{f(v)^2} < \infty, \tag{14}$$

then M *has a Green's function.*

THEOREM 4. *If* M *satisfies the conditions of Theorem* 3, Ω *is a precompact open subset of* M, *and* $G(x)$ *is a Green's function with pole at a point* $O \in \Omega$, *then*

$$\min_{\overline{\Omega}} G \leqslant \int_{|\Omega|}^{|M|} \frac{dv}{f(v)^2}. \tag{15}$$

Theorems 3 and 4 can be proved from the following estimate of the capacity in terms of the measure.

LEMMA 2. *If* A *and* B *are open precompact subsets of* M *and* $\overline{A} \subset B$, *then*

$$\operatorname{cap}(A, B)^{-1} \leqslant \int_{|A|}^{|\overline{B}|} \frac{dv}{f(v)^2},$$

and, consequently,

$$(\operatorname{cap} A)^{-1} \leqslant \int_{|A|}^{|M|} \frac{dv}{f(v)^2}.$$

We give a proof of Lemma 2 for the case of smooth boundaries $\partial_0 A$ and $\partial_0 B$. In the general case one should approximate A from the inside and B from the outside by domains with smooth boundaries.

Let u be the capacity potential of A with respect to B. Let $B_t = \{x \mid u(x) > t\} \cup A$. It follows from (10) that for almost all t

$$-\frac{d}{dt}|B_t| \geqslant \int_{\partial_0 B_t} |\nabla u|^{-1}\, d\mu_1 \geqslant \frac{|\partial_0 B_t|^2}{\int_{\partial_0 B_t} |\nabla u|\, d\mu_1}.$$

By (5),

$$\int_{\partial_0 B_t} |\nabla u|\, d\mu_1 = \int_{\partial_0 B_t} \frac{\partial u}{\partial \nu}\, d\mu_1 = \operatorname{cap}(A, B),$$

and $|\partial_0 B_t| \geqslant f(|B_t|)$ by the isoperimetric inequality. Therefore,

$$-\frac{d}{dt}|B_t| \geqslant f(|B_t|)^2 \operatorname{cap}(A, B)^{-1}, \qquad \operatorname{cap}(A, B)^{-1} \leqslant \frac{1}{f(|B_t|)^2}\left(-\frac{d}{dt}|B_t|\right).$$

Integrating the last inequality with respect to dt from 0 to 1 and making the substitution $v = |B_t|$ in the integral, we get that

$$\operatorname{cap}(A, B)^{-1} \leqslant \int_{|B_1|}^{|B_0|} \frac{dv}{f(v)^2} = \int_{|A|}^{|B|} \frac{dv}{f(v)^2}.$$

REMARK. For complete manifolds condition (14) can be satisfied only if $|M| = \infty$ (by the Cheng-Yau theorem, a complete manifold of finite volume is parabolic). In this case condition (14) is sharp in the following sense. If $f(v)$ is a function such that the integral (14) diverges (and satisfies some regularity conditions), then there exists a complete parabolic manifold on which the isoperimetric inequality with function const $f(v)$ holds for large v (see §4).

§4. Examples

Let $\varphi(t)$ be a positive smooth upwards convex function defined on $(0, +\infty)$. Denote by M_φ the solid of revolution determined by rotating φ about the x_n-axis in $\mathbf{R}^n$. We regard M_φ as a manifold with boundary (the boundary is smoothed at the origin), and endow it with an arbitrary Riemannian metric whose ratios with the Euclidean metric are finite. Consider the exhaustion function $\rho(x) = x_n$ on the manifold M_φ. Then $|S_t| \asymp \varphi(t)^{n-1}$ and $|\Omega_t| \asymp t\varphi(t)^{n-1}$ as $t \to \infty$ in the notation of §2 (where the sign $\asymp$ means "has finite ratios with").

The sufficient conditions (11) and (13) for parabolicity are equivalent to the condition

$$\int^\infty \frac{dt}{\varphi(t)^{n-1}} = \infty. \tag{16}$$

It can be proved that the manifold M_φ satisfies the isoperimetric inequality with the function const $f(v)$, where f is determined from the condition

$$f\left(t\varphi(t)^{n-1}\right) = \varphi(t)^{n-1} \tag{17}$$

(for sufficiently large t).

We show that the sufficient condition (16) for parabolicity is equivalent in this case to the necessary condition

$$\int^\infty \frac{dv}{f(v)^2} = \infty. \tag{18}$$

Indeed, let $v = t\varphi(t)^{n-1}$. Then

$$f(v) = \varphi(t)^{n-1}, \qquad dv = \varphi^{n-1}\,dt + t(n-1)\varphi^{n-2}\varphi'\,dt,$$

$$\varphi^{n-1}\,dt \leqslant dv \leqslant n\varphi^{n-1}dt, \qquad \frac{dt}{\varphi^{n-1}} \leqslant \frac{dv}{f(v)^2} \leqslant n\frac{dt}{\varphi(t)^{n-1}},$$

which implies that (16) and (18) are equivalent.

What has been proved implies that both the sufficient conditions (11) and (13) and the necessary condition (18) are sharp. Indeed, if f is a function such that

$$\int^{\infty} \frac{dv}{f(v)^2} = \infty$$

(and such that certain regularity conditions hold under which the function $\varphi(t)$ determined from (17) is convex for sufficiently large n), then M_φ (where φ is determined from (17)) is parabolic and has isoperimetric function const f.

It can be proved similarly that (11) and (13) are sharp.

As an example of the application of Theorems 2 and 4 we get estimates of the Green's function $G(x)$ of M_φ in the case when M_φ is not parabolic. We remark that in Euclidean coordinates G is a fundamental solution of the uniformly elliptic equation in divergence form with conormal condition on the boundary ∂M_φ.

Let

$$M(t) = \max_{x_n = t} G(x), \qquad m(t) = \min_{x_n = t} G(x).$$

Then it follows from Theorems 2 and 4 that

$$M(t) \geqslant \text{const} \int_t^{\infty} \frac{dt}{\varphi(t)^{n-1}}, \qquad m(t) \leqslant \text{const} \int_t^{\infty} \frac{dt}{\varphi(t)^{n-1}}$$

(for large t). After using Harnack's inequality [9] and the convexity of M_φ, we get that $M(t) \leqslant \text{const}\, m(t)$. Consequently,

$$G(x) \asymp \int_{x_n}^{\infty} \frac{dt}{\varphi(t)^{n-1}} \qquad (x_n \to \infty).$$

In particular, $G(x) \asymp x_n^{1-(n-1)\gamma}$ for $\varphi(t) = t^\gamma$ $(1 \geqslant \gamma > 1/(n-1))$.

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